Bounds for degrees of minimal $\mu-$bases of parametric surfaces

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ABSTRACT

By adapting the effective version of Quillen-Suslin Theorem given in [7], we show that if the ideal defining a rational parametrization of degree d of an algebraic surface in 3-dimensional space is radical and has D points, then a μ-basis of this parametrization can be found of degree bounded by $5\max(1, D-1)^4(2d+1)^4$. This bound improves those obtained recently in [3] in our setup, and it is also sensitive to the number of base points.

KEYWORDS

μ-bases, syzygies, parametrization, Quillen-Suslin Theorem, effective bounds

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1 INTRODUCTION

The concept of $\mu-$basis was introduced in [4] in the case of parametrized rational curves. Let $\mathbb{K}$ be a field, s an indeterminate over $\mathbb{K}$ and $n \in \mathbb{N}$. An $(n+1)$-tuple $P(s) = (a_1(s), \ldots, a_{n+1}(s)) \in \mathbb{K}[s]^{n+1}$ can be regarded as the parametrization of a rational curve in $\mathbb{K}^n$ via the map $\mathbb{K} \dashrightarrow \mathbb{K}^n$ given by $\left(\frac{a_1(s)}{a_{n+1}(s)}, \ldots \frac{a_n(s)}{a_{n+1}(s)}\right)$. With this in mind, we can assume w.l.o.g. that $gcd(a_i(s)) = 1$. The sygyzy module of P over $\mathbb{K}[s]$ is defined as

$$Syz(P) = \{(A_1(s), \ldots, A_{n+1}(s)) \in \mathbb{K}[s]^{n+1} : \sum_{i=1}^{n+1} A_i(s)a_i(s) = 0\}.$$

This module is free of rank n.

Assume that $d = \max(\deg(a_i)) \geq 1$. By applying the Extended Euclidean Algorithm to the input (see [4] and [9]) one can find a basis $\{p_1(s), \ldots, p_n(s)\}$ of $Syz(P)$ such that $\deg(p_i(s)) = \mu_i$ with $\mu_1 + \ldots + \mu_n = d$. Such a basis is called a *μ-basis* in the literature. So, essentially, μ-bases are bases of $Syz(P)$ of controlled degree. For algorithms to compute $\mu-$bases of curves, see [4, 9]. Note that by Hilbert-BurchTheorem, the degree of a μ-basis is sharp in the sense that the condition $\sum_{i=1}^{n} \deg(p_i(s)) < d$ can never happen for any basis $\{p_1(s), \ldots, p_n(s)\}$ of $Syz(P)$ because of the condition $P(s) = p_1(s) \wedge \ldots \wedge p_n(s)$. The wedge product notation here only means that the coordinates of $P(s)$ are -up to a nonzero constant in $\mathbb{K}$- equal to the signed maximal minors of the $(n+1) \times n$ matrix having in the i-th column the coordinates of $p_i(s)$.

For parametric surfaces the situation is more complicated as no Euclidean Algorithm is possible in more than one variable. Yet μ-bases exist in this case. Let t be another indeterminate over $\mathbb{K}$. An $(n+1)$-tuple $P(s,t) = (a_1(s,t), \ldots, a_{n+1}(s,t)) \in \mathbb{K}[s,t]^{n+1}$ can be regarded as the parametrization of a surface in $\mathbb{K}^n$ as before. So, we can assume again that $gcd(a_i(s,t)) = 1$. In [2, Appendix], it is shown that the syzygy module

$$Syz(P) = \{(A_1(s,t), \ldots, A_{n+1}(s,t)) \in \mathbb{K}[s,t]^{n+1} :$$
$$A_1(s,t)a_1(s,t) + \ldots + A_{n+1}(s,t)a_{n+1}(s,t) = 0\}$$

is also free of rank n. In that paper, a *μ-basis* of $P(s,t)$ was defined as *any* basis of $Syz(P)$. No were neither required nor deduced on the degrees of any of these bases. A *minimal μ-basis* was defined as a basis $\{p_1(s,t), \ldots, p_n(s,t)\}$ of $Syz(P)$ such that $\sum_{i=1}^{n} \deg(p_i(s,t))$ is minimal among all the bases of $Syz(P)$, and the question on explicit bounds on the degree of such a minimal μ-basis was raised. Algorithms to compute μ-bases for this case can be found in [5], but no bounds on the degree of these elements can be easily derived from these algorithms.

In [3] the first of such bounds is produced for surfaces in $\mathbb{K}^3$, i.e. when $n = 3$. Indeed, it is shown in [3, Theorem A] that a minimal μ-basis in this situation has degree bounded by $O(d^{33})$. Several subcases were considered with better bounds in all of them. However, it is not clear yet whether these bounds are sharp, and there is definitely a lot of room for improvements.

In this paper, we present one of such sharpenings. To keep the notation simple and also to compare with previous results, we set ourselves in the case $n = 3$, but the generalization to any n is straightforward. So, for the rest of the text, we will deal with a parametrization

$$P(s,t) = (a_1(s,t), a_2(s,t)a_3(s,t), a_4(s,t)) \in \mathbb{K}[s,t]^4 \quad (1)$$

with $\deg(P) = \max(\deg(a_i(s,t))) = d$ and $gcd(a_i(s,t)) = 1$. We denote with $I_P = \langle a_1(s,t), a_2(s,t), a_3(s,t), a_4(s,t)\rangle \subset \mathbb{K}[s,t]$ the ideal defined by these polynomials, and $V_P \subset \overline{\mathbb{K}}^3$ the variety defined by this ideal in the algebraic closure of $\mathbb{K}$. Note that the condition $gcd(a_i(s,t)) = 1$ implies that V_P is a finite set. Let D be its *degree*, meaning the number of points in V_P counted with their corresponding multiplicities.

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It is known that a radical zero-dimensional ideal $I \subset \mathbb{K}[s,t]$ has -after possibly a linear change of coordinates- system of generators of the form $\{p(s), t-q(s)\}$. The converse of course holds for a larger class of ideals containing those radical and zero-dimensional, and have been characterized geometrically in [1]. These ideals are said to have a *shape basis*.

We will use the effective version of Quillen-Suslin Theorem given in [7] to prove the following result:

Theorem 1.1. *Let $P(s,t)$ be as in (1) with $d = \deg(P)$ and $D = \deg(V_P)$. If I_P has a shape basis, then a $\mu-$basis of $P(s,t)$ can be found with degree bounded by*

$$5\max(1, D-1)^4(2d+1)^4. \tag{2}$$

As a consequence of this result, if $V_P = \emptyset$ (i.e. $I_P = \mathbb{K}[s,t]$ thanks to Hilbert's Nullstellensatz), we have that $D = 0$ and hence (2) boils down to a bound of the size $O(d^4)$ which is the one appearing in [7] for this situation, and refines the amount $O(d^{22})$ obtained in [3] for this case.

In order to obtain a bound only depending on d, note that by Bézout Theorem we always have $D \leq d^2$, and hence (2) is always bounded by a quantity of the size of $O(d^{12})$. This is also a major improvement over the results in [3]. It should be said however, that our results are restricted to the case of I_P having a shape basis, and our techniques depend strongly on the particular properties of this kind of ideals, so no extensions to the general case seem to be deduced from our approach and extra ideas are needed to improve the bounds already known.

