Subresultants and the Shape Lemma

David Cox & **Carlos D'Andrea**

MEGA 2022 Krakow June 2022

Computational Algebra and Geometry: A special issue in memory and honor of Agnes Szanto

June 2022

Agnes Szanto passed away on March 21st 2022 at the age of 55. She served for many years on the Editorial Board of the *Journal of Symbolic Computation*. Agnes made significant contributions on the development and analysis of symbolic and numerical algorithms for problems in algebra and geometry. She was an extraordinary person. A devoted teacher and mentor, she enthusiastically committed to the community and became an inspiring model of leadership. This special issue is to honor her memory.

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Please refer to the Guide for Authors to prepare your manuscript, and select the article type of "VSI: Agnes Szanto's special issue" when submitting your manuscript online.

Tentative Schedule:

Submission Open Date: June 15, 2022

Submission Deadline: September 30, 2023

Editorial Acceptance Deadline: March 31, 2024

In addition of submitting the manuscript through the EM, the submission file and cover letter have to be simultaneously sent to jsc.si.szanto@gmail.com.

Arxiv paper

Cox, David – D'Andrea, Carlos
Subresultants and the Shape lemma

arXiv:2112.10306

Starting example

In $\mathbb{C}[x_1, x_2]$ set

$$f = x_1^2 - x_2 - 1,\ g = x_1^2 + x_1x_2 - 2$$

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A lex Gröbner basis of $\langle f, g \rangle$ with $x_2 \prec x_1$ is $\{x_2^3 + 2x_2 - 1,\ x_1 - x_2^2 + 1\}$

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- One generator depending only on x_2
- One generator of degree one in x_1

Resultants and Subresultants

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$$I = \langle f, g \rangle = \langle x_2^3 + 2x_2 - 1,\ x_1 - x_2^2 - 1 \rangle$$

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$$x_1 - x_2^2 - 1 \stackrel{?}{=} \boxed{\mathrm{Sres}_1(f, g, x_1)}$$

Resultants

$$f(x_1) = f_m x_1^m + \ldots + f_1 x_1 + f_0$$
$$g(x_1) = g_n x_1^n + \ldots + g_1 x_1 + g_0$$

Resultants

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$$g(x_1) = g_n x_1^n + \ldots + g_1 x_1 + g_0$$
$$\boxed{\mathrm{Res}(f, g, x_1) = \det\left(Sylv(f, g)\right)}$$

$$\begin{pmatrix} f_m & \cdots & f_0 & & & \\ & f_m & \cdots & f_0 & & \\ & & \ddots & & \ddots & \\ & & & f_m & \cdots & f_0 \\ g_n & \cdots & g_0 & & & \\ & g_n & \cdots & g_0 & & \\ & & \ddots & & \ddots & \\ & & & g_n & \cdots & g_0 \end{pmatrix}$$

Subresultants

For $0 \leq t < \min\{m, n\}$,

$$\mathrm{Sres}_t(f, g, x_1) = \det \begin{array}{c} m+n-2t \\ \left|\begin{array}{ccccc} f_m & \cdots & \cdots & f_{t+1-(n-t-1)} & x_1^{n-t-1} f(x_1) \\ & \ddots & & \vdots & \vdots \\ & & f_m \quad \cdots & f_{t+1} & x_1^0 f(x_1) \\ \hline g_n & \cdots & \cdots & g_{t+1-(m-t-1)} & x_1^{m-t-1} g(x_1) \\ & \ddots & & \vdots & \vdots \\ & & g_n \quad \cdots & g_{t+1} & x_1^0 g(x_1) \end{array}\right| \end{array} \begin{array}{c} \\ n-t \\ \\ m-t \end{array}$$

In our case...

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But

$$\langle f, g\rangle = \langle \mathrm{Res}(f, g, x_1), \mathrm{Sres}_1(f, g, x_1)\rangle$$

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- When is $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g, x_1)\rangle$?

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Suppose $I = \langle f(x_1, x_2),\ g(x_1, x_2) \rangle$

- When has I a Shape Lemma?
- When is
  $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g, x_1) \rangle$?
  $I = \langle \mathrm{Res}(f, g, x_1), \mathrm{Sres}_1(f, g, x_1) \rangle$?

