



# FoCM 2017

## Foundations of Computational Mathematics

### Barcelona, July 10th-19th, 2017

<http://www.ub.edu/focm2017>

Organized in  
partnership with

#### Workshops

Approximation Theory  
Computational Algebraic Geometry  
Computational Dynamics  
Computational Harmonic Analysis and Compressive Sensing  
Computational Mathematical Biology with emphasis on the Genome  
Computational Number Theory  
Computational Geometry and Topology  
Continuous Optimization  
Foundations of Numerical PDEs  
Geometric Integration and Computational Mechanics  
Graph Theory and Combinatorics  
Information-Based Complexity  
Learning Theory  
Mathematical Foundations of Data Assimilation and Inverse Problems  
Multiresolution and Adaptivity in Numerical PDEs  
Numerical Linear Algebra  
Random Matrices  
Real-Number Complexity  
Special Functions and Orthogonal Polynomials  
Stochastic Computation  
Symbolic Analysis



#### Plenary Speakers

Karim Adiprasito  
Jean-David Benamou  
Alexei Borodin  
Mireille Bousquet-Mélou  
Mark Braverman  
Claudio Canuto  
Martin Hairer  
Pierre Lairez  
Monique Laurent  
Melvin Leok  
Lek-Heng Lim  
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Bruno Salvy  
Sylvia Serfaty  
Steve Smale  
Andrew Stuart  
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# **FoCM 2017**

**Foundations of Computational Mathematics**

**Barcelona, July 10th–19th, 2017**

**Books of abstracts**



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## Presentation

The conference on the Foundations of Computational Mathematics (FoCM 2017, <http://www.ub.edu/focm2017/>) took place in Barcelona between July 10th to 19th, 2017. It was organized by the Society for the Foundations of Computational Mathematics in partnership with the Local Organizing Committee with members from Universitat de Barcelona (UB), Universitat Politècnica de Catalunya (UPC), Universitat Pompeu Fabra (UPF) and ICREA, and administrative and logistic support from the UB Institut de Matemàtiques (IMUB) and the Centre de Recerca Matemàtica (CRM). The conference also received the institutional support of the Barcelona Graduate School of Mathematics (BGSMath).

This conference was the ninth in a sequence that started with the Park City meeting in 1995, organized by Steve Smale and where the idea of the FoCM Society was born, and followed by the FoCM conferences in Rio de Janeiro (1997), Oxford (1999), Minneapolis (2002), Santander (2005), Hong Kong (2008), Budapest (2011), and Montevideo (2014). Each of these conferences had several hundred participants from all branches of computational mathematics.

The conference took a format tried and tested to a great effect in the former FoCM conferences: plenary invited lectures in the mornings, and theme-centered parallel workshops in the afternoons. It consisted of three periods of three days each.

Each workshop extended over a period. It included two semi-plenary lectures, of interest to a more general audience, as well as (typically shorter) talks aimed at a more technical audience. The choice of these speakers was the responsibility of the workshop organizers, and all of these workshop talks were by invitation. There were also contributed poster presentations associated to the workshops.

Although some participants chose to attend just one or two periods, on past experience the greatest benefit resulted from attending the conference for its full nine days: the entire idea of FoCM is that we strive to break out of narrow boundaries of our specific research areas and open our minds to the broad range of exciting developments in computational mathematics.

FoCM conferences currently appear to be a unique meeting point of workers in computational mathematics and of theoreticians in mathematics and in computer science. While presenting plenary talks by foremost world authorities and maintaining the highest technical level in the workshops, the emphasis is on multidisciplinary interaction across subjects and disciplines, in an informal and friendly atmosphere, giving the possibility to meet colleagues from different subject-areas and identify the wide-ranging (and often surprising) common denominator of research.





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The FoCM Executive Committee was composed of:

Wolfgang Dahmen, RWTH Aachen, Germany (chair)  
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Javier Peña, Carnegie Mellon University, USA (treasurer)

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## Plenary talks

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Plenary talk - July 10, 10:00 – 11:00

### STOCHASTIC PDES AND THEIR APPROXIMATIONS

**Martin Hairer**

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Many stochastic partial differential equations that arise naturally from first principles by physical considerations are ill-posed in the sense that they involve formal expressions containing products of distributions. The theory of regularity structures sometimes allows to give such equations a mathematical meaning, but it is not clear a priori how this should be interpreted from a computational point of view. We will address this question in a rather general framework.

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Plenary talk - July 10, 11:30 – 12:30

### INTERPOLATION, RUDIMENTARY GEOMETRY OF SPACES OF LIPSCHITZ FUNCTIONS AND COMPLEXITY

**Shmuel Weinberger**

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This talk will interweave (and hopefully motivate) three themes. The first theme is interpolation. Lipschitz functions are built to be well approximated by interpolations from samples. Analogous, but more difficult, is topological interpolation: inferring a manifold (or its properties) from samples. The second theme is geometric: what do spaces of Lipschitz functions look like, especially when they are nonlinear, i.e., the target is not a vector space? What can we say about Gromov-Hausdorff spaces of manifolds? The final theme is quantitative geometric topology, i.e., the complexity of objects or homeomorphisms that topologists infer.

*Joint work with Greg Chambers (Rice University, USA), Sasha Dranishnikov (University of Florida, USA), Dominic Dotterer (Stanford University, USA), Steve Ferry (Rutgers University, USA), and Fedor Manin (Ohio State University, USA).*

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Plenary talk - July 11, 9:30 – 10:30

### LARGE GRAPH LIMITS OF LEARNING ALGORITHMS

**Andrew Stuart**

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Many problems in machine learning require the classification of high dimensional data. One methodology to approach such problems is to construct a graph whose vertices are identified

with data points, with edges weighted according to some measure of affinity between the data points. Algorithms such as spectral clustering, probit classification and the Bayesian level set method can all be applied in this setting. The goal of the talk is to describe these algorithms for classification, and analyze them in the limit of large data sets. Doing so leads to interesting problems in the calculus of variations, in stochastic partial differential equations and in Monte Carlo Markov Chain, all of which will be highlighted in the talk. These limiting problems give insight into the structure of the classification problem, and algorithms for it.

*Joint work with Matt Dunlop (Caltech, USA), Dejan Slepcev (CMU, USA), and Matt Thorpe (CMU, USA).*

Plenary talk - July 11, 11:30 – 12:30

$$T < 4E$$

**Karim Adiprasito**

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Descartes proved that a graph embedding into the plane on  $V$  vertices and  $E$  edges satisfies

$$E < 3V.$$

Quite disappointingly, no similarly beautiful numerical inequality exists to limit the number of faces of, say, 2-dimensional simplicial complexes embedding in dimension 4, and topological techniques seem to be limited in their reach to provide us with such a result. I will instead use algebra and combinatorics to generalize Descartes' result.

Plenary talk - July 12, 9:30 – 10:30

# VARIATIONAL DISCRETIZATIONS OF GAUGE FIELD THEORIES USING GROUP-EQUIVARIANT INTERPOLATION SPACES

**Melvin Leok**

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Variational integrators are geometric structure-preserving numerical methods that preserve the symplectic structure, satisfy a discrete Noether's theorem, and exhibit excellent long-time energy stability properties. An exact discrete Lagrangian arises from Jacobi's solution of the Hamilton-Jacobi equation, and it generates the exact flow of a Lagrangian system. By approximating the exact discrete Lagrangian using an appropriate choice of interpolation space and quadrature rule, we obtain a systematic approach for constructing variational integrators. The convergence rates of such variational integrators are related to the best approximation properties of the interpolation space.

Many gauge field theories with local symmetries can be formulated variationally using a multisymplectic Lagrangian formulation, and we will present a characterization of the exact generating functionals that generate the multisymplectic relation. By discretizing these using

group-equivariant spacetime finite element spaces, we obtain methods that exhibit a discrete multimomentum conservation law. We will then briefly describe an approach for constructing group-equivariant interpolation spaces that take values in the space of Lorentzian metrics that can be efficiently computed using a generalized polar decomposition. The goal is to eventually apply this to the construction of variational discretizations of general relativity, which is a second-order gauge field theory whose configuration manifold is the space of Lorentzian metrics.

*Joint work with Evan Gawlik (University of California, San Diego, USA) and Joris Vankerschaver (Enthought).*

Plenary talk - July 12, 11:30 – 12:30

### DYNAMIC FORMULATION OF OPTIMAL TRANSPORTATION AND VARIATIONAL RELAXATION OF EULER EQUATIONS

**Jean-David Benamou**

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We will briefly recall the classical Optimal Transportation Framework and its Dynamic relaxations. We will show the link between these Dynamic formulation and the so-called Multi-Marginal extension of Optimal Transportation. We will then describe the so-called Iterative Proportional Fitting Procedure (IPFP aka Sinkhorn method) which can be efficiently applied to the multi-marginal OT setting. Finally we will show how this can be used to compute generalized Euler geodesics due to Brenier. This problem can be considered as the oldest instance of Multi-Marginal Optimal Transportation problem.

*Joint work with Guillaume Carlier (Ceremade, Université Paris Dauphine, France) and Luca Nenna (Mokaplan INRIA, France).*

Plenary talk - July 13, 10:00 – 11:00

### FUNCTIONAL EQUATIONS IN ENUMERATIVE COMBINATORICS

**Mireille Bousquet-Mélou**

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A basic idea in enumerative combinatorics is to translate the recursive structure of a class of objects into recurrence relations defining the sequence of numbers that counts them. In many cases, these recurrence relations translate further into functional equations, of various types, defining the associated generating functions.

Such equations give an answer to the counting problem, but it is not completely satisfactory if one cannot decide whether their solution belongs, by any chance, to a more classical family of functions: could it be rational, or algebraic? Could it satisfy a polynomial differential equation? Or even a linear one?

These are recurring questions in enumerative combinatorics, and they raise attractive problems at the intersection of algebra, analysis and computer algebra. We will present a few results in this direction, and illustrate them with many examples.

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Plenary talk - July 13, 11:30 – 12:30

APPLIED RANDOM MATRIX THEORY

**Joel Tropp**

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Plenary talk - July 14, 9:30 – 10:30

ADAPTIVE HIGH-ORDER METHODS FOR ELLIPTIC PROBLEMS: CONVERGENCE AND OPTIMALITY

**Claudio Canuto**

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Adaptive algorithms for  $h$ -type finite element discretizations of elliptic problems are by now well understood, as far as their convergence and optimality properties are concerned.

The design and analysis of adaptive algorithms for  $hp$ -type discretizations poses new challenges. Indeed, the choice between applying a mesh refinement or a polynomial enrichment is a delicate stage in the adaptive process, since early decisions in one of the two directions should be lately amenable to a correction in order to guarantee the final near-optimality of the adaptive discretization for a prescribed accuracy. Furthermore, the optimality of the approximation should be assessed with respect to specific functional classes in which the best  $N$ -term approximation error is allowed to decay exponentially, as opposed to the more familiar classes of algebraic decay which are natural for  $h$ -type, finite-order methods.

Building on the experience gained on adaptive spectral (Fourier, Legendre) discretizations, we will highlight the results obtained in the last few years on the analysis of adaptive discretizations of  $hp$ -type. In particular, we will describe an abstract framework ( $hp$ -AFEM) in which such methods can be casted. It is based on alternating a solution stage, which provides a new approximate solution with guaranteed error reduction, and an adaptation stage, which yields a new  $hp$ -near best partition at the expense of a mild increase of the error. Under reasonable assumptions, this general algorithm is proven to be convergent with geometric rate and instance optimal. Several practical realizations of  $hp$ -AFEM will be discussed. Particular attention will be devoted to the issue of  $p$ -robustness, i.e., the independence from the polynomial degree of the constants involved in the analysis.

*Joint work with Ricardo H. Nochetto (University of Maryland, USA), Rob Stevenson (University of Amsterdam, The Netherlands), and Marco Verani (Politecnico di Milano, Italy).*

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Plenary talk - July 14, 11:30 – 12:30

LINEAR DIFFERENTIAL EQUATIONS AS A DATA-STRUCTURE

**Bruno Salvy**

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Many informations concerning solutions of linear differential equations can be computed directly from the equation. It is therefore natural to consider these equations as a data-structure, from which mathematical properties can be computed. A variety of algorithms has thus been designed in recent years that do not aim at “solving”, but at computing with this representation. The talk will survey some of these results.

---

Plenary talk - July 15, 9:30 – 10:30

FINDING ONE ROOT OF A POLYNOMIAL SYSTEM: SMALE’S 17TH PROBLEM

**Pierre Lairez**  
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How many operations are sufficient to numerically compute one root of a polynomial system of equations? Smale’s 17th problem asks whether a polynomial complexity is possible for random systems. We now have a thorough answer. Based on work by Beltrán, Bürgisser, Cucker, Pardo, Shub, Smale and myself, I will explain the main ideas that underlie it, from effective Newton’s method estimates to the sampling of random system-solution pairs.

---

Plenary talk - July 15, 11:30 – 12:30

MEAN ESTIMATION: MEDIAN-OF-MEANS TOURNAMENTS

**Gábor Lugosi**  
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One of the most basic problems in statistics is how to estimate the expected value of a distribution, based on a sample of independent random draws. When the goal is to minimize the length of a confidence interval, the usual empirical mean has a sub-optimal performance, especially for heavy-tailed distributions. In this talk we discuss some estimators that achieve a sub-Gaussian performance under general conditions. The multivariate scenario turns out to be more challenging. We present an estimator with near-optimal performance. We also discuss how these ideas extend to regression function estimation.

*Joint work with Shahar Mendelson (Technion, Israel), Luc Devroye (Mcgill University, Canada), Matthieu Lerasle (CNRS, France), and Roberto Imbuzeiro Oliveira (IMPA, Brazil).*

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Plenary talk - July 17, 10:00 – 11:00

MATHEMATICS OF CELL DIVISION

**Stephen Smale**  
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The problem of a mathematical structure for the biological process of cell differentiation will be addressed.

---

Plenary talk - July 17, 11:30 – 12:30

INFORMATION COMPLEXITY AND APPLICATIONS

**Mark Braverman**

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Over the past two decades, information theory has reemerged within computational complexity theory as a mathematical tool for obtaining unconditional lower bounds in a number of models, including streaming algorithms, data structures, and communication complexity. Many of these applications can be systematized and extended via the study of information complexity — which treats information revealed or transmitted as the resource to be conserved.

In this talk we will discuss the two-party information complexity and its properties — and the interactive analogues of classical source coding theorems. We will then discuss applications to exact communication complexity bounds, multi-party communication, and quantum communication complexity.

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Plenary talk - July 18, 9:30 – 10:30

FOURIER-LIKE BASES AND INTEGRABLE PROBABILITY

**Alexei Borodin**

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Plenary talk - July 18, 11:30 – 12:30

STRUCTURE TENSORS

**Lek-Heng Lim**

University of Chicago, USA  
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We show that in many instances, at the heart of a problem in numerical computation sits a special 3-tensor, the structure tensor of the problem that uniquely determines its underlying algebraic structure. For example, the Grothendieck constant, which plays an important role in unique games conjecture and SDP relaxations of NP-hard problems, arises as the spectral norm of such a structure tensor. In matrix computations, a decomposition of the structure tensor into rank-1 terms gives an explicit algorithm for solving the problem, its tensor rank gives the speed of the fastest possible algorithm, and its nuclear norm gives the numerical stability of the stablest algorithm. As an explicit example, we determine the fastest algorithms for the basic operation underlying Krylov subspace methods — the structured matrix-vector products for sparse, banded, triangular, symmetric, circulant, Toeplitz, Hankel, Toeplitz-plus-Hankel, BTTB matrices — by analyzing their structure tensors. Our method is a generalization of the Cohn–Umans method, allowing for arbitrary bilinear operations in place of matrix-matrix product,

and arbitrary algebras (e.g., coordinate rings of schemes, cohomology rings of manifolds, PI algebras) in place of group algebras. The second part is joint work with Ke Ye.

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Plenary talk - July 19, 9:30 – 10:30

COMPLETELY POSITIVE SEMIDEFINITE MATRICES: CONIC APPROXIMATIONS AND  
MATRIX FACTORIZATION RANKS

**Monique Laurent**

CWI, Amsterdam, and Tilburg University, Netherlands  
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The completely positive semidefinite cone is a new matrix cone, which consists of all the symmetric matrices (of a given size) that admit a Gram factorization by positive semidefinite matrices of any size. This cone can thus be seen as a non-commutative analogue of the classical completely positive cone, replacing factorizations by nonnegative vectors (aka diagonal psd matrices) by arbitrary psd matrices. It permits to model bipartite quantum correlations and quantum analogues of classical graph parameters like colouring and stable set numbers, which arise naturally in the context of nonlocal games and the study of entanglement in quantum information.

We will consider various questions related to the structure of the completely positive semidefinite cone. In particular, we will discuss how semidefinite optimization and tracial non-commutative polynomial optimization can be used to design conic approximations, to model the dual cone, and to design lower bounds for matrix factorization ranks, also for the related notions of cp-rank and nonnegative rank.

*Joint work with David de Laat (CWI, Amsterdam) and Sander Gribling (CWI, Amsterdam).*

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Plenary talk - July 19, 11:30 – 12:30

MICROSCOPIC DESCRIPTION OF SYSTEMS OF POINTS WITH COULOMB-TYPE  
INTERACTIONS

**Sylvia Serfaty**

Université Pierre et Marie Curie Paris 6, France

We are interested in systems of points with Coulomb, logarithmic or Riesz interactions (i.e., inverse powers of the distance). They arise in various settings: an instance is the study of Fekete points in approximation theory, another is the classical log gas or Coulomb gas which in some cases happens to be a random matrix ensemble, another is vortices in superconductors, superfluids or Bose-Einstein condensates, where one observes the emergence of densely packed point vortices forming perfect triangular lattice patterns named Abrikosov lattices.

After reviewing the motivations, we will take a point of view based on the detailed expansion of the interaction energy to describe the behavior of the systems. In particular a Central Limit Theorem for fluctuations and a Large Deviations Principle for the microscopic point processes are given. This allows to observe the effect of the temperature as it gets very large or very small, and to connect with crystallization questions. The main results are joint with Thomas Leblé and also based on previous works with Etienne Sandier, Nicolas Rougerie and Mircea Petrache.



# Workshops

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## Workshop A1 Approximation Theory

Organizers: Albert Cohen – Ron Devore – Peter Binev

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A1 - July 10, 14:30 – 15:20

### NONLINEAR $n$ -TERM APPROXIMATION OF HARMONIC FUNCTIONS FROM SHIFTS OF THE NEWTONIAN POTENTIAL

**Pencho Petrushev**

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A basic building block in Classical Potential Theory is the fundamental solution of the Laplace equation in  $\mathbb{R}^d$  (Newtonian potential). Our main goal is to study the rates of nonlinear  $n$ -term approximation of harmonic functions on the unit ball  $B^d$  from shifts of the Newtonian potential with poles outside  $\overline{B^d}$  in the harmonic Hardy spaces. Optimal rates of approximation are obtained in terms of harmonic Besov spaces. The main vehicle in establishing these results is the construction of highly localized frames for Besov and Triebel-Lizorkin spaces on the sphere whose elements are linear combinations of a fixed number of shifts of the Newtonian potential.

*Joint work with Kamen Ivanov (Bulgarian Academy of Sciences, Bulgaria).*

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A1 - July 10, 15:30 – 16:20

### GOEDEL AND TURING IN APPROXIMATION THEORY – ON SAMPLING, RECONSTRUCTION, SPARSITY AND COMPUTATIONAL BARRIERS

**Anders Hansen**

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The problem of approximating a function from a finite set of samples of linear measurements has been a core part of approximation theory over the last decades. The key issue is to design a decoder that takes the samples and produces an approximation to the function in an accurate and robust way. When the decoder is designed, one is then faced with the problem of finding a robust algorithm that can compute the output of the decoder or at least an approximation to it. And herein lies the crucial question: do such algorithms always exist? For linear decoders the answer is always yes, however, for non-linear decoders we show that the answer is often no. Moreover, and perhaps surprisingly, the answer is no for many of the popular non-linear decoders that are used in practice. In fact, a positive answer to the question would imply decidability of famous non-decidable problems such as the Halting problem, and hence the problems are formally

non-computable. In this talk I will discuss this paradox and demonstrate how, by adding sparsity as a key ingredient, one can assure the existence of robust algorithms in non-linear approximation theory, and explain the success of their use in practice. Moreover, I will show how the above paradox opens up for a rich classification theory of non-linear decoders that contains many surprises.

*Joint work with Alex Bastounis (University of Cambridge, UK), Laura Terhaar (University of Cambridge, UK), and Verner Vlacic (University of Cambridge, UK).*

A1 - July 10, 17:00 – 17:35

#### TENSORIZATION FOR DATA-SPARSE APPROXIMATION

**Lars Grasedyck**

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In this talk we will review techniques for the tensorization of functions or vectors that are given as low dimensional objects. The tensorization artificially casts them into high dimensional objects where we are able to apply low rank tensor approximation techniques. The rationale behind this is the fact that low rank tensors can be represented with a complexity linear in the dimension, thus allowing a logarithmic complexity for the representation of objects that — when stored in a naive way — would require a linear complexity. We characterize the approximability in terms of subspaces and give examples for classes of functions that allow for such a compression of data.

A1 - July 10, 17:40 – 18:15

#### RESCALED PURE GREEDY ALGORITHM FOR HILBERT AND BANACH SPACES AND BEYOND

**Guergana Petrova**

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We show that a very simple modification of the Pure Greedy Algorithm for approximating functions by sparse sums from a dictionary in a Hilbert or more generally a Banach space has optimal convergence rates. Moreover, this greedy strategy can be applied to convex optimization in Banach spaces. We prove its convergent rates under a suitable behavior of the modulus of uniform smoothness of the objective function and test our algorithm on several data sets involving the log-likelihood function for logistic regression.

*Joint work with Zheming Gao (North Carolina State University, USA).*

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A1 - July 10, 18:20 – 18:55

DATA ASSIMILATION AND SAMPLING IN BANACH SPACES

**Przemek Wojtaszczyk**  
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We present a new framework to interpolation in general Banach spaces. It is based on recent study on data assimilation in the context of reduced basis method see P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, P. Wojtaszczyk, “Data Assimilation in Reduced Modeling.” SIAM UQ. Our framework unifies in particular the earlier approaches to sampling using point values and Fourier coefficients.

*Joint work with Ron DeVore (Texas A&M University, USA) and Guergana Petrova (Texas A&M University, USA).*

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A1 - July 11, 14:30 – 15:20

MULTISCALE METHODS FOR DICTIONARY LEARNING, REGRESSION AND OPTIMAL  
 TRANSPORT FOR DATA NEAR LOW-DIMENSIONAL SETS

**Mauro Maggioni**  
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We discuss a family of ideas, algorithms, and results for analyzing various new and classical problems in the analysis of high-dimensional data sets. These methods we discuss perform well when data is (nearly) intrinsically low-dimensional. They rely on the idea of performing suitable multiscale geometric decompositions of the data, and exploiting such decompositions to perform a variety of tasks in signal processing and statistical learning. In particular, we discuss the problem of dictionary learning, where one is interested in constructing, given a training set of signals, a set of vectors (dictionary) such that the signals admit a sparse representation in terms of the dictionary vectors. We then discuss the problem of regressing a function on a low-dimensional unknown manifold. For both problems we introduce a multiscale estimator, fast algorithms for constructing it, and give finite sample guarantees for its performance, and discuss its optimality. Finally, we discuss an application of these multiscale decompositions to the fast calculation of optimal transportation plans, introduce a multiscale version of optimal transportation distances, and discuss preliminary applications.

*Joint work with Sam Gerber (University of Oregon, USA), Wenjing Liao (Johns Hopkins University, USA), and Stefano Vigogna (Johns Hopkins University, USA).*

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A1 - July 11, 15:30 – 16:20

SPACE-PARAMETER-ADAPTIVE APPROXIMATION OF AFFINE-PARAMETRIC ELLIPTIC PDES

**Markus Bachmayr**

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We consider the approximation of PDEs with parameter-dependent coefficients by sparse polynomial approximations in the parametric variables, combined with suitable discretizations in the spatial domain. Here we focus on problems with countably many parameters, as they arise when coefficients with uncertainties are modelled as random fields. For the resulting fully discrete approximations of the corresponding solution maps, we obtain convergence rates in terms of the total number of degrees of freedom. In particular, in the case of affine parametrizations, we find that independent adaptive spatial approximation for each term in the polynomial expansion yields improved convergence rates. Moreover, we discuss new operator compression results showing that standard adaptive solvers for finding such approximations can be made to converge at near-optimal rates.

*Joint work with Albert Cohen (UPMC – Paris VI, France), Wolfgang Dahmen (RWTH Aachen, Germany), Dinh Dung (Vietnam National University), Giovanni Migliorati (UPMC – Paris VI, France), and Christoph Schwab (ETH Zürich, Switzerland).*

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A1 - July 11, 17:00 – 17:35

SPARSE APPROXIMATION WITH RESPECT TO THE FABER-SCHAUDER SYSTEM

**Tino Ullrich**

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We consider approximations of multivariate functions using  $m$  terms from its tensorized Faber-Schauder expansion. The univariate Faber-Schauder system on  $[0, 1]$  is given by dyadic dilates and translates (in the wavelet sense) of the  $L_\infty$  normalized simple hat function with support in  $[0, 1]$ . We obtain a hierarchical basis which will be tensorized over all levels (hyperbolic) to get the dictionary  $\mathcal{F}$ . The worst-case error with respect to a class of functions  $\mathbf{F} \hookrightarrow X$  is measured by the usual best  $m$ -term widths denoted by  $\sigma_m(\mathbf{F}, \mathcal{F})_X$ , where the error is measured in  $X$ . We constructively prove the following sharp asymptotical bound for the class of Besov spaces with small mixed smoothness (i.e.,  $1/p < r < \min\{1/\theta - 1, 2\}$ ) in  $L_\infty$

$$\sigma_m(\mathbf{B}_{p,\theta}^r, \mathcal{F})_\infty \asymp m^{-r}.$$

Note, that this asymptotical rate of convergence does not depend on the dimension  $d$  (only the constants behind). In addition, the error is measured in  $L_\infty$  and to our best knowledge this is the first sharp result involving  $L_\infty$  as a target space. We emphasize two more things. First, the selection procedure for the coefficients is a level-wise constructive greedy strategy which only touches a finite prescribed number of coefficients. And second, due to the use of



the Faber–Schauder system, the coefficients are finite linear combinations of discrete Function values. Hence, this method can be considered as a nonlinear adaptive sampling algorithm leading to a pure polynomial rate of convergence for any  $d$ .