The paper is organized as follows: in Section 2 we will revisit the effective version of Quillen-Suslin Theorem given in [7] to obtain bounds for a minimal μ-basis in the case $V_P = \emptyset$, and in Section 3 we will prove the general case by reducing it to the situation of Quillen-Suslin. It should be mentioned that our techniques are elementary and computable, so they can be implemented in any computacional algebra software. We illustrate with examples the algorithms posted there.

2 THE UNIMODULAR CASE

Recall that a matrix in $\mathbb{K}[s,t]^{n\times m}$ is said to be *unimodular* if the ideal generated by its maximal minors is the whole ring $\mathbb{K}[s,t]$. We will consider P as a 1×4 matrix, and consider the situation when P is unimodular which equivalently means that $I_P = \mathbb{K}[s,t]$ or $V_P = \emptyset$.

Theorem 2.1. *[7] If $P(s,t)$ being as in (1) is a unimodular matrix, there exists a unimodular matrix $M \in \mathbb{K}[s,t]^{4\times 4}$ of degree $O(d^4)$, such that*

$$P(s,t)\,M = (1,0,0,0). \tag{3}$$

Corollary 2.2. *Let M be the matrix of above, and write $M = (M^1\, M^2\, M^3\, M^4)$, with M^i being the i-th column of M. Then, $\{M^2, M^3, M^4\}$ is a $\mu-$basis of $Syz(P)$.*

Proof. We clearly have $P \cdot M^i = 0$ for $i = 2,3,4$. Also, these columns are $\mathbb{K}[s,t]$-linearly independent as they are part of a matrix of full rank. To show that they generate $Syz(P)$, let A be any element in this module, as M is unimodular, we have that $\{M^1, M^2, M^3, M^4\}$ generates $\mathbb{K}[s,t]^4$, and hence we get $A = \sum_{j=1}^4 p_j M^j$ with $p_j \in \mathbb{K}[s,t]$, $j = 1,2,3,4$. As $P \cdot A = 0$, from (3) we deduce straightforwardly that $p_1 = 0$, which then implies that $A \in \langle M^2, M^3, M4\rangle$. This concludes with the proof of the claim. □

We will review now the algorithm proposed in [7] to compute such a matrix M. This will give explicit bounds for the degrees of the elements of a μ-basis thanks to Corollary 2.2. We will assume that $\mathbb{K}$ is infinite, otherwise we can work in an extension of it. Suppose w.l.o.g. that $\deg_t(a_1) = d$. If this is not the case, we can redefine a_1 with this property after a linear combination of the a_i's, and if necessary also after applying the change of variable $\tilde{s} = s + \lambda t$ with $\lambda \in \mathbb{K} \setminus \{0\}$.

Lemma 2.3. *If $1 \in I_P$, then there exist $\alpha_3, \alpha_4 \in \mathbb{K}$ and $\beta_2, \beta_3, \beta_4 \in \mathbb{K}$, such that by setting $\tilde{a}_2 = a_2 + \alpha_3 a_3 + \alpha_4 a_4$, $\tilde{a}_3 = \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4$, the polynomials $r_{12}(s) = Res_t(a_1(s,t), \tilde{a}_2(s,t))$ and $r_{13}(s) = Res_t(a_1(s,t), \tilde{a}_3(s,t))$ are coprimes.*

Proof. If $r_{12}(s) = Res_t(a_1(s,t), \tilde{a}_2(s,t)) \equiv 0\, \forall \alpha_3, \alpha_4$ then $a_1(s,t)$ and $\tilde{a}_2(s,t)$ have a common factor $g(s,t)$ of positive degree in t which divides a_1, a_2, a_3 and a_4 which is a contradiction because we are assuming $V(a_1, a_2, a_3, a_4) = \emptyset$. So, there are $\alpha_3, \alpha_4 \in \mathbb{K}$ such that $r_{12}(s) \neq 0$, and this implies that $V(a_1, \tilde{a}_2) = \{(s_i, t_i), i = 1, \ldots, l\}$. Set now $S = \{(s_i, t_{ij}) \mid a_1(s_i, t_{ij}) = 0,\ i = 1, \ldots, l\}$. Since $V(a_1, \tilde{a}_2, a_3, a_4) = \emptyset$ and $\mathbb{K}$ is infinite, there must exist $(\beta_2, \beta_3, \beta_4)$such that $(\beta_2, \beta_3, \beta_4) \notin \left\langle \left(\tilde{a}_2(s_i, t_{ij}), a_3(s_i, t_{ij}), a_4(s_i, t_{ij})\right)\right\rangle^{\perp}\ \forall (s_i, t_{ij}) \in S$. For these values of the β_i's, the claim follows straitghtforwardly. □

Theorem 2.4. *Let $P(s,t)$ be as in (1) with $1 \in I_P$. Then, a $\mu-$basis of $Syz(P)$ has degree bounded by $4d^4$.*

Proof. Assume w.l.o.g. that a_1, a_2 and a_3 have already been modified according to the hypothesis of Lemma 2.3. We will apply the constructive method given in [7] for the proof of Theorem 2.1 with two steps:

$$\begin{aligned}&(a_1(s,t), a_2(s,t), a_3(s,t), a_4(s,t))\\ &\xrightarrow{step1} (a_1(s,B'), a_2(s,B'), a_3(s,B'), a_4(s,B'))\\ &\xrightarrow{step2} (a_1(s,0), a_2(s,0), a_3(s,0), a_4(s,0))\end{aligned}$$

For the first step we compute

$$r_{12}(s) = Res_t(a_1(s,t), a_2(s,t)) \text{ and } r_{13}(s) = Res_t(a_1(s,t), a_3(s,t)).$$

By Bézout's Identity, there exist $A_1(s,t),\ A_2(s,t)\ A_1^*(s,t),\ A_2^*(s,t) \in \mathbb{K}[s,t]$ such that

$$A_1(s,t)a_1(s,t) + A_2(s,t)a_2(s,t) = r_{12}(s)$$

and

$$A_1^*(s,t)a_1(s,t) + A_2^*(s,t)a_3(s,t) = r_{13}(s)$$

with

$$\boxed{\deg(r_{12}), \deg(r_{13}) \leq d^2, \quad \deg(A_i(s,t)), \deg(A_i^*(s,t)) \leq d^2 - d.}$$

By Lemma 2.3, $r_{12}(s)$ and $r_{13}(s)$ are coprimes. So, again by Bézout's Identity, there exist $R_{12}(s), R_{13}(s) \in \mathbb{K}[s]$ such that

$$R_{12}(s)r_{12}(s) + R_{13}(s)r_{13}(s) = 1, \tag{4}$$

and multiplying by t the two sides we get

$$R_{12}(s)r_{12}(s)t + R_{13}(s)r_{13}(s)t = t.$$

Therefore in Step 1 we can take $\begin{cases} B = t, \\ B' = R_{13}(s)r_{13}(s)t \end{cases}$

