

Sparse Eliminants vs Sparse resultants

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Algebra, Geometry and Proofs in Symbolic Computation
Toronto, Dec 2015



Computational Algebra, Algebraic Geometry & Applications



A Conference in honor of Alicia Dickenstein
Buenos Aires, Argentina, August 1–3 2016
<http://mate.dm.uba.ar/~coalaga/>



Elimination Theory

Important for both **algorithmic** and **complexity** aspects of polynomial system solving



The Example: Determinants

Find “the condition” on
 $a_{00}, a_{01}, a_{10}, a_{11}$ so that the system

$$\begin{cases} a_{00}x_0 + a_{01}x_1 = 0 \\ a_{10}x_0 + a_{11}x_1 = 0 \end{cases}$$

has a solution different from $(0, 0)$

Elimination Theory

$$a_{00}x_0 + a_{01}x_1, a_{10}x_0 + a_{11}x_1$$

$$\in \mathbb{K}[a_{00}, a_{01}, a_{10}, a_{11}, x_0, x_1]$$

↓

$$a_{00}a_{11} - a_{10}a_{01} \in \mathbb{K}[a_{00}, a_{01}, a_{10}, a_{11}]$$

Another more general

Let $d_0, d_1 \in \mathbb{N}$. Find “the condition”
for the system of polynomials

$$\begin{cases} a_{00}x_0^{d_0} + a_{01}x_0^{d_0-1}x_1 + \dots = 0 \\ a_{10}x_0^{d_1} + a_{11}x_0^{d_1-1}x_1 + \dots = 0 \end{cases}$$

to have a solution different from
 $(0, 0)$

Elimination: The general problem

For $\mathbf{a} = (a_1, \dots, a_N)$, $k, n \in \mathbb{N}$ let
 $f_1(\mathbf{a}, x_1, \dots, x_n), \dots, f_k(\mathbf{a}, x_1, \dots, x_n) \in$
 $\mathbb{K}[\mathbf{a}, x_1, \dots, x_n]$. Find conditions on \mathbf{a} such that

$$\left\{ \begin{array}{l} f_1(\mathbf{a}, x_1, \dots, x_n) = 0 \\ f_2(\mathbf{a}, x_1, \dots, x_n) = 0 \\ \vdots \\ f_k(\mathbf{a}, x_1, \dots, x_n) = 0 \end{array} \right.$$

has a solution

Solution?

- Depends on the ground field

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- There is not necessarily a “closed” condition

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- Depends on the ground field
- There is not necessarily a “closed” condition
- The computation of the conditions may be out of control

Easy example

$$\left\{ \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{kn}x_1 + \dots + a_{kn}x_n = 0 \end{array} \right.$$

with $k \geq n$

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with $k \geq n$

Conditions: all maximal minors of $(a_{ij})_{1 \leq i \leq k, 1 \leq j \leq n}$
equal to zero

Another “easy” example

$$k = n = 1,$$

$$a_0 + a_1x_1 + a_2x_1^2 + \dots + a_dx_1^d = 0$$

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Conditions?

Geometry

$$V = \{(\mathbf{a}, x_1, \dots, x_n) : f_1(\mathbf{a}, x_1, \dots, x_n) = 0, \dots, f_k(\mathbf{a}, x_1, \dots, x_n) = 0\}$$

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$$V \subset \mathbb{K}^N \times \mathbb{K}^n$$

$$\downarrow \pi_1$$

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$$\pi_1(V) \subset \mathbb{K}^N$$

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The set of conditions is $\pi_1(V)$, not necessarily described by zeroes of polynomials

Elimination Theorem

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$$\pi_1(V) = \{p_1(\mathbf{a}) = 0, \dots, p_\ell(\mathbf{a}) = 0\}$$

“The” Condition

$$V = \{(\mathbf{a}, x_0, x_1, \dots, x_n) : f_1(\mathbf{a}, x_0, x_1, \dots, x_n) = 0, \dots, f_{n+1}(\mathbf{a}, x_0, x_1, \dots, x_n) = 0\}$$

