Auctions with Interdependent Values

The value of the object is a function of all bidders' signals, i.e.,

$$V_i = v_i(X_1, X_2, ..., X_n),$$

where v_i is non-decreasing in all its variables and strictly increasing in X_i. A particular case of the interdependent model is the *common value* where $V = v_i(X_1, X_2, ..., X_n)$, and where each bidder signal is an unbiased estimator of the value, $E(X_i/V = v) = v$.

The interdependence of values has important implications for bidders:

1. The winner's curse. Prior to the auction the only information of bidder i is his estimate of the value $E\left[V/X_1 = x\right]$. Conditional upon learning that he is the winner his estimated value decreases as it is given by $E\left[V/X_1 = x, Y_1 < x\right]$, which milieus that winning brings bad news. The estimate of the winner, conditional of the knowledge of being the winner, is upwards biased as

   $$E\left(\max X_i/V = v\right) > \max E(X_i/V = v) = v.$$

   This is the winner-curse, "I have won because I overestimated the value".

2. Non-equivalence of English and SPA. In the EA bidders get more information than in a SPA. Since information on others' signals affect v_i, these two auction formats are no longer equivalent.

3. The Revenue equivalence may no longer hold.

1 Affiliation

We will allow for correlation among bidders' signals. We will assumed that they are positively affiliated (a larger signals by my competitors makes it more likely that I will also have a large signal).

Suppose that the random variables $X_1, ...X_n$ are distributed on some product of intervals according to a joint density f. The variables $Z = (X_1...X_n)$ are affiliated if they are pairwise correlated on all rectangles in R^n. Formally:

$$f(z)f(z') \leq f(z \wedge z')f(z \vee z')$$

where $z \wedge z'$ is the pairwise minimum and $z \vee z'$ is the pairwise maximum. If the density is strictly positive in the interior of the intervals and twice continuously differentiable then f is affiliated iff $\frac{\partial^2 \ln(f(z))}{\partial z_i \partial z_j} \geq 0$ for all i, j (f is affiliated iff $\ln f$ is supermodular).

Affiliation requires that the product of the weights at the points (x, y) and (x', y') (where both values are high or both are low) is greater than (x, y') and (x', y) (where they are high and low, alternatively).

Implications of affiliation:

1. (a) If $X_1, X_2, ..., X_n$ are affiliated then $X_1, Y_1, ..., Y_{n-1}$ are also affiliated.

   (b) For any increasing function h, if $x' > x$ then $E\left[h(Y_1) \ / \ X_1 = x'\right] > E\left[h(Y_1) \ / \ X_1 = x\right].$

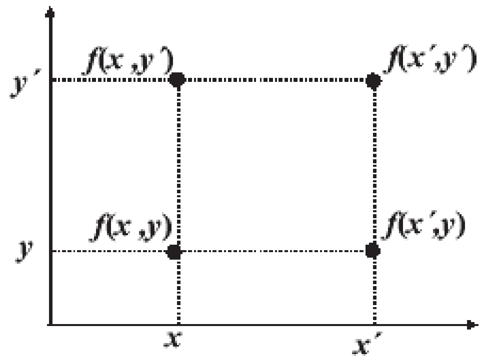


Figure 1: Definition of affiliation (see the text).

Some results concerning affiliation between two variables:

Since affiliation requires $f(x,y)f(x',y') \geq f(x,y')f(x',y)$ whenever $x' > x$ and $y' > y$ this implies

$$\frac{f(x',y')}{f(x',y)} \geq \frac{f(x,y')}{f(x,y)} \rightarrow \frac{f(y'/x')f(x')}{f(y/x')f(x')} \geq \frac{f(y'/x)f(x)}{f(y/x)f(x)} \rightarrow \frac{f(y'/x')}{f(y/x')} \geq \frac{f(y'/x)}{f(y/x)}$$

which implies that the likelihood ratio is increasing as

$$\frac{f(y'/x')}{f(y/x')} \geq \frac{f(y'/x)}{f(y/x)} \rightarrow \frac{f(y'/x')}{f(y'/x)} \geq \frac{f(y/x')}{f(y/x)},$$

or, equivalently, that $F(./x')$ dominates $F(./x)$ in term of the likelihood ratio, which is the strongest stochastic order.

