Determinants and Resultants in Elimination Theory

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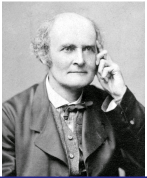
Gröbner free methods and their applications

Elimination Theory

Elimination Theory

In the problem of **elimination**, one seeks the relationship that must exist between the coefficients of a function or system of functions in order that some particular circumstance (or singularity) can occur.

Cayley, 1864 Nouvelles recherches sur l'élimination et la théorie des courbes



Elimination Theory 2

A system of arbitrarily many algebraic equations for $z^0, z', z'', \ldots, z^{(n-1)}$, in which the coefficients belong to the rationality domain $(R, R', R'', \ldots .)$, define the algebraic relations between z and R, whose knowledge and representation are the purpose of the **theory of elimination**.

Kronecker, 1882 Grundzüge einer arithmetischen Theorie der algebraischen Grössen

The baby example in elimination

The baby example in elimination

Find "the conditions" on

$a_{10}, a_{11}, a_{20}, a_{21}$ so that the system

$$\begin{cases} a_{10}x_0 + a_{11}x_1 = 0 \\ a_{20}x_0 + a_{21}x_1 = 0 \end{cases}$$

has a solution different from $(0,0)$

"Elimination"

$$a_{10}x_0 + a_{11}x_1, \; a_{20}x_0 + a_{21}x_1$$

$$\in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}, x_0, x_1]$$

“Elimination”

$$a_{10}x_0 + a_{11}x_1,\ a_{20}x_0 + a_{21}x_1$$

$$\in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}, x_0, x_1]$$

$$\downarrow$$

$$a_{10}a_{21} - a_{20}a_{11} \in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}]$$

($\mathbb{K}$ any field)

More general

Find "the conditions" for the system

$$\begin{cases} a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n & = 0 \\ a_{20}x_0 + a_{21}x_1 + \ldots + a_{2n}x_n & = 0 \\ \vdots & \vdots \quad \vdots \\ a_{(n+1)0}x_0 + a_{(n+1)1}x_1 + \ldots + a_{(n+1)n}x_n & = 0 \end{cases}$$

More general

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to have a solution different from

$$(0, 0, \ldots, 0)$$

Another more general

Let $d_1, d_2 \in \mathbb{N}$. Find "the conditions" for the system of polynomials

$$\begin{cases} a_{10}{x_0}^{d_1} + a_{11}{x_0}^{d_1-1}x_1 + \ldots = 0 \\ a_{20}{x_0}^{d_2} + a_{21}{x_0}^{d_2-1}x_1 + \ldots = 0 \end{cases}$$

to have a solution different from $(0,0)$

More more more general...

Let $n \in \mathbb{N}$, and $d_1, \ldots, d_{n+1} \in \mathbb{N}$, find the condition for

$$\left\{\begin{array}{rcl} \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} & = & 0 \\ \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} & = & 0 \\ \vdots & \vdots & \vdots \\ \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} & = & 0 \end{array}\right.$$

to have a solution different from $(0, 0, \ldots, 0)$

Elimination: The general problem

For $\mathbf{a} = (a_1, \ldots, a_N)$, $k,\ n \in \mathbb{N}$ let $f_1(\mathbf{a}, x_1, \ldots, x_n), \ldots, f_k(\mathbf{a}, x_1, \ldots, x_n) \in \mathbb{K}[\mathbf{a}, x_1, \ldots, x_n]$. Find conditions on $\mathbf{a}$ such that

$$\left\{\begin{array}{rcl} f_1(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \\ f_2(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ f_k(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \end{array}\right.$$

has a solution

Solution?

- Depends on the ground field

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- There is not necessarily a “closed” condition

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- Depends on the ground field
- There is not necessarily a “closed” condition
- Algorithms??

Easy example

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{kn}x_1 + \ldots + a_{kn}x_n & = & 0 \end{cases}$$

with $k \geq n$

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with $k \geq n$

Conditions: all maximal minors of $\left(a_{ij}\right)_{1\leq i\leq k,\,1\leq j\leq n}$ equal to zero

Another "easy" example

$$k = n = 1,$$

$$a_0 + a_1 x_1 + a_2 {x_1}^2 + \ldots + a_d {x_1}^d = 0$$

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Conditions?