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Example 2

$$\begin{cases} f_0 = a_{00}x_0^{d_0} + a_{01}x_0^{d_0-1}x_1 + \dots + a_{0d_0}x_1^{d_0} \\ f_1 = a_{10}x_0^{d_1} + a_{11}x_0^{d_1-1}x_1 + \dots + a_{1d_1}x_1^{d_1} \end{cases}$$

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$$\text{Res}(f_0, f_1) = \det \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0d_0} & 0 & \dots \\ 0 & a_{00} & \dots & a_{0d_0-1} & a_{0d_0} & \dots \\ \vdots & \vdots & \ddots & \dots & \dots & \ddots \\ 0 & 0 & \dots & a_{00} & \dots & a_{0d_0} \\ a_{10} & a_{11} & \dots & a_{1d_1} & 0 & \dots \\ 0 & a_{10} & \dots & a_{1d_0-1} & a_{1d_1} & \dots \\ \vdots & \vdots & \ddots & \dots & \dots & \ddots \\ 0 & 0 & \dots & a_{10} & \dots & a_{1d_1} \end{pmatrix}$$

Example 3

$$\left\{ \begin{array}{l} f_0 = \sum_{\alpha_0 + \dots + \alpha_n = d_0} \mathbf{a}_{0, \alpha_0, \dots, \alpha_n} X_0^{\alpha_0} \dots X_n^{\alpha_n} \\ f_1 = \sum_{\alpha_0 + \dots + \alpha_n = d_1} \mathbf{a}_{1, \alpha_0, \dots, \alpha_n} X_0^{\alpha_0} \dots X_n^{\alpha_n} \\ \vdots \\ f_n = \sum_{\alpha_0 + \dots + \alpha_n = d_n} \mathbf{a}_{n, \alpha_0, \dots, \alpha_n} X_0^{\alpha_0} \dots X_n^{\alpha_n} \end{array} \right.$$

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$$\text{Res}(f_0, f_1, \dots, f_n)$$

“Univariate” Resultant

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What is the resultant of f_0 and f_1 ?



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Algebraic Definitions

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- the “condition” such that $\text{gcd}(f_0, f_1) = 1$
(Sylvester)
- The determinant of the “Sylvester” Matrix
(Sylvester)
- One of the (two) generators of the ideal

$$\langle f_0(x_0, 1), f_1(x_0, 1) \rangle \cap \mathbb{Z}[a_0, \dots, a_m, b_0, \dots, b_k]$$

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- bihomogeneous of bidegree (d_1, d_0)
- weighted homogeneous of degree $d_0 d_1$
- One of its terms is $a_{00}^{d_1} a_{1d_1}^{d_0}$

Algebra meets Geometry

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$$\text{Res}(f_0, f_1) = 0$$



$$\exists p \in \mathbb{P}^1 \mid f_0(p) = f_1(p) = 0$$

Poisson's formula and additivity

$$\begin{aligned}\operatorname{Res}(f_0, f_1) &= a_{0d_0}^{d_1} \prod_{f_0(\xi)=0} f_1(\xi) \\ &= \pm a_{1d_1}^{d_0} \prod_{f_1(\nu)=0} f_0(\nu)\end{aligned}$$

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$$\text{Res}(f_0 \cdot f_0', f_1) = \text{Res}(f_0, f_1) \cdot \text{Res}(f_0', f_1)$$

Determinantal Formulae

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$$\text{Res}(f_0, f_1) = \det \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0d_0} & 0 & \dots \\ 0 & a_{00} & \dots & a_{0d_0-1} & a_{0d_0} & \dots \\ \vdots & \vdots & \ddots & \dots & \dots & \ddots \\ 0 & 0 & \dots & a_{00} & \dots & a_{0d_0} \\ a_{10} & a_{11} & \dots & a_{1d_1} & 0 & \dots \\ 0 & a_{10} & \dots & a_{1d_0-1} & a_{1d_1} & \dots \\ \vdots & \vdots & \ddots & \dots & \dots & \ddots \\ 0 & 0 & \dots & a_{10} & \dots & a_{1d_1} \end{pmatrix}$$

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■ Bézout's formulas

Determinantal Formulae

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- Bézout's formulas
- Hybrid formulas

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■ ...