*Joint work with Glenn Byrenheid (University of Bonn, Germany).*

A1 - July 11, 17:40 – 18:15

## PRINCIPAL COMPONENT ANALYSIS FOR THE APPROXIMATION OF HIGH-DIMENSIONAL FUNCTIONS USING TREE-BASED LOW-RANK FORMATS

**Anthony Nouy**

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We present an algorithm for the approximation of high-dimensional functions using tree-based low-rank approximation formats (tree tensor networks). A multivariate function is here considered as an element of a Hilbert tensor space of functions defined on a product set equipped with a probability measure, the function being identified with a multidimensional array when the product set is finite. The algorithm only requires evaluations of functions (or arrays) on a structured set of points (or entries) which is constructed adaptively. The algorithm is a variant of higher-order singular value decomposition which constructs a hierarchy of subspaces associated with the different nodes of a dimension partition tree and a corresponding hierarchy of interpolation operators. Optimal subspaces are estimated using empirical principal component analysis of interpolations of partial random evaluations of the function. The algorithm is able to provide an approximation in any tree-based format with either a prescribed rank or a prescribed relative error, with a number of evaluations of the order of the storage complexity of the approximation format.

## Bibliography

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A1 - July 11, 18:20 – 18:55

## OPTIMAL APPROXIMATION WITH SPARSELY CONNECTED DEEP NEURAL NETWORKS

**Gitta Kutyniok**

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Despite the outstanding success of deep neural networks in real-world applications, most of the related research is empirically driven and a mathematical foundation is almost completely missing. One central task of a neural network is to approximate a function, which for instance

encodes a classification task. In this talk, we will be concerned with the question, how well a function can be approximated by a deep neural network with sparse connectivity. We will derive fundamental lower bounds on the connectivity and the memory requirements of deep neural networks guaranteeing uniform approximation rates for arbitrary function classes, also including functions on low-dimensional immersed manifolds. Additionally, we prove that our lower bounds are achievable for a broad family of function classes, thereby deriving an optimality result. Finally, we present numerical experiments demonstrating that the standard stochastic gradient descent algorithm generates deep neural networks providing close-to-optimal approximation rates at minimal connectivity. Moreover, surprisingly, these results show that stochastic gradient descent actually learns approximations that are sparse in the representation systems optimally sparsifying the function class the network is trained on.

*Joint work with Helmut Bölcskei (ETH Zürich, Switzerland), Philipp Grohs (Universität Wien, Austria), and Philipp Petersen (Technische Universität Berlin, Germany).*

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A1 - July 12, 14:30 – 15:20

## COMPUTING A QUANTITY OF INTEREST FROM OBSERVATIONAL DATA

**Simon Foucart**

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Scientific problems often feature observational data received in the form  $w_1 = l_1(f), \dots, w_m = l_m(f)$  of known linear functionals applied to an unknown function  $f$  from some Banach space  $\mathcal{X}$ , and it is required to either approximate  $f$  (the full approximation problem) or to estimate a quantity of interest  $Q(f)$ . In typical examples, the quantities of interest can be the maximum/minimum of  $f$  or some averaged quantity such as the integral of  $f$ , while the observational data consists of point evaluations. To obtain meaningful results about such problems, it is necessary to possess additional information about  $f$ , usually as an assumption that  $f$  belongs to a certain model class  $\mathcal{K}$  contained in  $\mathcal{X}$ . This is precisely the framework of optimal recovery, which produced substantial investigations when the model class is a ball of a smoothness space, e.g., when it is a Lipschitz, Sobolev, or Besov class. This presentation is concerned with other model classes described by approximation processes. Its main contributions are:

- (i) for the estimation of quantities of interest, the production of numerically implementable algorithms which are optimal over these model classes,
- (ii) for the full approximation problem, the construction of linear algorithms which are optimal or near optimal over these model classes in case of data consisting of point evaluations.

Regarding (i), when  $Q$  is a linear functional, the existence of linear optimal algorithms was established by Smolyak, but the proof was not numerically constructive. In classical recovery settings, it is shown here that such linear optimal algorithms can be produced by constrained minimization methods, and examples involving the computations of integrals from the given data are examined in greater details. Regarding (ii), it is shown that linearization of optimal algorithms can be achieved for the full approximation problem, too, in the important situation

where the  $l_j$  are point evaluations and  $\mathcal{X}$  is a space of continuous functions equipped with the uniform norm. It is also revealed how the quasi-interpolation theory allows for the construction of linear algorithms which are near optimal.

*Joint work with Ron DeVore (Texas A&M University, USA), Guergana Petrova (Texas A&M University, USA), and Przemysław Wojtaszczyk (University of Warsaw, Poland).*

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A1 - July 12, 15:30 – 16:20

## STABLE GABOR PHASE RETRIEVAL AND SPECTRAL CLUSTERING

**Philipp Grohs**

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We consider the problem of reconstructing a signal  $f$  from its spectrogram, i.e., the magnitudes  $|V_\varphi f|$  of its Gabor transform

$$V_\varphi f(x, y) := \int_{\mathbb{R}} f(t) e^{-\pi(t-x)^2} e^{-2\pi i y t} dt, \quad x, y \in \mathbb{R}.$$

Such problems occur in a wide range of applications, from optical imaging of nanoscale structures to audio processing and classification.

While it is well-known that the solution of the above Gabor phase retrieval problem is unique up to natural identifications, the stability of the reconstruction has remained wide open. The present paper discovers a deep and surprising connection between phase retrieval, spectral clustering and spectral geometry. We show that the stability of the Gabor phase reconstruction is bounded by the inverse of the *Cheeger constant* of the flat metric on  $\mathbb{R}^2$ , conformally multiplied with  $|V_\varphi f|$ . The Cheeger constant, in turn, plays a prominent role in the field of spectral clustering, and it precisely quantifies the ‘disconnectedness’ of the measurements  $V_\varphi f$ .

It has long been known that a disconnected support of the measurements results in an instability — our result for the first time provides a converse result in the sense that there are no other sources of instabilities.

Due to the fundamental importance of Gabor phase retrieval in coherent diffraction imaging, we also provide a new understanding of the stability properties of these imaging techniques: Contrary to most classical problems in imaging science whose regularization requires the promotion of smoothness or sparsity, the correct regularization of the phase retrieval problem promotes the ‘connectedness’ of the measurements in terms of bounding the Cheeger constant from below. Our work thus, for the first time, opens the door to the development of efficient regularization strategies.

*Joint work with Martin Rathmair (University of Vienna, Austria).*

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A1 - July 12, 17:00 – 17:35

ON THE ENTROPY NUMBERS OF THE MIXED SMOOTHNESS FUNCTION CLASSES

**Vladimir Temlyakov**

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Behavior of the entropy numbers of classes of multivariate functions with mixed smoothness is studied here. This problem has a long history and some fundamental problems in the area are still open. The main goal of this talk is to present a new method of proving the upper bounds for the entropy numbers. This method is based on recent developments of nonlinear approximation, in particular, on greedy approximation. This method consists of the following two steps strategy. At the first step we obtain bounds of the best  $m$ -term approximations with respect to a dictionary. At the second step we use general inequalities relating the entropy numbers to the best  $m$ -term approximations. For the lower bounds we use the volume estimates method, which is a well known powerful method for proving the lower bounds for the entropy numbers.

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A1 - July 12, 17:40 – 18:15

APPROXIMATION OF SET-VALUED FUNCTIONS BY ADAPTATION OF CLASSICAL  
APPROXIMATION OPERATORS TO SETS

**Nira Dyn**

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In this talk we consider approximation of univariate set-valued functions, mapping a closed interval to compact sets in a Euclidean space. Since the collection of such sets is not a vector space, the classical approximation operators have to be adapted to this setting. One way is to replace operations between numbers, by operations between sets. When the approximation error is measured in the Hausdorff metric, the operations between sets, designed by us, lead to error bounds expressed in terms of the regularity properties of the approximated set-valued function.

An example of a possible application of the theory to the approximation of a 3D object from its parallel 2D cross-sections concludes the talk.

*Joint work with Elza Farkhi (Tel-Aviv University, Israel) and Alona Mokhov (Afeka College, Tel-Aviv, Israel).*

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A1 - July 12, 18:20 – 18:55

POLYNOMIAL APPROXIMATION OF SMOOTH, MULTIVARIATE FUNCTIONS ON  
IRREGULAR DOMAINS

**Ben Adcock**

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Smooth, multivariate functions defined on tensor domains can be approximated using simple orthonormal bases formed as tensor products of one-dimensional orthogonal polynomials. On the other hand, constructing orthogonal polynomials on irregular domains is a difficult and computationally intensive task. Yet irregular domains arise in many practical problems, including uncertainty quantification, model-order reduction, optimal control and numerical PDEs. In this talk I will introduce a method for approximating smooth, multivariate functions on irregular domains, known as polynomial frame approximation. Importantly, this method corresponds to approximation in a frame, rather than a basis; a fact which leads to several key differences, both theoretical and numerical in nature. However, this method requires no orthogonalization or parametrization of the boundary, thus making it suitable for very general domains. I will discuss theoretical results for the approximation error, stability and sample complexity of this algorithm, and show its suitability for high-dimensional approximation through independence (or weak dependence) of the guarantees on the ambient dimension  $d$ . I will also present several numerical results, and highlight some open problems and challenges.

*Joint work with Daan Huybrechs (K.U. Leuven, Belgium), Juan Manuel Cardenas (Universidad de Concepcion, Chile), and Sebastian Scheuermann (Universidad de Concepcion, Chile).*

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A1 - Poster

#### OPTIMAL RECOVERY OF INTEGRAL OPERATORS AND APPLICATIONS

**Yuliya Babenko**

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In this talk we present the solution to a problem of recovering a rather arbitrary integral operator based on incomplete information with error. We apply the main result to obtain optimal (asymptotically optimal) methods of recovery and compute the optimal (asymptotically optimal) error for the solutions to certain integral equations as well as boundary and initial value problems for various PDE's. In particular, to illustrate the method, we present results for various boundary value problems for wave, heat, and Poisson's equations. Nevertheless, the developed method is more general and can be applied to other similar problems.

*Joint work with Vladyslav Babenko (Oles Honchar Dnipropetrovsk National University, Ukraine), Natalia Parfinovych (Oles Honchar Dnipropetrovsk National University, Ukraine), and Dmytro Skorokhodov (Oles Honchar Dnipropetrovsk National University, Ukraine).*

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A1 - Poster

#### THE BEURLING-SELBERG BOX MINORANT PROBLEM

**Jacob Carruth**

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Let  $F$  be a function on  $\mathbb{R}^d$  which is bandlimited to the unit cube and which minorizes the characteristic function of the unit cube. How large can the integral of  $F$  be? Selberg first asked

this question to solve a problem in number theory but it has since found many applications including to signal recovery, crystallography, and sphere packing. We show that for a sufficiently large dimensions  $d^*$  the answer to this question is zero. Furthermore, we discuss a numerical method which gives an explicit upper bound on  $d^*$ .

*Joint work with Noam Elkies (Harvard University, USA), Felipe Goncalves (University of Alberta, Canada), and Michael Kelly (Institute for Defense Analysis, Princeton, NJ).*

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A1 - Poster

SPARSE POLYNOMIAL TECHNIQUES IN HIGH DIMENSION FOR NON INTRUSIVE  
TREATMENT OF PARAMETRIC PDES

**Moulay Abdellah Chkifa**

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Motivated by the development of non-intrusive methods parametric PDE's in high dimension, we present an overview of various polynomial schemes for approximation in high dimension. The parametric PDEs that we consider are formulated by

$$\mathcal{D}(u, y) = 0,$$

where  $\mathcal{D}$  is a partial differential operator,  $u$  is the unknown solution and  $y = (y_j)$  is a parameter vector of high dimension  $d \gg 1$  where every  $y_j$  ranges in  $[-1, 1]$ ,

$$y \in U := [-1, 1]^d.$$

If we assume well-posedness in a fixed Banach space  $V$  for all  $y$ , the solution map is given by

$$y \in U \mapsto u(y) \in V.$$

We are interested in constructing approximations  $u_\Lambda$  to  $u$  in polynomial spaces of the form

$$\mathbb{V}_\Lambda = V \otimes \mathbb{P}_\Lambda := \left\{ \sum_{\nu \in \Lambda} c_\nu y_1^{\nu_1} \dots y_d^{\nu_d} : c_\nu \in V \right\}, \quad \Lambda \subset \mathbb{N}^d,$$

where  $\Lambda$  is downward closed, i.e.,

$$\mu \leq \nu \in \Lambda \implies \mu \in \Lambda.$$

with

$$\mu \leq \nu \quad \equiv \quad \mu_j \leq \nu_j, \quad j = 1, \dots, d.$$

To this end, we often use non-intrusive collocation type polynomial schemes which build the approximations  $u_\Lambda$  based on queries  $u^1 = u(y^1)$ ,  $u^2 = u(y^2), \dots \in V$  of the map  $u$ , at well chosen points  $y^1, y^2, \dots \in U$ , that are obtained through a black box type solver (e.g., a legacy solver). We provide an overview of some of these methods, namely

Sparse grids.

Sparse polynomial interpolation.

Sparse approximation based on LaVallée Poussin type operators.

Sparse least square.

Compressive sensing via  $l_1$ -minimization.

The computability, cost, robustness and stability of the various scheme are compared. These criteria are strongly tied to the sampling technique of the parameters  $y^j$ . In others words, how the locations  $y^1, y^2, \dots$  should be placed in the hypercube  $U$  while the criteria for the scheme are sustained. For instance, cost saving can be achieved using hierarchical sampling for which the sampling set is progressively enriched together with the polynomial space  $\mathbb{V}_\Lambda$  as new basis elements are added (as  $\Lambda$  gets enriched with new multi-indices  $\nu \in \mathbb{N}^d$ ). We show how structured hierarchical sampling based on Clenshaw–Curtis rules, or more generally  $\mathfrak{R}$ -Leja sequences, yields satisfactory results for this purpose.

*Joint work with Albert Cohen (Université Pierre et Marie Curie, France), Christoph Schwab (ETH Zürich, Switzerland), Giovanni Migliorati (Université Pierre et Marie Curie, France), Fabio Nobile (École Polytechnique Fédérale de Lausanne, Switzerland), Raul Tempone (King Abdullah University of Science and Technology, Saudi Arabia), Clayton Webster (Oak Ridge National Laboratory, USA), and Hoang Tran (Oak Ridge National Laboratory, USA).*

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A1 - Poster

## MULTI-SCALE DECOMPOSITION OF TRANSFORMATIONS

**Paul Escande**

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In many applications, transformations between two domains are defined through point-wise mappings. These functions can be costly to store and compute, but also hard to interpret in a geometric fashion. In this work, we propose a way to overcome these difficulties. The main idea is a novel multi-scale decomposition of complex transformations into a cascade of elementary, user-specified, transformations.

This methods allows to:

- (i) Construct efficient approximations for elements of large spaces of complex transformations using simple understandable blocks,
- (ii) use transformations to measure similarities between complex objects,
- (iii) deal with invariance under certain transformations,
- (iv) perform statistical inference tasks on sets of transformations.

In this poster we will describe the method as well as provide theoretical guarantees on the quality of the multi-scale approximations. Then we will present some numerical experiments that show its computational efficiency.

*Joint work with Mauro Maggioni (Johns Hopkins University, USA).*

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A1 - Poster

# FRAMES AND NUMERICAL APPROXIMATION

**Daan Huybrechs**

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A frame is a generalisation of a basis which can be redundant (i.e., linearly dependent). This redundancy provides a wealth of flexibility and enables rapid approximation of classes of functions that cannot be well approximated by commonly used bases. Examples include functions with a singularity, or multivariate functions defined on irregular domains. On the other hand, the redundancy also leads to extremely ill-conditioned linear systems. Yet, it can be shown that the computation of best approximations in a frame is numerically stable, in spite of the ill-conditioning. Furthermore, several frames of practical interest give rise to linear systems with a particular singular value structure — a plunge region — that can be exploited to achieve fast approximation algorithms.

In this poster we demonstrate by example how to approximate functions using several different types of frames. The stability of numerical approximations in frames was recently shown in [1]. The fast algorithms relate to the literature on bandlimited extrapolation in sampling theory and signal processing. For example, fast approximation in the so-called Fourier extension frame is achieved via implicit projection onto discrete prolate spheroidal wave sequences [2]. This allows the efficient approximation of functions defined on very irregular sets, including fractals.

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- [2] R. Matthysen & D. Huybrechs, “Fast algorithms for the computation of Fourier extensions of arbitrary length.” SIAM J. Sci. Comput. 38 (2016), no. 2, 899–922.

*Joint work with Ben Adcock (Simon Fraser University, Canada), Vincent Coppé (KU Leuven, Belgium), Roel Matthysen (KU Leuven, Belgium), and Marcus Webb (KU Leuven, Belgium).*

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A1 - Poster

# LEBESGUE CONSTANTS FOR CONVEX POLYHEDRA AND POLYNOMIAL INTERPOLATION ON LISSAJOUS-CHEBYSHEV NODES

**Yurii Kolomoitsev**

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Let  $W$  be a bounded set in  $\mathbb{R}^d$ . The following integral is called the Lebesgue constant for  $W$

$$\mathcal{L}(W) = \int_{[0,2\pi)^d} \left| \sum_{k \in W \cap \mathbb{Z}^d} e^{i(k,x)} \right| dx.$$



In the talk, we will present new upper and lower estimates of the Lebesgue constants for convex polyhedra. These estimates we apply to analyze the absolute condition number of multivariate polynomial interpolation on Lissajous–Chebyshev node points. The proof is based on a relation between the Lebesgue constant for the polynomial interpolation problem and the Lebesgue constant  $\mathcal{L}(W)$ , where  $W$  is some specific convex polyhedron.

*Joint work with P. Dencker (University of Lübeck, Germany), W. Erb (University of Lübeck, Germany), and T. Lomako (Institute of Applied Mathematics and Mechanics of NAS of Ukraine).*

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A1 - Poster

# THE GEOMETRICAL DESCRIPTION OF FEASIBLE SINGULAR VALUES IN THE TENSOR TRAIN FORMAT

**Sebastian Kraemer**

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Tensors have grown in importance and are applied to an increasing number of fields. Crucial in this regard are tensor formats, such as the widespread Tensor Train (TT) decomposition, which represent low rank tensors. This multivariate TT-rank and accordant  $d-1$  tuples of singular values are based on different matricizations of the same  $d$ -dimensional tensor. While the behavior of these singular values is as essential as in the matrix case ( $d = 2$ ), here the question about the *feasibility* of specific TT-singular values arises: for which prescribed tuples exist correspondent tensors and how is the topology of this set of feasible values?

This work is largely based on a connection that we establish to eigenvalues of sums of hermitian matrices. After extensive work spanning several centuries, that problem, known for the Horn Conjecture, was basically resolved by Knutson and Tao through the concept of so called *honeycombs*. We transfer and expand their and earlier results on that topic and thereby find that the sets of squared, feasible TT-singular values are geometrically described by polyhedral cones, resolving our problem setting to the largest extend. Besides necessary inequalities, we also present a linear programming algorithm to check feasibility as well as a simple heuristic, but quite reliable, parallel algorithm to construct tensors with prescribed, feasible singular values.

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A1 - Poster

# DICTIONARY MEASUREMENT SELECTION FOR STATE ESTIMATION WITH REDUCED MODELS

**Olga Mula**

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Parametric PDEs of the general form

$$\mathcal{P}(u, a) = 0$$

are commonly used to describe many physical processes, where  $\mathcal{P}$  is a differential operator,  $a$  is a high-dimensional vector of parameters and  $u$  is the unknown solution belonging to some Hilbert space  $V$ .

A typical scenario in state estimation is the following: for an unknown parameter  $a$ , one observes  $m$  independent linear measurements of  $u(a)$  of the form  $\ell_i(u) = (w_i, u)$ ,  $i = 1, \dots, m$ , where  $\ell_i \in V'$  and  $w_i$  are the Riesz representers, and we write  $W_m = \text{span}\{w_1, \dots, w_m\}$ . The goal is to recover an approximation  $u^*$  of  $u$  from the measurements.

Due to the dependence on  $a$  the solutions of the PDE lie in a manifold and the particular PDE structure often allows to derive good approximations of it by linear spaces  $V_n$  of moderate dimension  $n$ . In this setting, the observed measurements and  $V_n$  can be combined to produce an approximation  $u^*$  of  $u$  up to accuracy

$$\|u - u^*\| \leq \beta^{-1}(V_n, W_m) \text{dist}(u, V_n)$$

where

$$\beta(V_n, W_m) := \inf_{v \in V_n} \frac{\|P_{W_m} v\|}{\|v\|}$$

plays the role of a stability constant. For a given  $V_n$ , one relevant objective is to guarantee that  $\beta(V_n, W_m) \geq \gamma > 0$  with a number of measurements  $m \geq n$  as small as possible. We present results in this direction when the measurement functionals  $\ell_i$  belong to a complete dictionary.

*Joint work with Peter Binev (University of South Carolina, USA), Albert Cohen (Pierre et Marie Curie University, France), and James Nichols (Pierre et Marie Curie University, France).*

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A1 - Poster

## HIGHER ORDER TOTAL VARIATION, MULTISCALE GENERALIZATIONS, AND APPLICATIONS TO INVERSE PROBLEMS

**Toby Sanders**

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In the realm of signal and image denoising and reconstruction, L1 regularization techniques have generated a great deal of attention with a multitude of variants. In this work, we are interested in higher order total variation (HOTV) approaches, which are motivated by the idea of encouraging low order polynomial behavior in the reconstruction. A key component for their success is that under certain assumptions, the solution of minimum L1 norm is a good approximation to the solution of minimum L0 norm. In this work, we demonstrate that this approximation can result in artifacts that are inconsistent with desired sparsity promoting L0 properties, resulting in subpar results in some instances. Therefore we have developed a multi-scale higher order total variation (MHOTV) approach, which we show is closely related to the use of multiscale Daubechies wavelets. In the development of MHOTV, we confront a number of computational issues, and show how they can be circumvented in a mathematically elegant way, via operator decomposition and alternatively converting the problem into Fourier space. The relationship with wavelets, which we believe has generally gone unrecognized, is shown to hold in several numerical results, although subtle improvements in the results can be seen due

to the properties of MHOTV. The results are shown to be useful in a number of computationally challenging practical applications, including image inpainting, synthetic aperture radar, and 3D tomography.

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A1 - Poster

SUBDIVISION AND SPLINE SPACES

**Tatyana Sorokina**

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A standard construction in approximation theory is mesh refinement. For a simplicial or polyhedral mesh  $\Delta \subseteq R^k$ , we study the subdivision  $\Delta'$  obtained by subdividing a maximal cell of  $\Delta$ . We give sufficient conditions for the module of splines on  $\Delta'$  to split as the direct sum of splines on  $\Delta$  and splines on the subdivided cell. As a consequence, we obtain dimension formulas and explicit bases for several commonly used subdivisions and their multivariate generalizations.

*Joint work with Hal Schenck (University of Illinois Urbana-Champaign, USA).*

**Workshop A2****Computational Algebraic Geometry**

Organizers: Marta Casanellas – Agnes Szanto – Thorsten Theobald

A2 - July 10, 14:30 – 15:20

SYMMETRIC SUMS OF SQUARES OVER  $k$ -SUBSET HYPERCUBES**Rekha Thomas**

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Polynomial optimization over hypercubes has important applications in combinatorial optimization. We develop a symmetry-reduction method that finds sums of squares certificates for non-negative symmetric polynomials over  $k$ -subset hypercubes that improves on a technique due to Gatermann and Parrilo. For every symmetric polynomial that has a sos expression of a fixed degree, our method finds a succinct sos expression whose size depends only on the degree and not on the number of variables. Our results relate naturally to Razborov's flag algebra calculus for solving problems in extremal combinatorics. This leads to new results involving flags and their power in finding sos certificates in a finite setting, an angle that had not been explored before.

*Joint work with Annie Raymond (University of Washington, USA), James Saunderson (Monash University, Australia), and Mohit Singh (Georgia Institute of Technology, USA).*

A2 - July 10, 15:30 – 15:55

## SUBRESULTANTS IN MULTIPLE ROOTS

**Teresa Krick**

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I will introduce Sylvester sums for univariate polynomials with simple roots and their classical connection with polynomial subresultants (that can be viewed as a generalization of the Poisson formula for resultants), and then speak about the search of analogous expressions in the case of polynomials with multiple roots, with motivations and examples.

*Joint work with Alin Bostan (INRIA Saclay Ile de France, France), Carlos D'Andrea (Universitat de Barcelona, Spain), Agnes Szanto (North Carolina State University, USA), and Marcelo Valdettaro (Universidad de Buenos Aires, Argentina).*

A2 - July 10, 16:00 – 16:25

## QUADRATIC PERSISTENCE AND PYTHAGORAS NUMBER OF VARIETIES

**Greg Blekherman**

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Let  $X \subset \mathbf{P}^n$  be a variety. Quadratic persistence of  $X$  is the smallest integer  $k$  such that the ideal of projection away from  $k$  generic points of  $X$  contains no quadrics. I will motivate this definition and explain how quadratic persistence captures some geometric properties of a variety. Finally I will explain how quadratic persistence can be used to provide lower bounds for length of sums of squares decompositions.

*Joint work with Rainer Sinn (Georgia Tech, USA), Greg Smith (Queen's University, Canada), and Mauricio Velasco (Universidad de los Andes, Colombia).*

A2 - July 10, 17:00 – 17:25

### ALGEBRAIC GEOMETRY OF GAUSSIAN GRAPHICAL MODELS

**Seth Sullivant**

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Gaussian graphical models are statistical models widely used for modeling complex interactions between collections of linearly related random variables. A graph is used to encode recursive linear relationships with correlated error terms. These models are subalgebraic subsets of the cone of positive definite matrices, that generalize familiar objects in combinatorial algebraic geometry like toric varieties and determinantal varieties. I will explain how the study of the equations of these models is related to matrix Schubert varieties.

*Joint work with Alex Fink (Queen Mary University of London, UK) and Jenna Rajchgot (University of Saskatchewan, Canada).*

A2 - July 10, 17:30 – 17:55

### UNIVERSAL GROEBNER BASES AND CARTWRIGHT-STURMFELS IDEALS

**Elisa Gorla**

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I will introduce a family of multigraded ideals named after Cartwright and Sturmfels, and defined in terms of properties of the multigraded generic initial ideals. By definition, a multigraded ideal  $I$  is Cartwright–Sturmfels if it has a radical multigraded generic initial ideal. Our main technical result asserts that the family of Cartwright–Sturmfels ideals is closed under several natural operations, including multigraded linear sections and multigraded eliminations. Connection to universal Groebner bases for different families of ideals will be discussed.