Note that we have

$$\max(\deg(R_{12}(s)r_{12}(s)), \deg(R_{13}(s)r_{13}(s))) \leq 2d^2 - 1,$$

therefore

$$\boxed{\deg(B) = 1, \deg(B') \leq 2d^2}$$

Continuing with step 1 of the algorithm, we compute a unimodular matrix $N_1 \in \mathbb{K}[s,t]^{4\times 4}$ such that

$$P(s,t)N_1 = P(s,B').$$

This matrix will be obtained as a product of two matrices, $N_1 = E_1 S_1$, where $E_1 \in \mathbb{K}[s,t]^{4\times 4}$ satisfies

$$P(s,t)E_1 = (a_1(s,t), a_2(s,t), a_3(s,B'), a_4(s,B')),$$

and $S_1 \in \mathbb{K}[s,t]^{4\times 4}$ is such that

$$(a_1(s,t), a_2(s,t), a_3(s,B'), a_4(s,B'))S_1 = P(s,B').$$

To be more precise,

$$E_1 = \begin{pmatrix} I_2 & E_{12} \\ 0 & I_2 \end{pmatrix},$$

where I_2 is the identity 2×2 matrix, and

$$E_{12} = \begin{pmatrix} -\alpha(s,t)A_1(s,t) & -\beta(s,t)A_1(s,t) \\ -\alpha(s,t)A_2(s,t) & -\beta(s,t)A_2(s,t) \end{pmatrix},$$

with

$$\alpha(s,t) = \frac{1}{r_{12}(s)}(a_3(s,t) - a_3(s,B'))$$

and

$$\beta(s,t) = \frac{1}{r_{12}(s)}(a_4(s,t) - a_4(s,B')).$$

The fact that both $\alpha(s,t)$, $\beta(s,t) \in \mathbb{K}[s,t]$ can be deduced straightforwardly from (4) which implies that $t - B'$ is a multiple of $r_{12}(s)$.

Their degrees are

$$\boxed{\deg(\alpha) \leq d\deg(B') = 2d^3, \quad \deg(\beta) \leq d\deg(B') = 2d^3}$$

and therefore

$$\boxed{\deg(E_1) \leq 2d^3 + d^2 - d}.$$

The matrix $S_1 \in \mathbb{K}[s,t]^{4\times 4}$ is of the form

$$S_1 = \begin{pmatrix} S_{11} & 0 \\ 0 & I_2 \end{pmatrix},$$

with

$$S_{11} = \begin{pmatrix} \frac{A_1(s,t)a_1(s,B')+A_2(s,B')a_2(s,t)}{r_{12}(s)} & \frac{A_1(s,t)a_2(s,B')-A_1(s,B')a_2(s,t)}{r_{12}(s)} \\ \frac{A_2(s,t)a_1(s,B')-A_2(s,B')a_1(s,t)}{r_{12}(s)} & \frac{A_2(s,t)a_2(s,B')+A_1(s,B')a_1(s,t)}{r_{12}(s)} \end{pmatrix}.$$

Again the fact that the entries of S_{11} are polynomials can be deduced from (4). We compute

$$\boxed{\begin{array}{c} \deg(S_1) \leq \max\{d^2 - d + d\deg(B'), (d^2-d)\deg(B') + d\} \\ = \\ 2d^4 - 2d^3 + d \end{array}}$$

Finally, we have

$$N_1 = E_1 S_1 = \begin{pmatrix} S_{11} & E_{12} \\ 0 & I_2 \end{pmatrix},$$

and hence

$$\boxed{\deg(N_1) \leq 2d^4 - 2d^3 + d}$$

Now we pass to step 2 of the algorithm, where we compute a unimodular matrix $N_2 \in \mathbb{K}[s,t]^{4\times 4}$ such that

$$P(s,B')N_2 = P(s,0).$$

As before, this matrix is obtained as a product of two unimodular matrices $N_2 = E_2 S_2$, where $E_2 \in \mathbb{K}[s,t]^{4\times 4}$ satisfies

$$P(s,B')E_2 = (a_1(s,B'), a_2(s,0), a_3(s,B'), a_4(s,0)),$$

and $S_2 \in \mathbb{K}[s,t]^{4\times 4}$ is such that

$$(a_1(s,B'), a_2(s,0), a_3(s,B'), a_4(s,0))S_2 = P(s,0).$$

To be more precise, we have

$$E_2 = \begin{pmatrix} 1 & -\tilde{\alpha}(s,t)A_1^*(s,B') & 0 & -\tilde{\beta}(s,t)A_1^*(s,B') \\ 0 & 1 & 0 & 0 \\ 0 & -\tilde{\alpha}(s,t)A_2^*(s,B') & 1 & -\tilde{\beta}(s,t)A_2^*(s,B') \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\tilde{\alpha}(s,t) = \frac{1}{r_{13}(s,t)}(a_2(s,B') - a_2(s,0))$$

and

$$\tilde{\beta}(s,t) = \frac{1}{r_{13}(s,t)}(a_4(s,B') - a_4(s,0)),$$

Computing the degrees, we get

$$\boxed{\deg(\tilde{\alpha}(s,t)),\ \deg(\tilde{\beta}(s,t)) \leq d\deg(B') = 2d^3}$$

and therefore

$$\boxed{\deg(E_2) \leq (2d^4 - 2d^3) + 2d^3 = 2d^4}.$$

The other matrix is

$$S_2 = \begin{pmatrix} \frac{A_1^*(s,B')a_1(s,0)+A_2^*(s,0)a_3(s,B')}{r_{13}(s)} & 0 & \frac{A_1^*(s,B')a_3(s,0)-A_1^*(s,0)a_3(s,B')}{r_{13}(s)} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{A_2^*(s,B')a_1(s,0)-A_2^*(s,0)a_1(s,B')}{r_{13}(s)} & 0 & \frac{A_2^*(s,B')a_3(s,0)+A_1^*(s,0)a_1(s,B')}{r_{13}(s)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\deg(S_2) \leq 2d^4 + d$. So, we have that $N_2 = E_2 S_2$, and as before the coefficients do not get mixed in the product, so we have

$$\boxed{\deg(N_2) \leq 2d^4 + d.}$$

Therefore, we have that

$$P(s,t)N_1 N_2 = P(s,0)$$

with degree

$$\boxed{\deg(N_1 N_2) \leq 4d^4 - 2d^3 + 2d}.$$

Note that $P(s,0) \in \mathbb{K}[s]^4$ is also a unimodular matrix of degree d. It is known that (see for instance [9]) there exists a unimodular matrix $M \in \mathbb{K}[s]^{4\times 4}$ with $\deg(M) \leq d$ such that

$$P(s,t)N_1 N_2 M = P(s,0)M = (1,0,0,0).$$

As a consequence of Corollary 2.2, we get that the last three columns of $N_1 N_2 M$ are a $\mu-$basis of $Syz(P)$ of degree is bounded by

$$\boxed{\deg(N_1 N_2 M) \leq 4d^4 - 2d^3 + 3d \leq 4d^4.}$$

□

Example 2.5. Let us consider

$$P(s,t) = (s^2, t^2, s^2-1, s^2+1).$$

Applying the constructive proof of Theorem 2.4 we have that $B' = t - s^4 t$. As the last two polynomials do not depend on t, we get that $E_1 = I_4$,

$$S_1 = \begin{pmatrix} 1 & s^2(-2+s^4)t^2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and hence

$$N_1 = E_1\,S_1 = \begin{pmatrix} 1 & -2s^2t^2+s^6t^2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we compute E_2 and S_2 in the second step of the algorithm to get the matrix

$$N_2 = E_2\,S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t^2+s^2t^2-s^4t^2-s^6t^2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also compute

$$M = \begin{pmatrix} -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s^2+1 \\ 1 & 0 & 1 & -s^2+1 \end{pmatrix},$$

so we have that

$$N_1N_2M = \begin{pmatrix} -1 & -2s^2t^2+s^6t^2 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t^2+s^2t^2-s^4t^2-s^6t^2 & 1 & 1+s^2 \\ 1 & 0 & 1 & 1-s^2 \end{pmatrix}$$

and a μ−basis $\{p,\ q,\ r\}$ of $P(s,t)$ is given by the last three columns of the above matrix, that is $p = (-2s^2t^2+s^6t^2, 1, t^2+s^2t^2-s^4t^2-s^6t^2, 0)$, $q = (-2,0,1,1)$ and $r = (0,0,1+s^2,1-s^2)$.