General Resultants

$$\left\{ \begin{array}{l} f_0 = \sum_{\alpha_0 + \dots + \alpha_n = d_0} \mathbf{a}_{0, \alpha_0, \dots, \alpha_n} X_0^{\alpha_0} \dots X_n^{\alpha_n} \\ f_1 = \sum_{\alpha_0 + \dots + \alpha_n = d_1} \mathbf{a}_{1, \alpha_0, \dots, \alpha_n} X_0^{\alpha_0} \dots X_n^{\alpha_n} \\ \vdots \\ f_n = \sum_{\alpha_0 + \dots + \alpha_n = d_n} \mathbf{a}_{n, \alpha_0, \dots, \alpha_n} X_0^{\alpha_0} \dots X_n^{\alpha_n} \end{array} \right.$$

General Resultants

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$$\boxed{\text{Res}(f_0, f_1, \dots, f_n)}$$

Known case

If $d_0 = d_1 = \dots = d_n = 1$, then

$$\text{Res}(f_0, \dots, f_n) = \pm \det (a_{ij})_{0 \leq i, j \leq n}$$

A non trivial example

$$f_0 = a_{00}x_0 + a_{01}x_1 + a_{02}x_2$$

$$f_1 = a_{10}x_0 + a_{11}x_1 + a_{12}x_2$$

$$f_2 = a_{20}x_0^2 + a_{21}x_0x_1 + a_{22}x_0x_2 + a_{23}x_1^2 + a_{24}x_1x_2 + a_{25}x_2^2$$

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$$\begin{aligned} \text{Res}(f_0, f_1, f_2) = & a_{00}^2 a_{11}^2 a_{25} - a_{00}^2 a_{11} a_{12} a_{24} + a_{00}^2 a_{12}^2 a_{23} \\ & - 2a_{00} a_{01} a_{10} a_{11} a_{25} + a_{00} a_{01} a_{10} a_{12} a_{24} \\ & + a_{00} a_{01} a_{11} a_{12} a_{22} - a_{00} a_{01} a_{12}^2 a_{21} + a_{00} a_{02} a_{10} a_{11} a_{24} \\ & - 2a_{00} a_{02} a_{10} a_{12} a_{23} - a_{00} a_{02} a_{11}^2 a_{22} + a_{00} a_{02} a_{11} a_{12} a_{21} \\ & + a_{01}^2 a_{10}^2 a_{25} - a_{01}^2 a_{10} a_{12} a_{22} + a_{01}^2 a_{12}^2 a_{20} \\ & - a_{01} a_{02} a_{10}^2 a_{24} + a_{01} a_{02} a_{10} a_{11} a_{22} + a_{01} a_{02} a_{10} a_{12} a_{21} \\ & - 2a_{01} a_{02} a_{11} a_{12} a_{20} + a_{02}^2 a_{10}^2 a_{23} - a_{02}^2 a_{10} a_{11} a_{21} \\ & + a_{02}^2 a_{11}^2 a_{20} \end{aligned}$$

Properties of $\text{Res}(f_0, \dots, f_n)$

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Properties of $\text{Res}(f_0, \dots, f_n)$

- It is irreducible
- It is homogeneous in each group of variables, of degree $\frac{d_0 \cdot d_1 \cdot \dots \cdot d_n}{d_{i_0}}$
- It is “weighted” homogeneous with weight $d_0 \cdot d_1 \cdot \dots \cdot d_n$

More Properties

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- $\text{Res}(f_0, \dots, f_n) = 0 \iff \exists \xi \in \mathbb{P}^n$
such that $f_0(\xi) = \dots = f_n(\xi) = 0$

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such that $f_0(\xi) = \dots = f_n(\xi) = 0$
- Poisson Formula:

$$\begin{aligned} & \text{Res}(f_0, \dots, f_n) \\ & = \\ & \text{Res}(f_1^0, \dots, f_n^0)^{d_0} \prod_{\xi \in V(f_1^1, \dots, f_n^1)} f_0(\xi) \end{aligned}$$