*Joint work with Aldo Conca (University of Genoa, Italy) and Emanuela De Negri (University of Genoa, Italy).*

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A2 - July 10, 18:00 – 18:25

THE CHOW FORM OF A RECIPROCAL LINEAR SPACE

**Cynthia Vinzant**

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A reciprocal linear space is the image of a linear space under coordinate-wise inversion. This nice algebraic variety appears in many contexts and its structure is governed by the combinatorics of the underlying hyperplane arrangement. A reciprocal linear space is also an example of a hyperbolic variety, meaning that there is a family of linear spaces all of whose intersections with it are real. This special real structure is witnessed by a determinantal representation of its Chow form in the Grassmannian. In this talk, I will introduce reciprocal linear spaces and discuss the relation of their algebraic properties to their combinatorial and real structure.

*Joint work with Mario Kummer (Max Planck Institute, Leipzig, Germany).*

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A2 - July 10, 18:30 – 18:55

SYMMETRIZING THE MATRIX MULTIPLICATION TENSOR

**Jonathan Hauenstein**

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Determining the exponent of matrix multiplication is a central question in algebraic complexity theory. We propose a novel approach to bounding this exponent via the rank of cubic polynomials that are closely related to matrix multiplication. This allows us to exploit the vast literature of the algebraic geometry of cubic hypersurfaces and perform many numerical computations to study the exponent of matrix multiplication.

*Joint work with Luca Chiantini (Universita di Siena, Italy), Christian Ikenmeyer (Max Planck Institute for Informatics, Germany), Giorgio Ottaviani (Universita di Firenze, Italy), and J.M. Landsberg (Texas A&M University, USA).*

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A2 - July 11, 14:30 – 14:55

FACTORIZATION OF SPARSE BIVARIATE POLYNOMIALS

**Martín Sombra**

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It is expected that, for a given sparse univariate polynomial over the rationals, its non-cyclotomic irreducible factors are also sparse. This is a vague principle that takes a more precise form in an old (and still open) conjecture of Schinzel, on the irreducible factors in families of polynomials with fixed coefficients and varying monomials.

In this talk, I will present a theorem giving an analogue of Schinzel conjecture for polynomials over a function field. This result gives a description of the irreducible factors in families of bivariate polynomials over a field of characteristic zero.

Its proof is based on a toric version of Bertini's theorem.

*Joint work with Francesco Amoroso (Université de Caen, France).*

A2 - July 11, 15:00 – 15:25

## LOCAL EQUATIONS FOR EQUIVARIANT EVOLUTIONARY MODELS

**Jesús Fernández-Sánchez**

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Phylogenetic varieties related to equivariant substitution models have been studied largely in the last years. One of the main objectives has been finding a set of generators of the ideal of these varieties, but this has not yet been achieved in some cases (for example, for the general Markov model this involves the open “salmon conjecture”) and it is not clear how to use all generators in practice. However, for phylogenetic reconstruction purposes, the elements of the ideal that could be useful only need to describe the variety around certain points of no evolution. With this idea in mind, we produce a collection of explicit equations that describe the variety on a neighborhood of these points. Namely, for any tree on any number of leaves (and any degrees at the interior nodes) and for any equivariant model on any set of states, we compute the codimension of the corresponding phylogenetic variety. We prove that this variety is smooth at general points of no evolution, and provide an algorithm to produce a complete intersection that describes the variety around these points.

*Joint work with Marta Casanellas (Universitat Politècnica de Catalunya, Spain) and Mateusz Michalek (Polish Academy of Sciences).*

A2 - July 11, 15:30 – 15:55

## ARITHMETICALLY-FREE RESOLUTIONS OF TORIC VECTOR BUNDLES

**Gregory G. Smith**

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To each torus-equivariant vector bundle over a smooth complete toric variety, we associated a representable matroid (essentially a finite collection of vectors). In this talk, we will describe how the combinatorics of this matroid encodes a resolution of the toric vector bundles by a complex whose terms are direct sums of toric line bundles. We will also outline some applications to the equations and syzygies of smooth projective toric varieties.

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A2 - July 11, 16:00 – 16:25

## BOUNDS ON THE DEGREE OF THE CENTRAL CURVE OF SEMIDEFINITE PROGRAMMING

**Elias Tsigaridas**

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We present bounds on the algebraic degree of the central curve of semidefinite programming (SDP). We derive the bounds based on the degree of polynomial systems that exploit the primal and dual formulation of SDP.

*Joint work with Jean-Charles Faugère (INRIA, LIP6/UPMC, France) and Mohab Safey el Din (LIP6/UPMC, INRIA, France).*

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A2 - July 11, 17:00 – 17:25

## AUTOMATED REASONING TOOLS IN GEOGEBRA

**Tomas Recio**

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GeoGebra (see Hohenwarter, 2002) is a dynamic geometry software with tenths of millions users worldwide. Despite its original merely graphical flavor, successful attempts were performed during the last years towards combining standard dynamic geometry approaches with automated reasoning methods using computer algebra tools. Since Automated Theorem Proving (ATP) in geometry has reached a rather mature stage, an Austro-Spanish group (see Botana & al., 2015) started in 2010 a project of incorporating and testing a number of different automated geometry provers in GeoGebra. This collaboration was built upon previous approaches and achievements of a large community of researches, involving different techniques from algebraic geometry and computer algebra. Moreover, various symbolic computation, open source, packages have been involved, most importantly the Singular (Decker & al., 2012) and the Giac (Parisse, 2013) computer algebra systems. See Kovács, 2015a and Kovács, 2015b for a more detailed overview. As a result of this collaboration, we have been able to recently announce the implementation (Abanades & al., 2016) of three automated reasoning tools (ART) in GeoGebra, all of them working in the desktop, web, tablet or smartphone versions of GeoGebra: the automated derivation of (numerical) properties in a given construction, by means of the Relation Tool; the verification of the symbolic truth of these properties, by means of the Prove and ProveDetails tools; and the discovery of missing hypotheses for a conjectural statement to hold true, through the LocusEquation tool. The Relation Tool, in its original form, allows selecting two geometrical objects in a construction, and then to check for typical relations among them, including perpendicularity, parallelism, equality or incidence. Finally, it shows a message box with the obtained information (yes/no the relation holds). GeoGebra version 5 now displays an extra button in the message box with the caption “More...” which results in some symbolic computations when pressed. That is, by pressing the “More...” button, GeoGebra’s Automatic Theorem Proving subsystem starts and selects (by some heuristics) an appropriate prover method to decide if the numerically obtained property is indeed absolutely true in general. The current version of GeoGebra is capable of choosing



- a) the Gröbner basis method,
- b) Wu's characteristic method,
- c) the area method, or
- d) sufficient number of exact checks, deterministic method (see Kovács & al., 2012, Weitzhofer, 2013), as the underlying ATP technique addressed by the Prove command. See Kovács 2015b for more details on this portfolio prover.

Moreover, if the conjectured relation does not (mathematically speaking) hold, the first two methods can determine some geometrical extra-conditions, which need to hold true in order to make the given statement generally correct, either using the ProveDetails tool (in the generally true case) or the LocusEquation tool (in the generally false case). In the talk I will outline, through examples, some features of the ART in GeoGebra, providing some details on the underlying algebraic methods and reporting on our current work-in-progress concerning this topic.

*Acknowledgement:* Partially supported by the Spanish Ministerio de Economía y Competitividad and by the European Regional Development Fund (ERDF), under the Project MTM2014-54141-P.

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A2 - July 11, 17:30 – 17:55

## DEGREE-OPTIMAL MOVING FRAMES FOR RATIONAL CURVES

**Irina Kogan**

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We present an algorithm that, for a given vector  $\mathbf{a}$  of  $n$  relatively prime polynomials in one variable over an arbitrary field, outputs an  $n \times n$  invertible matrix  $P$  with polynomial entries, such that it forms a degree-optimal moving frame for the rational curve defined by  $\mathbf{a}$ . The first column of the matrix  $P$  consists of a minimal-degree Bézout vector (a minimal-degree solution to the univariate effective Nullstellensatz problem) of  $\mathbf{a}$ , and the last  $n - 1$  columns comprise an optimal-degree basis, called a  $\mu$ -basis, of the syzygy module of  $\mathbf{a}$ . To develop the algorithm, we prove several new theoretical results on the relationship between optimal moving frames, minimal-degree Bézout vectors, and  $\mu$ -bases. In particular, we show how the degree bounds of these objects are related. Comparison with other algorithms for computing moving frames and Bézout vectors will be given, however, we are currently not aware of another algorithm that produces an optimal degree moving frame or a Bézout vector of minimal degree.

*Joint work with Hoon Hong (North Carolina State University, USA), Zachary Hough (North Carolina State University, USA), and Zijia Li (Joanneum Research, Austria).*

A2 - July 11, 18:00 – 18:25

## A POSITIVSTELLENSATZ FOR SUMS OF NONNEGATIVE CIRCUIT POLYNOMIALS

**Timo de Wolff**

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In 2014, Ilman and I introduced an entirely new nonnegativity certificate based on sums of non-negative circuit polynomials (SONC), which are independent of sums of squares. We successfully applied SONCs to global nonnegativity problems.

In 2016, Dressler, Ilman, and I proved a Positivstellensatz for SONCs, which provides a converging hierarchy of lower bounds for constrained polynomial optimization problems. These bounds can be computed efficiently via relative entropy programming.

In this talk, I will give an overview about sums of nonnegative circuit polynomials, introduce our Positivstellensatz, and if time permits, briefly explain the connection to relative entropy programming.

*Joint work with Mareike Dressler (Goethe University Frankfurt, Germany) and Sadik Ilman (Frankfurt am Main, Germany).*

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A2 - July 12, 14:30 – 14:55

COMPUTATIONAL GEOMETRY OF REAL PLANE CURVES

**Daniel Plaumann**

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Deriving the geometric and topological properties of a curve in the real projective plane from its defining equation leads to many interesting and challenging computational problems in real algebraic geometry. Such problems concern the configuration of the ovals, singularities, the flex and bitangent lines, as well as various representation theorems. In this talk, we will focus on the case of quartics and sextics and highlight some recent experimental results.

*Joint work with Nidhi Kaihnsa (MPI Leipzig, Germany), Mario Kummer (MPI Leipzig, Germany), Mahsa Sayyary Namin (MPI Leipzig, Germany), and Bernd Sturmfels (UC Berkeley, USA / MPI Leipzig, Germany).*

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A2 - July 12, 15:00 – 15:25

ON THE SOLUTIONS TO POLYNOMIALS SYSTEMS ARISING FROM CHEMICAL REACTION NETWORKS

**Elisenda Feliu**

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Under the law of mass-action, the dynamics of the concentration of biochemical species over time is modelled using polynomial dynamical systems, which often have parameterised coefficients. The steady states, or equilibrium points, of the system are the solutions to a parameterised family of systems of polynomial equations. Because only nonnegative concentrations are meaningful in applications, one aims at finding the nonnegative solutions to these systems. In the talk I will present a result on finding parameter regions for which the steady state equations admit multiple solutions, and discuss the existence of positive parameterisations of the set of steady states.

*Joint work with Carsten Conradi (HTW Berlin, Germany), Maya Mincheva (Northern Illinois University, USA), Meritxell Sáez (University of Copenhagen, Denmark), and Carsten Wiuf (University of Copenhagen, Denmark).*

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A2 - July 12, 15:30 – 15:55

COMPUTING SIMPLE MULTIPLE ZEROS OF POLYNOMIAL SYSTEMS

**Lihong Zhi**

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Given a polynomial system  $f$  associated with a simple multiple zero  $x$  of multiplicity  $\mu$ , we give a computable lower bound on the minimal distance between the simple multiple zero  $x$  and

other zeros of  $f$ . If  $x$  is only given with limited accuracy, we propose a numerical criterion that  $f$  is certified to have  $\mu$  zeros (counting multiplicities) in a small ball around  $x$ . Furthermore, for simple double zeros and simple triple zeros whose Jacobian is of normalized form, we define modified Newton iterations and prove the quantified quadratic convergence when the starting point is close to the exact simple multiple zero. For simple multiple zeros of arbitrary multiplicity whose Jacobian matrix may not have a normalized form, we perform unitary transformations and modified Newton iterations, and prove its non-quantified quadratic convergence and its quantified convergence for simple triple zeros.

*Joint work with Zhiwei Hao (Chinese Academy of Sciences, China), Wenrong Jiang (Chinese Academy of Sciences, China), and Nan Li (Tianjin University, China).*

A2 - July 12, 16:00 – 16:25

#### DISTINGUISHING PHYLOGENETIC NETWORKS

**Elizabeth Gross**

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Phylogenetic networks are increasingly becoming popular in phylogenetics since they have the ability to describe a wider range of evolutionary events than their tree counterparts. In this talk, we discuss Markov models on phylogenetic networks, i.e., directed acyclic graphs, and their associated algebra and geometry. In particular, assuming the Jukes–Cantor model of evolution and restricting to one reticulation vertex, using tools from computational algebraic geometry we show that the semi-directed network topology of  $k$ -cycle networks with  $k \geq 4$  is generically identifiable.

*Joint work with Colby Long (Mathematical Biosciences Institute, USA).*

A2 - July 12, 17:00 – 17:50

#### THE EUCLIDEAN DISTANCE DEGREE OF ORTHOGONALLY INVARIANT MATRIX VARIETIES

**Giorgio Ottaviani**

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The closest orthogonal matrix to a real matrix  $A$  can be computed by the Singular Value Decomposition of  $A$ . Moreover, all the critical points for the Euclidean distance function from  $A$  to the variety of orthogonal matrices can be found in a similar way, by restriction to the diagonal case. The number of these critical points is the Euclidean distance degree. We generalize this result to any orthogonally invariant matrix variety. This gives a new perspective on classical results like the Eckart-Young Theorem and also some new results, e.g., on the essential variety in computer vision. At the end of the talk we will touch the case of tensors.

*Joint work with Dmitriy Drusvyatskiy (University of Washington, Seattle, USA), Hon-Leung Lee (University of Washington, Seattle, USA), and Rekha R. Thomas (University of Washington, Seattle, USA).*

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A2 - July 12, 18:00 – 18:25

## SOLVING THE RATIONAL HERMITE INTERPOLATION PROBLEM

**Carlos D'Andrea**
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The Rational Hermite Interpolation Problem is an extension of the classical Polynomial Interpolation one, and there are several approaches to it: Euclidean algorithm, Linear Algebra with structured matrices, barycentric coordinates, orthogonal polynomials, computation of syzygies,... We will review some of these methods along with the study of their complexity.

*Joint work with Teresa Cortadellas (Universitat de Barcelona, Spain) and Eulàlia Montoro (Universitat de Barcelona, Spain).*

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A2 - Poster

## ANGLES AND DIMENSION: ESTIMATING LOCAL DIMENSIONS FROM A DATASET

**Mateo Diaz**
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Given a subvariety  $\mathcal{Z}$  of  $\mathbb{R}^n$ , we want to determine its dimension from an independent sample of points  $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$  according to some probability distribution supported on  $\mathcal{Z}$ . Recall that there are many varieties which are connected but not equidimensional (think for instance of the union of a plane and a line which intersects the plane transversely). When considering these examples it is natural to think of dimension as a local concept. In this work, we address the following problem: How to estimate the dimension  $d$  of a variety at a given point  $\mathbf{p}$  from the sample? Assuming that  $\mathbf{p}$  is not a singularity point of  $\mathcal{Z}$ , we define a statistic based on pairwise angles between data points, that only depends on  $d$ . Due to its simple description we are able to compute its mean and variance, in closed-form, for a specific dimension. Which allows us to develop a test to estimate the dimension, we prove that under mild assumptions our estimate is correct with high probability if the sample size is at least a constant multiple of the dimension at  $\mathbf{p}$ . We also provide experimental results that support our claims.

*Joint work with Adolfo Quiroz (Universidad de los Andes, Colombia) and Mauricio Velasco (Universidad de los Andes, Colombia).*

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A2 - Poster

## GRÖBNER BASES OF NEURAL IDEALS

**Rebecca E. Garcia**
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A major area in neuroscience is the study of how the brain processes spatial information. Neurons in the brain represent external stimuli via neural codes. These codes often arise from

regions of space called receptive fields: each neuron fires at a high rate precisely when the animal is in the corresponding receptive field. Much research in this area has focused on understanding what features of receptive fields can be extracted directly from a neural code. In particular, Curto, Itskov, Veliz-Cuba, and Youngs recently introduced the concept of neural ideal, which is an algebraic object that encodes the full combinatorial data of a neural code. Every neural ideal has a particular generating set, called the canonical form, that directly encodes a minimal description of the receptive field structure intrinsic to the neural code. On the other hand, for a given monomial order, any polynomial ideal is also generated by its unique (reduced) Gröbner basis with respect to that monomial order. How are these two types of generating sets, canonical forms and Gröbner bases, related? Our main result states that if the canonical form of a neural ideal is a Gröbner basis, then it is the universal Gröbner basis (that is, the union of all reduced Gröbner bases). Furthermore, we prove that this situation — when the canonical form is a Gröbner basis — occurs precisely when the universal Gröbner basis contains only pseudo-monomials (certain generalizations of monomials). Our results motivate two questions:

- (1) When is the canonical form a Gröbner basis?
- (2) When the universal Gröbner basis of a neural ideal is not a canonical form, what can the non-pseudo-monomial elements in the basis tell us about the receptive fields of the code?

We give partial answers to both questions. Along the way, we develop a representation of pseudo-monomials as hypercubes in a Boolean lattice.

*Joint work with Luis David Garcia Puente (Sam Houston State University, USA), Ryan Kruse (Central College, USA), Jessica Liu (Bard College, USA), Dane Miyata (Willamette University, USA), Ethan Petersen (Rose-Hulman Institute of Technology, USA), Kaitlyn Phillipson (St. Edward's University, USA), and Anne Shiu (Texas A&M University, USA).*

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A2 - Poster

## MULTILINEAR ALGEBRA FOR PHYLOGENETIC RECONSTRUCTION

**Marina Garrote-López**

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The objective of phylogenetic reconstruction is recovering the ancestral relationships among a group of current species. In order to reconstruct phylogenetic trees it is necessary to model evolution adopting a parametric statistic model which allows us to define a joint distribution at the leaves of the trees. Using these models one is able to deduce polynomial relationships between the parameters, known as phylogenetic invariants. One can study these polynomials and the geometry of the algebraic varieties that arise from them and use it to reconstruct phylogenetic trees.

The framework of our research is to study and analyse some transformations of these joint distributions, the characterizations of stochasticity of the parameters for the general Markov model and understand the relationship between phylogenetics and algebraic techniques to recover phylogenetic trees from real data. Our main goal is to use phylogenetic invariants over the joint

distribution transformations to give new information to phylogenetic reconstruction methods and to infer new methods for phylogenetic inference.

A2 - Poster

# IMAGINARY PROJECTIONS OF (HOMOGENEOUS) POLYNOMIALS

**Thorsten Jörgens**

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We introduce the imaginary projection of a general multivariate polynomial  $f \in \mathbb{C}[\mathbf{z}]$  as the projection of the variety of  $f$  onto its imaginary part,  $\mathcal{I}(f) = \{\text{Im}(\mathbf{z}) : \mathbf{z} \in \mathcal{V}(f)\}$ . Since a polynomial  $f$  is stable if and only if  $\mathcal{I}(f) \cap \mathbb{R}_{>0}^n = \emptyset$ , the notion offers a novel geometric view underlying stability questions of polynomials. As a first fundamental geometric property we show that the connected components of the complement of the imaginary projections are convex.

For homogeneous (and hyperbolic) polynomials, we study the relation between imaginary projections and hyperbolicity cones. Hyperbolic polynomials became of interest in hyperbolic programming, where the set of feasible solutions is a hyperbolicity cone of a hyperbolic polynomial. It is a natural generalization of semidefinite programming.

Building upon this, for the homogeneous case we characterize the boundary of imaginary projections and we discuss a connection between hyperbolicity cones and certain components in the complement of the imaginary projection of inhomogeneous polynomials.

*Joint work with Thorsten Theobald (Goethe University Frankfurt, Germany) and Timo de Wolff (Texas A&M University, USA).*

A2 - Poster

# THE LEXICOGRAPHIC DEGREE OF THE FIRST TWO-BRIDGE KNOTS

**Pierre-Vincent Koseleff**

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We study the degree of polynomial representations of knots. We give here the lexicographic degree of all two-bridge knots with 11 or fewer crossings. The proof uses the braid theoretical method developed by Orevkov to study real plane curves [BKP2], isotopies on trigonal space curves and explicit parametrizations obtained by perturbing a triple point.

The lexicographic degree of a knot  $K$  is the minimal multidegree, for the lexicographic order, of a polynomial knot whose closure in  $S_3$  is isotopic to  $K$ .

We show that this degree is not necessarily obtained for curves that have the minimal number of real crossings. For example the knot  $8_6$  has minimal degree  $(3, 10, 14)$  that corresponds to a diagram with 9 crossings while the alternating diagram with 8 crossings has degree at least  $(3, 11, 13)$ .

Here we study two-bridge knots, namely those that admit a trigonal parametrization, that is to say a polynomial parametrization in degree  $(3, b, c)$ .

We show the sharp lower bound  $c \geq 3N - b$ , where  $N$  is the crossing number of  $K$ .

Lower bounds for  $b$  are obtained by considering first Bézout-like conditions on the real crossings of the plane projection.

We also use the associated 3-strings braid associated to any trigonal algebraic plane curve that must be quasi-positive [Or2].

Upper bounds are given by previous constructions on Chebyshev diagrams [KP3] and slide isotopies on knot diagrams: isotopies for which the number of crossings never increases [BKP1].

In addition, we introduce a new reduction  $R$  on trigonal plane curves. In many cases, the use of this reduction allows to subtract three to the number of crossings of the diagram and to the degree  $b$ , thanks to a result on real pseudoholomorphic curves deduced from [Or2]. On the other hand we show that we can explicitly give a polynomial curve of degree  $(3, d + 3)$  from a polynomial curve of degree  $(3, d)$  by adding a triple point and perturbing the singularity.

We obtain the lexicographic degree for some infinite families of two-bridge knots (the torus knots and generalized twist-knots) and for all two-bridge knots with 11 or fewer crossings. For every considered knot, we compute first an upper bound  $b_C$  for  $b$ , using Chebyshev diagrams. We compute all diagrams with  $b_C - 1$  or fewer crossings that cannot be reduced by slide isotopies. Most of these diagrams may be reduced using  $R$ -reductions. For those that cannot be reduced, we consider all possible schemes corresponding to admissible rational algebraic curve of degree  $(3, b) < (3, b_C)$  and compute their associated braids, that must be quasi-positive.

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*Joint work with Erwan Brugallé (Centre de Mathématiques Laurent Schwartz, CMLA, École Polytechnique, France) and Daniel Pecker (UPMC – Sorbonne Universités, Paris 6, France).*

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A2 - Poster

## THE EMBEDDING PROBLEM FOR KIMURA NUCLEOTIDE SUBSTITUTION MODELS

**Jordi Roca-Lacostena**

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In this poster we present the embedding problem for nucleotide substitution models from a biological and algebraic point of view. More precisely, we focus on the family of Kimura 3-parameters model and sub-models and we give a complete characterization of embeddable matrices for biological meaningful matrices.



*Joint work with Jesús Fernández-Sánchez (Universitat Politècnica de Catalunya, Spain) and Marta Casanellas (Universitat Politècnica de Catalunya, Spain).*

A2 - Poster

## MULTIPROJECTIVE WITNESS SETS AND A TRACE TEST

**Jose Rodriguez**

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In the field of numerical algebraic geometry, positive-dimensional solution sets of systems of polynomial equations are described by witness sets. We define multiprojective witness sets which will encode the multidegree information of an irreducible multiprojective variety. Our main results generalize the regeneration solving procedure, a trace test, and numerical irreducible decomposition to the multiprojective case. Examples in tensor decomposition and kinematics are used to illustrate the approach.

*Joint work with Jonathan Hauenstein (University of Notre Dame, USA).*

A2 - Poster

## ELIMINATION IN STEADY STATE EQUATIONS OF CHEMICAL REACTION NETWORKS

**Meritxell Sáez Cornellana**

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The steady states of a chemical reaction network with mass-action kinetics are solutions to a system of polynomial equations. Even for small systems, finding the steady states of the system is a very demanding task and therefore methods that reduce the number of variables are desirable. In [2] the authors give one such method, in which so-called non-interacting species are eliminated from the system of steady state equations.

We extend this method for the elimination of what we call reactant non-interacting species and give some conditions that ensure the positivity of the elimination obtained, that is, for positive values of the reaction rate constants and the non-eliminated species, we ensure that the eliminated species are positive as well. In particular, if enough species can be eliminated, then this method provides a parametrisation of the positive part of the steady state variety. Such a parametrisation can be used, for instance, to study the number of steady states in each linear invariant subspace defined by the conservation laws [1].

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*Joint work with Elisenda Feliu (University of Copenhagen, Denmark) and Carsten Wiuf (University of Copenhagen, Denmark).*

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## Workshop A3 Computational Number Theory

Organizers: Christophe Ritzenhaler – Enric Nart – Tanja Lange

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A3 - July 10, 15:30 – 16:20

### 3-RANKS OF CUBIC CLASS GROUPS

**Peter Stevenhagen**

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A lot is known about the 2-part of class groups of quadratic number fields, with fundamental results as old as Gauss's description of the genera of binary quadratic forms, and of their ambiguous classes. Results for 4-ranks and 8-ranks of quadratic class groups were proved by Redei (1939), and his methods have been used to prove density results for quadratic discriminants with 2-class groups of prescribed behaviour, and even some cases of the largely unproved Cohen-Lenstra heuristics for class groups.

In this talk, we explore analogues of the quadratic results in the context of 3-parts of class groups of cubic number fields.

*Joint work with Erick Knight (Max Planck Institut für Mathematik, Bonn, Germany) and Abtien Javanpeykar (Universiteit Leiden, The Netherlands).*

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A3 - July 10, 17:00 – 17:40

### RATIONAL POINTS ON HIGH GENUS CURVES OVER FINITE FIELDS

**Alp Bassa**

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In this talk we will be concerned with rational points on algebraic curves over finite fields. The central result in this area is without doubt the Hasse–Weil Theorem, which implies strong bounds on the number of rational points. As noticed by Ihara, for curves of large genus this bound can be improved significantly. We will be interested in the question of how many rational points a high genus curve over a finite field can have. We will introduce several approaches to this problem and present a recent result (joint work with Beelen, Garcia, Stichtenoth) which provides strong lower bounds for the maximal possible number of rational points on curves over non-prime finite fields.

