Equations for the Flex locus of a hypersurface

Laurent Busé, **Carlos D'Andrea**, Martín Sombra, Martin Weimann

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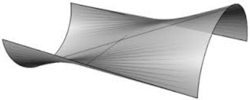
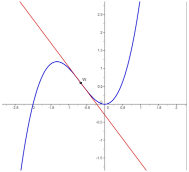
UNIVERSITAT DE BARCELONA



Flexes of curves

Let $C \subset \mathbb{K}^2$ be a graph of a function



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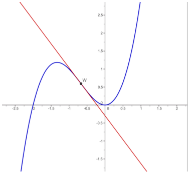

A **flex** or **inflexion** point of C is a point $p \in C$ such that its tangent line has contact order ≥ 3

In general...

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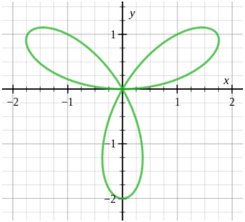

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$$\mathrm{Flex}(C) = \{f = H_f = 0\} \subset \mathbb{P}^2$$

- $f = f(x_0, x_1, x_2)$ the homogeneous irreducible equation of C
- $H_f =$ the Hessian of f

As a consequence...

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If a degree d curve C does not contain any line,

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If a degree d curve C does not contain any line, its flex locus has $d(3d-6)$ points counted with multiplicities

Surfaces

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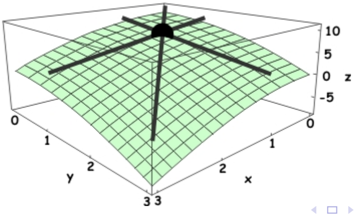
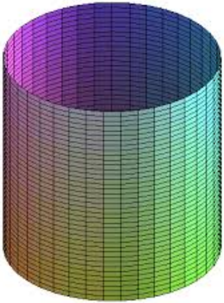


Warning

$p \in \mathrm{Flex}(S)$ if there is a line through p contained in S



Questions on Flex(S)

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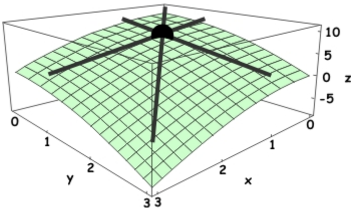
- Equations?



A bit of history

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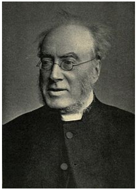
Rev. George Salmon



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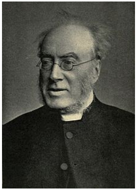

A Treatise on the analytic geometry of three dimensions

Longmans, Green & Co., 1862

Theorem (Salmon, 1862)

For a reduced $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ of degree d

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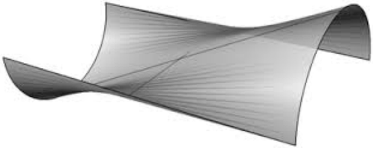
For a reduced $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ of degree d there exists $P_f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ of degree $\leq 11d - 24$ defining $\mathrm{Flex}(V(f))$



Corollary 1 (Salmon, 1862)

For a general degree d algebraic surface $S \subset \mathbb{P}^3$,

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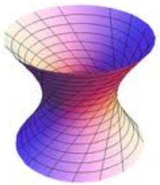
For a general degree d algebraic surface $S \subset \mathbb{P}^3$, $\mathrm{Flex}(S)$ is a one dimensional variety of degree $\leq d(11d - 24)$

Corollary 2 (Salmon, 1862)

If $P_S(x, y, z)$ vanishes in S, then it is a ruled surface



Corollary 3 (Salmon, 1862)

If S of degree d contains more than $d(11d-24)$ lines, then it has a ruled component

Back to the 21st century...

Janos Kollár (2014)

"I get a polynomial of degree $11d - 18$. Salmon claims that in fact the degree should be $11d - 24$. I have not checked this"

Terence Tao (blog, 2014)

"The original proof of the Cayley-Salmon theorem, dating back to at least 1915, is not easily accessible and not written in modern language"

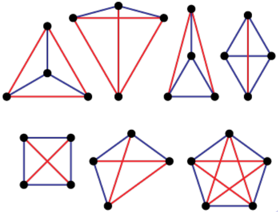
Nets Katz (ICM 2014)

"One of the motives for this lecture is to defend Salmon's honor and explain his original proof"

Why do we care more than 150 years after???

A conjecture by Erdös (1946)

Given n different points in the plane, there are at least $\mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ different distances among them



Sharpest result so far

Larry Guth & Nets Katz

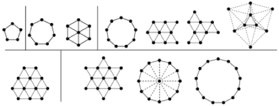
Annals of Mathematics (2015)

$$\mathcal{O}\left(\frac{n}{\log(n)}\right)$$

Conjecture 1 *(Erdős) The minimum number of distinct distances determined by n points in the Euclidean plane is $\Theta\left(\frac{n}{\sqrt{\log n}}\right)$.*

The first few exact values of the function $v(n)$ were determined in [ErF96]:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$v(n)$	0	1	1	2	2	3	3	4	4	5	5	5	6



Sets with k distinct distances, $2 \le k \le 6$, and maximum number of points

Via Incidence Geometry..

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Let $\mathcal{L}$ be a set of N^2 lines in $\mathbb{R}^3$, with no more than $\mathcal{O}(N)$ of them lying either in the same plane or in a double-ruled surface.

