Subresultants and the Shape Lemma

David Cox & **Carlos D'Andrea**

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Solving over-determined systems by the subresultant method (with an appendix by Marc Chardin)

Agnes Szanto [1]

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Abstract

A general *subresultant method* is introduced to compute elements of a given ideal with few terms and bounded coefficients. This subresultant method is applied to solve over-determined polynomial systems by either finding a triangular representation of the solution set or by reducing the problem to eigenvalue computation. One of the ingredients of the subresultant method is

Resultant matrices

$$f(\xi) = f_m\xi^m + \ldots + f_1\xi + f_0 = 0$$
$$g(\xi) = g_n\xi^n + \ldots + g_1\xi + g_0 = 0$$

Resultant matrices

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$$\begin{pmatrix} f_m & \cdots & f_0 & & & \\ & f_m & \cdots & f_0 & & \\ & & \ddots & & \ddots & \\ & & & f_m & \cdots & f_0 \\ g_n & \cdots & g_0 & & & \\ & g_n & \cdots & g_0 & & \\ & & \ddots & & \ddots & \\ & & & g_n & \cdots & g_0 \end{pmatrix}$$

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$$\begin{pmatrix} f_m & \ldots & f_0 & & & \\ & f_m & \ldots & f_0 & & \\ & & \ddots & & \ddots & \\ & & & f_m & \ldots & f_0 \\ g_n & \ldots & g_0 & & & \\ & g_n & \ldots & g_0 & & \\ & & \ddots & & \ddots & \\ & & & g_n & \ldots & g_0 \end{pmatrix} \cdot \begin{pmatrix} \xi^{m+n-1} \\ \xi^{m+n-2} \\ \vdots \\ \vdots \\ \vdots \\ \xi \\ 1 \end{pmatrix} = 0$$

Great Idea

- Hilbert-Burch

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- Cramer's rule

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- Minors of resultant matrices

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- Minors of resultant matrices
- Subresultants

Great Idea

- Hilbert-Burch
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- Minors of resultant matrices
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- Overdetermined systems

Used in

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- **D**, Khetan: *Macaulay style formulas for toric residues*. Compos. Math. 05
- **D**, Jeronimo: *Subresultants and generic monomial bases*. J. Symbolic Comput. 05
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- **D**, Chipalkatti: *On the Jacobian ideal of the binary discriminant*. Collect. Math. 07
- **D** Sombra: *A Poisson formula for the sparse resultant*. Proc. Lond. Math. Soc. 15
- Buse, **D** ; Sombra, Weimann: *The geometry of the flex locus of a hypersurface*. Pacific J. Math. 20

And also in

Cox, **D**: *Subresultants and the Shape Lemma*. Math. Comp. 92 (2023), no. 343, 2355–2379

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Cox, **D**: *Subresultants and the Shape Lemma*. Math. Comp. 92 (2023), no. 343, 2355–2379

Starting example

In $\mathbb{C}[x_1, x_2]$ set

$$f = x_1^2 - x_2 - 1, \; g = x_1^2 + x_1 x_2 - 2$$

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$$f = x_1^2 - x_2 - 1,\ g = x_1^2 + x_1x_2 - 2$$

A lex Gröbner basis of $\langle f, g\rangle$ with $x_2 \prec x_1$ is $\{x_2^3 + 2x_2 - 1,\ x_1 - x_2^2 + 1\}$

Shape Lemma

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- A generator depending only on x_2

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- A generator depending only on x_2
- A monic generator linear in x_1

Resultants and Subresultants

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$$I = \langle f, g\rangle = \langle x_2^3 + 2x_2 - 1,\ x_1 - x_2^2 - 1\rangle$$

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$$I = \langle f, g \rangle = \langle x_2^3 + 2x_2 - 1,\ x_1 - x_2^2 - 1 \rangle$$

$$x_2^3 + 2x_2 - 1 = \boxed{\mathrm{Res}(f, g, x_1)}$$

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$$x_1 - x_2^2 - 1 \stackrel{?}{=} \boxed{\mathrm{Sres}_1(f, g, x_1)}$$