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$$f = x_2x_1^2 + x_1 + x_2^2 + x_2, \; g = x_2x_1 + 1$$

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$$I = \langle x_2^3(x_2 + 1), x_2x_1 + 1\rangle$$

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$$\langle x_2 + 1, x_1 - 1\rangle$$

Geometry of the Shape Lemma

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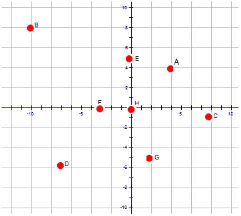
$$I = \langle p(x_2), x_1 - q(x_2) \rangle$$



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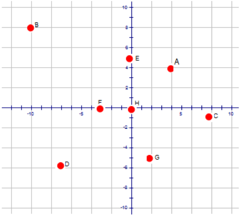

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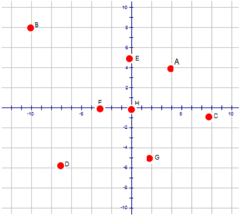

- Projection onto the x_2 axis should be injective
- How about multiplicities?

Equivalences

If $I \cap \mathbb{C}[x_2] = \langle r(x_2) \rangle$ and the projection $V(I) \to \mathbb{C}$ onto the x_2-axis injective, TFAE:

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- $\forall \xi \in V(I)$, $T_\xi(V(I)) \xrightarrow{\simeq} T_{\xi_2}(V(r))$

Geometry of Resultants

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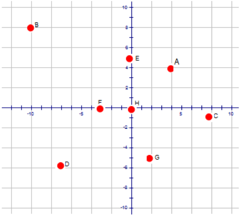

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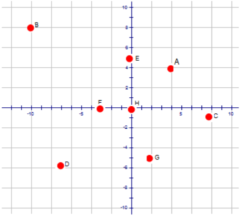
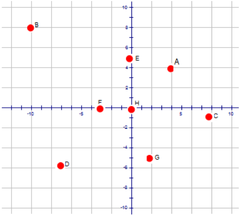


Poisson formula:

Geometry of Resultants



Poisson formula:

$$\mathrm{Res}(f, g, x_1) = c^* \cdot \prod_{(\xi_1,\xi_2)\in V(I)} (x_2 - \xi_2)^{m_\xi}$$

Resultants and Shape Lemma

(Cox-**D**)

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- $I = \langle \mathrm{Res}(f, g), \mathrm{Sres}_1(f, g)) \rangle$ and $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g) \rangle$

Works in several variables!

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Same results for zero-dim ideals

$$I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$$

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using multivariate **resultants** and **subresultants**

Multivariate Resultants

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Multivariate Resultants

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- $f_1, \ldots, f_n \in \mathbb{K}[x_n][x_1, \ldots, x_{n-1}]$

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- $f_1, \ldots, f_n \in \mathbb{K}[x_n][x_1, \ldots, x_{n-1}]$
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  their homogenizations
- $d_i = \deg_{x_1,\ldots,x_{n-1}}(f_i), i = 1, \ldots, n$

$$\boxed{\begin{array}{c}\mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n) := \\ \mathrm{Res}_{d_1,\ldots,d_n}(f_1^h, \ldots, f_n^h) \in \mathbb{K}[x_n]\end{array}}$$

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2. $f_1, \dots, f_n$ have no solutions at ∞

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Any two of the following three conditions imply the third:

1. $\langle f_1, \ldots, f_n \rangle$ has a Shape Lemma
2. $f_1, \ldots, f_n$ have no solutions at ∞
3. $\langle f_1, \ldots, f_n \rangle \cap \mathbb{K}[x_n] = \langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n) \rangle$

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  $\dim_{\mathbb{K}(x_n)} \left(\mathbb{K}(x_n)[x_0, \ldots, x_{n-1}]/\langle f_1^h, \ldots, f_n^h\rangle\right)_\rho = 1$

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  $s_\alpha(x_n) \in \mathbb{K}[x_n]$

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- For x^α in $x_0, \ldots, x_{n-1}$ of degree ρ the *scalar subresultant* $s_\alpha(x_n) \in \mathbb{K}[x_n]$ measures whether this monomial is nonzero above

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$$s_i(x_n) := s_{\alpha(i)}(x_n)$$

$$p_i(x_i, x_n) := s_0(x_n) x_i - s_i(x_n) \in \mathbb{K}[x_i, x_n]$$

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- $I = \langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n), p_1(x_1, x_n), \ldots, p_{n-1}(x_{n-1}, x_n) \rangle$ and $I \cap \mathbb{K}[x_n] = \langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n) \rangle$

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$$\gcd(\mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n), s_0(x_n), \ldots, s_{n-1}(x_n)) = 1$$

and I has a Shape Lemma

Thanks!

arXiv:2112.10306

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