Example 2.6. Let us consider

$$P(s,t) = (2st, 2t, 2s, s^2+t^2+1).$$

Note that in this case $V(a_1,a_2,a_3) \neq \emptyset$, but we can resort the sequence and get

$$P(s,t) = (s^2+t^2+1, 2t, 2s, 2st).$$

which suits better to our computations. Now $B' = -s^2t$ and we compute

$$N_1 = \begin{pmatrix} s^2t^2+1 & -2t & 0 & -2st \\ -\frac{1}{2}t\left(s^2t^2+s^2+1\right) & t^2+1 & 0 & st^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2}s^3t^2 & st & 1 & s^2t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 1 & 0 & 0 & -2s \\ 0 & 1 & 0 & 0 \\ -s/2 & 0 & 0 & s^2+1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore the three last columns of N_1N_2M are a μ−basis of $P(s,t)$, that is, last three columns of

$$\begin{pmatrix} s^2t^2+1 & -2t & -2st & -2s^3t^2-2s \\ -\frac{1}{2}s^2t^3-\frac{s^2t}{2}-\frac{t}{2} & t^2+1 & st^2 & s^3t^3+s^3t+st \\ -\frac{1}{2}s^3t^2-\frac{s}{2} & st & s^2t & s^4t^2+s^2+1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

3 PROOF OF THEOREM 1.1

Now we will consider the case when I_P has -maybe after a linear change of coordinates- a shape basis, i.e.

$$\langle a_1(s,t), a_2(s,t), a_3(s,t), a_4(s,t)\rangle = \langle p(s), t-q(s)\rangle \qquad (5)$$

with

$$\deg(p(s)) = D, \quad \deg(q(s)) \leq D-1.$$

By Bézout's Theorem, we have straightforwardly that $D \leq d^2$. This setup contains the case when I_P is a radical ideal.

From (5) we get that there exist $A_i(s,t), B_i(s)$, $i = 1,2,3,4$ such that

$$a_i(s,t) = A_i(s,t)(t-q(s)) + B_i(s)p(s). \qquad (6)$$

We can compute explicitely these polynomials and bound their degrees as follows. Set $D^* := \max(1, D-1)$, so that $\deg(t-q(s)) \leq D^*$. By applying the Division Algorithm between $a_i(s,t)$ and $t - q(s)$, we have

$$a_i(s,t) = A_i(s,t)(t-q(s)) + r_i(s)$$

with $r_i(s) = a_i(s, q(s))$, and then

$$\deg(r_i(s)) = \deg(a_i(s,q(s))) \leq dD^*.$$

Moreover, from (6) we deduce that $r_i(s) = p(s)B_i(s)$. Therefore,

$$\deg(B_i(s)) \leq \deg(r_i(s)) - \deg(p(s)) = dD^* - D \leq D^*(d-1),$$

and $\deg(A_i(s,t)) \leq dD^*$.

Then, we can write $P(s,t)$ as

$$(t-q(s)\ \ p(s)) \cdot \begin{pmatrix} A_1(s,t) & A_2(s,t) & A_3(s,t) & A_4(s,t) \\ B_1(s) & B_2(s) & B_3(s) & B_4(s) \end{pmatrix}$$

with

$$\boxed{\deg(A_i, B_i) \leq dD^*.}$$

From (5) we have also that

$$P(s,t) \cdot \begin{pmatrix} \alpha_1(s,t) & \beta_1(s,t) \\ \alpha_2(s,t) & \beta_2(s,t) \\ \alpha_3(s,t) & \beta_3(s,t) \\ \alpha_4(s,t) & \beta_4(s,t) \end{pmatrix} = (t-q(s)\ \ p(s))$$

for suitable polynomials $\alpha_i(s,t), \beta_i(s,t) \in \mathbb{K}[s,t]$, $i = 1,2,3,4$. So, we get

$$t - q(s) = \sum_{i=1}^{4} \alpha_i(s,t)a_i(s,t)$$

and

$$p(s) = \sum_{i=1}^{4} \beta_i(s,t)a_i(s,t) =$$

$$= \sum_{i=1}^{4} \beta_i(s,t)(A_i(s,t)(t-q(s)) + B_i(s)p(s)) \Rightarrow$$

$$\Rightarrow \left(1 - \sum_{i=1}^{4} \beta_i(s,t)B_i(s)\right)p(s) = \sum_{i=1}^{4} \beta_i(s,t)A_i(s,t)(t-q(s)) \Rightarrow$$

$$\Rightarrow \begin{cases} 1 - \sum_{i=1}^{4} \beta_i(s,t)B_i(s) = A(s,t)(t-q(s)) \\ \sum_{i=1}^{4} \beta_i(s,t)A_i(s,t) = A(s,t)p(s) \end{cases} \tag{7}$$

for a suitable $A(s,t) \in \mathbb{K}[s,t]$. We set $t = q(s)$ in the first equation of (7) and get

$$\sum_{i=1}^{4} \beta_i(s,q(s))B_i(s) = 1,$$

that is $\gcd(B_1(s), B_2(s), B_3(s), B_4(s)) = 1$ and by the results of [9] we can compute a unimodular matrix $M_B \in \mathbb{K}[s]^{4\times4}$ such that

$$\begin{pmatrix} A_1(s,t) & A_2(s,t) & A_3(s,t) & A_4(s,t) \\ B_1(s) & B_2(s) & B_3(s) & B_4(s) \end{pmatrix} \cdot M_B = \begin{pmatrix} \tilde{A}_1(s,t) & \tilde{A}_2(s,t) & \tilde{A}_3(s,t) & \tilde{A}_4(s,t) \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

with

$$\boxed{\deg(M_B) \le D^*(d-1),\ \deg(\tilde{A}_i) \le 2dD^*.}$$

Set now

$$(\tilde{a}_1(s,t), \tilde{a}_2(s,t), \tilde{a}_3(s,t), \tilde{a}_4(s,t)) := P(s,t) \cdot M_B. \tag{9}$$

From (8), we have that I_P is actually equal to the ideal generated by

$$\begin{array}{l} \tilde{A}_1(s,t)(t-q(s)) + p(s), \tilde{A}_2(s,t)(t-q(s)), \tilde{A}_3(s,t)(t-q(s)), \\ \tilde{A}_4(s,t)(t-q(s)), \end{array},$$

and therefore, there exist $\gamma_i(s,t) \in \mathbb{K}[s,t]$, $i = 1,2,3,4$, such that

$$(t-q(s)) = \gamma_1(s,t)(\tilde{A}_1(s,t)(t-q(s))+p(s)) + \sum_{i=2}^{4} \gamma_i(s,t)\tilde{A}_i(s,t)(t-q(s)).$$