1. (a) If $x' > x$ then $G(./x^{'})$ dominates $G(./x)$ in terms of the hazard rate.
   (b) If $x' > x$ then $G(./x^{'})$ dominates $G(./x)$ in terms of the reverse hazard rate.
   (c) If $x' > x$ then $G(./x^{'})$ dominates $G(./x)$ in terms of first order stochastic dominance.

2 The Symmetric Model

With interdependent values and affiliation there are two aspects to symmetry: in valuations (v_i) and in the distribution of signals.

We will assume symmetry in the valuations

$$v_i(X) = u(X_i, X_{-i})$$

with u symmetric in the last $n-1$ components and with $u(0) = 0$. We will further assume that all signals are drawn from the same interval $[0, \omega]$.

To characterize the equilibria let us define the function

$$v(x,y) = E\left[V/\ X_1 = x,\ Y_1 = y\right],$$

which is non-decreasing in y (recall affiliation) and strictly increasing in x and with $v(0,0) = 0$.

SPA

Proposition 1 *Symmetric equilibrium strategies in a SPA are given by* $\beta^{II}(x) = v(x,x)$.

Proof. Suppose that all bidders $j \neq 1$ bid according to the purported strategy, then bidders 1 profit is given by

$$\begin{aligned} \Pi(b,x) &= \int_0^{\beta-1(b)} (v(x,y) - \beta(y))\, g(y/x)dy \\ &= \int_0^{\beta-1(b)} (v(x,y) - v(y,y))\, g(y/x)dy \end{aligned}$$

Since $v(x,y) > v(y,y)$ iff $x > y$ then bidder 1 maximizes his profit when choosing $\beta^{-1}(b) = x$ or $b = \beta(x)$. ■

Bidders' equilibrium strategies are such that they break even in a tie.

Examples:

1.- $V = \sum X_i$ with $X_i \sim U(0,1)$ and with 5 bidders

If i wins then

$$E[V/\ X_1 = x,\ Y_{1:4} < x] = x + 4\frac{x}{2} = 3x.$$

At equilibrium, $b(x) = v(x,x) = E[V/\ X_1 = x,\ Y_{1:4} = x] = 2x + 3\frac{x}{2} = 3.5x$

If i wins then

$$\begin{aligned} E[V/\ X_1 = x,\ Y_{1:4} < x] &= x + 4\frac{x}{2} = 3x. \\ p &= 3.5E[Y_{1:4}/\ Y_{1:4} < x] = \frac{14}{5}x \end{aligned}$$

Note that $3x > \frac{14}{5}x$.

2.- Three bidders, $V \sim U(0,1)$. Given $V = v$, bidders' signals are iid $U[0, 2v]$.

Let $X = (X_1, X_2, X_3)$ and define $Z = \max(X_1, X_2, X_3)$. Note that Z is a sufficient statistic for V as $P(X = x/Z = z)$ does not depend on v.

Since $f(X_i/V = v) = \frac{1}{2v}$ and $f(X,V) = \Pi_{i=1}^3 f(X_i/V = v)f(V) = \frac{1}{8v^3}$ on the set $\{(V,X)$ such that $X_i \leq 2V$ for all $i\}$, we have:

$$f(X) = \int_{z/2}^1 \frac{1}{8v^3}dv = \frac{4-z^2}{16z^2} \text{ and } f(v/X = x) = \frac{\frac{1}{8v^3}}{\frac{4-z^2}{16z^2}} = \frac{1}{8v^3} \times \frac{16z^2}{4-z^2}$$

Since Z is a sufficient statistic then

$$E[V/\ X = x] = E[V/\ Z = z] = \int_{z/2}^1 vf(v/X = x)dv = \frac{2z}{2+z}.$$

Now

$$\begin{aligned} v(x,y) &= E[V/\ Z = \max(x,y)] = \frac{2\max(x,y)}{2+\max(x,y)} \text{ and} \\ \beta^{II}(x) &= v(x,x) = \frac{2x}{2+x} \end{aligned}$$

English Auctions (EA)

Bidders can drop out at any point in time, but once they do so, they cannot reenter the auction. The set of active bidders is commonly known.

A SES in an English auction is a collection of $N-1$ functions $\left(\beta^N, \beta^{N-1}, ..., \beta^2\right)$ such that $\beta^k(x, p_{k+1}, ..., p_n)$ is the price at which bidder 1 with signal x will drop is there are k active bidders, and the prices at which the other $N-k$ bidders dropped are $p_{k+1} \geq p_{k+2} \geq ...p_n$.