*Joint work with Peter Beelen (DTU, Denmark), Arnaldo Garcia (IMPA, Brazil), and Henning Stichtenoth (Sabanci University, Turkey).*

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A3 - July 10, 17:40 – 18:20

### FAST ARITHMETIC IN THE DIVISOR CLASS GROUP

**Jens Bauch**

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Let  $C$  be a smooth projective geometrically irreducible algebraic curve of genus  $g$  over a field  $k$ . The Jacobian variety  $J$  of  $C$  is a  $g$ -dimensional algebraic group that parametrizes the degree zero divisors on  $C$ , up to linear equivalence. Khuri-Makdisi showed that the basic arithmetic in  $J$  can be realized in an asymptotic complexity of  $O(g^{3+\epsilon})$  field operations in  $k$ . Denote by  $F = k(C)$  the function field of the curve. Then the elements of  $J$  can be identified with divisor classes  $[D]$  of the function field  $F/k$  where  $D$  can be represented by a lattice  $L_D$  over the polynomial ring  $k[t]$ . In fact, the class  $[D]$  can be parametrized in terms of invariants of the lattice  $L_D$ . The basic arithmetic (addition and inversion) in  $J$  can then be realized asymptotically in  $O(g^3)$  field operations in  $k$  and beats the one of Khuri-Makdisi. Under reasonable assumptions the runtime can be reduced to  $O(g^2)$  field operations.

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A3 - July 11, 14:50 – 15:30

### A KILOBIT HIDDEN SNFS DISCRETE LOGARITHM COMPUTATION

**Nadia Heninger**

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We perform a special number field sieve discrete logarithm computation in a 1024-bit prime field. To our knowledge, this is the first kilobit-sized discrete logarithm computation ever reported for prime fields. This computation took a little over two months of calendar time on an academic cluster using the open-source CADO-NFS software.

Our chosen prime  $p$  looks random, and  $p - 1$  has a 160-bit prime factor, in line with recommended parameters for the Digital Signature Algorithm. However, our  $p$  has been trapdoored in such a way that the special number field sieve can be used to compute discrete logarithms in  $\mathbb{F}_p^*$ , yet detecting that  $p$  has this trapdoor seems out of reach. Twenty-five years ago, there was considerable controversy around the possibility of backdoored parameters for DSA. Our computations show that trapdoored primes are entirely feasible with current computing technology. We also describe special number field sieve discrete log computations carried out for multiple conspicuously weak primes found in use in the wild.

*Joint work with Joshua Fried (University of Pennsylvania, US), Pierrick Gaudry (INRIA – CNRS, Université de Lorraine, France), and Emmanuel Thomé (INRIA – CNRS, Université de Lorraine, France).*

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A3 - July 11, 15:30 – 16:20

SHORT GENERATORS WITHOUT QUANTUM COMPUTERS: THE CASE OF  
MULTIQUADRATICS

**Daniel J. Bernstein**

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Finding a short element  $g$  of a number field, given the ideal generated by  $g$ , is a classic problem in computational algebraic number theory. Solving this problem recovers the private key in cryptosystems introduced by Gentry, Smart–Vercauteren, Gentry–Halevi, Garg–Gentry–Halevi, & al. Work over the last few years has shown that for some number fields this problem has a surprisingly low \*post-quantum\* security level. The point of this talk is that for some number fields this problem has a surprisingly low \*pre-quantum\* security level.

*Joint work with Jens Bauch (Simon Fraser University, Canada), Henry de Valence (Technische Universiteit Eindhoven, The Netherlands), Tanja Lange (Technische Universiteit Eindhoven, The Netherlands), and Christine van Vredendaal (University of Illinois at Chicago, USA).*

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A3 - July 11, 17:00 – 17:40

IMPROVEMENTS TO POINT COUNTING ON HYPERELLIPTIC CURVES OF GENUS TWO

**Maike Massierer**

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Schoof’s algorithm is a standard method for counting points on elliptic curves defined over finite fields of large characteristic, and it was extended by Pila to higher-dimensional abelian varieties. Improvements by Elkies and Atkin lead to an even faster method for elliptic curves, known as the SEA algorithm. This is the current state of the art for elliptic curves. Motivated by the fact that Jacobians of hyperelliptic curves of genus two have been found to be good alternatives to elliptic curves in cryptography, we investigate the possibility of applying the improvements of Elkies and Atkin to Pila’s point counting algorithm for such varieties. We prove analogous theoretical results for genus two Jacobians with real multiplication by maximal orders, and we discuss the challenges involved in the practical implementation, such as the computation of suitable modular ideals.

*Joint work with S. Ballentine (University of Maryland, USA), A. Guillevis (INRIA, France), E. Lorenzo García (Université de Rennes 1, France), C. Martindale (Universiteit Leiden, The Netherlands), B. Smith (INRIA and LIX, France), and J. Top (University of Groningen, The Netherlands).*

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A3 - July 11, 17:40 – 18:20

COMPUTING TWISTS OF HYPERELLIPTIC CURVES

**Elisa Lorenzo García**

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In this talk we will see an algorithm for computing twists of hyperelliptic curves. We will see in details some of the key steps for computing the twists as well as some illustrative examples.

*Joint work with D. Lombardo (University of Hannover, Germany).*

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A3 - July 11, 18:20 – 19:00

KUMMER ARITHMETIC AND COMPACT SIGNATURE SCHEMES

**Benjamin Smith**

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The Kummer surfaces of certain genus 2 curves offer fast scalar multiplication algorithms with a measure of built-in resistance to basic side-channel attacks. The corresponding algorithms for the Jacobians of these curves are far slower, more complicated, and are ultimately not competitive with elliptic curve-based cryptographic implementations. When implementing discrete logarithm-based signature schemes such as Schnorr signatures (and generalizations of ECDSA), the group structure on the Jacobian appears essential.

In recent work with Chung and Costello, we proposed algorithms to compute Jacobian-based signatures while exploiting the speed and safety of Kummer arithmetic. Moving from theory to practice, these algorithms were subsequently implemented for microcontroller platforms (with very limited computing resources) in joint work with Renes, Schwabe, and Batina; the results provided easily the most efficient secure signatures for microcontrollers to date. However, the increase in speed comes at the cost of heavy memory usage, largely due to the complicated formulae relating fast Kummer surfaces with their Jacobians.

In this talk we explain how to remove Jacobians from the picture entirely, presenting the first signature schemes requiring only Kummer arithmetic. The result is faster, simpler signatures with a much smaller memory footprint, suitable for use in constrained environments.

*Joint work with Joost Renes (Radboud University Nijmegen, The Netherlands).*

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A3 - July 12, 14:50 – 15:30

CONGRUENCES, GRAPHS AND MODULAR FORMS

**Samuele Anni**

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 samuele.anni@gmail.com

The theory of congruences between modular forms is a central topic in contemporary number theory, lying at the basis of the proof of Mazur's theorem on torsion in elliptic curves, Fermat's

Last Theorem, and Sato-Tate, amongst others. Congruences are a display of the interplay between geometry and arithmetic. In order to study them, we are constructing graphs encoding congruence relations between newforms. These graphs have extremely interesting features: they help our understanding of the structure of Hecke algebras, and they are also a new tool in the study of numerous conjectures. In this talk I will describe these new objects, show examples and explain some of the possible applications.

*Joint work with Vandita Patel (University of Warwick, UK).*

A3 - July 12, 15:30 – 16:20

### PARITY OF RANKS OF ABELIAN SURFACES

**Celine Maistret**

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Let  $K$  be a number field and  $A/K$  an abelian surface. By the Mordell–Weil theorem, the group of  $K$ -rational points on  $A$  is finitely generated and as for elliptic curves, its rank is predicted by the Birch and Swinnerton–Dyer conjecture. A basic consequence of this conjecture is the parity conjecture: the sign of the functional equation of the L-series determines the parity of the rank of  $A/K$ . Under suitable local constraints and finiteness of the Shafarevich-Tate group, we prove the parity conjecture for principally polarized abelian surfaces. We also prove analogous unconditional results for Selmer groups.

A3 - July 12, 17:00 – 17:40

### RIGOROUS COMPUTATION OF THE ENDOMORPHISM RING OF A JACOBIAN

**Jeroen Sijsling**

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Consider a curve  $X$ , defined by an explicit equation, over a number field. Let  $J$  be the Jacobian of  $X$ . The endomorphism ring  $E$  of  $J$  is an important arithmetic invariant of  $X$ ; for example, the Sato–Tate group of  $X$  can be recovered from the Galois module structure of  $E$ .

Heuristically, the ring  $E$  can be determined quite quickly. In this talk, we consider the rigor of such calculations. In particular, we give an algorithm that verifies whether a putative tangent representation of an endomorphism in fact globalizes. It does not use complex approximations and is, to our knowledge, the first general-purpose algorithm over number fields that provably terminates. We then discuss how this algorithm can be combined with other techniques to rigorously compute the ring  $E$ .

*Joint work with Edgar Costa (Dartmouth College, USA), Nicolas Mascot (University of Warwick, UK), and John Voight (Dartmouth College, USA).*

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A3 - July 12, 17:40 – 18:20

GENETICS OF POLYNOMIALS OVER LOCAL FIELDS

**Jordi Guàrdia**

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Let  $(K, v)$  be a discrete valued field with valuation ring  $\mathcal{O}$ , and let  $\mathcal{O}_v$  be the completion of  $\mathcal{O}$  with respect to the  $v$ -adic topology. In our talk we will discuss the advantages of manipulating polynomials in  $\mathcal{O}_v[x]$  in a computer by means of OM representations of prime polynomials. An OM representation supports discrete data that are a kind of DNA sequence common to all individuals in the Okutsu equivalence class of the prime polynomial. These discrete parameters contain relevant arithmetic information about the polynomial and the extension of  $K_v$  it determines.

*Joint work with Enric Nart (Universitat Autònoma de Barcelona, Spain).*

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A3 - July 12, 18:20 – 19:00

RATIONAL POINT COUNT DISTRIBUTIONS FOR PLANE QUARTIC CURVES OVER FINITE FIELDS

**Nathan Kaplan**

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We use ideas from coding theory, specifically the MacWilliams theorem, to study rational point count distributions for quartic curves in the projective plane over a finite field. We explain how the set of all homogeneous quartic polynomials in three variables gives rise to an evaluation code, and how low-weight coefficients of the weight enumerator of the dual code give information about rational point count distributions for quartic curves. No previous familiarity with coding theory will be assumed.

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## Workshop A4 Computational Geometry and Topology

Organizers: Joel Hass – Herbert Edelsbrunner – Gunnar Carlsson

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A4 - July 10, 14:30 – 14:55

### GENERALIZED PERSISTENCE DIAGRAMS

**Amit Patel**

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We generalize the persistence diagram of Cohen–Steiner, Edelsbrunner, and Harer to the setting of constructible persistence modules valued in a symmetric monoidal category. We call this the type A persistence diagram of a persistence module. If the category is also abelian, then we define a second type B persistence diagram. In addition, we define a new metric between persistence diagrams we call the erosion distance which closely resembles the interleaving distance between persistence modules. We show that our assignment of a type B persistence diagram to a constructible persistence module is 1-Lipschitz.

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A4 - July 10, 15:00 – 15:25

### SUPERVISED LEARNING OF LABELED POINTCLOUD DIFFERENCES VIA COVER-TREE ENTROPY REDUCTION

**Paul Bendich**

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We introduce a new algorithm, called CDER, for supervised machine learning that merges the multi-scale geometric properties of Cover Trees with the information-theoretic properties of entropy. CDER applies to a training set of labeled pointclouds embedded in a common Euclidean space. If typical pointclouds corresponding to distinct labels tend to differ at any scale in any sub-region, CDER can identify these differences in (typically) linear time, creating a set of distributional coordinates which act as a feature extraction mechanism for supervised learning. We describe theoretical properties and implementation details of CDER, and illustrate its benefits on several synthetic examples

*Joint work with Abraham Smith (University of Wisconsin-Stout, Geometric Data Analytics, USA), John Harer (Duke University and Geometric Data Analytics, USA), and Jay Hineman (Geometric Data Analytics, USA).*



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A4 - July 10, 15:30 – 15:55

## LOCAL TOPOLOGICAL CONVERGENCE OF GRAIN GROWTH SYSTEMS

**Jeremy Mason**

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Most materials are composed of individual crystallites or grains joined along a network of grain boundaries. The grain boundary network is dynamic, with an evolution governed by an energy that is directly proportional to the surface area. This system is experimentally and computationally observed reach a dynamic steady state known as the grain growth microstructure that is defined by the convergence of all scale invariant quantities. Surprisingly, precise values for all but the most basic of these quantities are not known, and even the existence of the steady state has yet to be proven. This talk describes our simulations of grain boundary network evolution, and the use of local topological convergence to more precisely define and establish the existence of the grain growth microstructure.

*Joint work with Benjamin Schweinhart (Ohio State University, USA) and Robert MacPherson (Institute for Advanced Study, USA).*

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A4 - July 10, 16:00 – 16:25

## LOCAL GEOMETRIC CONVERGENCE OF EMBEDDED GRAPHS

**Benjamin Schweinhart**

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We introduce a notion of local geometric convergence for embedded graphs in  $\mathbb{R}^n$ , developed in analogy with Benjamini–Schramm convergence of abstract graphs. It is defined in terms of weak convergence of probability measures on the space of embedded graphs with a smooth topology. We provide hypotheses under which it is implied by convergence of the first Wasserstein metric induced by the local Hausdorff distance. These concepts are used to state a universality conjecture for the long-term behavior of graphs evolving by curvature flow, and to test it computationally.

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A4 - July 10, 17:00 – 17:25

## UNIVERSALITY OF THE HOMOTOPY INTERLEAVING DISTANCE

**Michael Lesnick**

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We introduce and study homotopy interleavings between filtered topological spaces. These are homotopy-invariant analogues of interleavings, objects commonly used in topological data analysis to articulate stability and inference theorems. Intuitively, whereas a strict interleaving between filtered spaces  $X$  and  $Y$  certifies that  $X$  and  $Y$  are approximately isomorphic,

a homotopy interleaving between  $X$  and  $Y$  certifies that  $X$  and  $Y$  are approximately weakly equivalent.

The main results of this paper are that homotopy interleavings induce an extended pseudometric  $d_{HI}$  on filtered spaces, and that this is the universal pseudometric satisfying natural stability and homotopy invariance axioms. To motivate these axioms, we also observe that  $d_{HI}$  (or more generally, any pseudometric satisfying these two axioms and an additional “homology bounding” axiom) can be used to formulate lifts of fundamental TDA theorems from the algebraic (homological) level to the level of filtered spaces.

*Joint work with Andrew J. Blumberg (The University of Texas at Austin, USA).*

A4 - July 10, 17:30 – 17:55

#### MINIMUM ACTION PRINCIPLES AND SHAPE DYNAMICS

**Patrice Koehl**

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The functions of all life forms depend on organization in space and time. Spatial organization involves the interactions of shapes with their surroundings, while organization in time concerns changes in shape. Geometric methods therefore play an essential role in the analyses and simulations of biological systems. Geometry and topology are now used regularly for representing, searching, simulating, analyzing, and comparing biological systems. In this talk, I will focus on the latter. I will present a new method for computing a distance between two shapes embedded in three dimensional space. Instead of comparing directly the geometric properties of the two shapes, we measure the cost of deforming one of the two shapes into the other. The deformation is computed as the geodesic between the two shapes in the space of shapes. The geodesic is found as a minimizer of the Onsager Machlup action, based on an elastic energy for shapes. I will illustrate applications of this method to geometric morphometrics using data sets representing bones and teeth of primates. Experiments on these data sets show that the method performs remarkably well both in shape recognition and in identifying evolutionary patterns, with success rates similar to, and in some cases better than, those obtained by expert observers.

A4 - July 10, 18:00 – 18:25

#### STABILIZING THE UNSTABLE OUTPUT OF PERSISTENT HOMOLOGY COMPUTATIONS

**Peter Bubenik**

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For data filtered by time or scale, persistent homology provides a succinct summary, the persistence diagram or bar code, which encodes the births and deaths of topological features. Crucially, this summary is stable with respect to perturbations of the data. Persistent homology computations may also provide the locations of these topological features, something of particular interest to practitioners, but these are unfortunately unstable. I will present a general framework for providing stable versions of such calculations.

*Joint work with Paul Bendich (Duke University, USA) and Alexander Wagner (University of Florida, USA).*

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A4 - July 10, 18:30 – 18:55

### THE REEB GRAPH EDIT DISTANCE IS UNIVERSAL

**Ulrich Bauer**

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We consider Reeb graphs in a general topological setting, and the construction of distances between Reeb graphs that are stable, meaning that similar functions in the supremum norm have similar Reeb graphs. We define a graph edit distance and show that it is universal, providing an upper bound to any other stable distance.

*Joint work with Claudia Landi (University of Modena and Reggio Emilia, Italy).*

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A4 - July 11, 14:30 – 14:55

### LOCAL COHOMOLOGY AND STRATIFICATION

**Vidit Nanda**

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I will outline an algorithm to recover the canonical (or, coarsest possible) stratification of a given regular CW complex into cohomology manifolds, each of which is a union of cells. The construction proceeds by iteratively localizing the poset of cells about a family of subposets; these subposets are in turn determined by a collection of cosheaves which capture variations in local cohomology across cells in the underlying complex. The result is a finite sequence of categories whose colimit recovers the canonical strata via isomorphism classes of its objects. The entire process is amenable to efficient distributed computation.

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A4 - July 11, 15:00 – 15:25

### INTRINSIC 1-DIMENSIONAL PERSISTENCE OF GEODESIC SPACES

**Ziga Virk**

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Given a nice compact geodesic space  $X$ , such as a compact Riemannian manifold, we present the theory of intrinsic 1-dimensional persistence, i.e., persistence on all points of  $X$  induced by the geodesic distance. While the induced complexes have uncountably many vertices, the mentioned persistence turns out to be pretty tame and has very precise geometric interpretation. It also gives a specific understanding on how the size of holes is measured by persistence.

It turns out that persistence via Rips or Čech construction contains precisely the same information in both cases. This information represents the collection of lengths of the shortest'

generating set. Furthermore, critical triangles can be used to localize minimal representatives of homology classes, and the approximation by finite samples is much more stable than suggested by standard stability theorems.

In the end we will outline how one may extract one-dimensional geometric information about  $X$  from certain higher-dimensional persistence features.

A4 - July 11, 15:30 – 15:55

#### ON THE COMPLEXITY OF OPTIMAL HOMOTOPIES

**Arnaud de Mesmay**

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We provide new structural results and algorithms for the Homotopy Height problem. In broad terms, this problem quantifies how much a curve on a surface needs to be stretched to sweep continuously between two positions. More precisely, given two homotopic curves  $\gamma_1$  and  $\gamma_2$  on a surface, we want to compute a homotopy between  $\gamma_1$  and  $\gamma_2$  where the length of the longest intermediate curve is minimized. Such optimal homotopies are relevant for a wide range of purposes, from very mathematical questions in quantitative homotopy theory to more practical applications such as similarity measures and graph searching problems.

Our main contribution is a structural theorem showing that there exist optimal homotopies which are very well behaved. It builds on earlier results in Riemannian geometry by Chambers and Liokumovitch and Chambers and Rotman. Leveraging on this theorem allows us to derive new exact and approximation algorithms to compute the homotopy height, and to draw connections with other problems in the same vein like homotopic Fréchet distance.

*Joint work with Erin W. Chambers (Saint Louis University, USA), Gregory R. Chambers (Rice University, USA), Tim Ophelders (TU Eindhoven, The Netherlands), and Regina Rotman (University of Toronto, Canada).*

A4 - July 11, 16:00 – 16:25

#### ESTIMATING THE REACH OF A MANIFOLD

**Frédéric Chazal**

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Various problems in manifold estimation, and topological and geometric inference make use of the so-called the reach (also known as the conditioning number or feature size) which is a measure of the regularity of the manifold. In this talk, we will investigate into the problem of how to estimate the reach of a manifold  $M$  from point clouds randomly sampled on  $M$ . We propose an estimator of the reach (in the framework where the tangent spaces of  $M$  are known) and we obtain upper and lower bounds on the minimax rates for estimating the reach.

*Joint work with E. Aamari (INRIA and Université Paris-Saclay, France), J. Kim (Carnegie Mellon University, USA), B. Michel (École Centrale de Nantes, France), A. Rinaldo (Carnegie Mellon University, USA), and L. Wasserman (Carnegie Mellon University, USA).*

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A4 - July 11, 17:00 – 17:25

TELLING 3-MANIFOLDS APART: NEW ALGORITHMS TO COMPUTE TURAEV–VIRO  
INVARIANTS

**Jonathan Spreer**

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The Turaev-Viro invariants are a powerful family of topological invariants. Although difficult to compute in general, they are the method of choice for computationally distinguishing between two given triangulations of 3-manifolds. I will discuss this family of invariants, and present an algorithm to compute them at level four, which runs in polynomial time for 3-manifolds with bounded first Betti number.

*Joint work with Clément Maria (The University of Queensland, Brisbane, Australia).*

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A4 - July 11, 17:30 – 17:55

EMBEDDABILITY IN  $\mathbb{R}^3$  IS **NP**-HARD

**Eric Sedgwick**

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Let  $\text{EMBED}_{k \rightarrow d}$  be the problem of deciding whether a given  $k$ -dimensional complex admits a piecewise-linear embedding in  $\mathbb{R}^d$ . The complexity of solving this problem varies widely in the parameters  $k$  and  $d$ :  $\text{GRAPH PLANARITY}$  ( $\text{EMBED}_{1 \rightarrow 2}$ ) is solvable in linear time, but in higher dimensions, depending on the codimension, the problem can range from polynomial-time solvable ( $d \geq 4, k < (2d - 2)/3$ ), to **NP**-hard ( $d \geq 4, d \geq k \geq (2d - 2)/3$ ), to undecidable ( $k \geq d - 1 \geq 4$ ).

We show that  $\text{EMBED}_{2 \rightarrow 3}$  and  $\text{EMBED}_{3 \rightarrow 3}$  are **NP**-hard, which helps to clarify the picture in dimension 3. These two problems were only recently shown to be decidable and, like that work, our proof relies heavily on the theory of 3-manifolds.

*Joint work with Arnaud de Mesmay (CNRS, GIPSA-Lab, France), Yo'av Rieck (University of Arkansas, USA), and Martin Tancer (Charles University, Czech Republic).*

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A4 - July 11, 18:00 – 18:25

RANDOM WALKS ON GROUPS WITH NEGATIVE CURVATURE

**Joseph Maher**

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We will discuss random walks on groups satisfying various types of negative curvature conditions. A simple example is the nearest neighbour random walk on the 4-valent tree, also known as the Cayley graph of the free group on two generators. A typical random walk moves away from

the origin at linear speed, and converges to one of the ends of the tree. We will discuss how to generalize this result to more general settings, such as hyperbolic groups, or acylindrical groups.

*Joint work with Giulio Tiozzo (University of Toronto, Canada).*

A4 - July 11, 18:30 – 18:55

#### THE COMPLEXITY OF UNKNOT RECOGNITION

**Marc Lackenby**

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The problem of deciding whether a given knot diagram represents the unknot is known to be solvable, but its complexity class is not fully understood. It is known to be in NP and co-NP, but it is not known whether there is a polynomial-time algorithm. I will briefly discuss some of the existing algorithms and will explain why there is little hope that they can be made to run in anything faster than exponential time. I'll then describe a new unknot recognition algorithm. I conjecture that there is a version of this algorithm that runs in quasi-polynomial time.

A4 - July 12, 15:00 – 15:25

#### TOPOLOGY ON TINY MACHINES: LOGSPACE AND FINITE-STATE COMPLEXITY

**Benjamin Burton**

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Instead of the usual focus on time complexity, here we discuss the space complexity of topological algorithms. We show that you can test 2-manifolds for homeomorphism in logspace (joint with Elder, Kalka & Tillmann), and we prove a variety of positive and negative results for 3-dimensional problems on finite state tree automata (joint with Fellows).

A4 - July 12, 15:30 – 15:55

#### DECIDING CONTRACTIBILITY OF A NON-SIMPLE CURVE ON THE BOUNDARY OF A 3-MANIFOLD

**Éric Colin de Verdière**

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We present an algorithm for the following problem: Given a triangulated 3-manifold  $M$  and a (possibly non-simple) closed curve on the boundary of  $M$ , decide whether this curve is contractible in  $M$ . Our algorithm is combinatorial and runs in exponential time. This is the first algorithm that is specifically designed for this problem; its running time considerably improves upon the existing bounds implicit in the literature for the more general problem of contractibility of closed curves in a 3-manifold. The proof of the correctness of the algorithm relies on methods of 3-manifold topology and in particular on those used in the proof of the Loop Theorem.

*Joint work with Salman Parsa (Mathematical Sciences Department, Sharif University of Technology, Tehran, Iran).*

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A4 - July 12, 16:00 – 16:25

MAXIMALLY PERSISTENT CYCLES IN RANDOM GEOMETRIC SIMPLICIAL COMPLEXES

**Matthew Kahle**

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One motivation for the emerging subject of stochastic topology is as a null hypothesis for topological statistics. There has been some earlier work studying the topology of random geometric simplicial complexes, but until recently not much was known about persistent homology of these complexes.

We prove upper and lower bounds on the persistence of the maximally persistent cycle, agreeing up to a constant factor. The results hold for persistent  $k$ -cycles in  $\mathbb{R}^d$ , for all  $d \geq 2$  and  $2 \leq k \leq d - 1$ , and for both Vietoris–Rips and Čech filtrations. This is an important step in the direction of quantifying topological inference, separating topological signal from noise.

*Joint work with Omer Bobrowski (Technion, Israel) and Primož Skraba (Jozef Stefan Institute, Slovenia).*

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A4 - July 12, 17:00 – 17:25

DISCRETE UNIFORMIZATION FOR POLYHEDRAL SURFACES AND ITS CONVERGENCE

**Feng Luo**

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We will discuss our work on discrete conformal geometry of polyhedral surfaces. A notion of discrete conformal equivalence of polyhedral surfaces is introduced and a discrete version of the uniformization theorem for compact polyhedral surfaces is established. We show that the discrete conformality is algorithmic and converges to the smooth conformal geometry. Applications to computer graphics will be addressed.

*Joint work with David Gu, Jian Sun, and Tianqi Wu.*

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A4 - July 12, 17:30 – 17:55

POLYHEDRAL REALIZATION OF HYPERBOLIC PUNCTURED SPHERES AND DISCRETE UNIFORMIZATION

**Boris Springborn**

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Two seemingly unrelated problems turn out to be equivalent. The first is a problem of 3-dimensional hyperbolic geometry: Given a complete hyperbolic surface of finite area that is homeomorphic to a sphere with punctures, find a realization as convex ideal polyhedron in hyperbolic space. The second is a problem of discrete complex analysis: Given a closed triangle mesh of genus zero, find a discretely conformally equivalent convex triangle mesh inscribed in a

sphere. The existence and uniqueness of a solution of the first (hence also the second) problem was shown by I. Rivin. His proof is not constructive. A variational principle leads to a new constructive proof.