Replacing $t = q(s)$ in the above equation, we obtain that $\gamma_1(s,q(s)) = 0$ and therefore we have $\gamma_1(s,t) = \tilde{\gamma}_1(s,t)(t-q(s))$, so

$$1 = \tilde{\gamma}_1(s,t)(\tilde{A}_1(s,t)(t-q(s)) + p(s)) + \sum_{i=2}^{4} \gamma_i(s,t)\tilde{A}_i(s,t),$$

that is, $\langle(\tilde{A}_1(s,t)(t-q(s))+p(s), \tilde{A}_2(s,t), \tilde{A}_3(s,t), \tilde{A}_4(s,t))\rangle = \mathbb{K}[s,t]$. So, we can apply Theorem 2.1 to

$$\tilde{A} = (\tilde{A}_1(s,t)(t-q(s)) + p(s), \tilde{A}_2(s,t), \tilde{A}_3(s,t), \tilde{A}_4(s,t)), \tag{10}$$

which has $\deg(\tilde{A}) \le 2d(D-1) - 1$, to get a unimodular matrix $M_{\tilde{A}} \in \mathbb{K}[s,t]^{4\times4}$ such that

$$(\tilde{A}_1(s,t)(t-q(s)) + p(s), \tilde{A}_2(s,t), \tilde{A}_3(s,t), \tilde{A}_4(s,t))M_{\tilde{A}} = (1,0,0,0) \tag{11}$$

with

$$\boxed{\deg(M_{\tilde{A}}) \le 4(2dD^* + D^*)^4 = 4D^{*4}(2d+1)^4.}$$

If we denote with $M^2_{\tilde{A}}$, $M^3_{\tilde{A}}$, $M^4_{\tilde{A}}$ the three last columns of $M_{\tilde{A}}$, then by the Hilbert-Burch Theorem we have that -up to a nonzero constant in $\mathbb{K}$-

$$M^2_{\tilde{A}} \wedge M^3_{\tilde{A}} \wedge M^4_{\tilde{A}} = (\tilde{A}_1(s,t)(t-q(s))+p(s), \tilde{A}_2(s,t), \tilde{A}_3(s,t), \tilde{A}_4(s,t)). \tag{12}$$

Write $M_{\tilde{A}} = \left(m_{ij}\right)_{1\le i,j\le 4}$, and set

$$M_{\tilde{P}} = \begin{pmatrix} m_{12}(t-q(s)) & m_{13}(t-q(s)) & m_{14}(t-q(s)) \\ m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{pmatrix}.$$

We clearly have

$$\boxed{\deg(M_{\tilde{P}}) \le 4D^{*4}(2d+1)^4 + D^*}.$$

and from (12) we deduce that

$$M^2_{\tilde{P}} \wedge M^3_{\tilde{P}} \wedge M^4_{\tilde{P}} = (\tilde{A}_1(s,t)(t-q(s)) + p(s), (t-q(s))\tilde{A}_2(s,t), (t-q(s))\tilde{A}_3(s,t), (t-q(s))\tilde{A}_4(s,t)).$$

So, by the converse of Hilbert-Burch Theorem ([6, Theorem 20.15])) we deduce that $\{M^2_{\tilde{P}}, M^3_{\tilde{P}}, M^4_{\tilde{P}}\}$ is a μ-basis of the parametrization

$$(\tilde{A}_1(s,t)(t-q(s))+p(s), (t-q(s))\tilde{A}_2(s,t), (t-q(s))\tilde{A}_3(s,t), (t-q(s))\tilde{A}_4(s,t)). \tag{13}$$

To conclude, we set $\tilde{M} = M_B M_{\tilde{P}} \in \mathbb{K}[s,t]^{4\times3}$. From (8) and (9) we deduce that

$$P(s,t) \cdot \tilde{M} = (0,0,0). \tag{14}$$

As M_B is unimodular, and the columns of $M_{\tilde{P}}$ a μ-basis of (13), we deduce straightforwardly that the columns of $\tilde{M}$ are a μ−basis of $P(s,t)$. Computing it straightforwardly we get

$$\boxed{\deg(\tilde{M}) \le D^*(d-1) + 4D^{*4}(2d+1)^4 + D^* \le 5D^{*4}(2d+1)^4,}$$

which concludes with the proof of the Theorem.

Example 3.1. Consider the following parametrization:

$$\begin{cases} a_1(s,t) &=& -st + s^2t \\ a_2(s,t) &=& s - 2s^2 + s^3 + t - 2t^2 + t^3 \\ a_3(s,t) &=& -s^2 + s^2t - t^2 + st^2 \\ a_4(s,t) &=& -st + st^2. \end{cases}$$

By computing a Gröbner Basis of these polynomials with respect to lex $t > s$, we get that $I_P = \langle t - (2s^2 - s), s^3 - s^2\rangle$, so we have $d = D = 3$ in this case, and $p(s) = s^3 - s^2$, $q(s) = 2s^2 - s$. We compute explicitly $A_i(s,t)$, $B_i(s)$, $1 \le i \le 4$, as in (6) to get

$$\begin{pmatrix} A_1(s,t) & B_1(s) \\ A_2(s,t) & B_2(s) \\ A_3(s,t) & B_3(s) \\ A_4(s,t) & B_4(s) \end{pmatrix} = \begin{pmatrix} -s + s^2 & -1 + 2s \\ 1 + 2s - 3s^2 - 4s^3 + 4s^4 - 2t - st + 2s^2t + t^2 & 2 - 6s - 4s^2 + 8s^3 \\ s - 2s^2 + 2s^3 - t + st & 2 - 2s + 4s^2 \\ -s - s^2 + 2s^3 + st & -1 + 4s^2 \end{pmatrix}$$

We now compute a matrix M_B as in (8), and get

$$M_B = \begin{pmatrix} 1/2 & -B_2(s)/2 & -B_3(s)/2 & -B_4(s)/2 \\ 0 & 1 & 0 & 0 \\ 1/2 & -B_2(s)/2 & -B_3(s)/2+1 & -B_4(s)/2 \\ -1/2 & B_2(s)/2 & B_3(s)/2 & B_4(s)/2+1 \end{pmatrix}.$$

So we have

$$\begin{gathered} \left(\tilde{A}_1(s,t), \tilde{A}_2(s,t), \tilde{A}_3(s,t), \tilde{A}_4(s,t)\right) \\ = \\ (A_1(s,t), A_2(s,t), A_3(s,t), A_4(s,t)) \cdot M_B \\ = \\ \begin{pmatrix} s/2 - t/2 \\ 1 + s - 2s^3 - t - 4st + 4s^3t + t^2 \\ -s^2 + 2s^2t \\ -s/2 - s^2 - t/2 + st + 2s^2t \end{pmatrix}^T, \end{gathered}$$

so $\tilde{A}$ from (10) is equal to

$$(-\frac{s^2}{2}+s^2t-\frac{t^2}{2}, 1+s-2s^3-t-4st+4s^3t+t^2, -s^2+2s^2t, -\frac{s}{2}-s^2-\frac{t}{2}+st+2s^2t),$$

which we can confirm (by computing Gröbner bases for instance) that it is a unimodular matrix. To conclude, we need to compute the matrix $M_{\tilde{A}}$ from (11) for this reduced parametrization. This can be done following the methods of Section 2, and we get finally the matrix $\tilde{M} = \left(m_{ij}\right)_{1\le i\le 4, 1\le j\le 3}$ from (14) such that

$$P(s,t) \cdot \tilde{M} = (0,0,0)$$

its three columns being a μ-basis of $P(s,t)$. Computing the coefficients explicitly we get