Proposition 2 *Symmetric equilibrium strategies in an EA are given by.*

$$\begin{aligned} \beta^N(x) &= u(x, ..., x) \\ \beta^{N-1}(x, p_n) &= u(x, ..., x_n) \\ \beta^k(x, p_{k+1}, ..., p_n) &= u(\underbrace{x, ...x}_{k}, \underbrace{p_{k+1}, ..., p_n}_{N-k}) \end{aligned}$$

Note that since bidders do not affect the price they pay when winning, they bid as to guarantee that they are always winners whenever winning yields a positive payoff.

Note that the EA equilibrium is regret-free (it is an ex-post equilibrium) whereas the equilibrium in the SPA is not an ex-post (once signals are known) equilibrium.

Recall Example 1 where $\beta^{II}(x) = 3.5x$. Let $x = (0.9, 0.8, 0.7, 0.7, 0.6)$

$V = 0.9 + 0.8 + 0.7 + 0.7 + 0.6 = 3.7$

$\beta_1^{II}(x) = 3.5(0.9) = 3.15$

Bidder 2 with signal 0.8 is losing and he regrets his losing (a bid of 3.2 is a winning bid that yields a positive expected payoff).

FPA

In a FPA:

$$\Pi(z, x) = \int_0^z (v(x, y) - \beta(z))g(y/x)dy$$

Proposition 3 *Symmetric equilibrium strategies in a FPA are given by*

$$\beta(x) = \int_0^x v(y, y)dL(y/x).$$

where $L(y/x) = \exp\left(-\int_y^x \frac{g(s/s)}{G(s/s)}ds\right)$.

Proof. Maximizing expected profits the foc becomes:

$$(v(x, z) - \beta(z))g(z/x) \mid_{z=x} - \int_0^z \beta'(z)g(y/x)dy \mid_{z=x} = 0.$$

Note that equilibrium bidding behavior is determined by two forces: a *value adjustment* for the information revealed by winning, and the *bidding trade-off* (higher cost higher probability of winning, so that

$$\underbrace{v(x, z)g(z/x) \mid_{z=x}}_{\text{Value adjustment}} \underbrace{- \beta(z))g(z/x) \mid_{z=x} - \int_0^z \beta'(z)g(y/x)dy \mid_{z=x}}_{\text{Bidding trade-off}} = 0$$

In equilibrium:

$$(v(x,x)-\beta(x))g(x/x)-\beta'(x)G(x/x)=0.$$

We further need $v(x,x)>\beta(x)$, which provides the boundary condition $\beta(0)=0$.

$$\begin{aligned} v(x,x)g(x/x) &= \beta'(x)G(x/x)+\beta(x)g(x/x) \\ \int_0^x v(y,y)g(y/x)dy &= \int_0^x d(\beta(z)G(z/x)) \end{aligned}$$

Integrating by parts the right-hand side of equality above we have

$$v(x,x)G(x/x)-0-\int_0^x G(y/x)dv(y,y)=\beta(x)G(x/x)$$

Thus

$$\beta(x)=v(x,x)-\int_0^x \frac{G(y/x)}{G(x/x)}dv(y,y)=v(x,x)-\int_0^x L(y/x)dv(y,y), \tag{1}$$

where $L(y/x)=\frac{G(y/x)}{G(x/x)}$.

Note that the equilibrium can also be written as

$$\beta(x)=\int_0^x v(y,y)dL(y/x). \tag{2}$$

as integration by parts in (2) yields (1).

To see that this is indeed an equilibrium we must verify two more things: 1) that β is an increasing function, and 2) that that profits are maximized by choosing $z=x$.

To see (1) note that since $v(y,y)$ is an increasing function, if $L(y/x)$ were a distribution function such that $L(y/x')\succeq_{FOSD} L(y,x)$ for $x'>x$ then the result will follow trivially from stochastic dominance as it would imply $\beta(x')>\beta(x)$.

