Kernels of matrices of bivariate polynomials

Carlos D'Andrea

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UNIVERSITAT DE BARCELONA

Computational Algebra and Geometry: A special issue in memory and honor of Agnes Szanto

June 2022

Agnes Szanto passed away on March 21st 2022 at the age of 55. She served for many years on the Editorial Board of the *Journal of Symbolic Computation*. Agnes made significant contributions on the development and analysis of symbolic and numerical algorithms for problems in algebra and geometry. She was an extraordinary person. A devoted teacher and mentor, she enthusiastically committed to the community and became an inspiring model of leadership. This special issue is to honor her memory.

Guest editors:

Carlos D'Andrea, Universitat de Barcelona & Centre de Recerca Matemàtica, Facultat de MatemàtiquesiInformàtica.

Hoon Hong, NC State University, Department of Mathematics.

Evelyne Hubert, Inria; Université Côte d'Azur.

Prof. Teresa Krick, Universidad de Buenos Aires & CONICET, Depto de Matemática, FCEN & IMAS.

The **Editorial Manager®** is now available for receiving submissions to this special issue. The submission portal could be found here: https://www.editorialmanager.com/jsco/default1.aspx

Please refer to the Guide for Authors to prepare your manuscript, and select the article type of "VSI: Agnes Szanto's special issue" when submitting your manuscript online.

Tentative Schedule:

Submission Open Date: June 15, 2022

Submission Deadline: September 30, 2023

Editorial Acceptance Deadline: March 31, 2024

In addition of submitting the manuscript through the EM, the submission file and cover letter have to be simultaneously sent to jsc.si.szanto@gmail.com.

The team

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Computational Algebra Group at UB

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- Amrutha Balachandran Nair
- Teresa Cortadellas
- Eulàlia Montoro
- Juan Carlos Naranjo

The problem

Input: $k < n$, $\mathbb{K}$ a field and $p_{ij}(s,t) \in \mathbb{K}[s,t]$, $1 \leq i \leq k$, $1 \leq j \leq n$, of degrees bounded by d

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$$P(s,t) := \begin{pmatrix} p_{11}(s,t) & \dots & p_{1n}(s,t) \\ p_{21}(s,t) & \dots & p_{2n}(s,t) \\ \vdots & \dots & \vdots \\ p_{k1}(s,t) & \dots & p_{kn}(s,t) \end{pmatrix}$$

Output:

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- A $\mathbb{K}[s,t]$-basis $N_1(s,t),\ldots,N_\ell(s,t)$ of $\mathrm{Ker}\Big(p_{ij}(s,t)\Big) \subset \mathbb{K}[s,t]^n$

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- A $\mathbb{K}[s,t]$-basis $N_1(s,t),\ldots,N_\ell(s,t)$ of $\mathrm{Ker}\Big(p_{ij}(s,t)\Big) \subset \mathbb{K}[s,t]^n$
- Bounds on $\deg(N_i(s,t))$

Why is the kernel free?

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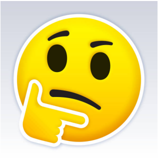
Hilbert's Syzygy Theorem

Why is the kernel free?



Hilbert's Syzygy Theorem

$$\mathbb{K}[s,t]^k \stackrel{P(s,t)}{\rightarrow} \mathbb{K}[s,t]^n \rightarrow \text{Coker} \rightarrow 0$$

Known cases

Univariate polynomials

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$$P(s) = \begin{pmatrix} p_{11}(s) & \ldots & p_{1n}(s) \\ \vdots & \ldots & \vdots \\ p_{k1}(s) & \ldots & p_{kn}(s) \end{pmatrix}$$

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Smith Normal Form

$$U(s) \cdot P(s) \cdot V(s) = \begin{pmatrix} * & \dots & * & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ * & \dots & * & 0 & \dots & 0 \end{pmatrix}$$

Univariate polynomials

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- The last columns of $V(s)$ are a basis of the kernel
- A $\mathbb{K}[s]$-basis can be found with degrees bounded by $4kd$

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Quillen Suslin Theorem

A unimodular $U(s,t) \in \mathbb{K}[s,t]^{n\times n}$ can be found

with $P(s,t) \cdot U(s,t) = \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}$

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- $U(s,t)$ can be found with degrees bounded by $12(kd)^4$
- Effective construction: uses univariate resultants and linear changes of coordinates

Known cases 3

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Parametrizations of surfaces

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- Cortadellas-**D**-Montoro 21: $\deg(N(s,t)) \in \mathcal{O}(d^{8})$ if the the base points are a complete intersection

Matrix Primitive Factorization

(Guiver-Bose 82)

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Degree bound: $\mathcal{O}(2^d)$!

Another approach

(following Almeida-D'Alfonso-Solernó 99)

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$$\mathcal{O}(d^{30})$$

And yet another approach!

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Linearization of polynomial matrices

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- $A, B \in \mathbb{K}^{L(kd)\times L(kd)+n-k}$

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$$\pi\Big(\mathrm{Ker}(A \cdot s + B)\Big) = \mathrm{Ker}(P(s))$$

Normal forms of pencils

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$$U \cdot (As + B) \cdot V = \begin{pmatrix} B_{k_1} & 0 & \dots & 0 & 0 \\ 0 & B_{k_2} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & \dots & B_{k_\ell} & 0 \\ 0 & 0 & \dots & 0 & M \end{pmatrix}$$

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$$B_{k_i} = \begin{pmatrix} s & 1 & 0 & \dots & 0 \\ 0 & s & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 \dots & s & 1 \end{pmatrix} \in \mathbb{K}[s]^{k_i \times (k_i+1)}$$

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M non singular

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The values of $k_1, \dots, k_\ell$ are unique

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The values of $k_1, \dots, k_\ell$ are unique

$\Longrightarrow$ A basis of $\mathrm{Ker}(A \cdot s + B)$ can be found with **minimal degrees** $k_1, \dots, k_\ell$

Can you do this in 2 variables?

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$$U(s,t)\cdot\begin{pmatrix}\mathbb{I} & 0\\ 0 & P(s,t)\end{pmatrix}\cdot V(s,t) = A\cdot s + B\cdot t + C$$

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"Canonical structure" of this matrix?

Partial results

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- Using Gröbner bases of modules

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- Syzygies over the syzygies

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Thanks!

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