$$\begin{aligned} m_{11} \;=\; & 512s^{16}t^2 + 512s^{15}t^3 - 1536s^{15}t^2 - 2048s^{14}t^3 - 480s^{14}t^2 \\ & +512s^{13}t^4 + 1568s^{13}t^3 + 6672s^{13}t^2 - 1536s^{12}t^4 \\ & +3568s^{12}t^3 - 7368s^{12}t^2 + 1056s^{11}t^4 - 6584s^{11}t^3 \\ & -2420s^{11}t^2 + 1296s^{10}t^4 + 1892s^{10}t^3 + 9736s^{10}t^2 \\ & -2536s^9t^4 + 3364s^9t^3 - 6500s^9t^2 + 32s^9t + 1740s^8t^4 \\ & -3440s^8t^3 + 1590s^8t^2 - 40s^8t - 720s^7t^4 + 1630s^7t^3 \\ & -609s^7t^2 - 60s^7t + 212s^6t^4 - 539s^6t^3 + 498s^6t^2 + 108s^6t \\ & -22s^5t^4 + 77s^5t^3 - 108s^5t^2 - 12s^5t - 5s^4t^4 + 5s^4t^3 \\ & +15s^4t^2 - 36s^4t - 2s^3t^3 + 7s^3t + 3s^2t^3 - 5s^2t^2 - st^3 \\ & +2st^2 - st + t^3 - 2t^2 + t + 1 \\ m_{12} \;=\; & -512s^{17}t^2 - 512s^{16}t^3 + 1792s^{16}t^2 + 2304s^{15}t^3 - 288s^{15}t^2 \\ & -512s^{14}t^4 - 2592s^{14}t^3 - 6912s^{14}t^2 + 1792s^{13}t^4 - 2784s^{13}t^3 \\ & +10704s^{13}t^2 - 1824s^{12}t^4 + 8368s^{12}t^3 - 1264s^{12}t^2 - 768s^{11}t^4 \\ & -5184s^{11}t^3 - 10946s^{11}t^2 + 3184s^{10}t^4 - 2418s^{10}t^3 \\ & +11368s^{10}t^2 - 32s^{10}t - 3008s^9t^4 + 5122s^9t^3 - 4840s^9t^2 \\ & +56s^9t + 1590s^8t^4 - 3350s^8t^3 + 1404s^8t^2 + 48s^8t - 572s^7t^4 \\ & +1354s^7t^3 - \tfrac{1589s^7t^2}{2} - 148s^7t + 128s^6t^4 - \tfrac{693s^6t^3}{2} + 339s^6t^2 \\ & +47s^6t - 6s^5t^4 + \tfrac{83s^5t^3}{2} - 70s^5t^2 + 70s^5t - \tfrac{5s^4t^4}{2} - \tfrac{11s^4t^3}{2} \\ & +\tfrac{49s^4t^2}{2} - \tfrac{89s^4t}{2} + s^3t^3 - \tfrac{11s^3t^2}{2} + \tfrac{11s^3t}{2} + \tfrac{s^2t^3}{2} - 2s^2t - st^3 \\ & +\tfrac{3st^2}{2} - s + \tfrac{t^3}{2} - t^2 + \tfrac{t}{2} \end{aligned}$$

$$\begin{aligned} m_{13} \;=\; & 32768t^3s^{26} + 32768t^4s^{25} - 155648t^3s^{25} - 188416t^4s^{24} \\ & +116736t^3s^{24} + 32768t^5s^{23} + 305152t^4s^{23} + 650752t^3s^{23} \\ & -155648t^5s^{22} + 247296t^4s^{22} - 1527552t^3s^{22} + 215040t^5s^{21} \\ & -1324288t^4s^{21} + 383744t^3s^{21} + 134656t^5s^{20} + 1148928t^4s^{20} \\ & +2557824t^3s^{20} - 689920t^5s^{19} + 878976t^4s^{19} - 3331104t^3s^{19} \\ & +2048t^2s^{19} + 655616t^5s^{18} - 2240928t^4s^{18} + 459664t^3s^{18} \\ & -5120t^2s^{18} + 51072t^5s^{17} + 1201200t^4s^{17} + 2203296t^3s^{17} \\ & -5504t^2s^{17} - 585504t^5s^{16} + 487408t^4s^{16} - 1997496t^3s^{16} \\ & +26240t^2s^{16} + 536592t^5s^{15} - 960136t^4s^{15} + 724408t^3s^{15} \\ & -8480t^2s^{15} - 264192t^5s^{14} + 574720t^4s^{14} - 195564t^3s^{14} \\ & -50096t^2s^{14} + 84552t^5s^{13} - 200492t^4s^{13} + 111648t^3s^{13} \\ & +61408t^2s^{13} - 14504t^5s^{12} + 39980t^4s^{12} - 26414t^3s^{12} \\ & +2212t^2s^{12} - 1212t^5s^{11} - 7290t^4s^{11} - 4516t^3s^{11} \\ & -54288t^2s^{11} + 64ts^{11} + 640t^5s^{10} + 12146t^4s^{10} \\ & -13480t^3s^{10} + 46788t^2s^{10} - 112ts^{10} + 50t^5s^9 - 11114t^4s^9 \\ & +19346t^3s^9 - 19280t^2s^9 - 80ts^9 + 5596t^4s^8 - 11956t^3s^8 \\ & +5665t^2s^8 + 324ts^8 - 1878t^4s^7 + 4537t^3s^7 - 2568t^2s^7 \\ & -240ts^7 + 406t^4s^6 - 1183t^3s^6 + 1228t^2s^6 - 82ts^6 \\ & -39t^4s^5 + 211t^3s^5 - 305t^2s^5 + 284ts^5 - 10t^4s^4 - 34t^3s^4 \\ & +98t^2s^4 - 220ts^4 + 23t^3s^3 - 54t^2s^3 + 67ts^3 - 5t^3s^2 \\ & +16t^2s^2 - 11ts^2 + s^2 - 2t^2s + 5ts - 2s + t^3 - t^2 - t + 2 \\ m_{21} \;=\; & -512s^{15}t^2 + 1024s^{14}t^2 + 512s^{13}t^3 + 480s^{13}t^2 \\ & -1536s^{12}t^3 - 2096s^{12}t^2 + 1056s^{11}t^3 \\ & +600s^{11}t^2 + 1296s^{10}t^3 + 1004s^{10}t^2 \\ & -2536s^9t^3 - 268s^9t^2 + 1740s^8t^3 - 384s^8t^2 - 32s^8t \\ & -720s^7t^3 + 234s^7t^2 + 8s^7t \\ & +212s^6t^3 - 105s^6t^2 + 36s^6t - 22s^5t^3 + 33s^5t^2 \\ & -16s^5t - 5s^4t^3 - 5s^4t^2 + 4s^4t - 2s^3t^2 - 2s^3t \\ & +3s^2t^2 - s^2t - st^2 + t^2 \\ m_{22} \;=\; & 512s^{16}t^2 - 1280s^{15}t^2 - 512s^{14}t^3 + 32s^{14}t^2 + 1792s^{13}t^3 \\ & +2336s^{13}t^2 - 1824s^{12}t^3 - 1648s^{12}t^2 - 768s^{11}t^3 - \\ & 704s^{11}t^2 + 3184s^{10}t^3 + 770s^{10}t^2 - 3008s^9t^3 + 250s^9t^2 \\ & +32s^9t + 1590s^8t^3 - 426s^8t^2 - 24s^8t - 572s^7t^3 \\ & +222s^7t^2 - 40s^7t + 128s^6t^3 - \tfrac{171s^6t^2}{2} + 36s^6t \\ & -6s^5t^3 + \tfrac{59s^5t^2}{2} - 7s^5t - \tfrac{5s^4t^3}{2} - \tfrac{21s^4t^2}{2} + 5s^4t + s^3t^2 \\ & -\tfrac{3s^3t}{2} + \tfrac{s^2t^2}{2} - st^2 - \tfrac{st}{2} + \tfrac{t^2}{2} \\ m_{23} \;=\; & -32768s^{25}t^3 + 122880s^{24}t^3 + 32768s^{23}t^4 - 59392s^{23}t^3 \\ & -155648s^{22}t^4 - 333312s^{22}t^3 + 215040s^{21}t^4 + 485632s^{21}t^3 \\ & +134656s^{20}t^4 + 107008s^{20}t^3 - 689920s^{19}t^4 - 603008s^{19}t^3 \\ & +655616s^{18}t^4 + 241312s^{18}t^3 - 2048s^{18}t^2 + 51072s^{17}t^4 \\ & +230160s^{17}t^3 + 3072s^{17}t^2 - 585504s^{16}t^4 - 155216s^{16}t^3 \\ & +6528s^{16}t^2 + 536592s^{15}t^4 - 56056s^{15}t^3 - 14592s^{15}t^2 \\ & -264192s^{14}t^4 + 75344s^{14}t^3 + 928s^{14}t^2 + 84552s^{13}t^4 \\ & -28964s^{13}t^3 + 15056s^{13}t^2 - 14504s^{12}t^4 + 9692s^{12}t^3 \\ & -10128s^{12}t^2 - 1212s^{11}t^4 - 9814s^{11}t^3 - 404s^{11}t^2 \\ & +640s^{10}t^4 + 13426s^{10}t^3 + 1540s^{10}t^2 - 64s^{10}t \\ & +50s^9t^4 - 11014s^9t^3 + 1024s^9t^2 + 112s^9t + 5596s^8t^3 \\ & -1576s^8t^2 + 16s^8t - 1878s^7t^3 + 859s^7t^2 - 148s^7t \\ & +406s^6t^3 - 351s^6t^2 + 128s^6t - 39s^5t^3 + 133s^5t^2 \\ & -46s^5t - 10s^4t^3 - 54s^4t^2 + 10s^4t + 23s^3t^2 \\ & -8s^3t - 5s^2t^2 + 4s^2t - 2st + s + t^2 + t \\ m_{31} \;=\; & -512s^{16}t^2 + 512s^{15}t^3 + 2048s^{15}t^2 - 2048s^{14}t^3 - 2080s^{14}t^2 \\ & +512s^{13}t^4 + 1568s^{13}t^3 - 2032s^{13}t^2 - 1536s^{12}t^4 \\ & +3568s^{12}t^3 + 5272s^{12}t^2 + 1056s^{11}t^4 - 6584s^{11}t^3 - 2292s^{11}t^2 \\ & +1296s^{10}t^4 + 1892s^{10}t^3 - 1676s^{10}t^2 - 2536s^9t^4 + 3364s^9t^3 \\ & +1156s^9t^2 - 32s^9t + 1740s^8t^4 - 3440s^8t^3 + 766s^8t^2 + 72s^8t \\ & -720s^7t^4 + 1630s^7t^3 - 1029s^7t^2 - 12s^7t + 212s^6t^4 \\ & -539s^6t^3 + 473s^6t^2 - 80s^6t - 22s^5t^4 + 77s^5t^3 \\ & -108s^5t^2 + 72s^5t - 5s^4t^4 + 5s^4t^3 + 15s^4t^2 - 26s^4t \\ & -2s^3t^3 + 7s^3t + 3s^2t^3 - 5s^2t^2 - st^3 + 2st^2 - st + t^3 - 2t^2 + t \end{aligned}$$

