Mixed volumes, sparse resultants and residues in the torus

Carlos D'Andrea

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UNIVERSITAT DE BARCELONA

Computational Algebra and Geometry: A special issue in memory and honor of Agnes Szanto

June 2022

Agnes Szanto passed away on March 21st 2022 at the age of 55. She served for many years on the Editorial Board of the *Journal of Symbolic Computation*. Agnes made significant contributions on the development and analysis of symbolic and numerical algorithms for problems in algebra and geometry. She was an extraordinary person. A devoted teacher and mentor, she enthusiastically committed to the community and became an inspiring model of leadership. This special issue is to honor her memory.

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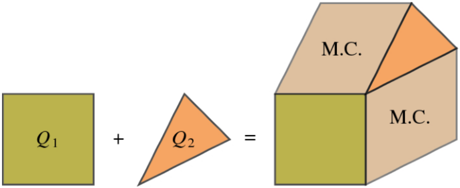

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- If Q_i is integral $\forall i, \; \mathrm{MV}(Q_j) \in \mathbb{Z}_{\geq 0}$

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$$\deg \left(V_{(\mathbb{C}^\times)^n}(f_1, \ldots, f_n)_0 \right) \leq \mathrm{MV}(\Delta_j)$$

Bernstein-Kushnirenko

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If $V := V_{(\mathbb{C}^\times)^n}(f_1, \ldots, f_n)$ is finite and each P_i n-dimensional:

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$$\mathrm{Res}^T_{f_1,\ldots,f_n}(h) := \sum_{p \in V} \frac{h(p)}{J^T_{f_1,\ldots,f_n}(p)}$$

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(Cox-Dickenstein 2005)

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if all P_i's are n-dimensional

(**D**-Dickenstein 2023)

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$\mathrm{Elim}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \in \mathbb{Z}[u_{i,a}]$ irreducible

defining the systems with solutions

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More homogeneities?

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$$\deg_\omega(\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1}) = 2$$

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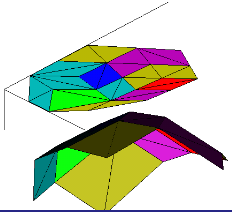

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$$\mu_i = \mathrm{MI}(x_i|_{\Delta_0}, \ldots, x_i|_{\Delta_n})$$

(**D**-Jeronimo-Sombra 2023)

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If this is the case

$$\mathrm{Res}_{\mathcal{A}}(\tilde{f}) = \pm \prod_{D \in \Pi_\rho} \mathrm{Res}_{\mathcal{A}_D}$$

(**D**-Jeronimo-Sombra 2023)

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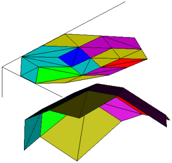
$$\omega = (\omega_0, \ldots, \omega_n) \in \mathbb{R}^{\mathcal{A}_0} \times \ldots \mathbb{R}^{\mathcal{A}_n}$$



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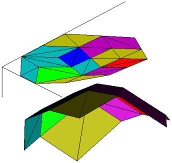

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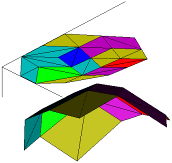

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$$\mathrm{init}_\omega(\mathrm{Res}_{\mathcal{A}}) = \pm \prod_{D \in S(\rho_\omega)} \mathrm{Res}_{\mathcal{A}_D}$$

(**D**-Jeronimo-Sombra 2023)

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References

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Thanks!

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