Resolution of systems of polynomials

$$P(u_0, u_i) = \text{Res}(u_i x_0 - u_0 x_i, f_1, \dots, f_n)$$

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$$P(u_0, u_i) = \text{Res}(u_i x_0 - u_0 x_i, f_1, \dots, f_n)$$

can be used to compute the
coordinates of the (finite) roots of
the system

$$f_1 = 0, \dots, f_n = 0$$

Computation

$$\mathcal{R}(f_0, f_1, f_2) = \det \begin{bmatrix} -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 \\ 0 & -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 \\ -c_4 & -c_5 & -c_8 & 0 & 0 & 0 & a_1 & 0 & a_3 & 0 \\ -c_1 & -c_3 & -c_7 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ -c_0 & -c_2 & -c_6 & 0 & 0 & 0 & 0 & a_0 & 0 & a_2 \\ 0 & 0 & 0 & -c_4 & -c_5 & -c_8 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & -c_1 & -c_3 & -c_7 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -c_0 & -c_2 & -c_6 & 0 & b_0 & 0 & b_2 \end{bmatrix}$$

A more general situation

In the “real world” systems of equations are neither homogeneous

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In the “real world” systems of equations are neither homogeneous nor all the monomials appear in the expansion

$$\begin{cases} f_0 = a_{01} + a_{02}x_1^2 x_2^2 + a_{03}x_1 x_2^3 \\ f_1 = a_{10} + a_{11}x_1^2 + a_{12}x_1 x_2^2 \\ f_2 = a_{20}x_1^3 + a_{21}x_1 x_2 \end{cases}$$

Sparse Resultants

- $\mathcal{A}_0, \dots, \mathcal{A}_n \subset \mathbb{Z}^n$
- For $i = 0, \dots, n$, $f_i = \sum_{\mathbf{a} \in \mathcal{A}_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}$

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What is $\text{Res}(f_0, \dots, f_n)$?

Sparse resultants

Sparse resultants

- A generator of

$$\langle f_0, \dots, f_n \rangle \cap \mathbb{Z}[c_{i,\mathbf{a}}, i = 0, \dots, n]?$$

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$$\langle f_0, \dots, f_n \rangle \cap \mathbb{Z}[c_{i,\mathbf{a}}, i = 0, \dots, n]?$$

- The defining equation of

$$\begin{array}{ccc} W = \{(\mathbf{c}_{i,\mathbf{a}}, \mathbf{x}) : f_i = 0\} & \subset & \mathbb{K}^N \times (\mathbb{C}^\times)^n \\ \downarrow & & \downarrow \pi_1 \\ \pi_1(W) & \subset & \mathbb{K}^N \end{array} \quad ?$$

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- $\langle f_0, f_1, f_2 \rangle \cap \mathbb{Z}[c_{ij}]$ is generated by the determinant of the coefficients
- $\pi(W)$ has codimension 1 **but** $\pi|_W$ is **not birational anymore**

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- **But you do not get neither satisfactory Poisson nor additivity formulae**

“The” Sparse Resultant

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(D-Sombra), PLMS 2015

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The sparse resultant of f_0, \dots, f_n is the defining equation of the direct image $\pi_* W$

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With this definition, Poisson's formula works!

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- Classically: $\text{Res}_{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2} = \det(c_{ij})$
- With the new definition, $\text{Res}_{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2} = \det(c_{ij})^4$

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- $\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n} = \prod_v \text{Res}_{\mathcal{A}_{1,v}, \dots, \mathcal{A}_{n,v}}^{-h_{\mathcal{A}_0}(v)} \cdot \left(\prod_{\xi} f_0(\xi)^{m_{\xi}} \right)$
 $\xi \in V(f_1, \dots, f_n)$