Since

$$L(y/x)=\frac{G(y/x)}{G(x/x)}=\exp\left(-\int_y^x d\left(\ln G(s/x)\right)\right)=\exp\left(-\int_y^x \frac{g(s/s)}{G(s/s)}ds\right)$$

then $L(y/x)$ can indeed be considered a distribution function with support $[0,x]$. Note that affiliation implies

$$\frac{g(s/s)}{G(s/s)}\geq\frac{g(s/0)}{G(s/0)} \text{ for all } s \text{ so that } -\int_0^x \frac{g(s/s)}{G(s/s)}ds\leq\int_0^x -\frac{g(s/0)}{G(s/0)}=\ln G(0/0)-\ln(x/0)=-\infty$$

Taking exponential in both sides we have: $L(0/x)\leq 0\rightarrow L(0/x)=0$ as the exponential is always non-negative. Similarly, $L(x/x)=1$. It is non-decreasing in y as $G(y/x)$ is non decreasing in y. These facts ensure that it is a distribution function. Note further that it is decreasing in x so that $L(y/x')\leq L(y/x)$ if $x'>x$ resulting in first order stochastic dominance of $L(y/x')$ over $L(y/x)$. As $v(y,y)$ is increasing in y this means that $\beta(x)$ is increasing in x.

Finally (2) does also hold as $\frac{\partial\Pi}{\partial z}>0$ if $z<x$ and $\frac{\partial\Pi}{\partial z}<0$ if $z>x$ so that profits are maximized by choosing $z=x$. ■

The iid private values model is a particular case of the general model. In (2) private values imply that there is no *value adjustment* so that $v(y,y)=y$, and iid imply $L(y/x)=\frac{G(y)}{G(x)}$ as x provides no information on y, so that $\beta(x)=\frac{\int_0^x yg(y)}{G(x)}=E(Y_1/Y_1<x)$.

3 Revenue Comparisons

3.1 English versus SPA

Proposition 4 *The expected revenue form an EA is at least as great as the expected revenue from a SPA.*

Proof. Since $\beta^{II}(x) = v(x,x)$ we have

$$\begin{aligned}
E\left[R^{II}\right] &= E\left[\beta^{II}(Y_1) \mid X_1 > Y_1\right] \\
&= E\left[v(Y_1,Y_1) \mid X_1 > Y_1\right] \\
&= E\left[E\left[u(Y_1,Y_1,Y_2,...,Y_{N-1}) \mid X_1 = y, Y_1 = y\right] \mid X_1 > Y_1\right] \\
&\leq E\left[E\left[u(Y_1,Y_1,Y_2,...,Y_{N-1}) \mid X_1 = x, Y_1 = y\right] \mid X_1 > Y_1\right] \text{ (by affiliation)} \\
&= E\left[u(Y_1,Y_1,Y_2,...,Y_{N-1}) \mid X_1 > Y_1\right] \\
&= E\left[\beta^2(Y_1,Y_1,Y_2,...,Y_{N-1})\right] \text{ (the price at which the second to last drops out)} \\
&= E\left[R^{EA}\right]
\end{aligned}$$

■

3.2 SPA versus FPA

Proposition 5 *The Expected revenue from a SPA is at least as great as the expected revenue from a FPA.*

Proof. A bidder with signal x winning in a FPA pays $\beta^I(x)$, whereas in a SPA pays $E\left[\beta^{II}(Y_1) \mid X_1 = x, Y_1 < x\right]$.

Since

$$\begin{aligned}
E\left[\beta^{II}(Y_1) \mid X_1 = x, Y_1 < x\right] &= E\left[v(Y_1,Y_1) \mid X_1 = x, Y_1 < x\right] \\
&= \frac{\int_0^x v(y,y)g(y/x)dy}{G(x/x)} = \int_0^x v(y,y)dK(y/x)
\end{aligned}$$

where $K(y/x) = \frac{G(y/x)}{G(x/x)}$, and

$$\beta^I(x) = \int_0^x v(y,y)dL(y/x),$$

if $K(y/x)$ stochastically dominates $L(y/x)$ then the results follows.

Because of affiliation we know that for all $t < x$

$$\begin{aligned}
\frac{g(t/t)}{G(t/t)} &\leq \frac{g(t/x)}{G(t/x)} \rightarrow \\
-\int_y^x \frac{g(t/t)}{G(t/t)}dt &\geq -\int_y^x \frac{g(t/x)}{G(t/x)}dt \text{ for all } y < x. \\
&= -\int_y^x \frac{d}{dt}\left(\ln(G(t/x)\right)dt \\
&= \ln G(y/x) - \ln G(x/x) \\
&= \ln\left(\frac{G(y/x)}{G(x/x)}\right) = \ln(K(y/x))
\end{aligned}$$