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A4 - July 12, 18:00 – 18:25

## EXTRINSIC CONFORMAL GEOMETRY PROCESSING

**Keenan Crane**

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This talk covers fundamental theory and algorithms for manipulating curved surfaces via conformal (i.e., angle-preserving) transformations. Conformal transformations are valuable in applications because they naturally preserve the integrity of geometric data. Until recently, however, there has been no theory of conformal transformations suitable for general-purpose geometry processing algorithms: previous methods for computing conformal maps have been purely intrinsic, with target domains restricted to the flat two-dimensional plane or other spaces of constant curvature. We instead consider a certain time-independent Dirac equation, which replaces the traditional Cauchy-Riemann equation for surfaces immersed in Euclidean 3-space. This equation leads to efficient numerical algorithms that enable processing of surface data via direct manipulation of extrinsic curvature. We will also discuss connections to the recent notion of discrete conformal equivalence based on length cross ratios, including recovery of shape from quantities like lengths and curvatures.

*Joint work with Ulrich Pinkall (Technische Universität Berlin, Germany) and Peter Schröder (Caltech, USA).*

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A4 - July 12, 18:30 – 18:55

## TOPOLOGICAL ANALYSES OF OLFACTION AND SEMANTIC LEARNING

**Chad Giusti**

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I will briefly describe two very different applications of persistent homology in the analysis of neural systems: one to a model of olfactory sensing, and the other to development of semantic networks in children. The former is a work in progress involving some light geometry with points and planes, the latter an application of network filtration by nodes, and will involve the observation that there are two cycles in a network involving turtles and peas.

*Joint work with Vladimir Itskov (Pennsylvania State University, USA), Ann Sizemore (University of Pennsylvania, USA), Elisabeth Karuza (University of Pennsylvania, USA), and Danielle Bassett (University of Pennsylvania, USA).*



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A4 - July 12, 19:00 – 19:25

## TROPICAL COORDINATES ON THE SPACE OF PERSISTENCE BARCODES

**Sara Kalisnik**

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In the last two decades applied topologists have developed numerous methods for ‘measuring’ and building combinatorial representations of the shape of the data. The most famous example of the former is persistent homology. This adaptation of classical homology assigns a barcode, i.e., a collection of intervals with endpoints on the real line, to a finite metric space. Unfortunately, barcodes are not well-adapted for use by practitioners in machine learning tasks. We can circumvent this problem by assigning numerical quantities to barcodes and these outputs can then be used as input to standard algorithms. I will talk about tropical functions that can be used as coordinates on the space of barcodes. All of these are stable with respect to the standard distance functions (bottleneck, Wasserstein) used on the barcode space.

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A4 - Poster

## VISUALISING NORMAL SURFACES

**Daniel Crane**

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Many modern algorithms in computational topology rely on normal surfaces as a tool to study 3-manifolds. Despite their pivotal role in the study of 3-manifolds, little work towards visualising normal surfaces exists in the literature. To this end, we present a program that visualises triangulations of normal and almost normal surfaces. This is achieved by post processing force-directed layouts of the triangulations’ layered primal/dual graphs. These can then be used to visualise 0-efficient triangulations of  $S^3$ , by tracking the normalisation of embedded almost normal spheres. An alternative and aesthetically pleasing — visualisation method for normal spheres using Lombardi drawings is also explored.

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A4 - Poster

## THE LEXICOGRAPHIC DEGREE OF THE FIRST TWO-BRIDGE KNOTS

**Pierre–Vincent Koseleff**

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We study the degree of polynomial representations of knots. We give here the lexicographic degree of all two-bridge knots with 11 or fewer crossings. The proof uses the braid theoretical method developed by Orevkov to study real plane curves [BKP2], isotopies on trigonal space curves and explicit parametrizations obtained by perturbing a triple point.

The lexicographic degree of a knot  $K$  is the minimal multidegree, for the lexicographic order, of a polynomial knot whose closure in  $S_3$  is isotopic to  $K$ .

We show that this degree is not necessarily obtained for curves that have the minimal number of real crossings. For example the knot  $8_6$  has minimal degree  $(3, 10, 14)$  that corresponds to a diagram with 9 crossings while the alternating diagram with 8 crossings has degree at least  $(3, 11, 13)$ .

Here we study two-bridge knots, namely those that admit a trigonal parametrization, that is to say a polynomial parametrization in degree  $(3, b, c)$ .

We show the sharp lower bound  $c \geq 3N - b$ , where  $N$  is the crossing number of  $K$ .

Lower bounds for  $b$  are obtained by considering first Bézout-like conditions on the real crossings of the plane projection.

We also use the associated 3-strings braid associated to any trigonal algebraic plane curve that must be quasi-positive [Or2].

Upper bounds are given by previous constructions on Chebyshev diagrams [KP3] and slide isotopies on knot diagrams: isotopies for which the number of crossings never increases [BKP1].

In addition, we introduce a new reduction  $R$  on trigonal plane curves. In many cases, the use of this reduction allows to subtract three to the number of crossings of the diagram and to the degree  $b$ , thanks to a result on real pseudoholomorphic curves deduced from [Or2]. On the other hand we show that we can explicitly give a polynomial curve of degree  $(3, d + 3)$  from a polynomial curve of degree  $(3, d)$  by adding a triple point and perturbing the singularity.

We obtain the lexicographic degree for some infinite families of two-bridge knots (the torus knots and generalized twist-knots) and for all two-bridge knots with 11 or fewer crossings. For every considered knot, we compute first an upper bound  $b_C$  for  $b$ , using Chebyshev diagrams. We compute all diagrams with  $b_C - 1$  or fewer crossings that cannot be reduced by slide isotopies. Most of these diagrams may be reduced using  $R$ -reductions. For those that cannot be reduced, we consider all possible schemes corresponding to admissible rational algebraic curve of degree  $(3, b) < (3, b_C)$  and compute their associated braids, that must be quasi-positive.

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*Joint work with Erwan Brugallé (Centre de Mathématiques Laurent Schwartz, CMLA, École Polytechnique, France) and Daniel Pecker (UPMC – Sorbonne Universités, Paris VI, France).*

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A4 - Poster

## STRING TOPOLOGICAL ROBOTICS II

**My Ismail Mamouni**

Maroc

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In a previous work, the authors claim to link two well known theories; namely the string topology (founded by M. Chas and D. Sullivan in 1999) and the topological robotics (founded by M. Farber some few years later, in 2003). For their purpose, they consider  $G$  a compact Lie group acting transitively on a path connected  $n$ -manifold  $X$ . On the set  $MLP(X)$  of the so-called loop motion planning algorithms, they define an string loop motion planning product on the shifted string loop motion planning homology  $\mathbb{H}_*(MLP(X)) := H_{*+2n}(MLP(X))$ . In this talk, we finish this work and extend the graded commutative and associative algebra structure on  $\mathbb{H}_*(MLP(X))$  to a structure of Gerstenhaber algebra and to that of Batalin-Vilkovisky algebra.

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A4 - Poster

## THE TOPOLOGY OF $D$ -STABILITY

**Grey Violet**

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The set of problems connected with  $D$ -stability: distribution of roots of a polynomial relative to a region  $\Omega \subset \mathbb{C}$  is one of the fundamental in control theory, studied for such special cases as Hurwitz stability ( $\Omega = \{Re z < 0\}$ ) and Schur stability ( $\Omega = \{|z| < 1\}$ ) since the emergence of the field.

General theory of  $D$ -stability, which development could be attributed to R.E. Kalman, B.R. Barmish & al., gave different algorithmic and algebraic criteria for determining distributions roots of sets of polynomials in relation to  $\Omega$ .

Despite that the geometric information about the structure of spaces of polynomials with fixed distribution of roots is very limited.

Author describe topology of sets of polynomials or homogeneous binary forms with fixed number of roots inside  $\Omega$ , on the border of  $\Omega$  and outside of  $\Omega$  using theory of symmetric products of topological spaces. Moreover author explains, the special position of 3 classical  $D$ -stability problems, namely, Hurwitz stability, Schur stability and hyperbolicity  $\Omega = \mathbb{R}$ .

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**Workshop A5****Geometric Integration and Computational Mechanics**Organizers: Fernando Casas – Elena Celledoni – David Martin de Diego

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A5 - July 10, 14:30 – 15:00

**IT TAKES A WAVE PACKET TO CATCH A WAVE PACKET****Arieh Iserles**University of Cambridge, United Kingdom  
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We are concerned with spectral methods for signals composed of wave packets, e.g., in quantum mechanics. The traditional approach is to use periodic boundary conditions, in which case standard Fourier methods are more than adequate, except that in long-term integration wave packets might reach the boundary and non-physical behaviour ensues. This motivates us to consider approximations on the entire real line. We consider and analyse in detail four candidates: Hermite polynomials, Hermite functions, stretched Chebyshev expansions and stretched Fourier expansions. In particular, we are concerned with the speed of convergence of standard and  $m$ -term approximations. And the winner is...

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A5 - July 10, 15:00 – 15:30**UNIFORMLY ACCURATE AVERAGING NUMERICAL SCHEMES FOR OSCILLATORY  
EVOLUTION EQUATIONS****Philippe Chartier**INRIA, France  
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In this talk, we consider differential equations with (possibly high) oscillations whose frequency scales like the inverse of a parameter with values in an interval of the form  $]0; 1]$ . In particular, we do not envisage the (possibly highly-) oscillating system in its asymptotic regime. Our aim is to design a numerical method which can solve such systems at a cost and with an accuracy remaining essentially insensitive to the value of the parameter.

*Joint work with Mohammed Lemou (Université de Rennes, France), Florian Mehats (Université de Rennes, France), and Gilles Vilmart (Université de Genève, Suisse).*

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A5 - July 10, 15:30 – 16:00**MAGNUS-TYPE INTEGRATORS COMBINED WITH OPERATOR SPLITTING METHODS AND  
THEIR AREAS OF APPLICATIONS****Mechthild Thalhammer**Leopold-Franzens-Universität Innsbruck, Austria  
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In this talk, I shall introduce the class of commutator-free quasi-Magnus exponential integrators for non-autonomous linear evolution equations and identify different areas of application.

Commutator-free quasi-Magnus exponential integrators are (formally) given by a composition of several exponentials that comprise certain linear combinations of the values of the defining operator at specified nodes. Avoiding the evaluation of commutators, they provide a favourable alternative to standard Magnus integrators.

Non-autonomous linear evolution equations also arise as a part of more complex problems, for instance in connection with nonlinear evolution equations of the form  $u'(t) = A(t)u(t) + B(u(t))$ . A natural approach is thus to apply commutator-free quasi-Magnus exponential integrators combined with operator splitting methods. Relevant applications include Schrödinger equations with space-time-dependent potential describing Bose-Einstein condensation or diffusion-reaction systems modelling pattern formation.

A5 - July 10, 16:00 – 16:30

#### ENERGY PRESERVING TIME INTEGRATORS FOR PDES ON MOVING GRIDS

**Brynjulf Owren**

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Energy preserving, or more generally, integral preserving schemes for ODEs can be derived by means of for instance discrete gradient methods. For PDEs, one may first discretize in space, using for instance finite difference methods or finite element methods, and then apply an integral preserving method for the corresponding ODEs. For PDEs discretized on moving grids, the situation is more complicated, it is not even clear exactly what should be meant by an integral preserving scheme in this setting. We shall propose a definition and then derive the resulting conservative schemes, both with finite difference schemes and with finite element schemes. We test the methods on problems with travelling wave solutions and demonstrate that they give remarkably good results, both compared to fixed grid and to non-conservative schemes

*Joint work with Sølve Eidnes (NTNU, Norway) and Torbjørn Ringholm (NTNU, Norway).*

A5 - July 10, 17:00 – 17:30

#### THE SPHERICAL MIDPOINT METHOD

**Klas Modin**

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The 2-sphere  $S^2$  is, perhaps, the most fundamental example of a non-canonical symplectic manifold. Yet, to construct symplectic integration schemes for Hamiltonian systems on  $S^2$  has been surprisingly cumbersome. In this talk I shall present a new integrator — the spherical midpoint method — for general Hamiltonian systems on  $S^2$ . The new method uses a minimal number of variables, is equivariant with respect to the homogeneous space structure of the 2-sphere, and is readily extendable to general Hamiltonian systems on  $(S^2)^m \times \mathbb{R}^{2n}$ . I shall also

discuss applications to atomistic spin dynamics in condensed matter physics (collaboration with physicists at Uppsala University), and a possible generalization of the method to other Kähler manifolds.

*Joint work with Robert McLachlan (Massey University, New Zealand) and Olivier Verdier (Western Norway University of Applied Sciences, Norway).*

A5 - July 10, 17:30 – 18:00

## HIGHER ORDER VARIATIONAL INTEGRATORS AND THEIR RELATION TO RUNGE-KUTTA METHODS FOR UNCONSTRAINED AND CONSTRAINED SYSTEMS

**Sina Ober-Blöbaum**

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In this talk higher order variational integrators are developed and their relation to Runge–Kutta methods is investigated. While it is well known that particular higher order variational integrators are equivalent to symplectic partitioned Runge–Kutta methods, this is not true for more generally constructed variational integrators. Based on existing results, modified schemes of Runge–Kutta integrators are derived and shown to be equivalent to a new class of higher order variational integrators which distinguishes in the dimension of the function space for approximating the solution curves compared to classical Galerkin variational integrators. Conditions are derived under which both variational integrators are identical and demonstrated numerically by simple mechanical examples. The results are extended to systems with holonomic constraints.

*Joint work with Sigrid Leyendecker (University of Erlangen–Nuremberg, Germany) and Theresa Wenger (University of Erlangen–Nuremberg, Germany).*

A5 - July 10, 18:00 – 18:30

## THE INVERSE PROBLEM FOR DISCRETE SYSTEMS

**Marta Farré Puiggalí**

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The classical inverse problem of the calculus of variations consists in determining whether or not a given system of explicit second order differential equations is equivalent to a system of Euler–Lagrange equations for some regular Lagrangian. We will develop a geometric version of the inverse problem of the calculus of variations for discrete mechanics and constrained discrete mechanics, using suitable Lagrangian and isotropic submanifolds. We will also provide a transition between the discrete and the continuous problems and propose variationality as an interesting geometric property to take into account in the design of geometric integrators.

*Joint work with María Barbero Liñán (Universidad Politécnica de Madrid, Spain), Sebastián Ferraro (Universidad Nacional del Sur, Argentina) and David Martín de Diego (ICMAT, Spain).*

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A5 - July 10, 18:30 – 19:00

A FAMILY OF MODIFIED TRIGONOMETRIC INTEGRATORS FOR HIGHLY OSCILLATORY PROBLEMS WITH GOOD GEOMETRIC PROPERTIES

**Antonella Zanna Munthe-Kaas**

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We propose a new family of modified trigonometric integrators for highly oscillatory ODEs. The family includes the IMEX method as a particular example. All the methods are symplectic, second order, and resonance free. Slow energy exchange and long time preservation of energy and oscillatory energy are analyzed.

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A5 - July 11, 14:30 – 15:20

THE CLEBSCH REPRESENTATION IN OPTIMAL CONTROL, INTEGRABLE SYSTEMS AND DISCRETE DYNAMICS

**Anthony Bloch**

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In this talk we discuss certain kinematic optimal control problems (the Clebsch problems) and their connection to classical integrable systems. In particular, we consider the rigid body problem and its low rank counterparts, the geodesic flows on Stiefel manifolds and their connection with the work of Moser, flows on symmetric matrices, and the Toda flows. We also discuss discrete formulations of these systems and their connection with numerical algorithms.

*Joint work with François Gay-Balmaz (École Normale Supérieure, France) and Tudor Ratiu (Shanghai Jiao Tong University and EPFL (Shanghai and Lausanne)).*

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A5 - July 11, 15:30 – 16:00

HOW DO NONHOLONOMIC INTEGRATORS WORK?

**Olivier Verdier**

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Nonholonomic systems are mechanical systems with constraints on the velocity. Their behaviour is quite different from that of mechanical systems with constraints on the positions (holonomic systems). There has been reports in the literature of integrators which behaved particularly well for some nonholonomic systems: near conservation of energy, or near conservation of other integrals.

We will explain the general mechanism behind those good properties. The main structure of the examples where the nonholonomic integrators work is that of a fibration over a reversible integrable system. The explaining theory, for descending integrators, is then the reversible Kolmogorov–Arnold–Moser theory.

We will explain how to design various systems which deviate from that pattern, in order to show experimentally that this structure of fibration over a reversible integrable system is necessary. Non-holonomic integrators do not preserve any of the features of those perturbed systems: neither the energy, nor the integrable structure.

We will also single out one non-holonomic integrator, which has two properties of interests: it is semi-implicit for a large range of systems, and it still has a good, completely unexplained, behaviour on some nonholonomic systems.

*Joint work with Klas Modin (Chalmers University of Technology, Sweden).*

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A5 - July 11, 16:00 – 16:30

## ON UNIT-QUATERNION BASED GALERKIN LIE GROUP VARIATIONAL INTEGRATORS

**Sigrid Leyendecker**

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Lie-group variational integrators are often used to simulate rigid body dynamics [4, 5]. Using unit quaternions is an efficient way to represent the rotational degrees of freedom of a rigid body [1]. This is in part due to the fact, that they facilitate a simple interpolation method presented in this talk. This interpolation method, together with an appropriate quadrature rule, is then used to approximate the action of the dynamical system with rotational degrees of freedom and a Lie-group variational integrator of arbitrary order is derived.

We present a numerical convergence analysis, both on the main grid, as well as for the whole Galerkin curve. Our convergence analysis for the whole Galerkin curve is in agreement with the results of Hall & al., [2, 3], but we also investigate the convergence on the main grid, where the convergence rate is considerably higher. Furthermore, we use unit quaternions instead of rotation matrices, which might be computationally more efficient. Our interpolation method does not require a change of coordinates in the momentum matching part of the discrete Euler–Lagrange equations as the one used in [2]. The computational efficiency is investigated by showing the relationship between the error and the CPU-time.

The same method can be extended to derive multisymplectic Galerkin Lie-group variational integrators, e.g., for the simulation of geometrically exact beam dynamics, by applying the interpolation method to the two-dimensional space time domain.

## Bibliography

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*Joint work with Thomas Leitz (University of Erlangen–Nuremberg, Germany).*

A5 - July 11, 17:00 – 17:30

#### ROUGH PATHS ON HOMOGENEOUS SPACES

**Hans Munthe–Kaas**

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We consider rough paths in the context of differential equations on homogenous manifolds. Our approach generalises both the Chen-type shuffle algebra of word series underlying Terry Lyons’ theory of rough paths as well as the branched rough paths of Massimiliano Gubinelli, which are based on B-series and the Connes–Kreimer Hopf algebra of rooted trees. The new approach contains these as special cases and extends geometrically to rough paths evolving on homogeneous manifolds.

#### Bibliography

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*Joint work with Charles Curry (NTNU, Norway), Kurusch Ebrahimi–Fard (NTNU, Norway), and Dominique Manchon (Clermont-Ferrand, France).*

A5 - July 11, 17:30 – 18:00

#### ACTION ALGEBROIDS ARE THE ONLY POST-LIE ALGEBROIDS

**Ari Stern**

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Despite their name, Lie group integrators may also be applied to certain manifolds that are not Lie groups: homogeneous spaces, for example. In general, Munthe–Kaas and Lundervold showed that Lie–Butcher series methods may be defined on vector bundles called post-Lie algebroids. We show that every post-Lie algebroid arises from a Lie algebra action, meaning that manifolds

with Lie algebra actions are essentially the most general spaces to which Lie–Butcher series methods may be applied.

*Joint work with Hans Z. Munthe–Kaas (University of Bergen, Norway) and Olivier Verdier (Bergen University College, Norway).*

A5 - July 11, 18:00 – 18:30

### PALINDROMIC 3-STAGE SPLITTING INTEGRATORS, A ROADMAP

**Cedric M. Campos**

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The implementation of multi-stage splitting integrators is essentially the same as the implementation of the familiar Strang/Verlet method. Therefore multi-stage formulas may be easily incorporated into software that now uses the Strang/Verlet integrator. We study in detail the two-parameter family of palindromic, three-stage splitting formulas and identify choices of parameters that may outperform the Strang/Verlet method. One of these choices leads to a method of effective order four suitable to integrate in time some partial differential equations. Other choices may be seen as perturbations of the Strang method that increase efficiency in molecular dynamics simulations and in Hybrid Monte Carlo sampling.

*Joint work with J.M. Sanz–Serna (Universidad Carlos III de Madrid, Spain).*

A5 - July 11, 18:30 – 19:00

### ANALYZING SPLITTING STOCHASTIC INTEGRATORS USING WORD SERIES

**Alfonso Álamo**

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Recently, word series expansion have been suggested as an alternative to B-series. Word series are similar to, but simpler than, the B-series used to analyze Runge–Kutta and other one-step integrators, although its scope of applicability is narrower than that of B-series. We shall show how to extend them to the stochastic case in which word series turn out to be again an extremely useful tool for the analysis of split-step integrators, yielding a systematic way to obtain, for example, modified equations and order conditions.

*Joint work with J.M. Sanz–Serna (Universidad Carlos III de Madrid, Spain).*

A5 - July 12, 14:30 – 15:20

### STOCHASTIC PARTIAL DIFFERENTIAL FLUID EQUATIONS AS A DIFFUSIVE LIMIT OF DETERMINISTIC LAGRANGIAN MULTI-TIME DYNAMICS

**Darryl Holm**

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In [Holm, Proc. Roy. Soc. A 471 (2015)] stochastic fluid equations were derived by employing a variational principle with an assumed stochastic Lagrangian particle dynamics. Here we show that the same stochastic Lagrangian dynamics naturally arises in a multi-scale decomposition of the deterministic Lagrangian flow map into a slow large-scale mean and a rapidly fluctuating small scale map. We employ homogenization theory to derive effective slow stochastic particle dynamics for the resolved mean part, thereby justifying stochastic fluid partial equations in the Eulerian formulation. To justify the application of rigorous homogenization theory, we assume mildly chaotic fast small-scale dynamics, as well as a centering condition. The latter requires that the mean of the fluctuating deviations is small, when pulled back to the mean flow.

*Joint work with Colin J. Cotter (Imperial College London, UK) and Georg A. Gottwald (University of Sydney, Australia).*

A5 - July 12, 15:30 – 16:00

#### VARIATIONAL INTEGRATORS IN GEOPHYSICAL FLUID DYNAMICS

**François Gay-Balmaz**

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We present an overview of structure-preserving variational discretizations of various equations of geophysical fluid dynamics, such as the anelastic, pseudo-incompressible, and shallow-water equations. These discretizations rely on finite dimensional approximations of the groups of diffeomorphisms underlying the dynamics. In particular, we extend previous approaches to the compressible case. As descending from variational principles, the discussed variational schemes exhibit discrete versions of Kelvin circulation theorems, are applicable to irregular meshes, and show excellent long term energy behavior.

*Joint work with Werner Bauer (Imperial College London, UK).*

A5 - July 12, 16:00 – 16:30

#### INTERPOLATION OF MANIFOLD-VALUED FUNCTIONS VIA THE GENERALIZED POLAR DECOMPOSITION

**Evan Gawlik**

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We construct interpolation operators for manifold-valued functions, with an emphasis on functions taking values in symmetric spaces and Lie groups. A key role in our construction is played by the polar decomposition — the well-known factorization of a real nonsingular matrix into the product of a symmetric positive-definite matrix times an orthogonal matrix — and its generalization to Lie groups. We demonstrate that this factorization can be leveraged to carry out a number of seemingly disparate tasks, including the design of finite elements for numerical relativity, the interpolation of subspaces for reduced-order modeling, and the approximation of Riemannian cubics on the special orthogonal group.

*Joint work with Melvin Leok (University of California, San Diego, USA).*

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A5 - July 12, 17:00 – 17:30

COMMUTATOR-FREE MAGNUS BASED METHODS

**Karolina Kropielnicka**

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In this talk, I shall introduce the class of commutator-free quasi-Magnus exponential integrators for non-autonomous linear evolution equations and identify different areas of application.

Commutator-free quasi-Magnus exponential integrators are (formally) given by a composition of several exponentials that comprise certain linear combinations of the values of the defining operator at specified nodes. Avoiding the evaluation of commutators, they provide a favourable alternative to standard Magnus integrators.

Non-autonomous linear evolution equations also arise as a part of more complex problems, for instance in connection with nonlinear evolution equations of the form  $u'(t) = A(t)u(t) + B(u(t))$ . A natural approach is thus to apply commutator-free quasi-Magnus exponential integrators combined with operator splitting methods. Relevant applications include Schrödinger equations with space-time-dependent potential describing Bose-Einstein condensation or diffusion-reaction systems modelling pattern formation

*Joint work with Philipp Bader (La Trobe University, Australia), Iserles Arieh (University of Cambridge, UK), Pranav Singh (University of Oxford, UK).*

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A5 - July 12, 17:30 – 18:00

GEOMETRIC INTEGRATORS AND OPTIMAL CONTROL FOR REAL-TIME CONTROL OF  
ROBOTIC SYSTEMS

**Marin Kobilarov**

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This talk will focus on geometric integrators and trajectory optimization of physical systems consisting of rigid bodies and actuators such as propellers, joint motors, or wheels. Since the integrators in general are implicit, the time-step is chosen in order to guarantee convergence. Therefore, even with a large time step one could preserve properties such as momentum conservation while still being able to solve the system dynamics correctly. These integrators are then used for optimal control, using a projected stage-wise Newton method which using standard regularization and line-search techniques provides a locally optimal solution. The methods are applied to robotic systems including aerial and ground vehicles, resulting in millisecond run-times suitable for on-board implementation.

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A5 - July 12, 18:00 – 18:30

AN IMPROVED ALGORITHM TO COMPUTE THE EXPONENTIAL OF A MATRIX

**Sergio Blanes**

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Exponential integrators have shown to be highly efficient geometric integrators, in particular as Lie group methods. However, for matrices of moderate size, their performance depends on an efficient computation of the matrix exponential. This can be achieved by the scaling-squaring technique with an appropriate approximation to the Taylor expansion of the exponential. Padé approximants allows to approximate polynomial matrix functions with relatively few products of matrices (the cost to compute the inverse of a matrix is  $4/3$  the cost of one product). Recently, the Paterson-Stockmeyer scheme has shown to be superior to Padé in many cases, since only requires  $2(k - 1)$  products to compute a polynomial of degree  $k^2$ .

Taking into account that with  $k$  products one can build matrix polynomials of degree  $2^k$ , we consider the inverse problem. Given a matrix polynomial, we analyse how to factorize it with the minimum number of products. For example, a polynomial of degree 8 can be obtained with only 3 products. We consider the decomposition of higher order polynomials and its application to approximate the exponential of a matrix and its performance with respect to the algorithm “expm” used in Matlab.