Algebra meets Geometry

(if $\pi(W)$ has codimension one)

$$\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}(f_0, \dots, f_n) = 0$$



$$\exists p \in X_{\mathcal{A}} \mid f_0(p) = \dots = f_n(p) = 0$$

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$X_{\mathcal{A}}$ is the toric variety defined by $\mathcal{A}_0, \dots, \mathcal{A}_n$ with respect to $(\mathbb{C}^\times)^n$

More properties

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- $\deg_{C_{i,a}} (\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}) = MV(Q_0, \dots, \widehat{Q}_i, \dots, Q_n)$

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 $Q_i = \text{chull}(\mathcal{A}_i)$
- “hidden variables”:

$$\text{Res}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{x_n}(f_1, \dots, f_n) = \pm x_n^d \text{Res}_{\{\mathbf{0}, \mathbf{e}_n\}, \mathcal{A}_1, \dots, \mathcal{A}_n}(z - x_n, f_1, \dots, f_n) \Big|_{z=x_n}$$

for some $d \in \mathbb{Z}$

And there is more...

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(**D**-Jeronimo-Sombra... In progress)

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of $\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}$

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For $\omega \in \prod_{i=0}^n \mathbb{R}^{\mathcal{A}_i}$, $\omega = (\omega_{i,a})$

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$$\text{init}_{\omega}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}) = \pm \prod_{\mathbf{v}} \text{Res}_{\mathcal{A}_0, \mathbf{v}, \dots, \mathcal{A}_n, \mathbf{v}}(f_{0, \mathbf{v}} \dots, f_{n, \mathbf{v}}),$$

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the product is over all primitive $\mathbf{v} \in \mathbb{Z}^{n+1}$ inner normals to a facet of the lower envelope of the “lifted” polytopes $Q_{0, \omega_0} + \dots + Q_{n, \omega_n}$

Example

$$\mathcal{A}_0 = \{(0, 0), (1, 3), (2, 2)\}, \mathcal{A}_1 = \{(0, 0), (1, 2), (2, 0)\},$$