Geometry

$$V = \{(\mathbf{a}, x_1, \ldots, x_n) : f_1(\mathbf{a}, x_1, \ldots, x_n) = 0, \ldots f_k(\mathbf{a}, x_1, \ldots, x_n) = 0\}$$

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$$\begin{array}{ccc} V & \subset & \mathbb{K}^N \times \mathbb{K}^n \\ \pi_1|_V \downarrow & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array}$$

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The set of conditions is $\pi_1(V)$ is not necessarily described by zeroes of polynomials

Projective Elimination

(homogeneous polynomials in an algebraically closed field)

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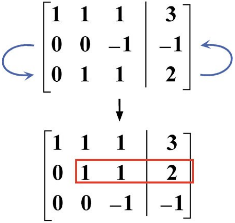
$$\pi_1(V) = \{p_1(\mathbf{a}) = 0, \ldots, p_\ell(\mathbf{a}) = 0\}$$

Homogeneous vs non homogeneous

Homogeneous vs non homogeneous

Analogy with Linear Algebra...



Generalization of the determinant?

$$V = \{(\mathbf{a}, x_0, x_1, \ldots, x_n) : f_1(\mathbf{a}, x_0, x_1, \ldots, x_n) = 0, \ldots, f_{n+1}(\mathbf{a}, x_0, x_1, \ldots, x_n) = 0\}$$

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One hopes!

Example 1

$$\left\{\begin{array}{rcl} a_{00}x_0 + a_{01}x_1 + \ldots + a_{0n}x_n & = & 0 \\ a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{n0}x_0 + a_{n1}x_1 + \ldots + a_{nn}x_n & = & 0 \end{array}\right.$$

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$$\boxed{p_1(a) = \det(a_{ij})}$$

Example 2

$$\begin{cases} f_1 = a_{10}{x_0}^{d_1} + a_{11}{x_0}^{d_1-1}x_1 + \ldots + a_{1d_1}{x_1}^{d_1} \\ f_2 = a_{20}{x_0}^{d_2} + a_{21}{x_0}^{d_2-1}x_1 + \ldots + a_{2d_2}{x_1}^{d_2} \end{cases}$$

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$$\mathrm{Res}(f_1, f_2) = \det \begin{pmatrix} a_{10} & a_{11} & \ldots & a_{1d_1} & 0 & \ldots & 0 \\ 0 & a_{10} & \ldots & a_{1d_1-1} & a_{1d_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{10} & \ldots & \ldots & a_{1d_1} \\ a_{20} & a_{21} & \ldots & a_{2d_2} & 0 & \ldots & 0 \\ 0 & a_{20} & \ldots & a_{2d_2-1} & a_{2d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{20} & \ldots & \ldots & a_{2d_2} \end{pmatrix}$$

Example 3

$$\begin{cases} f_1 = \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \\ f_2 = \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \\ \vdots \\ f_{n+1} = \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \end{cases}$$

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$$\mathrm{Res}_{d_1,\ldots,d_n}(f_1, f_2, \ldots, f_{n+1})$$

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$\mathrm{Res}_{d_1,\ldots,d_n}(f_1, f_2, \ldots, f_{n+1})$

Macaulay/dense/classical resultant

No-example

No-example

$$\begin{cases} a_{00}{x_0}^4 + a_{01}x_0^2x_1x_2 + a_{02}x_1^2{x_2}^2 & = & 0 \\ a_{10}{x_0}^4 + a_{11}x_0^2x_1x_2 + a_{12}x_1^2{x_2}^2 & = & 0 \\ a_{20}{x_0}^4 + a_{21}x_0^2x_1x_2 + a_{22}x_1^2{x_2}^2 & = & 0 \end{cases}$$

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$$\boxed{p_1(\boldsymbol{a}) \neq \det(a_{ij})}$$

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$$p_3(\mathbf{a}) = a_{20}^2a_{12}^2 - a_{20}a_{21}a_{11}a_{12} - 2a_{20}a_{22}a_{10}a_{12} + a_{20}a_{22}a_{11}^2 + a_{21}^2a_{10}a_{12} - a_{21}a_{22}a_{10}a_{11} + a_{22}^2a_{10}^2$$

The dictionary of elimination

Linear **Polynomial**

The dictionary of elimination

Linear	**Polynomial**
Gauss elimination	Gröbner Bases

The dictionary of elimination

Linear	**Polynomial**
Gauss elimination	Gröbner Bases
Triangulation	Triangular sets

The dictionary of elimination

Linear	**Polynomial**
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Determinants	Resultants

The dictionary of elimination

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Cramer's rule	u-resultants
...	...