Resultants

$$f(x_1) = f_m x_1^m + \ldots + f_1 x_1 + f_0$$
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Resultants

$$f(x_1) = f_m x_1^m + \ldots + f_1 x_1 + f_0$$
$$g(x_1) = g_n x_1^n + \ldots + g_1 x_1 + g_0$$

$$\mathrm{Res}(f, g, x_1) = \det \begin{pmatrix} f_m & \cdots & f_0 & & & \\ & f_m & \cdots & f_0 & & \\ & & \ddots & & \ddots & \\ & & & f_m & \cdots & f_0 \\ g_n & \cdots & g_0 & & & \\ & g_n & \cdots & g_0 & & \\ & & \ddots & & \ddots & \\ & & & g_n & \cdots & g_0 \end{pmatrix}$$

First subresultant

$$\mathrm{Sres}_1(f, g, x_1) = \det \begin{pmatrix} f_m & \dots & f_0 & & & & \\ & f_m & \dots & f_0 & & & \\ & & \ddots & & \ddots & & \\ & & & f_m & \dots & f_0 + f_1 x \\ g_n & \dots & g_0 & & & & \\ & g_n & \dots & g_0 & & & \\ & & \ddots & & \ddots & & \\ & & & g_n & \dots & g_0 + g_1 x \end{pmatrix}$$

In our case...

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$$\mathrm{Sres}_1(f, g, x_1) = x_2 x_1 + x_2 - 1$$

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But

$$\langle f, g \rangle = \langle \mathrm{Res}(f, g, x_1), \mathrm{Sres}_1(f, g, x_1) \rangle$$

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Questions

Suppose $I = \langle f(x_1, x_2), g(x_1, x_2) \rangle$

- When has I a Shape Lemma?
- When $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g, x_1) \rangle$?
  $I = \langle \mathrm{Res}(f, g, x_1), \mathrm{Sres}_1(f, g, x_1) \rangle$?

Warning!

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$$f = x_2 x_1^2 + x_1 + x_2^2 + x_2, \; g = x_2 x_1 + 1$$

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$$f = x_2x_1^2 + x_1 + x_2^2 + x_2,\ g = x_2x_1 + 1$$

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$$\langle x_2 + 1, x_1 - 1\rangle$$

Geometry of the Shape Lemma

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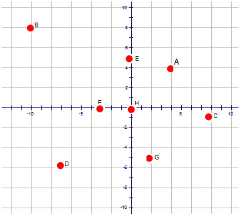
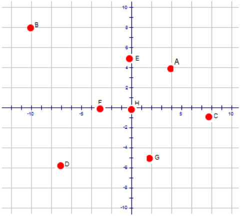
$$I = \langle p(x_2), x_1 - q(x_2) \rangle$$



Geometry of the Shape Lemma

$$I = \langle p(x_2),\, x_1 - q(x_2)\rangle$$



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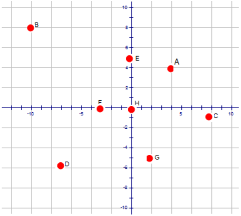

- Projection onto the x_2 axis should be injective
- How about multiplicities?

Equivalences

If $I \cap \mathbb{C}[x_2] = \langle r(x_2) \rangle$ and the projection $V(I) \to \mathbb{C}$ onto the x_2-axis injective, TFAE:

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- $\forall \underline{\xi} \in V(I), m_V(\underline{\xi}) = m_r(\xi_2)$
- $\forall \underline{\xi} \in V(I), T_{\underline{\xi}}(V(I)) \xrightarrow{\simeq} T_{\xi_2}(V(r))$

Geometry of Resultants

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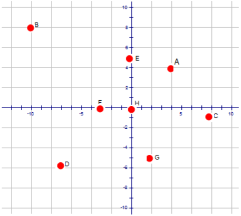

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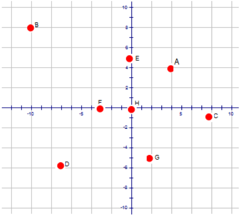
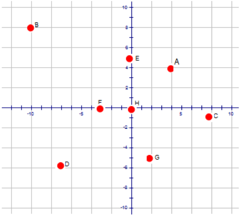