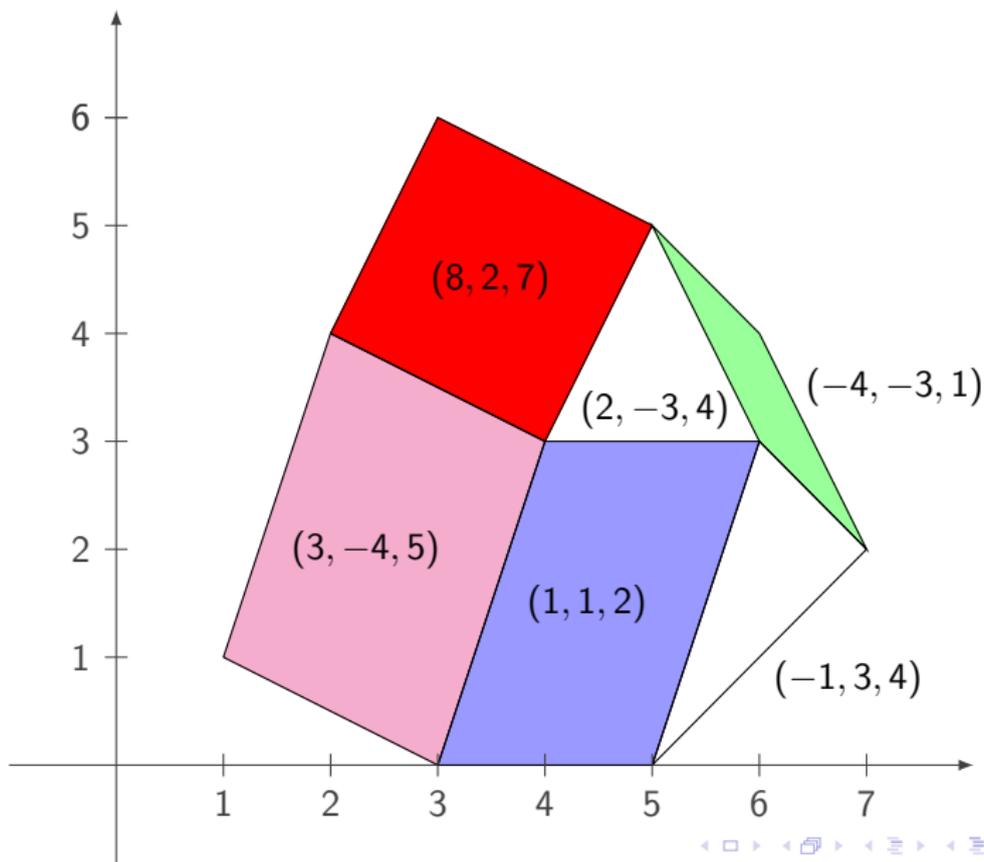
$$\mathcal{A}_2 = \{(1, 1), (3, 0)\}$$

$$\omega = (\omega_0, \omega_1, \omega_2), \omega_0 = (1, -1, 0), \omega_1 = (0, 1, -1), \omega_2 = (1, -1)$$

$$\begin{aligned} \text{Res}_{\mathcal{A}} = & -u_{1,12} u_{1,00} u_{0,22} u_{0,13}^2 u_{1,20}^5 u_{2,11}^5 u_{2,30}^2 u_{0,00}^2 + 3 u_{1,12}^3 u_{0,22}^2 u_{1,20}^4 u_{2,11}^5 u_{2,30}^2 \\ & + 5 u_{1,12}^3 u_{1,00}^4 u_{0,13}^2 u_{0,22} u_{2,11} u_{2,30}^6 u_{0,00}^2 - 7 u_{1,12} u_{1,00}^5 u_{0,13}^4 u_{1,20} u_{2,11} u_{2,30}^6 u_{0,00} \\ & + 2 u_{1,12} u_{1,00}^4 u_{0,13}^2 u_{0,22}^2 u_{1,20}^3 u_{2,11}^4 u_{2,30} u_{0,00} - 2 u_{1,12} u_{1,00}^3 u_{0,22}^4 u_{1,20}^3 u_{2,11}^5 u_{2,30}^2 u_{0,00} \\ & + u_{1,12}^7 u_{2,11} u_{2,30}^6 u_{0,00}^5 - 13 u_{0,13} u_{0,22} u_{1,00}^2 u_{1,12}^4 u_{1,20} u_{2,11}^2 u_{2,30}^5 u_{0,00}^3 \\ & - 2 u_{0,13}^3 u_{0,22} u_{1,00}^3 u_{1,20}^4 u_{2,11}^3 u_{2,30}^3 u_{0,00} + u_{1,12} u_{1,00}^6 u_{0,22}^5 u_{2,11}^3 u_{2,30}^4 + \\ & 6 u_{1,12}^3 u_{1,00}^3 u_{0,22}^3 u_{1,20} u_{2,11}^3 u_{2,30}^4 u_{0,00}^2 - 7 u_{1,12}^3 u_{1,00}^2 u_{0,13}^3 u_{1,20} u_{2,11}^3 u_{2,30}^4 u_{0,00}^3 + u_{1,12}^5 \\ & + u_{1,12} u_{0,22}^3 u_{1,20}^6 u_{2,11}^7 u_{0,00}^2 - 5 u_{0,13} u_{0,22}^5 u_{1,00}^3 u_{1,12}^2 u_{2,11}^5 u_{2,30} u_{0,00} \\ & + u_{0,13}^3 u_{0,22}^2 u_{1,00}^6 u_{1,20} u_{2,11}^2 u_{2,30}^5 + 14 u_{0,13}^3 u_{1,00}^3 u_{1,12}^2 u_{1,20}^2 u_{2,11}^5 u_{2,30}^2 u_{0,00} \\ & - u_{0,13} u_{0,22}^2 u_{1,00}^2 u_{1,12}^3 u_{1,20}^4 u_{2,11}^3 u_{2,30}^2 u_{0,00}^2 + u_{0,13}^3 u_{1,20}^7 u_{2,11}^6 u_{2,30} u_{0,00}^2 \\ & + 3 u_{1,12}^5 u_{0,22} u_{1,20}^2 u_{2,11}^3 u_{2,30}^4 u_{0,00}^4 \end{aligned}$$