$$
\begin{aligned}
m_{32} \quad = \quad & 512s^{17}t^2 - 512s^{16}t^3 - 2304s^{16}t^2 + 2304s^{15}t^3 + 3104s^{15}t^2 \\
& -512s^{14}t^4 - 2592s^{14}t^3 + 992s^{14}t^2 + 1792s^{13}t^4 \\
& -2784s^{13}t^3 - 6288s^{13}t^2 - 1824s^{12}t^4 + 8368s^{12}t^3 \\
& +4928s^{12}t^2 - 768s^{11}t^4 - 5184s^{11}t^3 + 530s^{11}t^2 \\
& +3184s^{10}t^4 - 2418s^{10}t^3 - 1994s^{10}t^2 + 32s^{10}t \\
& -3008s^9t^4 + 5122s^9t^3 - 188s^9t^2 - 88s^9t + 1590s^8t^4 \\
& -3350s^8t^3 + 1412s^8t^2 + 40s^8t - 572s^7t^4 + 1354s^7t^3 \\
& -\frac{1959s^7t^2}{2} + 92s^7t + 128s^6t^4 - \frac{693s^6t^3}{2} \\
& +\frac{653s^6t^2}{2} - 119s^6t - 6s^5t^4 + \frac{83s^5t^3}{2} \\
& -70s^5t^2 + 55s^5t - \frac{5s^4t^4}{2} - \frac{11s^4t^3}{2} \\
& +\frac{49s^4t^2}{2} - \frac{37s^4t}{2} + s^3t^3 - \frac{11s^3t^2}{2} + 8s^3t \\
& +\frac{s^2t^3}{2} - 2s^2t - st^3 + \frac{3st^2}{2} + \frac{t^3}{2} - t^2 + \frac{t}{2}
\end{aligned}
$$

$$
\begin{aligned}
m_{33} \quad = \quad & -32768t^3s^{26} + 32768t^4s^{25} + 188416t^3s^{25} - 188416t^4s^{24} \\
& -337920t^3s^{24} + 32768t^5s^{23} + 305152t^4s^{23} - 91648t^3s^{23} \\
& -155648t^5s^{22} + 247296t^4s^{22} + 1092864t^3s^{22} + 215040t^5s^{21} \\
& -1324288t^4s^{21} - 1197568t^3s^{21} + 134656t^5s^{20} + 1148928t^4s^{20} \\
& -331392t^3s^{20} - 689920t^5s^{19} + 878976t^4s^{19} + 1554336t^3s^{19} \\
& -2048t^2s^{19} + 655616t^5s^{18} - 2240928t^4s^{18} - 853424t^3s^{18} \\
& +7168t^2s^{18} + 51072t^5s^{17} + 1201200t^4s^{17} - 381392t^3s^{17} \\
& -1664t^2s^{17} - 585504t^5s^{16} + 487408t^4s^{16} + 484152t^3s^{16} \\
& -24576t^2s^{16} + 536592t^5s^{15} - 960136t^4s^{15} + 63472t^3s^{15} \\
& +36640t^2s^{15} - 264192t^5s^{14} + 574720t^4s^{14} - 280668t^3s^{14} \\
& -1392t^2s^{14} + 84552t^5s^{13} - 200492t^4s^{13} + 150468t^3s^{13} \\
& -39312t^2s^{13} - 14504t^5s^{12} + 39980t^4s^{12} - 20114t^3s^{12} \\
& +34908t^2s^{12} - 1212t^5s^{11} - 7290t^4s^{11} - 4266t^3s^{11} \\
& -7780t^2s^{11} - 64ts^{11} + 640t^5s^{10} + 12146t^4s^{10} \\
& -13480t^3s^{10} - 2332t^2s^{10} + 240ts^{10} + 50t^5s^9 - 11114t^4s^9 \\
& +19346t^3s^9 - 2436t^2s^9 - 272ts^9 + 5596t^4s^8 - 11956t^3s^8 \\
& +5227t^2s^8 - 68ts^8 - 1878t^4s^7 + 4537t^3s^7 \\
& -3333t^2s^7 + 440ts^7 + 406t^4s^6 - 1183t^3s^6 + 1178t^2s^6 \\
& -450ts^6 - 39t^4s^5 + 211t^3s^5 - 305t^2s^5 + 230ts^5 - 10t^4s^4 \\
& -34t^3s^4 + 98t^2s^4 - 74ts^4 + 23t^3s^3 - 54t^2s^3 + 30ts^3 \\
& -5t^3s^2 + 16t^2s^2 - 16ts^2 + s^2 - 2t^2s \\
& +5ts - 2s + t^3 - t^2 - t + 1
\end{aligned}
$$