Elimination via "triangulation"

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Two ways:

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Two ways:

- Gröbner Bases (w.r.t. "lex")

Elimination via “triangulation”

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- Gröbner Bases (w.r.t. “lex”)
- Triangular Decomposition

Elimination via "triangulation"

Two ways:

- Gröbner Bases (w.r.t. "lex")
- Triangular Decomposition

Both use some kind of **division** of polynomials

$$
\begin{array}{rc}
a_1: & y \\
a_2: & -1 \\
xy+1 & \\
y+1 & \sqrt{xy^2+1} \\
 & xy^2+y \\
\hline
 & -y+1 \\
 & -y-1 \\
\hline
 & 2
\end{array}
$$

Resultants

Resultants

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  $\iff\ \deg(\gcd(\tilde{f}_1, \tilde{f}_2)) > 0$

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- bi-homogeneous of bidegree (d_2, d_1)
- irreducible
- vanishes on $\tilde{f}_1,\ \tilde{f}_2 \in \mathbb{K}[x_0, x_1]$

$$\iff \deg(\gcd(\tilde{f}_1, \tilde{f}_2)) > 0$$

$$\iff \exists \boldsymbol{\xi} \in \mathbb{P}^1_{\mathbb{K}} : \ \tilde{f}_1(\boldsymbol{\xi}) = \tilde{f}_2(\boldsymbol{\xi}) = 0$$

Geometry

Geometry

$$V = \{(\alpha_{10}, \ldots, \alpha_{2d_2}; p_0 : p_1) : f_1(\boldsymbol{\alpha}, p_0, p_1) = f_2(\boldsymbol{\alpha}, p_0, p_1) = 0\}$$

$$\begin{array}{ccc} V & \subset & \mathbb{K}^{d_1+d_2+2} \times \mathbb{P}^1_{\mathbb{K}} \\ \pi_1|_V \downarrow & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^{d_1+d_2+2} \end{array}$$

Geometry

$$V = \{(\alpha_{10}, \ldots, \alpha_{2d_2}; p_0 : p_1) : f_1(\boldsymbol{\alpha}, p_0, p_1) = f_2(\boldsymbol{\alpha}, p_0, p_1) = 0\}$$

$$\begin{array}{ccc} V & \subset & \mathbb{K}^{d_1+d_2+2} \times \mathbb{P}^1_{\mathbb{K}} \\ \pi_1|_V \downarrow & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^{d_1+d_2+2} \end{array}$$

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  $\operatorname{Res}(f_1 \cdot f_1', f_2) = \operatorname{Res}(f_1, f_2) \cdot \operatorname{Res}(f_1', f_2)$

Multivariate Resultants

$$\begin{cases} f_1 = \sum_{|\alpha|=d_1} a_{1,\alpha} x_0{}^{\alpha_0} \dots x_n{}^{\alpha_n} \\ f_2 = \sum_{|\alpha|=d_2} a_{2,\alpha} x_0{}^{\alpha_0} \dots x_n{}^{\alpha_n} \\ \vdots \\ f_{n+1} = \sum_{|\alpha|=d_{n+1}} a_{n+1,\alpha} x_0{}^{\alpha_0} \dots x_n{}^{\alpha_n} \end{cases}$$

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What is $\text{Res}(f_1, \ldots, f_{n+1}) = \text{Res}_{d_1,\ldots,d_{n+1}}$?

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- The defining equation of

$$\begin{array}{ccc} V = \{(u_{i,\alpha},\xi) : f_i(u_{i,\alpha},\xi) = 0\} & \subset & \mathbb{K}^N \times \mathbb{P}^n \\ \downarrow & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array} \quad ?$$

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- A condition for the f_i's to have a common factor? A common zero?
- Can we compute it?