$$\text{init}_\omega(\text{Res}_A) = u_{0,13}^5 u_{1,00}^7 u_{2,30}^7$$

\mathbf{v}	$\text{Res}_{\mathcal{A}_{0,\mathbf{v}}, \mathcal{A}_{1,\mathbf{v}}, \mathcal{A}_{2,\mathbf{v}}}(F_{0,\mathbf{v}}, F_{1,\mathbf{v}}, F_{2,\mathbf{v}})$
(1, 1, 2)	$u_{2,30}^6$
(-4, -3, 1)	$u_{2,30}^1$
(3, -4, 5)	$u_{1,00}^7$
(8, 2, 7)	$u_{0,13}^5$
(2, -3, 4)	1
(-1, 3, 4)	1



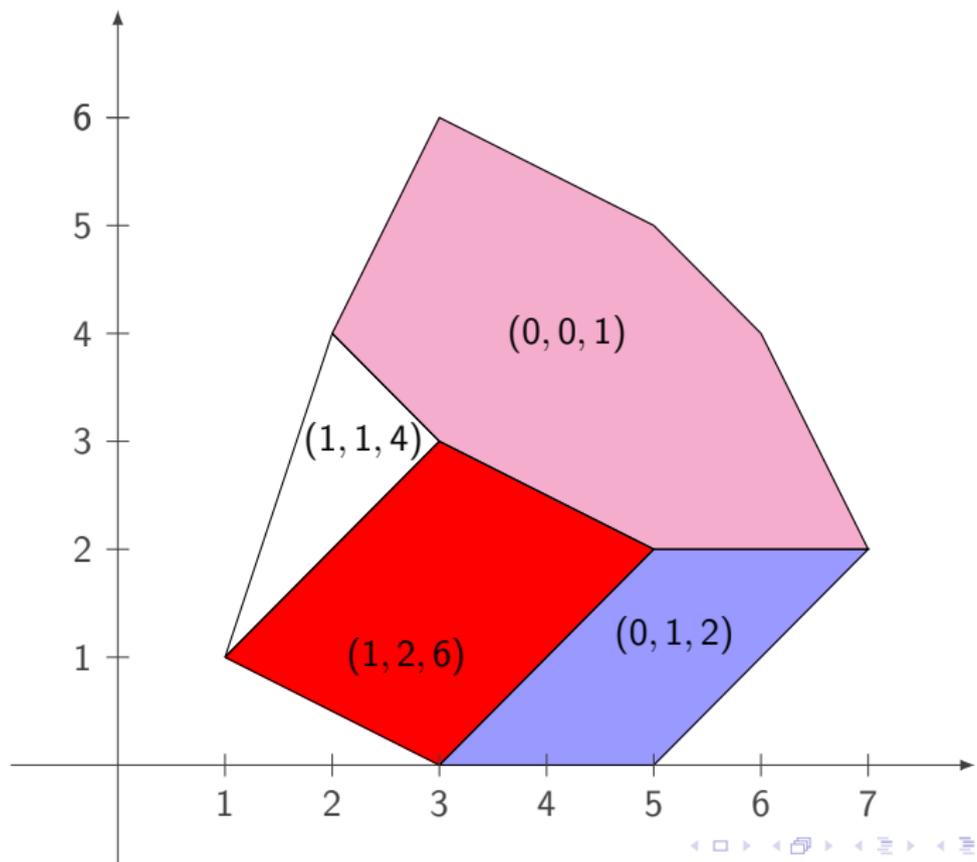
Changing the weight

$$\omega_0 = (1, 0, 0), \omega_1 = (0, 0, 0), \omega_2 = (0, 0)$$

$$\text{init}_\omega(\text{Res}_A)$$

$$= u_{1,0}^6 u_{2,30}^4 (u_{1,00} u_{0,13}^5 u_{2,30}^3 + u_{0,13}^3 u_{0,22}^2 u_{1,20} u_{2,11}^2 u_{2,30} + u_{2,11}^3 u_{0,22}^5 u_{1,12})$$