Taking exponents in both sides we get

$$\exp\left(-\int_y^x \frac{g(t/t)}{G(t/t)}dt\right) = L(y/x) \geq K(y/x) \rightarrow K(y/x) \text{ s.d. } L(y/x)$$

■

The conclusions from above propositions can be summarized as:

Proposition 6 *In the symmetric model with interdependent values and affiliated signals, the E, SP and FP can be ranked in terms of expected revenues as follows.*

$$E\left[R^{EA}\right] \geq E\left[R^{SPA}\right] \geq E\left[R^{FPA}\right]$$

Intuitions:

1. It is due to the winner curse: the winner curse is less severe under SPA as, after all, if a bidder overestimates values in her bid, she will pay a lower price. WRONG! In equilibrium bidders take into account this. Note that the winner curse comes from the fact that with interdependent values, i must realize that j's bid is informative about i's own value. This leads to the winner's curse. It is hence present in a common value model with independent signals but there RE holds!
2. Is due to signal correlation. Bidders profits are due to information rents. When signals are correlated, information rents can be reduced if "more information" is used in setting the price.

The linkage Principle

Main Result

Suppose that A is a standard auction in which the highest bid wins the object and that it has a symmetric equilibrium β^A. Let $W^A(z,x)$ denote the expected price paid by bidder 1 if he is the winning bidder, his signal is x and he bids as if his type were z.

Proposition 7 *Let A and B be two auction in which the highest bidder wins and only he pays a positive amount. Suppose that each has a symmetric and increasing equilibrium such that*

1. *For all* $x, W_2^A(x,x) \geq W_2^B(x,x)$
2. $W^A(0,0) = W^B(0,0) = 0$

Then the expected revenue in A is at least as large as in B.

Proof. In a symmetric equilibrium

$$x = \arg\max_z \left(\int_0^z v(x,y)g(y/x)dy - G(z/x)W^F(z,x)\right),$$

where $F = A, B$.

If auctions A and B have a symmetric and increasing equilibrium then $\int_0^z v(x,y)g(y/x)dy$ coincides in both auctions. Consequently, the f.o.c. imply

$$g(x/x)W^A(x,x) + G(x/x)W_1^A(x,x) = g(x/x)W^B(x,x) + G(x/x)W_1^B(x,x)$$

where (1) denotes the partial derivative of W with respect to its first argument.

Thus,

$$W_1^A(x,x) - W_1^B(x,x) = -\frac{g(x/x)}{G(x/x)}\left[W^A(x,x) - W^B(x,x)\right]$$

Let us define $\Delta(x) = W^A(x,x) - W^B(x,x)$. Taking derivatives we get

$$\begin{aligned}\Delta'(x) &= W_1^A(x,x) - W_1^B(x,x) + W_2^A(x,x) - W_2^B(x,x) \\ &= -\frac{g(x/x)}{G(x/x)}\left[W^A(x,x) - W^B(x,x)\right] + \underbrace{W_2^A(x,x) - W_2^B(x,x)}_{\geq 0 \text{ by hypothesis}}\end{aligned}$$

Thus, if $\Delta(x) \geq 0$ then $\Delta'(x) \leq 0$. Furthermore, $\Delta(0) = 0$ by hypothesis. It hence follows that $\Delta(x) \geq 0$ for all x. ■

We are here using the following lemma

Lemma 8 *Let f, g be two differentiable functions on $[0,1]$ If*

i) $f(0) = g(0)$ and ii) $f(x) = g(x)$ implies $f'(x) > g'(x)$.

Then $f(x) > g(x)$ for all $x \in [0,1]$.

When comparing the FPA with the SPA we have

$$\begin{aligned}W^I(z,x) &= \beta^I(z) \rightarrow W_2^I(x,x) = 0 \\ W^{II}(z,x) &= E\left[\beta^{II}(Y_1) \mid X_1 = x, Y_1 < z\right] \rightarrow W_2^{II}(x,x) \geq 0 \text{ by affiliation.}\end{aligned}$$

We can hence conclude that the revenue in the SPA is no smaller than the one in the FPA

Intuition: Revenue is larger the larger is the statistical linkage between a bidder's own signal and the price he would pay upon winning. Notice that the larger is that linkage, the less severe is the winner's curse and the more aggressive can hence bidders be without regretting.