*Joint work with Fernando Casas (Universitat Jaume I, Spain) and Philipp Bader (La Trobe University, Australia).*

A5 - Poster

#### HAMEL’S FORMALISM FOR INFINITE-DIMENSIONAL MECHANICAL SYSTEMS

**Yakov Berchenko–Kogan**

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This poster presents results of a paper of the same title. We introduce Hamel’s formalism for infinite-dimensional mechanical systems and in particular consider its applications to the dynamics of nonholonomically constrained systems. This development is a nontrivial extension of its finite-dimensional counterpart. The analysis is applied to several continuum mechanical systems of interest, including coupled systems and systems with infinitely many constraints.

*Joint work with Donghua Shi (Beijing Institute of Technology, China), Dmitry V. Zenkov (North Carolina State University, USA), and Anthony M. Bloch (University of Michigan, USA).*

A5 - Poster

#### TIME-AVERAGE MAGNUS-DECOMPOSITION METHODS FOR SOLVING NON-AUTONOMOUS LINEAR WAVE EQUATIONS

**Nikita Kopylov**

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The work presented addresses the numerical integration of the second-order time-dependent linear partial differential equation

$$u_{tt}(x, t) = f(t, x) u(x, t), \quad x \in \mathbb{R}^d, \quad t \geq 0,$$

equipped with the initial conditions  $u(x, 0) = u_0(x)$  and  $u_t(x, 0) = u'_0(x)$ .

After spatial discretization, the equation can be expressed as

$$y''(t) = M(t)y(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

where  $t \in \mathbb{R}$ ,  $y \in \mathbb{C}^r$ .

In general, the solution is oscillatory, and exponential integrators show a good performance since they, in turn, provide numerical solutions which reflect the oscillatory nature of the exact solution.

However, due to the large dimension of the problem, one has to consider thoroughly the computational cost of the schemes. As  $M(t)$  originates from the discretization of a PDE, methods that employ only matrix–vector products are computationally reasonable. Thus, one of the promising approaches is to approximate exponentials in Magnus-decomposition methods by employing Krylov-type subspace methods.

We compared the solutions of non-autonomous problems obtained by the new methods, by their predecessors and by a set of well-known methods. The results show that Krylov-type exponential computation facilitates application of Magnus-decomposition methods, which were initially designed for solving matrix differential equations, to large-dimensional problems.

*Joint work with Philipp Bader (La Trobe University, Australia), Sergio Blanes (Universitat Politècnica de València, Spain), and Fernando Casas (Universitat Jaume I, Spain).*

A5 - Poster

## HIGHER-ORDER GEOMETRIC NONHOLONOMIC INTEGRATORS ON VECTOR SPACES AND LIE GROUPS

**Rodrigo Takuro Sato Martín de Almagro**

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We have obtained geometrically consistent arbitrarily high-order partitioned Runge–Kutta integrators for nonholonomic systems both on vector spaces and Lie groups. These methods can be understood as a natural extension of symplectic and variational integrators in the realm of nonholonomic systems and fit nicely into a nonholonomic Hamilton–Jacobi framework. These methods differ from those of J. Cortés and S. Martínez in that we do not require the discretisation of the constraint. Our methods preserve the continuous constraint exactly and can be seen to extend those of M. de León, D. Martín de Diego, and A. Santamaría.

*Joint work with David Martín de Diego (ICMAT, Spain).*

A5 - Poster

## STRUCTURE PRESERVING INTEGRATION OF NON-SPHERICAL PARTICLES IN TURBULENT FLOWS

**Benjamin Tapley**

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The dynamics of point, non-spherical particles in turbulent flows are considered. The flows are simulated with a direct numerical simulation (DNS) strategy and the particles are modeled as rigid bodies with torques calculated from the Jeffrey equations [1]. The particle dimensions are assumed to be lower than the Kolmogorov scale, the minimum length scale at which vortices remain stable. Typical large-scale computations involve systems with many particles (e.g.,  $10^6$ ), whose trajectories are obtained by integrating in time 13 ordinary differential equations per particle [2]; hence, computational cost becomes a significant factor when deciding on an appropriate numerical scheme. Implementing appropriate geometric numerical integration techniques can improve the performance of the simulation codes thus allowing for larger time-steps in the integration of the particle trajectories whilst maintaining similar levels of accuracy when compared to conventional integrators. In this poster, we implement structure preserving splitting methods that use Jacobi elliptic functions on the free rigid body equations to model the particle torques. This solution is compared against conventional integrators (e.g., Runge–Kutta or multi-step type integrators) and a reference solution. Such a method will lead to more stable and accurate numerical solutions when compared to conventional methods, and therefore reduce computational cost.

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*Joint work with Helge Andersson (NTNU, Norway), Elena Celledoni (NTNU, Norway), Laurel Ohm (University of Minnesota, USA), and Brynjulf Owren (NTNU, Norway).*

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A5 - Poster

## MIXED ORDER VARIATIONAL INTEGRATORS FOR MULTISCALE PROBLEMS

**Theresa Wenger**

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The numerical treatment of mechanical systems showing dynamics on different time scales, caused e.g., by different types or stiffnesses in potentials, is challenging. The required accuracy for a stable integration is dictated by the fast motion, leading to unacceptable high computational costs. To construct a more efficient integrator, the idea here is to separate the unknown configurations  $q$  into fast and slow degrees of freedom and use different polynomials to approximate each. Furthermore, the contributions of the action are approximated via quadrature rules of different order. Applying Hamilton’s principle provides the variationally derived integration scheme. If the system is holonomically constrained, the action is augmented by the integral of  $g(q) \cdot \lambda$ , with  $\lambda$  being the Lagrange-multiplier. The constraints  $g$

are split into a part depending only on slow degrees of freedom and another part coupling the whole configuration. This distinction enables different approximations, which are subject to some restrictions that ensure the solvability of the discrete equations and prevent a drift off the constraint manifold. The properties of the presented integrators of mixed order are considered such as energy and momentum maps preservation, time reversibility and stability.

*Joint work with Sigrid Leyendecker (University of Erlangen-Nuremberg, Germany) and Sina Ober-Blöbaum (University of Oxford, UK).*

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A5 - Poster

# LIE GROUP INTEGRATORS COMMUTE UP TO ORDER FOUR

**Michele Zadra**

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Let  $\mathcal{M}$  be a manifold,  $\mathcal{G}$  a Lie group and  $\mathfrak{g}$  its associated Lie algebra. We wish to integrate systems of the form

$$\begin{cases} \rho_x = Q^x \rho \\ \rho_y = Q^y \rho \end{cases}$$

where  $\rho: \mathcal{M} \rightarrow \mathcal{G}$  is a moving frame and  $Q^x, Q^y: \mathcal{M} \rightarrow \mathfrak{g}$ . The compatibility condition for the system above can be expressed as

$$R(Q^x, Q^y) := \frac{\partial}{\partial y} Q^x - \frac{\partial}{\partial x} Q^y - [Q^y, Q^x] = 0$$

We investigate whether the Lie group integrators discussed in [2] commute when applied in the two different directions. Specifically we seek to show that if the integrals for  $\rho$  are calculated along the two paths represented by

$$(x_0, y_0) \longrightarrow (x_0 + h, y_0) \longrightarrow (x_0 + h, y_0 + k)$$

and

$$(x_0, y_0) \longrightarrow (x_0, y_0 + k) \longrightarrow (x_0 + h, y_0 + k),$$

where  $h, k$  are the stepsizes in the two directions of a discretised region of  $\mathbb{R}^2$ , then the results are the same.

To this end, we write  $Q^x$  and  $Q^y$  as symbolic Taylor expansions around the point  $(x_0, y_0)$  and insert those into the relevant Magnus and BCH expansions. We show the integrators commute up to order 4 in the sense that the coefficients of  $h^i k^j$  ( $i + j \leq 4$ ) are linear differential expressions of  $R(Q^x, Q^y)$  and are hence zero. The commutation is a testament to the geometry built into the Lie group integrators [2].

Our application is to the study of Lie group invariant variational systems, where the moving frame,  $\rho$ , plays a pivotal role [1, 3].



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*Joint work with Elizabeth Mansfield (University of Kent, UK).*

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**Workshop A6****Mathematical Foundations of Data Assimilation and Inverse Problems**Organizers: Jean-Frédéric Gerbeau – Sebastian Reich – Karen Willcox

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A6 - July 10, 14:30 – 15:00

**UNLOCKING DATASETS BY CALIBRATING POPULATIONS OF MODELS TO DATA  
DENSITY: A STUDY IN ATRIAL ELECTROPHYSIOLOGY****Kevin Burrage**Queensland University of Technology, the University of Oxford, Australia, UK  
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The understanding of complex physical or biological systems nearly always requires a characterisation of the variability that underpins these processes. In addition, the data used to calibrate such models may also often exhibit considerable variability. A recent approach to deal with these issues has been to calibrate populations of models (POMs), that is multiple copies of a single mathematical model but with different parameter values. To date this calibration has been limited to selecting models that produce outputs that fall within the ranges of the dataset, ignoring any trends that might be present in the data. We present here a novel and general methodology for calibrating POMs to the distributions of a set of measured values in a dataset. We demonstrate the benefits of our technique using a dataset from a cardiac atrial electrophysiology study based on the differences in atrial action potential readings between patients exhibiting sinus rhythm (SR) or chronic atrial fibrillation (cAF) and the Courtemanche-Ramirez-Nattel model for human atrial action potentials. Our approach accurately captures the variability inherent in the experimental population, and allows us to identify the differences underlying stratified data as well as the effects of drug block.

*Joint work with Brodie A.J. Lawson (Queensland University of Technology, Brisbane, Australia), Christopher C. Drovandi (Queensland University of Technology, Australia), Nicole Cusimano (Basque Center for Applied Mathematics, Spain), Pamela Burrage (Queensland University of Technology, Australia) and Blanca Rodriguez (University of Oxford, UK).*

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A6 - July 10, 15:00 – 15:30

**REDUCED BASIS' ACQUISITION BY A LEARNING PROCESS FOR RAPID ON-LINE  
APPROXIMATION OF SOLUTION TO PDE'S: LAMINAR FLOW PAST A BACKSTEP****Yvon Maday**Université Pierre et Marie Curie, Laboratoire J.-L. Lions, France  
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Reduced Basis Methods for the approximation to parameter dependent Partial Differential Equations are now well developed and start to be used in industrial framework. The classical implementation of the Reduced Basis Method goes through two stages: in the the first one, offline

and time consuming, from standard approximation methods a reduced basis is constructed, then in a second stage, online and very cheap, a small problem, of the size of the reduced basis, is solved.

The offline stage is a learning one from which the online stage can proceed efficiently. In this presentation we propose to complement the offline stage with some statistical learning ingredients in order to build more knowledge in the process so as to tackle either different classes of problems or even speed more the online approximation. The method is presented on a simple flow problem governed by the Navier Stokes equation and illustrated on the test case of a flow pas a backward step.

*Joint work with P. Gallinari and O. Schwander (Sorbonne Universités, UPMC – Paris VI and CNRS, UMR 7606, LIP6), Y. Maday (Sorbonne Universités, UPMC – Paris VI and CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, Institut Universitaire de France, Division of Applied Mathematics, Brown University, USA), M. Sanguier (Sorbonne Universités, UPMC – Paris VI, France), and T. Taddei (Sorbonne Universités, UPMC – Paris VI and CNRS, UMR 7598, Laboratoire Jacques-Louis Lions).*

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A6 - July 10, 15:30 – 16:00

#### ENKF-BASED INTERACTING PARTICLE FILTER FORMULATIONS

**Jana de Wiljes**

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In a nonlinear setting the filtering distribution can be approximated via the empirical measure provided that an ensemble of samples is available. A computationally feasible option to generate these particles for each time instance is to define an appropriate modified evolution equation that describes the dynamics of the particles with respect to the incoming data. The most famous example of such interacting particle filter formulations is the ensemble Kalman filter (EnKF). Although it works remarkably well in practice its success is not well understood from a mathematical point of view. In a recent study we were able to derive stability and accuracy results for a specific variant of the EnKF in a continuous setting with a small number of particles. Inspired by the EnKF we explored more general interacting particle filter formulations that allow to overcome weaknesses of the EnKF as well as drawbacks of classical sequential resampling schemes. More precisely, we consider the recently proposed ensemble transform particle filter (ETPF) which is an adaption of the standard particle filter where the resampling step is replaced by a linear transformation. However, the transformation step is computationally expensive and leads to an underestimation of the ensemble spread for small and moderate ensemble sizes.

These shortcomings have recently been addressed by developing second-order accurate extensions of the ETPF. It is also demonstrated that the nonlinear ensemble transform filter (NETF) arises as a special case of our general framework. Numerical results for the Lorenz-63 and Lorenz-96 models demonstrate the effectiveness of the proposed modified particle filters.

*Joint work with Wilhelm Stannat (Technical University Berlin, Germany), Sebastian Reich (University Potsdam, Germany and University of Reading, UK), and Walter Acevedo (University Potsdam, Germany).*

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A6 - July 10, 16:00 – 16:30

DATA-DRIVEN OPERATOR INFERENCE FOR NON-INTRUSIVE PROJECTION-BASED  
MODEL REDUCTION

**Karen Willcox**  
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This talk presents a non-intrusive projection-based model reduction approach for full models based on time-dependent partial differential equations. Projection-based model reduction constructs the operators of a reduced model by projecting the equations of the full model onto a reduced space. Traditionally, this projection is intrusive, which means that the full-model operators are required either explicitly in an assembled form or implicitly through a routine that returns the action of the operators on a given vector; however, in many situations the full model is given as a black box that computes trajectories of the full-model states and outputs for given initial conditions and inputs, but does not provide the full-model operators. Our non-intrusive operator inference approach solves an optimization problem to infer approximations of the reduced operators from the initial conditions, inputs, trajectories of the states, and outputs of the full model, without requiring the full-model operators. The inferred operators are the solution of a least-squares problem and converge, with sufficient state trajectory data, in the Frobenius norm to the reduced operators that would be obtained via an intrusive projection of the full-model operators.

## Bibliography

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*Joint work with Benjamin Peherstorfer (University of Wisconsin-Madison, USA).*

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A6 - July 10, 17:00 – 18:00

WAVES AND IMAGING IN RANDOM MEDIA

**Josselin Garnier**  
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In sensor array imaging an unknown medium is probed by waves emitted by an array of sources and recorded by an array of receivers. Sensor array imaging in a randomly scattering medium is usually limited because coherent signals recorded by the receiver array and coming from a reflector to be imaged are weak and dominated by incoherent signals coming from multiple scattering by the medium. Stochastic and multiscale analysis allows to understand the direct problem and helps solving the inverse problem. We will see in this talk how correlation-based imaging techniques can mitigate or even sometimes benefit from the multiple scattering of waves.

Applications to seismic interferometry, non-destructive testing, and intensity correlation imaging in optics will be discussed.

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A6 - July 11, 14:30 – 15:00

OBSERVER STRATEGIES FOR INVERSE PROBLEMS ASSOCIATED WITH WAVE-LIKE EQUATIONS

**Philippe Moireau**

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We present a novel strategy to perform estimation for wave-like systems and more general evolution PDEs with uncertain initial conditions and parameters. We adopt a filtering approach on the dynamical system formulation to construct a joint state-parameter estimator that uses some measurements available in standard operating conditions. Namely, the aim is to obtain a modified dynamical system converging to the reference by incorporating correction terms using the data. First, in the case of known parameters, state estimation is performed using a Luenberger observer inspired from feedback control theory. This type of state estimator is chosen for its particular effectiveness and robustness. In particular, unlike the classical Kalman approach, this filter is computationally tractable for numerical systems arising from the discretization of PDEs and — although optimality is lost — the exponential stability of the corresponding error system gives exponential convergence of the estimator/observer. With uncertain parameters we extend the estimator by incorporating the parameters in an augmented dynamical system. The effect of the first stage state filter then consists in essence in circumscribing the uncertainty to the parameter space — which is usually much “smaller” than the parameter space — and allows for an  $H^2$  type filter in the resulting low rank space. This second step is related to “reduced rank filtering” procedures in data assimilation. The convergence of the resulting joint state-parameter estimator can be mathematically established, and we demonstrate its effectiveness by identifying localized parameters. We propose to illustrate every aspect of this strategy from the observer formulation, analysis, robustness to noise, discretization up to the numerical implementation with various examples based on wave-like models in bounded but also unbounded domains.

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A6 - July 11, 15:00 – 15:30

OPTIMAL EXPERIMENTAL DESIGN FOR LARGE-SCALE PDE-CONSTRAINED BAYESIAN INVERSE PROBLEMS

**Omar Ghattas**

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We address the problem of optimal experimental design (OED) for infinite-dimensional nonlinear Bayesian inverse problems. We seek an A-optimal design, i.e., we aim to minimize the average variance of a Gaussian approximation to the inversion parameters at the MAP point, i.e., the Laplace approximation. The OED problem includes as constraints the optimality condition PDEs defining the MAP point as well as the PDEs describing the action of the posterior

covariance. A randomized trace estimator along with low rank approximations of the Hessian of the data misfit lead to efficient OED cost function computation, and an adjoint approach leads to efficient gradient computation for the OED problem. We provide numerical results for the optimal sensor locations for inference of the permeability field in a porous medium flow problem. The results indicate that the cost of the proposed method (measured by the number of forward PDE solves) is independent of the parameter and data dimensions.

*Joint work with Alen Alexanderian (North Carolina State University, USA), Noemi Petra (University of California, Merced, USA), and George Stadler (New York University, USA).*

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A6 - July 11, 15:30 – 16:00

# INVERSE PROBLEM OF ELECTROCARDIOGRAPHY: A MULTIPHYSICS DATA ASSIMILATION APPROACH

**Jean-Frédéric Gerbeau**

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We address the inverse problem of electrocardiography by combining electrical and mechanical measurements. Our strategy, proposed in [1], relies on a model of the electromechanical contraction which is registered on ECG data [2, 3] but also on measured mechanical displacements of the heart tissue typically extracted from medical images. In this respect, we establish the convergence of a sequential estimator which combines for such coupled problems various state of the art sequential data assimilation methods in an efficient framework [4]. We aggregate a Luenberger observer for the mechanical state and a Reduced Order Unscented Kalman Filter applied on the parameters to be identified, and a POD projection of the electrical state. Then using synthetic data we show the benefits of our approach for the estimation of the electrical state of the ventricles along the heart beat compared with more classical strategies which only consider an electrophysiological model with ECG measurements. Our numerical results actually show that the mechanical measurements improve the identifiability of the electrical problem allowing to reconstruct the electrical state of the coupled system more precisely. Therefore, this work is intended to be a first proof of concept, with theoretical justifications and numerical investigations, of the advantage of using available multi-modal observations for the estimation and identification of an electromechanical model of the heart.

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*Joint work with Philippe Moireau (INRIA, France) and Cesare Corrado (King’s College London, UK).*

A6 - July 11, 17:00 – 18:00

#### BAYESIAN INFERENCE VIA LOW-DIMENSIONAL COUPLINGS

**Youssef Marzouk**

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Integration against an intractable probability measure is among the fundamental challenges of statistical inference, particularly in the Bayesian setting. A principled approach to this problem seeks a deterministic coupling of the measure of interest with a tractable “reference” measure (e.g., a standard Gaussian). This coupling is induced by a transport map, and enables direct simulation from the desired measure simply by evaluating the transport map at samples from the reference. Yet characterizing such a map — e.g., representing, constructing, and evaluating it — grows challenging in high dimensions.

We will present links between the conditional independence structure of the target measure and the existence of certain low-dimensional couplings, induced by transport maps that are sparse or decomposable. We also describe conditions, common in Bayesian inverse problems, under which transport maps have a particular low-rank structure. Our analysis not only facilitates the construction of couplings in high-dimensional settings, but also suggests new inference methodologies. For instance, in the context of nonlinear and non-Gaussian state space models, we describe new variational algorithms for online filtering, smoothing, and parameter estimation. These algorithms implicitly characterize — via a transport map — the full posterior distribution of the sequential inference problem using only local operations while avoiding importance sampling or resampling.

*Joint work with Alessio Spantini (Massachusetts Institute of Technology, USA) and Daniele Bigoni (Massachusetts Institute of Technology, USA).*

A6 - July 12, 14:30 – 15:00

#### ESTIMATION FOR FRONT PROPAGATION MODELS WITH FRONT LEVEL-SET DATA USING OBSERVERS. APPLICATIONS IN MEDICINE AND IN FIRE PROPAGATION

**Annabelle Collin**

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In this presentation, we present a sequential — observer based — data assimilation strategy for front propagation models (as reaction-diffusion, eikonal or reaction-transport equations), and for data corresponding to propagation isochrones. First, an original similarity measure between the computed front and the observed front is introduced. Then an efficient feedback

adapted to this similarity measure is presented based on shape derivatives leading in fine to a so-called Luenberger observer. This shape observer can be completed with the introduction of the topological derivative of the measure in order to take into account the breakthrough of new fronts. Mathematical justifications of the stabilization property brought by the feedback are detailed. We also discuss the extension to joint state-parameter estimation by using a Kalman based strategy such as RO-UKF. Finally, numerical illustrations are presented on synthetic data and on real data in three applications domains (cardiac electrophysiology, tumor growth and fire propagation) revealing the potential of the approach.

*Joint work with Dominique Chapelle (INRIA, Université Paris Saclay, France) and Philippe Moireau (INRIA, Université Paris Saclay, France).*

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A6 - July 12, 15:00 – 15:30

#### LOCALIZED MODEL REDUCTION FOR DATA ASSIMILATION AND INVERSE PROBLEMS

**Mario Ohlberger**

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Data assimilation and large scale inverse problems pose sever challenges to numerical methods due to their high computational complexity. In this contribution we present novel model reduction approaches to handle such problems. In particular, we focus on localized model reduction with online enrichment, as well as on combined parameter and state reduction.

*Joint work with Christian Himpe (MPI Magdeburg, Germany), Stephan Rave (University of Muenster, Germany), Felix Schindler (University of Muenster, Germany).*

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A6 - July 12, 15:30 – 16:00

#### CONTINUOUS TIME FILTERING ALGORITHMS

**Sebastian Reich**

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I will review recent work on interacting particle representations of the continuous time filtering problem. The starting point is provided by the feedback particle filter and its particle approximations. The ensemble Kalman-Bucy filter emerges as a special and I will summarize results on its accuracy and stability. I will conclude with some open problems.

*Joint work with Jana de Wiljes (University of Potsdam, Germany), Wilhelm Stannat (TU Berlin, Germany), Prashant Mehta (University of Illinois at Urbana – Champaign, USA), and Amirhossein Taghvaei (University of Illinois at Urbana – Champaign, USA).*



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**Workshop A7**  
**Stochastic Computation**

Organizers: Tony Lelièvre – Arnulf Jentzen

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A7 - July 10, 14:30 – 15:20

**WEAK ORDER ANALYSIS FOR SPDES**

**Arnaud Debussche**

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The numerical analysis of Stochastic Partial Differential Equations has known a lot of progress. The study of the weak order involves a lot of difficulties and started to be understood only recently. In this talk, I will present a new result of regularity for the solution of the Kolmogorov equations associated to a SPDE with a nonlinear diffusion coefficient and a Burger's type nonlinearity. This allows a complete study of the weak order of a standard semi implicit Euler scheme. I will explain why Malliavin calculus and two sided stochastic integrals are useful.

*Joint work with Charles-Edouard Bréhier (Université de Lyon 1, France).*

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A7 - July 10, 15:30 – 15:55

**WEAK CONVERGENCE RATES FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS  
WITH NONLINEAR DIFFUSION COEFFICIENTS**

**Ryan Kurniawan**

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Strong convergence rates for numerical approximations of semilinear stochastic partial differential equations (SPDEs) with smooth and regular nonlinearities are well understood in the literature. Weak convergence rates for numerical approximations of such SPDEs have been investigated for about two decades and are still not yet fully understood. In particular, it has been an open problem to establish essentially sharp weak convergence rates for numerical approximations of space-time white noise driven SPDEs with nonlinear multiplication operators in the diffusion coefficients. In this talk we overcome this weak convergence problem. In particular, we establish essentially sharp weak convergence rates for numerical approximations of the continuous version of the parabolic Anderson model. Key ingredients of our approach are applications of the mild Ito type formula in UMD Banach spaces with type 2.

*Joint work with Daniel Conus (Lehigh University, USA), Mario Hefter (TU Kaiserslautern, Germany) and Arnulf Jentzen (ETH Zürich, Switzerland).*

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A7 - July 10, 16:00 – 16:25

## WEAK ERROR ANALYSIS VIA FUNCTIONAL ITÔ CALCULUS

**Felix Lindner**
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Weak errors of numerical approximations of SDEs are relatively well understood in the case where the considered functional of the solution process depends only on an evaluation of the solution at a given time  $T$ . In contrast, the number of available results in the literature on weak convergence rates for approximations of path-dependent functionals of SDEs is quite limited. In this talk, I present a new approach to analyzing weak approximation errors for path-dependent functionals of SDEs, based on tools from functional Itô calculus such as the functional Itô formula and functional Kolmogorov equation. It leads to a general representation formula for weak errors of the form  $\mathbb{E}(f(X) - f(\tilde{X}))$ , where  $X$  and  $\tilde{X}$  are the solution process and its approximation and the functional  $f: C([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$  is assumed to be sufficiently regular. The representation formula can be used to derive explicit convergence rates, such as rate 1 for the linearly time-interpolated explicit Euler method. Finally, I will outline how the presented approach can be extended to numerical methods for mild solutions of SPDEs.

*Joint work with Mihály Kovács (Chalmers University of Technology, Sweden) and Saeed Hadjizadeh (University of Kaiserslautern, Germany).*

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A7 - July 10, 17:00 – 17:25

## ON STOCHASTIC DIFFERENTIAL EQUATIONS WITH ARBITRARILY SLOW CONVERGENCE RATES FOR STRONG APPROXIMATION IN TWO SPACE DIMENSIONS

**Diyora Salimova**
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In this talk we show that for every arbitrarily slow convergence speed and every natural number  $d \in \{2, 3, \dots\}$  there exist stochastic differential equations with infinitely often differentiable and globally bounded coefficients such that no approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than the given speed of convergence.

*Joint work with Máté Gerencsér (Institute of Science and Technology, Austria) and Arnulf Jentzen (ETH Zürich, Switzerland).*

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A7 - July 10, 17:30 – 17:55

## APPROXIMATION OF BSDEs USING RANDOM WALK

**Christel Geiss**
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For the FBSDE

$$X_t = x + \int_0^t b(r, X_r) dr + \int_0^t \sigma(r, X_r) dB_r$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T$$

Briand, Delyon and Memin have shown in [1] a Donsker-type theorem: If one approximates the Brownian motion  $B$  by a random walk  $B^n$ , the according solutions  $(X^n, Y^n, Z^n)$  converge weakly to  $(X, Y, Z)$ .