$$
\begin{aligned}
m_{41} \quad = \quad & -512s^{16}t^2 - 512s^{15}t^3 + 1536s^{15}t^2 + 2048s^{14}t^3 + 480s^{14}t^2 \\
& -512s^{13}t^4 - 1568s^{13}t^3 - 6672s^{13}t^2 + 1536s^{12}t^4 - 3568s^{12}t^3 \\
& +7368s^{12}t^2 - 1056s^{11}t^4 + 6584s^{11}t^3 + 2420s^{11}t^2 - 1296s^{10}t^4 \\
& -1892s^{10}t^3 - 9736s^{10}t^2 + 2536s^9t^4 - 3364s^9t^3 + 6500s^9t^2 \\
& -32s^9t - 1740s^8t^4 + 3440s^8t^3 - 1590s^8t^2 + 40s^8t \\
& +720s^7t^4 - 1630s^7t^3 + 609s^7t^2 + 60s^7t - 212s^6t^4 \\
& +539s^6t^3 - 498s^6t^2 - 108s^6t + 22s^5t^4 - 77s^5t^3 \\
& +108s^5t^2 + 12s^5t + 5s^4t^4 - 5s^4t^3 - 15s^4t^2 \\
& +36s^4t + 2s^3t^3 - 7s^3t - 3s^2t^3 + 5s^2t^2 + st^3 - 2st^2 + st \\
& -t^3 + 2t^2 - t - 1
\end{aligned}
$$

$$
\begin{aligned}
m_{42} \quad = \quad & 512s^{17}t^2 + 512s^{16}t^3 - 1792s^{16}t^2 - 2304s^{15}t^3 + 288s^{15}t^2 \\
& +512s^{14}t^4 + 2592s^{14}t^3 + 6912s^{14}t^2 - 1792s^{13}t^4 \\
& +2784s^{13}t^3 - 10704s^{13}t^2 + 1824s^{12}t^4 - 8368s^{12}t^3 \\
& +1264s^{12}t^2 + 768s^{11}t^4 + 5184s^{11}t^3 + 10946s^{11}t^2 \\
& -3184s^{10}t^4 + 2418s^{10}t^3 - 11368s^{10}t^2 + 32s^{10}t \\
& +3008s^9t^4 - 5122s^9t^3 + 4840s^9t^2 - 56s^9t - 1590s^8t^4 \\
& +3350s^8t^3 - 1404s^8t^2 - 48s^8t + 572s^7t^4 - 1354s^7t^3 \\
& +\frac{1589s^7t^2}{2} + 148s^7t - 128s^6t^4 + \frac{693s^6t^3}{2} - 339s^6t^2 \\
& -47s^6t + 6s^5t^4 - \frac{83s^5t^3}{2} + 70s^5t^2 - 70s^5t \\
& +\frac{5s^4t^4}{2} + \frac{11s^4t^3}{2} - \frac{49s^4t^2}{2} + \frac{89s^4t}{2} - s^3t^3 + \frac{11s^3t^2}{2} \\
& -\frac{11s^3t}{2} - \frac{s^2t^3}{2} + 2s^2t + st^3 - \frac{3st^2}{2} - \frac{t^3}{2} + t^2 - \frac{t}{2}
\end{aligned}
$$

$$
\begin{aligned}
m_{43} \quad = \quad & -32768t^3s^{26} - 32768t^4s^{25} + 155648t^3s^{25} + 188416t^4s^{24} \\
& -116736t^3s^{24} - 32768t^5s^{23} - 305152t^4s^{23} - 650752t^3s^{23} \\
& +155648t^5s^{22} - 247296t^4s^{22} + 1527552t^3s^{22} - 215040t^5s^{21} \\
& +1324288t^4s^{21} - 383744t^3s^{21} - 134656t^5s^{20} - 1148928t^4s^{20} \\
& -2557824t^3s^{20} + 689920t^5s^{19} - 878976t^4s^{19} + 3331104t^3s^{19} \\
& -2048t^2s^{19} - 655616t^5s^{18} + 2240928t^4s^{18} - 459664t^3s^{18} \\
& +5120t^2s^{18} - 51072t^5s^{17} - 1201200t^4s^{17} - 2203296t^3s^{17} \\
& +5504t^2s^{17} + 585504t^5s^{16} - 487408t^4s^{16} + 1997496t^3s^{16} \\
& -26240t^2s^{16} - 536592t^5s^{15} + 960136t^4s^{15} - 724408t^3s^{15} \\
& +8480t^2s^{15} + 264192t^5s^{14} - 574720t^4s^{14} + 195564t^3s^{14} \\
& +50096t^2s^{14} - 84552t^5s^{13} + 200492t^4s^{13} - 111648t^3s^{13} \\
& -61408t^2s^{13} + 14504t^5s^{12} - 39980t^4s^{12} + 26414t^3s^{12} \\
& -2212t^2s^{12} + 1212t^5s^{11} + 7290t^4s^{11} + 4516t^3s^{11} \\
& +54288t^2s^{11} - 64ts^{11} - 640t^5s^{10} - 12146t^4s^{10} \\
& +13480t^3s^{10} - 46788t^2s^{10} + 112ts^{10} - 50t^5s^9 + 11114t^4s^9 \\
& -19346t^3s^9 + 19280t^2s^9 + 80ts^9 - 5596t^4s^8 + 11956t^3s^8 \\
& -5665t^2s^8 - 324ts^8 + 1878t^4s^7 - 4537t^3s^7 + 2568t^2s^7 \\
& +240ts^7 - 406t^4s^6 + 1183t^3s^6 - 1228t^2s^6 + 82ts^6 + 39t^4s^5 \\
& -211t^3s^5 + 305t^2s^5 - 284ts^5 + 10t^4s^4 + 34t^3s^4 - 98t^2s^4 \\
& +220ts^4 - 23t^3s^3 + 54t^2s^3 - 67ts^3 + 5t^3s^2 - 16t^2s^2 \\
& +11ts^2 - s^2 + 2t^2s - 5ts + 2s - t^3 + t^2 + t.
\end{aligned}
$$

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