3.3 Public information

Proposition 9 *If a seller must decide, after observing a signal s, whether to report it, and his report is verifiable, then at a perfect equilibrium he always reports s regardless of its value.*

This is just an application of the linkage principle. Think of a FPA. Without revelation $W^I(z,x) = \beta^I(z)$, so that $W_2^I(z,x) = 0$. With revelation, $\hat{W}^I(z,x) = E\left(\hat{\beta}(S,z)/X_1 = x\right)$ so that $\hat{W}_2^I(z,x) \geq 0$ as s and x are affiliated.

An alternative Linkage Principle

Proposition 10 *Let A and B be two auction in which the highest bidder wins. Suppose that each has a symmetric and increasing equilibrium such that*

1. *For all x, $M_2^A(x,x) \geq M_2^B(x,x)$*

2. $M^A(0,0) = M^B(0,0) = 0$

Then the expected revenue in A is at least as large as in B.

Proof. The expected payoff of a bidder with signal x who bids $\beta^A(z)$ is

$$\int_0^z v(x,y)g(y/x)dy - M^A(z,x).$$

In equilibrium:

$$M_1^A(x,x) = v(x,x)g(x/x) = M_1^B(x,x)$$

Let us define $\Delta(x) = M^A(x,x) - M^B(x,x)$.Taking derivatives we get

$$\Delta'(x) = \underbrace{M_1^A(x,x) - M_1^B(x,x)}_{0 \text{ by foc}} + \underbrace{M_2^A(x,x) - M_2^B(x,x)}_{\geq 0 \text{ by hypothesis}} \geq 0$$

Since $\Delta(0) = 0$ then for all x, $\Delta(x) \geq 0$. ∎

Ranking All-Pay Auctions

$$\begin{aligned} M^I(z,x) &= G(z/x)\beta^I(z) \rightarrow M_2^I(z,x) < 0 \\ M^{AP}(z,x) &= \beta^{AP}(z) \rightarrow M_2^I(z,x) = 0 \end{aligned}$$

Thus, expected revenue in APA is larger than in FPA. Note that

$$M_1^{AP}(x,x) = v(x,x)g(x/x) \rightarrow \frac{d}{dz}\left(\beta^{AP}(z)\right)|_{z=x} = v(x,x)g(x/x),$$

which provides

$$\beta^{AP}(x) = \int_0^x v(y,y)g(y/y)dy$$

4 Efficiency

An auction allocates efficiently if the bidder with the highest value is awarded the object. Since that bidder needs not to be the one with the higher signal, the three auction formats discussed here may be inefficient as next example shows.

Example 11 *Symmetric equilibria may be inefficient*

Let valuations in a two-bidder symmetric situation be

$$v_1(x_1,x_2) = \frac{1}{3}x_1 + \frac{2}{3}x_2 \text{ and } v_2(x_1,x_2) = \frac{2}{3}x_1 + \frac{1}{3}x_2$$

where $v_1 \geq v_2$ iff $x_2 > x_1$. In this example the bidder with the higher signal has the lower value and all three auction formats, almost always, allocate the object inefficiently.

A sufficient condition for an efficient equilibria is for valuations to satisfy the *single crossing condition.*

Definition 12 *Valuations satisfy the single crossing conditions if for all i and all $j \neq i$ and for all $\mathbf{x}$:*

$$\frac{\partial v_i}{\partial x_i}(\mathbf{x}) \geq \frac{\partial v_j}{\partial x_i}(\mathbf{x}).$$

As a function of x_i, the function v_i is steeper that v_j so they cross at most once.

The SCC ensures that the ex-post values of different bidders are ordered in the same way as their signals.

If $x_i > x_j$, and $\alpha(t) = (1-t)(x_j, x_i, x_{-ij}) + t(x_i, x_j, x_{-ij})$ is the path connecting (x_j, x_i, x_{-ij}) and (x_i, x_j, x_{-ij}) the Fundamental Theorem of Calculus for line integrals states that:

$$\int_C \nabla u dx = \int_0^1 \nabla u(\alpha(t))\alpha'(t)dt = u(q) - u(p)$$

where p and q are the ending points in the path C. For $C = \alpha(t)$ we have that $p = (x_j, x_i, x_{-ij}) = \alpha(0)$ and $q = (x_i, x_j, x_{-ij}) = \alpha(1)$, so that

$$\begin{aligned} u(q) &= u(p) + \int_0^1 \nabla u(\alpha(t))\alpha'(t)dt \\ u\left(x_i, x_j, x_{-ij}\right) &= u(x_j, x_i, x_{-ij}) + \int_0^1 \underbrace{u_1^{'}(\alpha(t))\left(x_i - x_j\right) + u_2^{'}(\alpha(t))\left(x_j - x_i\right)}_{\geq 0} dt \geq u(x_j, x_i, x_{-ij}) \end{aligned}$$