We investigate under which conditions  $(Y_t^n, Z_t^n)$  converges to  $(Y_t, Z_t)$  in  $L_2$  and compute the rate of convergence in dependence of the Hölder continuity of the terminal condition function  $g$ .

### Bibliography

- [1] P. Briand, B. Delyon, & J. Memin, “Donsker-type theorem for BSDEs.” *Electron. Comm. Probab.* 6 (2001), 1–14.

*Joint work with Céline Labart (Université de Savoie, France) and Antti Luoto (University of Jyväskylä, Finland).*

A7 - July 10, 18:00 – 18:25

#### LOWER ERROR BOUNDS FOR STRONG APPROXIMATION OF SCALAR SDEs WITH NON-LIPSCHITZIAN COEFFICIENTS

**Thomas Mueller–Gronbach**

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We study the problem of pathwise approximation of the solution of a scalar SDE, either at the final time or globally in time, based on  $n$  sequential evaluations of the driving Brownian motion on average. We present lower error bounds in terms of  $n$  under mild local smoothness assumptions on the coefficients of the SDE. This includes SDEs with superlinearly growing or piecewise Lipschitz continuous coefficients and also certain types of CIR-processes.

*Joint work with Mario Hefter (Technical University of Kaiserslautern, Germany) and Andre Herzwurm (Technical University of Kaiserslautern, Germany).*

A7 - July 11, 14:30 – 14:55

#### BEYOND WELL-TEMPERED METADYNAMICS ALGORITHMS FOR SAMPLING MULTIMODAL TARGET DENSITIES

**Gersende Fort**

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In many situations, sampling methods are considered in order to compute expectations of given observables with respect to a distribution  $\pi d\lambda$  on  $X \subseteq \mathbb{R}^D$ , when  $\pi$  is highly multimodal. Free-energy based adaptive importance sampling techniques have been developed in

the physics and chemistry literature to efficiently sample from such a target distribution: the auxiliary distribution  $\pi_\star d\lambda$  from which the samples are drawn, is defined, given a partition  $\{X_i, i \leq d\}$  of  $X$ , as a local biasing of the target  $\pi$  such that each element  $X_i$  has the same weight under  $\pi_\star d\lambda$ . These methods are casted in the class of adaptive Markov chain Monte Carlo (MCMC) samplers since the local biasing is unknown: it is therefore learnt on the fly and the importance function evolves along the run of the sampler. As usual with importance sampling, expectations with respect to  $\pi$  are obtained from a weighted mean of the samples returned by the sampler.

Examples of such approaches are Wang-Landau algorithms, the Self-Healing Umbrella Sampling, adaptive biasing forces methods, the metadynamic algorithm or the well-tempered metadynamics algorithm. Nevertheless, the main drawback of most of these methods is that two antagonistic phenomena are in competition: on one hand, to overcome the multimodality issue, the sampler is forced to visit all the strata  $\{X_i, i \leq d\}$  equally; on the other hand, the algorithm spends the same time in strata with high and low weight under  $\pi d\lambda$  which makes the Monte Carlo approximation of expectations under  $\pi d\lambda$  quite inefficient.

We present a new algorithm, which generalizes all the examples mentioned above: this novel algorithm is designed to reduce the two antagonistic effects. We will show that the estimation of the local bias can be seen as a Stochastic Approximation algorithm with random step-size sequence; and the sampler as an adaptive MCMC method. We will analyze its asymptotic behavior and discuss numerically the role of some design parameters.

*Joint work with Benjamin Jourdain (ENPC, France), Tony Lelièvre (ENPC, France), and Gabriel Stöltz (ENPC, France).*

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A7 - July 11, 15:00 – 15:25

## SELF-REPELLING PROCESSES AND METADYNAMICS

**Pierre-André Zitt**

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A usual drawback of Markov Chain Monte Carlo algorithms is their inherent difficulty to overcome potential barriers, which may lead to a poor exploration of the sampling space, and large sampling errors. The “metadynamics” algorithm introduced by Bussi, Laio and Parrinello in the 00s exemplifies one of the ideas to tackle this difficulty: by keeping track of the past trajectory of the sampling process, one can use it to bias the process so that it avoids the regions it has already visited, leading to a better sampling. The processes describing the evolution of the algorithm turn out to be quite difficult to analyze rigorously. We present two toy models that are amenable to such an analysis, using results from the self-interacting processes literature.

*Joint work with Benjamin Jourdain (École des Ponts ParisTech, France) and Tony Lelièvre (École des Ponts ParisTech, France).*

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A7 - July 11, 15:30 – 16:20

COMPETING SOURCES OF VARIANCE REDUCTION IN PARALLEL REPLICA MONTE CARLO, AND OPTIMIZATION IN THE LOW TEMPERATURE LIMIT

**Paul Dupuis**

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Computational methods such as parallel tempering and replica exchange are designed to speed convergence of more slowly converging Markov processes (corresponding to lower temperatures for models from the physical sciences), by coupling them through a Metropolis type swap mechanism with higher temperature processes that explore the state space more quickly. It has been shown that the sampling properties are in a certain sense optimized by letting the swap rate tend to infinity. This “infinite swapping limit” can be realized in terms of a process which evolves using a symmetrized version of the original dynamics, and then one produces approximations to the original problem by using a weighted empirical measure. The weights are needed to transform samples obtained under the symmetrized dynamics into distributionally correct samples for the original problem.

After reviewing the construction of the infinite swapping limit, we focus on the sources of variance reduction which follow from this construction. As will be discussed, some variance reduction follows from a lowering of energy barriers and consequent improved communication properties. A second and less obvious source of variance reduction is due to the weights used in the weighted empirical measure that appropriately transform the samples of the symmetrized process. These weights are analogous to the likelihood ratios that appear in importance sampling, and play much the same role in reducing the overall variance. A key question in the design of the algorithms is how to choose the ratios of the higher temperatures to the lowest one. As we will discuss, the two variance reduction mechanisms respond in opposite ways to changes in these ratios. One can characterize in precise terms the tradeoff and explicitly identify the optimal temperature selection for certain models when the lowest temperature is sent to zero, i.e., when sampling is most difficult.

*Joint work with Jim Doll (Brown University, USA), Guo-Jhen Wu (Brown University, USA), and Michael Snarski (Brown University, USA).*

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A7 - July 11, 17:00 – 17:25

MODELING AGGREGATION PROCESSES OF LENNARD–JONES PARTICLES VIA STOCHASTIC NETWORKS

**Maria Cameron**

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An isothermal aggregation process of particles/atoms interacting according to the Lennard–Jones pair potential is modeled by mapping the energy landscapes of each cluster size  $N$  onto stochastic networks, computing transition probabilities for the network for an  $N$ -particle cluster

to the one for  $N + 1$ , and connecting these networks into a single joint network. The attachment rate is a control parameter. The resulting network representing the aggregation and dynamics of up to 14 Lennard–Jones particles contains 6417 vertices. It is not only time-irreversible but also reducible. To analyze its transient dynamics, we introduce the sequence of the expected initial and pre- attachment distributions and compute them for a wide range of attachment rates and three values of temperature. As a result, we find the most likely to observe configurations in the process of aggregation for each cluster size. We examine the attachment process and conduct a structural analysis of the sets of local energy minima for every cluster size. We show that both processes taking place in the network, attachment and relaxation, lead to the dominance of icosahedral packing in small (up to 14 atom) clusters.

*Joint work with Yakir Forman (Yeshiva University, USA).*

A7 - July 11, 17:30 – 17:55

#### A COUPLING APPROACH TO THE KINETIC LANGEVIN EQUATION

**Andreas Eberle**

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The (kinetic) Langevin equation is an SDE with degenerate noise that describes the motion of a particle in a force field subject to damping and random collisions. It is also closely related to Hamiltonian Monte Carlo methods. An important question is, why in certain cases kinetic Langevin diffusions seem to approach equilibrium faster than overdamped Langevin

So far, convergence to equilibrium for kinetic Langevin diffusions has almost exclusively been studied by analytic techniques. In this talk, I present a new probabilistic approach that is based on a specific combination of reflection and synchronous coupling of two solutions of the Langevin equation. The approach yields rather precise bounds for convergence to equilibrium at the borderline between the overdamped and the underdamped regime, and it may help to shed some light on the open question mentioned above.

*Joint work with Arnaud Guillin (Université Blaise Pascal, Clermont-Ferrand, France) and Raphael Zimmer (Universität Bonn, Germany).*

A7 - July 11, 18:00 – 18:25

#### THE COMPLEXITY OF BEST-ARM IDENTIFICATION

**Aurélien Garivier**

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We consider the problem of finding the highest mean among a set of probability distributions that can be sampled sequentially. We provide a complete characterization of the complexity of this task in simple parametric settings: we give a tight lower bound on the sample complexity, and we propose the ‘Track-and-Stop’ strategy, which we prove to be asymptotically optimal. This algorithm consists in a new sampling rule (which tracks the optimal proportions of arm draws highlighted by the lower bound) and in a stopping rule named after Chernoff, for which we give a new analysis.

*Joint work with Emilie Kaufmann (CNRS, team CRISAL, France).*

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A7 - July 11, 18:30 – 18:55

FLUCTUATION ANALYSIS OF FLEMING–VIOT PARTICLE SYSTEMS

**Arnaud Guyader**

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The distribution of a Markov process with killing, conditioned to be still alive at a given time, can be approximated by a Fleming–Viot type particle system. In such a system, each particle is simulated independently according to the law of the underlying Markov process, and branches onto another particle at each killing time. The consistency of this method in the large population limit was the subject of several recent works. The purpose of this talk is to present a central limit theorem for the law of the Fleming–Viot particle system at a given time. We will illustrate this result on an application in molecular dynamics.

*Joint work with Frederic Cerou (INRIA, France), Bernard Delyon (University of Rennes, France), and Mathias Rousset (INRIA, France).*

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A7 - July 12, 14:30 – 14:55

MULTILEVEL MONTE CARLO FOR SDEs WITH RANDOM BITS

**Klaus Ritter**

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Let  $X$  be the solution to a stochastic differential equation (SDE), and let  $\varphi$  be a real-valued functional on the path space. We study the approximation of the expectation of  $\varphi(X)$  by means of randomized algorithms that may only use random bits. We provide upper and lower bounds on the complexity of the problem, as well as a multilevel algorithm that achieves the upper bound.

*Joint work with M. Giles (University of Oxford, UK), M. Heister (Kaiserslautern, Germany), and L. Mayer (Kaiserslautern, Germany).*

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A7 - July 12, 15:00 – 15:25

STRONG CONVERGENCE PROPERTIES OF THE NINOMIYA VICTOIR SCHEME AND APPLICATIONS TO MULTILEVEL MONTE CARLO

**Benjamin Jourdain**

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We prove that the strong convergence rate of the Ninomiya–Victoir scheme is  $1/2$ . The normalized error converges stably to the solution of an affine SDE with a source term involving the commutators between the Brownian vector fields. When the Brownian vector fields commute, this limit vanishes and we show that the strong convergence rate improves to 1. We also show that averaging the order of integration of the Brownian fields leads to a coupling with

strong order 1 with the scheme proposed by Giles and Szpruch (2014) in order to achieve the optimal complexity in the multilevel Monte Carlo method. Last, we are interested in the error introduced by discretizing the ordinary differential equations involved in the Ninomiya–Victoir scheme. We prove that this error converges with strong order 2 so that the convergence properties of our multilevel estimators are preserved when an explicit Runge–Kutta method with order 4 (resp. 2) is used for the ODEs corresponding to the Brownian (resp. Stratonovich drift) vector fields. We thus relax the order 5 for the Brownian ODEs needed by Ninomiya and Ninomiya (2009) to obtain the same order of strong convergence.

*Joint work with Anis Al Gerbi (École des Ponts, CERMICS) and Emmanuelle Clément (École Centrale Paris, MICS).*

A7 - July 12, 15:30 – 15:55

## ON THE APPROXIMATION OF PATHDEPENDENT BSDEs DRIVEN BY THE BROWNIAN MOTION

**Stefan Geiss**

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In this talk we present recent results about the  $L_p$ -approximation of the  $Y$ -process of path-dependent quadratic and sub-quadratic backwards stochastic differential equations driven by the Brownian motion of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s.$$

In this approximation we use adapted time-nets where the adaptation is based on certain quantitative properties of the initial data  $(\xi, f)$  (the terminal condition and the generator) that are related to differential properties (in the Malliavin sense) of  $(\xi, f)$ . The talk is mainly based on [1].

## Bibliography

- [1] S. Geiss and J. Ylinen, “Decoupling on the Wiener space, related Besov spaces, and applications to BSDEs.” [ArXiv:1409.5322v3](#).

*Joint work with Juha Ylinen (University of Jyväskylä, Finland).*

A7 - July 12, 16:00 – 16:25

## EXISTENCE, UNIQUENESS, AND NUMERICAL APPROXIMATION FOR STOCHASTIC BURGERS EQUATIONS

**Sara Mazzonetto**

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The talk is about a recently introduced explicit full-discrete numerical approximation scheme for some stochastic partial differential equations with additive noise and non-globally Lipschitz continuous nonlinearities. The scheme allows to prove simultaneously existence and uniqueness of the mild solution and strong convergence of the numerical approximation for some classes of equations, e.g., the stochastic Burgers equation with additive white noise.

*Joint work with Arnulf Jentzen and Diyora Salimova (ETH Zürich, Switzerland).*

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A7 - July 12, 17:00 – 17:25

#### RECENT ADVANCES ON STOCHASTIC METHODS IN DATA SCIENCE

**Sotirios Sabanis**

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Some recent advances on stochastic algorithms in data science will be discussed.

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A7 - July 12, 17:30 – 17:55

#### THE STABILITY OF STOCHASTIC GRADIENT DESCENT

**Benjamin Recht**

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The most widely used optimization method in machine learning practice is the Stochastic Gradient Method (SGM). This method has been used since the fifties to build statistical estimators, iteratively improving models by correcting errors observed on single data points. SGM is not only scalable, robust, and simple to implement, but achieves the state-of-the-art performance in many different domains. In contemporary systems, SGM powers enterprise analytics systems and is the workhorse tool used to train complex pattern-recognition systems in speech and vision.

In this talk, I will explore why SGM has had such staying power, focusing on the notion of generalization. I will show that any model trained with a few SGM iterations has vanishing generalization error and performs as well on unseen data as on the training data. The analysis will solely employ elementary tools from convex and continuous optimization. Applying the results to the convex case provides new explanations for why multiple epochs of stochastic gradient descent generalize well in practice, and give new insights into minibatch sizes in SGM. In the nonconvex case, I will describe a new interpretation of common practices in neural networks, and provide a formal rationale for stability-promoting mechanisms in training large, deep models.

*Joint work with Moritz Hardt (Google Brain/UC Berkeley, USA), Yoram Singer (Google Brain, USA).*

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A7 - July 12, 18:00 – 18:25

STOCHASTIC COMPOSITE LEAST-SQUARES REGRESSION WITH CONVERGENCE RATE  
 $O(1/n)$

**Francis Bach**

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We consider the minimization of composite objective functions composed of the expectation of quadratic functions and an arbitrary convex function. We study the stochastic dual averaging algorithm with a constant step-size, showing that it leads to a convergence rate of  $O(1/n)$  without strong convexity assumptions. This thus extends earlier results on least-squares regression with the Euclidean geometry to (a) all convex regularizers and constraints, and (b) all geometries represented by a Bregman divergence. This is achieved by a new proof technique that relates stochastic and deterministic recursions.

*Joint work with Nicolas Flammarion (École Normale Supérieure, France).*

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A7 - Poster

NUMERIC FOR STOCHASTIC STATE-DEPENDENT DELAY DIFFERENTIAL EQUATIONS

**Bahareh Akhtari**

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The stochastic equations of delay-type, including system memory, describe the model in a more accurate manner. The numerical solution for stochastic delay differential equations (SDDEs) have been relatively adequately discussed in the recent years. But all works deal with the case in which delay term is constant or time-dependent. In this poster, we consider a new one: **State – dependent**. Under the sufficiently smooth conditions on drift and diffusion coefficients, a new interpolation based on split-step scheme for approximating history in the presence of state-dependent delay developed and then mean-square convergence of the scheme investigated.

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A7 - Poster

STRONG STABILITY IN PHASE-TYPE QUEUEING SYSTEMS

**Yasmina Djabali**

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Phase-type queueing systems are used to approximate queues with general distributions. In this work, we provide the justification of this approximation method by employing the strong stability method. We prove the robustness of the underlying Markov chain and estimate an upper bound of the deviation of the stationary vector, resulting from the perturbation of the inter-arrival distribution of GI/M/1 queueing system. We provide numerical examples and compare the perturbation bounds obtained in this paper with the estimates of the real deviation of the stationary vector obtained by simulation.

*Keywords:* Queueing systems, Markov chain, Phase-type distributions, Perturbation, Strong stability.

*Joint work with Rabta Boualem (Research Unit LaMOS, University of Bejaia, Algeria) and Aïssani Djamil (Research Unit LaMOS, University of Bejaia, Algeria).*

A7 - Poster

# PERFORMANCE MODELING OF FINITE-SOURCE PRIORITY QUEUE WITH VACATIONS VIA GENERALIZED STOCHASTIC PETRI NETS

**Sedda Hakmi**

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Our study deals with performance modeling and analysis of finite-source non-preemptive priority queue with vacations via Generalized Stochastic Petri Nets (GSPN). Indeed, we considered two classes of non-preemptive priority: high priority and low priority request. The service times and the inter-arrival times are assumed to be exponentially distributed. The corresponding GSPN model of the studied system provided us a formal method for generating Markov Chain (MC) which allowed us to develop the formulas of the main stationary performance measures. Therefore, we provide an exact analysis for the performance indices of both requests classes. Through numerical examples, we discuss the impact of vacation and arrival rates on the network performances.

*Keywords:* Generalized Stochastic Petri Nets, Modeling, Performance Evaluation, Priority Queue.

*Joint work with Ouiza Lekadir (Research Unit LaMOS (Modeling and Optimization of Systems), University of Bejaia, Algeria) and Djamil Aïssani (Research Unit LaMOS (Modeling and Optimization of Systems), University of Bejaia, Algeria).*

A7 - Poster

# COMPARISON STUDY FOR RANDOM PDE OPTIMIZATION PROBLEMS BASED ON DIFFERENT MATCHING FUNCTIONALS

**Hyung-Chun Lee**

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In this poster, we consider an optimal control problem for a partial differential equation with random inputs. To determine an applicable deterministic control  $\hat{f}(x)$ , we consider the four cases which we compare for efficiency and feasibility. We prove the existence of optimal states, adjoint states and optimality conditions for each cases. We also derive the optimality systems for the four cases. The optimality system is then discretized by a standard finite element method and sparse grid collocation method for physical space and probability space, respectively. The numerical experiments are performed for their efficiency and feasibility.

*Joint work with Max D. Gunzburger (Florida State University, USA).*

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A7 - Poster

COUPLING SAMPLE PATHS TO THE PARTIAL THERMODYNAMIC LIMIT IN STOCHASTIC  
CHEMICAL REACTION NETWORKS

**Ethan Levien**

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Many stochastic biochemical systems have a multiscale structure so that they converge to piecewise deterministic Markov processes in a thermodynamic limit. While information about stochastic fluctuations is lost in the thermodynamic limit, studying the exact process is computationally expensive for most systems of interest. We present a new technique for accelerating the convergence of Monte Carlo estimators of the exact process that makes use of a probabilistic coupling between the exact process and the thermodynamic limit. In addition to rigorous results concerning the asymptotic computational complexity of our method, we apply our method to study various models of gene expression.

*Joint work with Paul C. Bressloff (University of Utah, USA).*

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A7 - Poster

COMPARISON OF THE EULER-MARUYAMA AND BACKWARD EULER METHODS FOR  
NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH TIME-DEPENDENT DELAY

**Marija Milosevic**

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Numerical solutions to a class of neutral stochastic differential equations with time-dependent delay are considered. Comparison of the explicit Euler-Maruyama and implicit backward Euler methods is presented with particular emphasis on the degree to which the numerical solutions inherit certain properties of the exact solution. The influences of the neutral term and time-dependent delay on the analysis of numerical solutions are stressed.

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A7 - Poster

STOCHASTIC IMAGE PROCESSING WITH AN APPLICATION

**Gamze Ozel Kadilar**

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Stochastic computation (SC) is important for the statistical nature of application-level performance metrics. The main benefit of SC is that complicated arithmetic computation can be performed by simple logic circuits since numbers are represented in SC by random bit-streams that are interpreted as probabilities. SC is categorized as an approximate computing technique. In this study, first we give a detailed discussion of the basic concepts of SC including accuracy and correlation. It also surveys the history of the field and highlights SC's applications. Then,

in this study, we provide the application of SC to image processing. We demonstrate the design of many representative image processing circuits and compare them to conventional binary counterparts.

*Joint work with Cem Kadilar (Hacettepe University, Turkey).*

A7 - Poster

## MINIMISATION OF RELATIVE ENTROPY TO EFFICIENTLY CAPTURE THE MACRO-SCALE BEHAVIOUR OF STOCHASTIC SYSTEMS

**Przemyslaw Zielinski**

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The purpose of this research is to develop, study and implement a micro-macro method to simulate observables of stiff SDEs. The technique exploits the separation between the fast time scale on which trajectories advance and the slow evolution of observables, by combining short bursts of paths simulation with extrapolation of a number of macroscopic states forward in time.

In the crucial step of the algorithm, we aim to obtain, after extrapolation, a new ensemble of particles/replicas compatible with given macroscopic states. To address and regularise this inference problem, we introduce the matching operator based on the minimisation of suitable distance between probability distributions — logarithmic relative entropy.

Particularly, the ongoing study focuses on establishing the convergence in the numerically weak sense, inquiring about the stability for appropriately chosen test models, and providing a convenient numerical approach. It also deals with the generic properties of the matching operator that allow to adopt other information theoretic distances besides the logarithmic entropy.

*Joint work with Kristian Debrabant (University of Southern Denmark, Denmark), Tony Lelièvre (École des Ponts ParisTech, France), and Giovanni Samaey (KU Leuven, Belgium).*

A7 - Poster

## INVERSE GAMMA KERNEL ESTIMATORS USING TWO MULTIPLICATIVE BIAS CORRECTION METHODS

**Nabil Zougab**

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Two multiplicative bias correction (MBC) approaches for nonparametric kernel estimators based on inverse gamma (IG) distribution for probability density function in the context of nonnegative supported data are proposed. We show some properties of the MBC-IG kernel estimators like bias, variance and mean integrated square error. A simulation study and an application on a real data set illustrate the performance of the MBC estimators based on the IG kernel in terms of the integrated squared error and integrated squared bias.

*Keywords:* Inverse gamma kernel, Integrated square error, Multiplicative bias correction, Squared bias.

*Joint work with Harfouche Lynda (Research Unit LaMOS, University of Bejaia, Algeria).*

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## Workshop B1 Computational Dynamics

Organizers: Àngel Jorba – Hiroshi Kokubu – Warwick Tucker

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B1 - July 13, 14:30 – 15:20

### FROM NORMAL FORMS TO KAM THEORY, FROM SPACE DEBRIS TO THE ROTATION OF THE MOON

**Alessandra Celletti**

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Celestial Mechanics is a test-bench for many theories of Dynamical Systems, most notably perturbation theory and KAM theory. Realistic results in concrete applications can be obtained through an accurate modeling and an appropriate study of the dynamics, which often requires a heavy computational effort. After a general discussion on perturbation theory and KAM theorem, I will consider two examples of implementations of normal forms and KAM theory in Celestial Mechanics. The first one concerns the dynamics of space debris, which can be successfully studied through averaging theory and normal forms computations. The second example analyzes the rotation of the Moon, whose stability can be investigated through a computer-assisted implementation of KAM theory. Perturbation theory and KAM theory can be used also to investigate dissipative systems. In this context, they can give interesting results on the dynamics of space debris at low altitude and possibly on the evolution of the Moon toward its present synchronous rotation.

*Joint work with Renato Calleja (National Autonomous University of Mexico, México), Christos Efthymiopoulos (Academy of Athens, Greece), Fabien Gachet (University of Rome Tor Vergata, Italy), Catalin Gales (Al. I. Cuza University, Romania), Rafael de la Llave (Georgia Institute of Technology, USA), and Giuseppe Pucacco (University of Rome Tor Vergata, Italy).*

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B1 - July 13, 15:30 – 16:00

### NUMERICAL ALGORITHMS AND A-POSTERIORI VERIFICATION OF PERIODIC ORBITS OF THE KURAMOTO-SIVASHINSKY EQUATION

**Jordi-Lluís Figueras**

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In this talk we will present a numerical algorithm for the computation of (hyperbolic) periodic orbits of the 1-D Kuramoto-Sivashinsky equation

$$u_t + \nu u_{xxxx} + u_{xx} + uu_x = 0,$$

with  $\nu > 0$ .

This numerical algorithm consists on applying a suitable quasi-Newton scheme. In order to do this, we need to rewrite the invariance equation that must satisfy a periodic orbit in a form that its linearization around an approximate solution is a bounded operator. We will also show how this methodology can be used to compute a-posteriori estimates of the errors of the solutions computed, leading to the rigorous verification of the existence of the periodic orbit.

If time permits, we will finish showing some numerical outputs of the algorithms presented along the talk.

*Joint work with Rafael de la Llave (Georgia Institute of Technology, USA).*

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B1 - July 13, 16:00 – 16:30

#### TOPOLOGICAL CHANGES IN SLOW-FAST SYSTEMS: CHAOTIC NEURON MODELS

**Roberto Barrio**

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The wide-range assessment of brain dynamics is one of the pivotal challenges of this century. To understand how an incredibly sophisticated system such as the brain functions dynamically, it is imperative to study the dynamics of its constitutive elements — neurons. Therefore, the design of mathematical models for neurons has arisen as a trending topic in science for a few decades. Mathematical neuron models are examples of fast-slow systems, and they exhibit several typical behaviours and bifurcations, like chaotic dynamics, spike-adding bifurcations and so on.

Two key open questions are how the chaotic behavior is organized [1, 2] and how spike-adding bifurcations influence chaotic behavior. In this talk we show how the orbit-flip (OF) codimension-2 bifurcation points, placed in homoclinic bifurcation curves and related with the spike-adding bifurcations [1, 2, 3], originate countable pencils of period-doubling and saddle-node (of limit cycles) bifurcation lines, but also of symbolic-flip bifurcations [4]. These bifurcations appear interlaced and generate the different symbolic sequences of periodic orbits, constituting the skeleton of the different chaotic attractors and determining their topological structure [4]. The study of the changes in the chaotic invariants is done by analyzing the gradual change in the spectrum of periodic orbits embedded in the invariant, and the onion-like structure in parameter space can be understood directly in terms of symbolic dynamics. The use of several numerical techniques, as continuation techniques, Lyapunov exponents, detection of unstable periodic orbits foliated to the chaotic invariants, template analysis and so on, has played a relevant role in the complete analysis of the problem [4].