Known cases

If $d_1 = d_2 = \ldots = d_{n+1} = 1$, then

$$\mathrm{Res}_{1,1,\ldots,1} = \det \left(a_{ij} \right)_{1 \leq i \leq n+1,\, 0 \leq j \leq n}$$

Example 2

$$f_1 = a_{10}x_0 + a_{11}x_1 + a_{12}x_2$$
$$f_2 = a_{20}x_0 + a_{21}x_1 + a_{22}x_2$$
$$f_3 = a_{30}x_0{}^2 + a_{31}x_0x_1 + a_{32}x_0x_2 + a_{33}x_1{}^2 + a_{34}x_1x_2 + a_{35}x_2{}^2$$

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$$\begin{aligned}
\mathrm{Res}_{1,1,2} = {} & a_{10}^2a_{21}^2a_{35} - a_{10}^2a_{21}a_{22}a_{34} + a_{10}^2a_{22}^2a_{33} \\
& -2a_{10}a_{11}a_{20}a_{21}a_{35} + a_{10}a_{11}a_{20}a_{22}a_{34} \\
& +a_{10}a_{11}a_{21}a_{22}a_{32} - a_{10}a_{11}a_{22}^2a_{31} + a_{10}a_{12}a_{20}a_{21}a_{34} \\
& -2a_{10}a_{12}a_{20}a_{22}a_{33} - a_{10}a_{12}a_{21}^2a_{32} + a_{10}a_{12}a_{21}a_{22}a_{31} \\
& +a_{11}^2a_{20}^2a_{35} - a_{11}^2a_{20}a_{22}a_{32} + a_{11}^2a_{22}^2a_{30} \\
& -a_{11}a_{12}a_{20}^2a_{34} + a_{11}a_{12}a_{20}a_{21}a_{32} + a_{11}a_{12}a_{20}a_{22}a_{31} \\
& -2a_{11}a_{12}a_{21}a_{22}a_{30} + a_{12}^2a_{20}^2a_{33} - a_{12}^2a_{20}a_{21}a_{31} \\
& +a_{12}^2a_{21}^2a_{30}
\end{aligned}$$

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$\mathrm{Res}_{d_1,\dots,d_{n+1}}=$ irreducible equation of $\pi_1(V)$

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$\forall d \in \mathbb{N}$ the linear map

$$\begin{array}{ccc} \mathbb{K}[x]_{d-d_1} \oplus \ldots \oplus \mathbb{K}[x]_{d-d_{n+1}} & \stackrel{\phi_d}{\rightarrow} & \mathbb{K}[x]_d \\ (g_1, \ldots, g_{n+1}) & \mapsto & \sum_{j=1}^{n+1} g_i \tilde{f}_i \end{array}$$

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(Proof: If $\xi \in \mathbb{P}^n$ is such a root,

$\xi_0 \neq 0 \implies {x_0}^d \notin \mathrm{Im}(\phi_d)$)

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Macaulay Matrices

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For $d \geq d_1 + \ldots + d_{n+1} - n$, ϕ_d is surjective as a linear map over $\mathbb{K}(a_{i,\alpha})$.

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$$\mathrm{Res}_{d_1,\ldots,d_{n+1}} = \frac{\det(\phi_d^M)}{\det(\phi_d^E)}$$

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with ϕ_d^E being a "submatrix" of ϕ_d^M

Example

$$\begin{cases} f_1 & = a_0 x_0{}^2 + a_1 x_0 x_1 + a_2 x_1{}^2 + a_3 x_0 x_2 + a_4\, x_1 x_2 + a_5\, x_2{}^2 \\ f_2 & = b_0 x_0{}^2 + b_1 x_0 x_1 + b_2 x_1{}^2 + b_3 x_0 x_2 + b_4\, x_1 x_2 + b_5 x_2{}^2 \\ f_3 & = c_0 x_0 + c_1 x_1 + c_2 x_2 \end{cases}$$

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\begin{array}{cccccccccc} x_0^3 & x_0^2x_1 & x_0^2x_2 & x_0x_1x_2 & \boxed{x_0x_1^2} & x_1^3 & x_1^2x_2 & \boxed{x_0x_2^2} & x_1x_2^2 & x_2^3 \end{array}$$

$$\begin{bmatrix} a_0 & a_1 & a_3 & a_4 & \boxed{a_2} & 0 & 0 & \boxed{a_5} & 0 & 0 \\ 0 & a_0 & 0 & a_3 & a_1 & a_2 & a_4 & 0 & a_5 & 0 \\ 0 & 0 & a_0 & a_1 & 0 & 0 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_3 & b_4 & \boxed{b_2} & 0 & 0 & \boxed{b_5} & 0 & 0 \\ 0 & b_0 & 0 & b_3 & b_1 & b_2 & b_4 & 0 & b_5 & 0 \\ 0 & 0 & b_0 & b_1 & 0 & 0 & b_2 & b_3 & b_4 & b_5 \\ c_0 & c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_0 & 0 & c_2 & c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 0 & c_0 & 0 & 0 & c_1 & 0 & c_2 & 0 \end{bmatrix}.$$