\mathbf{v}	$\text{Res}_{A_{0,\mathbf{v}}, A_{1,\mathbf{v}}, A_{2,\mathbf{v}}}(F_{0,\mathbf{v}}, F_{1,\mathbf{v}}, F_{2,\mathbf{v}})$
$(0, 0, 1)$	$u_{1,00} u_{0,13}^5 u_{2,30}^3 + u_{0,13}^3 u_{0,22}^2 u_{1,20} u_{2,11}^2 u_{2,30} + u_{2,11}^3 u_{0,22}^5 u_{1,12}$
$(1, 2, 6)$	$u_{1,00}^6$
$(0, 1, 2)$	$u_{2,30}^4$
$(1, 1, 4)$	1



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$$\omega_{i,a} = \begin{cases} 0 & \text{if } a \in \tilde{\mathcal{A}}_i, \\ 1 & \text{otherwise} \end{cases}$$

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$$\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}(\tilde{f}_0, \dots, \tilde{f}_n) = \pm \text{Res}_{\tilde{\mathcal{A}}_0, \dots, \tilde{\mathcal{A}}_n}(\tilde{f}_0, \dots, \tilde{f}_n) \cdot \prod_{\mathbf{v} \neq (\mathbf{0}, 1)} \text{Res}_{\mathcal{A}_{0,\mathbf{v}}, \dots, \mathcal{A}_{n,\mathbf{v}}}(\tilde{f}_{0,\mathbf{v}}, \dots, \tilde{f}_{n,\mathbf{v}})$$

Moreover...

For $0 \leq i \leq n$, $\sigma_i : Q_i \rightarrow \mathbb{R}$ is the “floor” function
of Q_{i,ω_i} above Q_i

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$$\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}(\tilde{f}_0, \dots, \tilde{f}_n) \neq 0 \iff Ml_n(-\sigma_0, \dots, -\sigma_n) = 0$$

If this is the case, then

$$\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}(\tilde{f}_0, \dots, \tilde{f}_n) = \pm \text{Res}_{\tilde{\mathcal{A}}_0, \dots, \tilde{\mathcal{A}}_n} \cdot \prod_{\mathbf{v} \neq (\mathbf{0}, 1)} \text{Res}_{\mathcal{A}_0, \mathbf{v}, \dots, \mathcal{A}_n, \mathbf{v}}$$

More Homogeneities

$$\begin{aligned} \text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n} (f_0(\lambda_1 x_1, \dots, \lambda_n x_n), \dots, f_n(\lambda_1 x_1, \dots, \lambda_n x_n)) \\ = \\ \lambda_1^{A_1} \dots \lambda_n^{A_n} \cdot \text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n} \end{aligned}$$

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with

$$A_i = MV_{n+1}(Q_{0,i}, \dots, Q_{n,i})$$

$$Q_{j,i} = \text{chull}(\{(\mathbf{a}, a_i), (\mathbf{a}, 0) \mid \mathbf{a} \in Q_j\}) \subset \mathbb{R}^{n+1} \\ (\text{if } Q_j \subset (\mathbb{R}_{\geq 0})^n)$$



“Determinantal” formulae

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Reformulation of the Macaulay style formulae given in (D - 2002)

$$\mathcal{R}(f_0, f_1, f_2) = \det \begin{bmatrix} -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 \\ 0 & -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 \\ -c_4 & -c_5 & -c_6 & 0 & 0 & 0 & a_1 & 0 & a_3 & 0 \\ -c_1 & -c_3 & -c_7 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ -c_0 & -c_2 & -c_6 & 0 & 0 & 0 & 0 & a_0 & 0 & a_2 \\ 0 & 0 & 0 & -c_4 & -c_5 & -c_6 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & -c_1 & -c_3 & -c_7 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -c_0 & -c_2 & -c_6 & 0 & b_0 & 0 & b_2 \end{bmatrix}$$

Thanks!



<http://mate.dm.uba.ar/~coalaga/>