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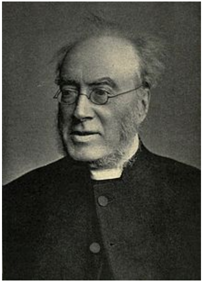
Let $\mathcal{L}$ be a set of N^2 lines in $\mathbb{R}^3$, with no more than $\mathcal{O}(N)$ of them lying either in the same plane or in a doubleruled surface. For $2 \leq k \leq N$, the number of points lying in at least in k lines is of order $\mathcal{O}(N^3 k^{-2})$

The proof of this Theorem uses the degree of the Salmon Polynomial!



Our contribution

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The *flex locus* of V is the set of all the flex points of V

Setup

Let $f_V \in K[x_0, \ldots, x_n]$ a squarefree homogeneous polynomial defining V, of degree d

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Set

$$f_V(\mathbf{x} + t\mathbf{y}) = \sum_{k=0}^{d} f_{V,k}(\mathbf{x}, \mathbf{y})t^k$$

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There is a homogeneous $\rho_V \in K[x_0, \ldots, x_n]$ with $\deg(\rho_V) = d \sum_{k=1}^{n} \frac{n!}{k} - (n+1)!$ defining the flex locus of V.

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There is a homogeneous $\rho_V \in K[x_0, \ldots, x_n]$ with $\deg(\rho_V) = d \sum_{k=1}^{n} \frac{n!}{k} - (n+1)!$ defining the flex locus of V. It is uniquely determined modulo f_V by the condition

$$\mathrm{Res}^{\boldsymbol{y}}(f_{V,1}(\boldsymbol{x}, \boldsymbol{y}), \ldots, f_{V,n}(\boldsymbol{x}, \boldsymbol{y}), \ell(\boldsymbol{y})) \equiv \ell^{n!} \rho_V \bmod f_V$$

for any linear form $\ell \in K[x_0, \ldots, x_n]$

Consequences

Corollary (Busé-D-Sombra-Weimann 18)

If V has no ruled irreducible components, then $\mathrm{Flex}(V)$ is a complete intersection subscheme of $\mathbb{P}^n$ of dimension $n-2$ and degree

$$d^2 \sum_{k=1}^{n} \frac{n!}{k} - d\,(n+1)!$$

In particular

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Flex(V) is set-theoretically defined by equations of degree at most $\max(d, d\sum_{k=1}^{n}\frac{n!}{k} - (n+1)!)$, and its degree is at most $d^2\sum_{k=1}^{n}\frac{n!}{k} - d(n+1)!$

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Application: lines $\subset$ varieties

Let $\mathcal{L}_V$ be the union of lines contained in V

When $d = n$, a flex line at a $p \in V$ has order of contact at least $n + 1$, and so it is necessarily contained in V.

In this case, $\mathcal{L}_V$ coincides with the flex locus of V

Corollary (Busé-D-Sombra-Weimann 18)

If V has degree n and no ruled components, then $\mathcal{L}_V$ is a ruled subvariety of V of dimension $n-2$ and degree at most $n^3(n-1)!\sum_{k=2}^{n-1}\frac{1}{k}$

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- For a generic flex point p of V, there is a unique flex line passing through it.
  If $d = n$, then this line is contained in V.
  If $d > n$, its order of contact with V at p is exactly $n+1$.

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Corollary (Busé-**D**-Sombra-Weimann 18)

If $V \subset \mathbb{P}^n$ is a generic hypersurface of degree n, then $\mathcal{L}_V$ is a ruled subvariety of V of dimension $n-2$ and degree

$$n^3\,(n-1)!\,\sum_{k=2}^{n-1}\frac{1}{k},$$

complete intersection of V with a hypersurface of degree $n^2\,(n-1)!\,\sum_{k=2}^{n-1}\frac{1}{k}$

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$$\rho_V = H_{f_V} = \begin{vmatrix} \frac{\partial^2 f_V}{\partial x_0^2} & \frac{\partial^2 f_V}{\partial x_0 \partial x_1} & \frac{\partial^2 f_V}{\partial x_0 \partial x_2} \\ \frac{\partial^2 f_V}{\partial x_0 \partial x_1} & \frac{\partial^2 f_V}{\partial x_1^2} & \frac{\partial^2 f_V}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f_V}{\partial x_0 \partial x_2} & \frac{\partial^2 f_V}{\partial x_1 \partial x_2} & \frac{\partial^2 f_V}{\partial x_2^2} \end{vmatrix}$$

$n = 3$

(Salmon 1862)

$$\rho_V = \Theta - 4H(\Phi + a\,\Psi)$$

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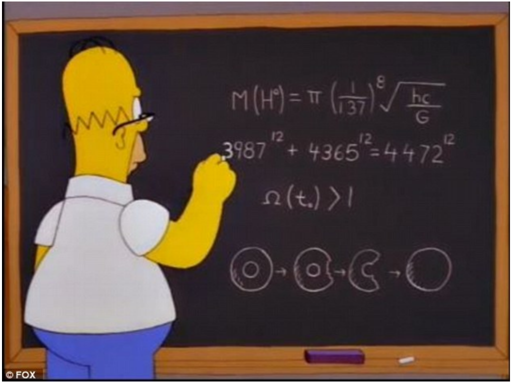
with $a \in \mathbb{Z}$ y Θ, H, Φ, Ψ covariants of f_V

Explicit formula for ρ_V??



All these and more...

arXiv.org

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arXiv.org

Busé, Laurent; D'Andrea, Carlos; Sombra, Martín, Weimann, Martin

The geometry of the flex locus of a hypersurface.

arXiV:1804.08025

Moltes Gràcies!!

http://www.ub.edu/arcades/cdandrea.html