Another “determinantal” formula

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For $d \in \mathbb{N}$ consider the Koszul Complex of $\mathsf{K} := \mathbb{K}(\boldsymbol{a}_{i,\alpha})$-e.v.

$$\cdots \to \bigoplus_{i<j} \mathsf{K}[\boldsymbol{x}]_{d-d_i-d_j} \to \bigoplus_i \mathsf{K}[\boldsymbol{x}]_{d-d_i} \stackrel{\phi_d}{\to} \mathsf{K}[\boldsymbol{x}]_d \to 0$$

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- For $d \geq \sum_{i=1}^{n+1} d_i - n$, the complex is exact.
- $\mathrm{Res}_{d_1,\ldots,d_n} = \det(\text{complex})$
  w.r.t. the monomial bases

What is det(complex)?

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$$\det(\text{complex}) = \frac{\det\left(M_{B_1^0,B_0}(f_1)\right)}{\det\left(M_{B_2,B_1^1}(f_2)\right)}, \quad B_1^0 \sqcup B_1^1 = B_1$$

In our case....

$$\cdots \to \bigoplus_{i<j} \mathsf{K}[x]_{d-d_i-d_j} \xrightarrow{\phi'_d} \bigoplus_i \mathsf{K}[x]_{d-d_i} \xrightarrow{\phi_d} \mathsf{K}[x]_d \to 0$$

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$$\det(\text{complex}) = \frac{M_1}{M_2}\frac{M_3}{M_4}\cdots$$

Other formulae

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Hybrid example

$$\begin{cases} f_1 &= a_0 x_0{}^2 + a_1 x_0 x_1 + a_2 x_1{}^2 + a_3 x_0 x_2 + a_4\, x_1 x_2 + a_5\, x_2{}^2 \\ f_2 &= b_0 x_0{}^2 + b_1 x_0 x_1 + b_2 x_1{}^2 + b_3 x_0 x_2 + b_4 x_1 x_2 + b_5 x_2{}^2 \\ f_3 &= c_0 x_0 + c_1 x_1 + c_2 x_2 \end{cases}$$

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$$J_f := \det\left(\frac{\partial f_i}{\partial x_j}\right) = j_0{x_0}^2 + j_1x_0x_1 + j_2{x_1}^2 + j_3x_0x_2 + j_4x_0x_2 + j_5{x_2}^2$$

$$\begin{array}{c} \\ f_1 \\ f_2 \\ J_f \\ x_0f_3 \\ x_1f_3 \\ x_2f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} {x_0}^2 & x_0x_1 & {x_1}^2 & x_0x_2 & x_1x_2 & {x_2}^2 \end{array} \\ \left[\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ j_0 & j_1 & j_2 & j_3 & j_4 & j_5 \\ c_0 & c_1 & 0 & c_2 & 0 & 0 \\ 0 & c_0 & c_1 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_0 & c_1 & c_2 \end{array}\right] \end{array}.$$

Application: systems solving

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Cramer's rule revisited

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Cramer's rule revisited

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n = b_2 \\ \quad\quad\vdots \quad\quad\quad\quad \vdots \quad \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n = b_n \end{cases}$$

U-Cramer

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Compute

$$\det \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ a_{21} & \dots & a_{2n} & b_2 \\ \vdots & \dots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & b_n \\ U_1 & 0 \dots & 0 & U_{n+1} \end{pmatrix} = A_1 U_1 + A_{n+1} U_{n+1}$$

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$$\det(\cdot) = 0 \implies U_1 \xi_1 + U_{n+1} \cdot 1 = 0$$

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$$\det(\cdot) = 0 \implies U_1\xi_1 + U_{n+1} \cdot 1 = 0$$
$$\implies -\frac{U_{n+1}}{U_1} = \xi_1 = \frac{A_1}{A_{n+1}}$$