Poisson formula:

Geometry of Resultants



Poisson formula:

$$\mathrm{Res}(f, g, x_1) = c^* \cdot \prod_{(\xi_1,\xi_2)\in V(I^h)} (x_2 - \xi_2)^{m_\xi}$$

Resultants and Shape Lemma

(Cox-**D**)

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If the projection $V(I) \to \mathbb{C}$ onto the x_2-axis is injective, any two of the following conditions imply the third:

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- I has a Shape Lemma
- $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g, x_1) \rangle$
- $\gcd(\mathrm{lc}_{x_1}(f), \mathrm{lc}_{x_1}(g)) = 1$

Example

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Subresultants and Shape Lemma

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- $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g, x_1) \rangle$ and $\gcd(\mathrm{Res}(f, g, x_1), \mathrm{Sres}_1(f, g)) = 1$

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- I has a Shape Lemma and $\gcd(\mathrm{lc}_{x_1}(f), \mathrm{lc}_{x_1}(g)) = 1$
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- $I = \langle \mathrm{Res}(f, g, x_1), \mathrm{Sres}_1(f, g, x_1) \rangle$ and $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g, x_1) \rangle$

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Same results for zero-dim ideals

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using multivariate **resultants** and **subresultants**

Multivariate Resultants

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- $\mathrm{Res}_{d_1,\ldots,d_n}(g_1, \ldots, g_n) \in A$ the multivariate resultant

Multivariate Resultants

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- $f_1, \ldots, f_n \in \mathbb{K}[x_n][x_1, \ldots, x_{n-1}]$

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- $f_1, \ldots, f_n \in \mathbb{K}[x_n][x_1, \ldots, x_{n-1}]$
- $f_1^h, \ldots, f_n^h \in \mathbb{K}[x_n][x_0, \ldots, x_{n-1}]$
  their homogenizations

Multivariate Resultants

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- $f_1^h, \ldots, f_n^h \in \mathbb{K}[x_n][x_0, \ldots, x_{n-1}]$
  their homogenizations
- $d_i = \deg_{x_1,\ldots,x_{n-1}}(f_i), i = 1, \ldots, n$

$$\mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n) := \mathrm{Res}_{d_1,\ldots,d_n}(f_1^h, \ldots, f_n^h) \in \mathbb{K}[x_n]$$

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2. $f_1, \ldots, f_n$ have no solutions at ∞

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Any two of the following three conditions imply the third:

1. $\langle f_1, \ldots, f_n \rangle$ has a Shape Lemma
2. $f_1, \ldots, f_n$ have no solutions at ∞
3. $\langle f_1, \ldots, f_n \rangle \cap \mathbb{K}[x_n] = \langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n) \rangle$

“First Subresultant Polynomials”

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Szanto, A. Multivariate subresultants using Jouanolou matrices. J. Pure Appl. Algebra 214 (2010), no. 8, 1347–1369

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If $I = \langle f_1, \ldots, f_n \rangle$ is zero-dimensional TFAE:

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- $I \cap \mathbb{K}[x_n] = \langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n) \rangle$ and $\gcd(\mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n), s_0(x_n)) = 1$
- $I = \langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n), p_1(x_1, x_n), \ldots, p_{n-1}(x_{n-1}, x_n) \rangle$ and $I \cap \mathbb{K}[x_n] = \langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n) \rangle$

Subresultants and Shape Lemma

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If $I = \langle f_1, \ldots, f_n \rangle =$
$\langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n), p_1(x_1, x_n), \ldots, p_{n-1}(x_{n-1}, x_n) \rangle$
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If $I = \langle f_1, \ldots, f_n \rangle = \langle \mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n), p_1(x_1, x_n), \ldots, p_{n-1}(x_{n-1}, x_n) \rangle$ is zero-dimensional, then:

$$\gcd(\mathrm{Res}^{x_n}_{d_1,\ldots,d_n}(f_1, \ldots, f_n), s_0(x_n), \ldots, s_{n-1}(x_n)) = 1$$

and I has a Shape Lemma

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Thanks!