Proposition 13 *With symmetric, interdependent values and affiliated signals, if the single crossing condition is satisfied then the SPA, FPA and EA all have symmetric equilibria that are efficient.*

5 An Example

Let S_1, S_2 and T be uniformly and independently distributed on [0,1]. Let $X_1 = S_1 + T$ and $X_2 = S_2 + T$. Finally, let $V = \frac{X_1 + X_2}{2}$.

Note that

$$F_1(x) = F_2(x) = \Pr(X_i \leq x) = \Pr(S_i \leq x - T) = \begin{cases} x^2/2 & \text{if} \quad x \leq 1 \\ 2x - 1 - x^2/2 & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$

as $\int_0^x (x-t)dt = x^2/2$ and $x - 1 + \int_{x-1}^1 (x-t)dt = 2x - 1 - \frac{1}{2}x^2$.

Thus,

$$f(x) = \begin{cases} x & \text{if} \quad x \leq 1 \\ 2 - x & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$

Further, (see proof below)

$$f(x_1, x_2) = \begin{cases} x_2 & 1 \geq x_2 \geq x_1 \geq 0 \\ x_1 & 1 \geq x_1 \geq x_2 \geq 0 \\ 1 - x_1 + x_2 & x_1 \geq 1 \geq x_2 \geq 0 \text{ and } x_1 - x_2 - 1 \leq 0 \\ 0 & x_1 \geq 1 \geq x_2 \geq 0 \text{ and } x_1 - x_2 - 1 \geq 0 \\ 2 - x_1 & x_1 \geq x_2 \geq 1 \\ 2 - x_2 & x_2 \geq x_1 \geq 1 \\ 1 + x_1 - x_2 & x_2 \geq 1 \geq x_1 \geq 0 \text{ and } x_2 - x_1 - 1 \leq 0 \\ 0 & x_2 \geq 1 \geq x_1 \geq 0 \text{ and } x_2 - x_1 - 1 \geq 0 \end{cases}$$

Thus

$$g(x/x) = \frac{f(x,x)}{f(x)} = \begin{cases} 1 & x \leq 1 \\ 1 & x \geq 1 \end{cases} \rightarrow G(x/x) = \frac{x}{2}$$

$$\frac{g(x/x)}{G(x/x)} = \frac{2}{x} \text{ so that } L(y/x) = \frac{y^2}{x^2}.$$

Consequently, since $v(x,y) = \frac{x+y}{2}$ we have

$$\begin{aligned} \beta^I(x) &= \int_0^x \left(\frac{y+y}{2}\right)\frac{2y}{x^2}dy = \frac{2}{3}x. \\ \beta^{II}(x) &= \frac{x+x}{2} = x \end{aligned}$$

The expected revenue in the SPA will be $E\left[\min(X_1, X_2)\right] = E\left[\min(S_1, S_2)\right] + E\left[T\right] = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

The expected revenue in the FPA will be $\frac{2}{3}E\left[\max(X_1, X_2)\right] = \frac{2}{3}\left(E\left[\max(S_1, S_2)\right] + E\left[T\right]\right) = \frac{4}{9} + \frac{1}{3} = \frac{7}{9}$

Thus $E\left[R^{II}\right] > E\left[R^I\right]$.

Theorem: Let $f(x,y)$ be the joint pdf of the random variables X and Y. If the functions $z_1 = w_1(x,y)$ and $z_2 = w_2(x,y)$ are partially differentiable with respect to both x and y and invertible so that $x = g_1(z_1, z_2)$ and $y = g_2(z_1, z_2)$ then

$$h(z_1, z_2) = f(g_1(z_1, z_2), g_2(z_1, z_2))\,|J|$$

where J is the Jacobian of the transformation.