U-resultant

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$$\begin{cases} F_1(x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ F_n(x_1, \ldots, x_n) & = & 0 \end{cases}$$

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$$\mathrm{Res}(f_1,\ldots,f_n,\mathrm{U}) =$$
$$c\cdot\prod_{F_i(\xi)=0,\,1\leq i\leq n}(U_0\xi_0+\ldots+U_n\xi_n)^{m_\xi}$$
$$(c\in\mathbb{K})$$

Hidden Cramer

Hidden Cramer

From

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n & = & b_n \end{cases}$$

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Compute

$$\det \begin{pmatrix} a_{11} & \ldots & a_{1n}x_n - b_1 \\ a_{21} & \ldots & a_{2n}x_n - b_2 \\ \vdots & \ldots & \vdots \\ a_{n1} & \ldots & a_{nn}x_n - b_n \end{pmatrix} = Ax_n + B.$$

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$$x_n = \frac{-B}{A}$$

Hidden variables - resultant

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$$\left\{ \begin{array}{rcl} F_1(x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ F_n(x_1, \ldots, x_n) & = & 0 \end{array} \right.$$

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$$\mathrm{Res}(f_1, \ldots, f_n) = c \cdot \prod_{F_i(\xi)=0} (\xi_0 x_n - \xi_n)^{m_\xi}$$

$$(c \in \mathbb{K})$$

Sparse Resultants

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In the "real world" systems of equations are neither homogeneous nor all the monomials appear in the expansion

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Sparse Resultants

- $\mathcal{A}_1, \ldots, \mathcal{A}_{n+1} \subset \mathbb{Z}^n$
- $F_i = \sum_{a \in \mathcal{A}_i} c_{i,a} x^a, \; i = 1, \ldots, n+1$

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What is
$\mathrm{Res}(F_1, \ldots, F_{n+1}) = \mathrm{Res}_{\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}}$?

Geometric definition

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(**D**-Sombra), PLMS 2015

$$\begin{array}{ccc} V = \{(u_{i,\mathrm{a}}, \boldsymbol{\xi}) : F_i(\mathrm{u}, \boldsymbol{\xi}) = 0 \,\forall i\} & \subset & \mathbb{K}^N \times (\mathbb{K}^\times)^n \\ \downarrow & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array}$$

$\mathrm{Res}_{\mathcal{A}}$ is the defining equation of $\pi_{1*}(V)$

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(could be identically 1)

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- With the new definition, $\mathrm{Res}_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3} = \det(c_{i,a})^4$

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- $$\text{Res}_{\mathcal{A}}(\tilde{F}_1,\ldots,\tilde{F}_{n+1}) = 0 \iff$$

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<u>Vanishing (essential)</u>

- $$\mathrm{Res}_{\mathcal{A}}(\tilde{F}_1,\ldots,\tilde{F}_{n+1}) = 0 \iff \exists \xi \in X_{\mathcal{A}} \supset (\mathbb{K}^\times)^n : \tilde{F}_i(\xi) = 0 \ \forall i$$

And there is more...

(**D**-Jeronimo-Sombra), JFoCM 2023

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- Initial forms & order of $\mathrm{Res}_{\mathcal{A}}$

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- Initial forms & order of $\mathrm{Res}_{\mathcal{A}}$
- Vanishing coefficients
- Macaulay formulae

$$\frac{\det(M_{\mathcal{A},\rho})}{\det(E_{\mathcal{A},\rho})} = \pm\,\mathrm{Res}_{\mathcal{A}}$$

Even More!

(**D**-Jeronimo 2024)

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- Applications to the Sparse Nullstellensatz

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Software available

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- Multires: a `Maple` package for the manipulation of multivariate polynomials, containing several tools for resultants, residues and the resolution of polynomial systems:
  https://www-sop.inria.fr/teams/galaad/software/multires/

  Authors: Laurent Busé and Bernard Mourrain

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- EliminationMatrices: a package for computing resultants in `Macaulay2`:

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- Resultants: a package for computation with resultants, discriminants, and Chow forms

  https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2-1.12/share/doc/Macaulay2/Resultants/html/

  Author: Giovanni Staglianò

Thanks!

Gröbner free methods and their applications

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