Let $X_1 = S_1 + T$, $X_2 = S_2 + T$ and $X_3 = S_1 + S_2$. Inverting the functions we get:

$$\begin{aligned} T &= \frac{X_1 + X_2 - X_3}{2} = g_1 \\ S_1 &= \frac{X_1 - X_2 + X_3}{2} = g_2 \\ S_2 &= \frac{-X_1 + X_2 + X_3}{2} = g_3 \end{aligned}$$

Note that $|J| = -0.5$

Thus

$$h(x_1, x_2, x_3) = \begin{cases} 1 \times 0.5 = 0.5 & \text{for } 0 \leq X_1 + X_2 - X_3 \leq 1, 0 \leq X_1 - X_2 + X_3 \leq 1, 0 \leq -X_1 + X_2 + X_3 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence, integrating out X_3 we get the desired *pdf*,

$$h(x_1, x_2) = \int_0^2 h(x_1, x_2, x_3)dx_3$$

.

Different cases have to be considered:

1. Let $1 \geq x_2 \geq x_1 \geq 0$ (A)

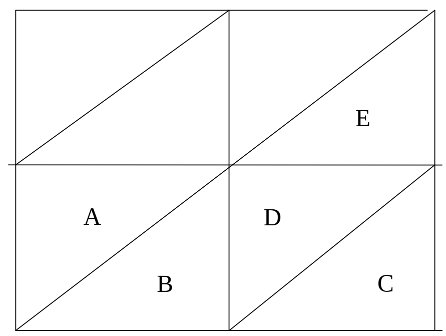
$0 \leq X_1 + X_2 - X_3 \leq 1 \rightarrow X_3 \leq X_1 + X_2$

$0 \leq X_1 - X_2 + X_3 \leq 1 \rightarrow 0 \leq X_2 - X_1 \leq X_3$

$0 \leq -X_1 + X_2 + X_3 \leq 1 \rightarrow$Non-binding

$$h(x_1, x_2) = \int_{x_2 - x_1}^{x_1 + x_2} 0.5 dx_3 = x_1$$

Appealing to symmetry, we have $h(x_1, x_2) = x_2$ if $1 \geq x_1 \geq x_2 \geq 0$ (B)



2.- Let $2 \geq x_1 \geq 1 \geq x_2 \geq 0$ (D)

Note further than in $D : x_1 - x_2 - 1 \leq 0 \rightarrow x_2 + 1 - x_1 \geq 0$

$0 \leq X_1 + X_2 - X_3 \leq 1 \rightarrow X_1 + X_2 \geq X_3 \geq X_1 + X_2 - 1$

$0 \leq X_1 - X_2 + X_3 \leq 1 \rightarrow X_2 - X_1 \leq 0 \leq X_3 \leq 1 + X_2 - X_1$

$0 \leq -X_1 + X_2 + X_3 \leq 1 \rightarrow 0 \leq X_1 - X_2 \leq X_3 \leq 1 + X_1 - X_2$

Since $1 + X_2 - X_1 \leq 1 + X_1 - X_2$ the binding constraints that ensure any other holds are:

$$h(x_1, x_2) = \int_{x_2 + x_1 - 1}^{x_2 + x_1} 0.5 dx_3 + \int_{x_1 - x_2}^{1 - x_1 + x_2} 0.5 dx_3 = x_2 - x_1 + 1$$

3.- Let $2 \geq x_1 \geq 1 \geq x_2 \geq 0$ and $x_1 - x_2 - 1 \geq 0$ (C)

Note that $x_1 \geq x_2 + 1$ implies $s_1 + t \geq s_2 + t + 1 \rightarrow s_1 \geq s_2 + 1$ which cannot be the case. the mass in this region must be 0.

4. Let $2 \geq x_1 \geq x_2 \geq 1 \rightarrow 1 \leq x_1 \leq 2$ and $2 - x_1 \leq x_2 \leq x_1$ (E)

$0 \leq X_1 + X_2 - X_3 \leq 1 \rightarrow X_3 \geq X_1 + X_2 - 1$

$X_3 \leq X_2 + X_1$ that is not binding as larger than 2

$0 \leq X_1 - X_2 + X_3 \leq 1 \rightarrow X_3 \geq X_2 - X_1 \rightarrow$Non-binding

$X_3 \leq 1 + X_2 - X_1 \rightarrow$Non-binding

$0 \leq -X_1 + X_2 + X_3 \leq 1 \rightarrow X_3 \geq X_1 - X_2$ and smaller than 1

$$X_3 \leq 1 - X_2 + X_1$$

$$\int_{x_1+x_2-1}^{2} 0.5dz + \int_{x_1-x_2}^{1} 0.5dz = 2 - x_1$$