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*Joint work with Marc Lefranc (Université de Lille 1, France), M. Angeles Martinez (University of Zaragoza, Spain), and Sergio Serrano (University of Zaragoza, Spain).*

B1 - July 13, 17:00 – 17:30

# ACCURATE MEASUREMENT OF ARNOLD DIFFUSION USING COMPUTER-ALGEBRAIC OPTIMAL NORMAL FORMS

**Christos Efthymiopoulos**

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The numerical measurement of Arnold diffusion, i.e., the slow drift of weakly chaotic orbits along resonances in nearly-integrable systems, has proven to be a particularly difficult task for computational dynamics. The talk will show how such measurement can be practically realized even for tiny values of the perturbation parameter, showing real examples in a model introduced in the literature some years ago by Froeschlé & al., (2000). Our technique combines three elements:

- i) A computer-algebraic determination of the resonant normal form which is optimal in the sense of minimizing the corresponding remainder.
- ii) Use of the normalizing transformation in order to remove all ‘deformation’ effects. This allows to clearly see the diffusion in ‘clean’ canonical variables, devoid of deformation noise.
- iii) A Melnikov-type determination of the most important remainder terms which drive the diffusion.

In step (iii) we implement a stationary or quasi-stationary phase approach for estimating the contribution of each term to the chaotic jumps of the action variables which take place along every individual homoclinic loop. This allows, in turn, to quantify the time evolution of these jumps, in excellent agreement with numerical results. Finally, the same technique allows to visualize Arnold diffusion, i.e., to show how the diffusion proceeds along the resonance in a sequence of consecutive, in time, homoclinic loops.

*Joint work with M. Guzzo (Dipt. di Matematica, Università degli Studi di Padova) and R.I. Paez (RCAAM, Academy of Athens).*

B1 - July 13, 17:30 – 18:00

# SPLITTING OF SEPARATRICES IN A HAMILTONIAN–HOPF BIFURCATION UNDER PERIODIC FORCING

**Carles Simó**

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We consider a Hamiltonian system given by a suitable truncation of the normal form of the Hamiltonian–Hopf bifurcation plus a concrete periodic non-autonomous perturbation. The goal is to study the behavior of the splitting of the 2-dimensional separatrices. Theoretical results are compared with direct computations of the invariant manifolds. An analysis of the associated Poincaré–Melnikov integral provides a description of the sequence of parameters corresponding to changes on the dominant harmonics of the splitting function.

*Joint work with Ernest Fontich (Universitat de Barcelona, Spain) and Arturo Vieiro (Universitat de Barcelona, Spain).*

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B1 - July 14, 14:30 – 15:00

#### PARAMETERIZATION METHODS FOR COMPUTING NORMALLY HYPERBOLIC INVARIANT MANIFOLDS

**Àlex Haro**

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We present a methodology for the computation of normally hyperbolic invariant tori in families of dynamical systems. The application of the parameterization method leads to solving invariance equations for which we use Newton-like method adapted to the dynamics and the geometry of the invariant manifold and its invariant bundles. The method computes the NHIT and its internal dynamics. We apply the method to several examples, and explore some mechanisms of breakdown of invariant tori.

*Joint work with Marta Canadell (Universitat de Barcelona, Spain).*

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B1 - July 14, 15:00 – 15:30

#### ON PARAMETER LOCI OF THE HÉNON FAMILY

**Yutaka Ishii**

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We characterize the hyperbolic horseshoe locus and the maximal entropy locus of the Hénon family defined on  $\mathbb{R}^2$ . More specifically, we show that

- i) the two parameter loci are both connected and simply connected,
- (ii) the closure of the hyperbolic horseshoe locus coincides with the maximal entropy locus, and
- (iii) their boundaries are identical and piecewise real analytic with two analytic pieces.

The strategy of our proof is first to extend the dynamical and the parameter spaces over  $\mathbb{C}$ , investigate their complex dynamical and complex analytic properties, and then reduce them to obtain the conclusion over  $\mathbb{R}$ . We also employ interval arithmetic together with some numerical

algorithms such as set-oriented computations and the interval Krawczyk method to verify certain numerical criteria which imply analytic, combinatorial and dynamical consequences.

*Joint work with Zin Arai (Chubu University, Japan).*

B1 - July 14, 15:30 – 16:00

#### NUMERICAL STUDY OF WIDE PERIODIC WINDOWS FOR THE QUADRATIC MAP

**Zbigniew Galias**

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Periodic windows for the map  $f(x) = ax(1 - x)$  are studied numerically.

An efficient method to find very accurate rigorous bounds for the endpoints of the periodic window with a given symbol sequence is presented. The method is based on applying the interval Newton method to find positions of bifurcation points of corresponding periodic orbits. The method is capable to handle periodic windows with periods up to several millions. We discuss how to select an initial point for the Newton method to improve the convergence speed. The method is applied to find all 1 966 957 258 periodic windows with periods  $2 \leq p \leq 36$  and show that the total width of these windows is above 0.611834003131. Positions of periodic windows' endpoints are found with the accuracy of more than 60 decimal digits.

A heuristic algorithm to locate wide periodic windows based on the results obtained for periodic windows with low periods is described. Periodic windows are classified as primary windows and period-tupling windows. Candidates for symbol sequences corresponding to wide primary windows are constructed iteratively from shorter symbol sequences by insertion of a single symbol and by substitution of a single symbol by two-symbol sequences. Symbol sequences corresponding to wide period-tupling windows are constructed iteratively from symbol sequences of primary windows and period-tupling windows found previously. The algorithm is used to find the majority of wide periodic windows with periods  $p \geq 37$ .

From the results concerning periodic windows found it follows that the measure of the set of regular parameters in the interval  $[3, 4]$  is above 0.613960137. Using these results, we estimate that the true value of the measure of the set of regular parameters is close to 0.6139603.

B1 - July 14, 16:00 – 16:30

#### SPECTRAL STABILITY FOR THE WAVE EQUATION WITH PERIODIC FORCING

**Hans Koch**

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We consider the spectral stability problem for Floquet-type systems such as the wave equation with periodic forcing. Our approach is based on a comparison with finite-dimensional approximations. Specific results are obtained for a system where the forcing is due to a coupling between the wave equation and a time-period solution of a nonlinear beam equation. We prove (spectral) stability for some period and instability for another. The finite-dimensional approximations are controlled via computer-assisted estimates.

*Joint work with Gianni Arioli (Politecnico di Milano, Italy).*

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 B1 - July 14, 17:00 – 17:30

## ORDER THEORY AND CONLEY'S CONNECTION MATRICES

**Konstantin Mischaikow**  
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Conley's decomposition theorem indicates that given any compact invariant set the dynamics off of the associated chain recurrent set is gradient-like. Furthermore, the chain recurrent set can be characterized in terms of the set of attractors and their dual repellers. However, in general invariant sets are not computable and their structure is sensitive to variation in parameters.

In this talk I will focus on efforts to develop a computationally robust theory by replacing invariant sets as the primary object of focus with order theoretic objects such as lattices and posets. In particular, I will discuss recent work on lattices of attracting neighborhoods and associated Morse tilings as a representative of the gradient like structure of dynamics, and attempts to develop an efficient computational framework for Conley's connection matrix to rigorously identify the structure of the gradient like invariant dynamics.

*Joint work with Shaun Harker (Rutgers University, USA), William Kalies (Florida Atlantic University, USA), Kelly Spendlove (Rutgers University, USA), and Robert Vandervorst (VU Amsterdam, The Netherlands).*

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 B1 - July 14, 17:30 – 18:00

 SYMBOLIC DYNAMICS FOR KURAMOTO–SIVASHINSKY PDE ON THE LINE —  
 A COMPUTER-ASSISTED PROOF

**Piotr Zgliczyński**  
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The Kuramoto-Sivashinsky PDE on the line with odd and periodic boundary conditions and with parameter  $\nu = 0.1212$  is considered. We give a computer-assisted proof the existence of symbolic dynamics and countable infinity of periodic orbits with arbitrary large periods.

The proof is based on the covering relations, the apparent existence of transversal heteroclinic connections between two periodic orbits and a new algorithm on rigorous integration of dissipative PDEs based on the automatic differentiation.

*Joint work with Daniel Wilczak (Jagiellonian University, Poland).*

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 B1 - July 15, 14:30 – 15:20

## VALIDATED NUMERICS FOR ROBUST SPACE MISSION DESIGN

**Mioara Joldes**  
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In this talk we overview several symbolic-numeric algorithms and software tools developed for robust space mission design. An application is the computation of validated impulsive optimal control for the rendezvous problem of spacecrafts, assuming a linear impulsive setting and a Keplerian relative motion. We combine theoretical tools in optimal control with computer algebra and validated numerics: this brings guarantees on numerical solutions, while preserving efficiency.

In particular, we focus on the a posteriori validation of trajectories solutions of the linear differential equations of the dynamics. These are computed as truncated Chebyshev series together with rigorously computed error bounds. A theoretical and practical complexity analysis of an a posteriori quasi-Newton validation method is presented. Several representative examples show the advantages of our algorithms as well as their theoretical and practical limits.

*This talk is based on joint works with D. Arzelier (LAAS-MAC, France), F. Bréhard (LAAS-CNRS, France), N. Brisebarre (CNRS, France), N. Deak (LAAS-MAC, France), J.-B. Lasserre (LAAS-CNRS, France), C. Louembet (LAAS-MAC, France), A. Rondepierre (IMT Toulouse, France), B. Salvy (INRIA and ENS Lyon, France), and R. Serra (LAAS-MAC, France).*

B1 - July 15, 15:30 – 16:00

#### VALIDATION OF KAM TORI USING A-POSTERIORI FORMULATION

**Alejandro Luque**

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In this talk we present a methodology to rigorously validate a given approximation of a quasi-periodic Lagrangian torus of an exact symplectic map. That is, we check the hypothesis of an a-posteriori KAM theorem and we prove the existence of a true invariant torus nearby. Our method is sustained in the a-posteriori KAM formulation developed in the last decade by R. de la Llave and collaborators.

To check the hypotheses of the theorem, we use rigorous fast Fourier transform in combination with a sharp control of the discretization error. An important consequence is that the rigorous computations are performed in a very fast way. Indeed, with the same asymptotic cost of using the parameterization method to obtain numerical approximations of invariant tori.

We will discuss the application of the method to the standard map and the Froeschlé maps.

*Joint work with Jordi-Lluís Figueras (Uppsala University, Sweden) and Àlex Haro (Universitat de Barcelona, Spain).*

B1 - July 15, 16:00 – 16:30

#### MODEL REJECTION AND PARAMETER REDUCTION VIA TIME SERIES

**Tomas Gedeon**

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We discuss a new approach to Dynamic Signatures Generated by Regulatory Networks (DSGRN) provides a queryable description of global dynamics over the entire parameter space.

We perform a model validation within this class of dynamical systems. We show how a graph algorithm for finding matching labeled paths in pairs of labeled directed graphs can be used to reject models that do not match experimental time series. In particular, we extract a partial order of events describing local minima and local maxima of observed quantities from experimental time-series data from which we produce a labeled directed graph we call the pattern graph for which every path from root to leaf corresponds to a plausible sequence of events. We then consider the regulatory network model, which can be itself rendered into a labeled directed graph we call the search graph via techniques previously developed in computational dynamics. Labels on the pattern graph correspond to experimentally observed events, while labels on the search graph correspond to mathematical facts about the model. We give a theoretical guarantee that failing to find a match invalidates the model. As an application we consider gene regulatory models for the yeast *S. cerevisiae*.

*Joint work with Bree Cummins (Montana State University, USA), Shaun Harker (Rutgers University, USA), and Konstantin Mischaikow (Rutgers University, USA).*

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B1 - July 15, 17:00 – 17:30

#### ON COMPUTATION OF MONODROMY OF THE HÉNON MAP AND ITS APPLICATIONS

**Zin Arai**

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We develop a computational method for the investigation of monodromy actions associated to loops in the parameter space of the complex Hénon map. When the loop is contained in the hyperbolic parameter region, we can determine the action rigorously as an automorphism of the shift space. The monodromy action is closely related to the structure of bifurcation curves in the parameter space and thus our method can be used to understand the bifurcations of the Julia sets.

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B1 - July 15, 17:30 – 18:00

#### THE GEOMETRY OF BLENDERS IN A THREE-DIMENSIONAL HÉNON-LIKE FAMILY

**Stefanie Hittmeyer**

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Blenders are a geometric tool to construct complicated dynamics in diffeomorphisms of dimension at least three and vector fields of dimension at least four. They admit invariant manifolds that behave like geometric objects which have dimensions higher than expected from the manifolds themselves. We consider an explicit family of three-dimensional Hénon-like maps that exhibit blenders in a specific regime in parameter space. Using advanced numerical techniques we compute stable and unstable manifolds in this system, enabling us to show one of the first numerical pictures of the geometry of blenders. We furthermore present numerical evidence suggesting that the regime of existence of the blenders extends to a larger region in parameter space.

*Joint work with Bernd Krauskopf (University of Auckland, New Zealand), Hinke Osinga (University of Auckland, New Zealand) and Katsutoshi Shinohara (Hitotsubashi University, Japan).*

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B1 - Poster

# APPROXIMATION OF BASIN OF ATTRACTION IN PIECEWISE-AFFINE SYSTEMS BY STAR-SHAPED POLYHEDRAL SETS

**Maxim Demenkov**

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We consider piecewise-affine system

$$\dot{x}(t) = A_i x(t) + f_i \text{ if } x(t) \in \mathbf{S}_i,$$

and suppose that regions  $\mathbf{S}_i, i = \overline{0, M}$  are polyhedral (i.e., defined by systems of linear inequalities) and the system state derivatives  $\dot{x}(t) \in R^n$  are continuous on the boundaries between regions. We assume that  $x_0 = 0$  is the stable equilibrium. In this case, an important system characteristic is the size of its basin of attraction for  $x_0$ .

We consider guaranteed estimation of the basin through the level sets of a Lyapunov function [1] — a positive definite continuous function  $V(x)$ , which is decreasing along the solution trajectories of the system. An estimation of the basin is given by a compact set defined as

$$\mathbf{\Omega}_\gamma = \{x | V(x) \leq \gamma\}.$$

We employ star-shaped polyhedral functions  $V(x)$  defined over union of convex cones  $\mathbf{C}_j, j = \overline{1, N}$ , having  $x_0$  as a common point, so that in each cone the function is linear:

$$V(x) = d_j^T x \text{ if } x \in \mathbf{C}_j, d_j \in R^n.$$

The estimation of maximum size in terms of inclusion is given by the following choice of  $\gamma$ :

$$\gamma = \min_{x \in \mathbf{H}} V(x), \quad \mathbf{H} = \{x : D^+V(x(t)) = 0, x \neq 0\},$$

where  $D^+V(x(t))$  is the upper Dini derivative of Lyapunov function [2] along system trajectories and  $\mathbf{H}$  appears to be piecewise-affine manifold. The manifold  $\mathbf{H}$  divides the state space into two (possibly disconnected) regions. We suppose that  $x_0$  is in the region where  $D^+V(x(t)) < 0$ , so to find maximum estimation we are looking for scaling factor  $\gamma$  so that the level set  $\mathbf{\Omega}_\gamma$  fits in the region and touches the manifold  $\mathbf{H}$ . The notion of Dini derivative is actually a replacement of usual derivative of smooth Lyapunov function along trajectories of smooth systems [3].

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B1 - Poster

NUMERICAL CONTINUATION STUDY OF INVARIANT SOLUTIONS OF THE COMPLEX  
GINZBURG-LANDAU EQUATION

**Vanessa López**

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We consider the problem of computation and deformation of group orbits of solutions of the complex Ginzburg-Landau equation (CGLE) with cubic nonlinearity in 1+1 space-time dimension invariant under the action of the three-dimensional Lie group of symmetries  $A(x, t) \rightarrow e^{i\theta} A(x + \sigma, t + \tau)$ . From an initial set of group orbits of such invariant solutions, for a particular point in the parameter space of the CGLE, we obtain new sets of group orbits of invariant solutions via numerical continuation along paths in the moduli space. The computed solutions along the continuation paths are unstable, and have multiple modes and frequencies active in their spatial and temporal spectra, respectively. Structural changes in the moduli space resulting in symmetry gaining / breaking associated mainly with the spatial reflection symmetry  $A(x, t) \rightarrow A(-x, t)$  of the CGLE were frequently uncovered in the parameter regions traversed.

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B1 - Poster

HALO ORBITS AND THEIR BIFURCATIONS — RIGOROUS NUMERICAL APPROACH

**Irmina Walawska**

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The Circular Restricted Three Body Problem (CR3BP) is a model equation for motion of a massless particle in the gravitational force of two large primaries. The system is given by

$$\begin{cases} \ddot{x} - 2\dot{y} = \frac{\partial \Omega(x, y, z)}{\partial x}, \\ \ddot{y} + 2\dot{x} = \frac{\partial \Omega(x, y, z)}{\partial y}, \\ \ddot{z} = \frac{\partial \Omega(x, y, z)}{\partial z}, \end{cases}$$

where

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{d_1} + \frac{\mu}{d_2}$$

and

$$d_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}, \quad d_2 = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2},$$

where  $\mu$  denotes the relative mass ratio of the two main bodies. It was observed by Robert Farquhar that there is a family of symmetric, periodic orbits, parameterized by the amplitude  $z$ . These orbits are called Halo orbits. We propose an algorithm for rigorous validation of bifurcation of family of periodic orbits. We also give an algorithm for rigorous continuation of these orbits. The method uses rigorous computation of higher order derivatives of well

chosen Poincaré map with symmetry properties of the system. As an application we give a computer assisted proof that the Halo orbits bifurcate from the family of Lyapunov orbits for wide range of the parameters  $\mu$ . For  $\mu$  corresponding to the Sun-Jupiter system we give a proof of the existence of a wide continuous branch of Halo orbits that undergo period doubling, tripling and quadrupling bifurcation for some amplitude  $z$ . The computer assisted proof uses rigorous ODE solvers and algorithms for computation of Poincaré maps from the CAPD library.

*Joint work with Daniel Wilczak (Jagiellonian University, Poland).*

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B1 - Poster

# SPECTRAL METHODS FOR TRANSFER OPERATORS IN ONE-DIMENSIONAL DYNAMICS

**John P. Wormell**

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Chaotic maps of the interval are important models in dynamical systems, and quantitative aspects of their long-time statistical properties (such as invariant measures) are key objects of study. These properties can be found by solving linear problems involving the transfer operator, but existing numerical methods converge very slowly and cannot capture many important statistical properties.

We present a rigorously justified Chebyshev spectral method for calculating statistical properties in the case of Markovian uniformly-expanding maps. This spectral method can be neatly integrated into an infinite-dimensional linear algebra framework. As a result, the accurate calculation of any statistical property of interest is not only quick and reliable, but highly convenient.

The spectral method can also be extended through inducing schemes to many other kinds of maps of the interval. We present the specific case of intermittent maps, which, though of much interest theoretically, have been hitherto been largely out of reach of numerical study.



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**Workshop B2**  
**Graph Theory and Combinatorics**

Organizers: Marc Noy – Jaroslav Nešetřil – Angelika Steger

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B2 - July 13, 14:30 – 15:20

STABILITY AND EXACTNESS RESULTS FROM FLAG ALGEBRA CALCULATIONS

**Oleg Pikhurko**

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Flag algebras provide a powerful framework for proving asymptotic inequalities between the possible densities of fixed subgraphs in a large graph. We develop general methods for establishing more refined results (namely, stability as well as exact inequalities) from flag algebra calculations and apply them to concrete examples (Turan function, inducibility problem, etc). In fact, some of our sufficient conditions are stated in a way that allows automatic verification by a computer. This gives a unifying way to obtain computer-assisted proofs of some known and new results.

*Joint work with Jakub Sliacan (The Open University, UK) and Kostas Tyros (Koç University, Turkey).*

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B2 - July 13, 15:30 – 15:55

SUBQUADRATIC ALGORITHMS FOR THE DIAMETER AND THE SUM OF PAIRWISE  
 DISTANCES IN PLANAR GRAPHS

**Sergio Cabello**

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We show how to compute for  $n$ -vertex planar graphs in roughly  $O(n^{11/6})$  expected time the diameter and the sum of the pairwise distances. These are the first algorithms for these problems using time  $O(n^c)$  for some constant  $c < 2$ , even when restricted to undirected, unweighted planar graphs.

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B2 - July 13, 16:00 – 16:25

CRITICAL PERCOLATION ON RANDOM REGULAR GRAPHS

**Guillem Perarnau**

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We show that for all  $d \in \{3, \dots, n-1\}$  the size of the largest component of a random  $d$ -regular graph on  $n$  vertices at the percolation threshold  $p = 1/(d-1)$  is  $\Theta(n^{2/3})$ , with high probability. This extends known results for fixed  $d \geq 3$  and for  $d = n-1$ , confirming a prediction of Nachmias and Peres on a question of Benjamini. In contrast to previous approaches, our proof is based on an application of the switching method.

*Joint work with Felix Joos (University of Birmingham, UK).*

B2 - July 13, 17:00 – 17:50

## MODELING LIMITS

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A sequence of graphs is FO-convergent if the probability of satisfaction of every first-order formula converges. A graph modeling is a graph, whose domain is a standard probability space, with the property that every definable set is Borel. It was known that FO-convergent sequences of graphs do not always admit a modeling limit, and it was conjectured that this is the case if the graphs in the sequence are sufficiently sparse. Precisely, two conjectures were proposed:

- 1) If a FO-convergent sequence of graphs is residual, that is, if for every integer  $d$  the maximum relative size of a ball of radius  $d$  in the graphs of the sequence tends to zero, then the sequence has a modeling limit.
- 2) A monotone class of graphs  $\mathcal{C}$  has the property that every FO-convergent sequence of graphs from  $\mathcal{C}$  has a modeling limit if and only if  $\mathcal{C}$  is nowhere dense, that is, if and only if for each integer  $p$  there is  $N(p)$  such that the  $p$ th subdivision of the complete graph on  $N(p)$  vertices does not belong to  $\mathcal{C}$ .

In this talk we present the proof of both conjectures. This solves some of the main problems in the area and among others provides an analytic characterization of the nowhere dense–somewhere dense dichotomy.

B2 - July 13, 18:00 – 18:25

## ON THE COMPUTATION OF NUMERICAL SEMIGROUPS

**Maria Bras**

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A numerical semigroup is a subset of the positive integers ( $\mathbb{N}$ ) together with 0, closed under addition, and with a finite complement in  $\mathbb{N} \cup \{0\}$ . The number of gaps is its *genus*. Numerical semigroups arise in algebraic geometry, coding theory, privacy models, and in musical analysis. It has been shown that the sequence counting the number of semigroups of each given genus  $g$ , denoted  $(n_g)_{g \geq 0}$ , has a Fibonacci-like asymptotic behavior. It is still not proved that, for each  $g$ ,  $n_{g+1} \geq n_g$ .

All algorithms used to compute  $n_g$  explore by brute force approach the tree that contains at each depth the semigroups of genus equal to that depth, and in which the parent of a semigroup is the semigroup obtained when adjoining to the child its largest gap. We present a new algorithm for descending the tree using the new notion of seed of a numerical semigroup.

*Joint work with Julio Fernández-González (Universitat Politècnica de Catalunya, Spain).*

B2 - July 13, 18:30 – 18:55

## BIJECTIONS FOR PLANAR MAPS WITH BOUNDARIES

**Eric Fusy**

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I will present a general bijective method for planar maps based on certain orientations. This method allows us to count planar maps with prescribed face-degrees and girth. I will then explain how the method and (partial) girth control can be adapted to the setting of planar maps with boundaries (i.e., planar maps where some faces are distinguished, so that these face-contours are vertex-disjoint simple cycles).

*Joint work with Olivier Bernardi (Brandeis University, USA).*

B2 - July 14, 14:30 – 15:20

## GAPS BETWEEN CLASSICAL SATISFIABILITY PROBLEMS AND THEIR QUANTUM RELAXATIONS

**Albert Atserias**

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An instance of the constraint satisfaction problem asks for an assignment of values to variables in such a way that a given set of local constraints are satisfied. One can think of such problems operationally in the form of a game: Alice provides value-assignments that satisfy given constraints on request, Bob provides value-assignments to the variables also on request, and a referee checks if Alice's and Bob's assignments are consistent with one another. Many problems of graph theory, combinatorics, logic and computer science fall within this abstract framework. Since the problem of deciding if Alice and Bob have a winning strategy is NP-hard to solve in general, a common approach to provide structure to the problem is to consider relaxations of it where the variables range over larger but more structured domains. We study a relaxation that is actually motivated by a modeling issue: the universe appears to be quantum mechanical, so Alice and Bob could use strategies that are correlated through quantum entanglement. In this relaxation, which is well-studied in quantum information theory, the constraints are represented by polynomials through their Fourier transform, and the variables range over bounded linear operators over a Hilbert space. Among other results we obtain a complete understanding of which types of Boolean constraints show a discrepancy between quantum and classical strategies.

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B2 - July 14, 15:30 – 15:55

AN APPLICATION OF HOFFMAN GRAPHS FOR SPECTRAL CHARACTERIZATIONS OF  
GRAPHS

**Aida Abiad**

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In this paper, we present the first application of Hoffman graphs for spectral characterizations of graphs. In particular, we show that the 2-clique extension of the  $(t + 1) \times (t + 1)$ -grid is determined by its spectrum when  $t$  is large enough. This result will help to show that the Grassmann graph  $J_2(2D, D)$  is determined by its intersection numbers as a distance regular graph, if  $D$  is large enough.

*Joint work with Qianqian Yang (University of Science and Technology of China, China) and Jack H. Koolen (University of Science and Technology of China and Wen-Tsun Wu Key Laboratory of CAS, China).*

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B2 - July 14, 16:00 – 16:25

COLORFUL SIMPLICIAL DEPTH, MINKOWSKI SUMS, AND GENERALIZED GALE  
TRANSFORMS

**Arnau Padrol**

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The colorful simplicial depth of a collection of  $d + 1$  finite sets of points in Euclidean  $d$ -space is the number of choices of a point from each set such that the origin is contained in their convex hull. We use methods from combinatorial topology to prove a tight upper bound on the colorful simplicial depth. This implies a conjecture of Deza & al., (2006). Furthermore, we introduce colorful Gale transforms as a bridge between colorful configurations and Minkowski sums. Our colorful upper bound then yields a tight upper bound on the number of totally mixed facets of certain Minkowski sums of simplices. This resolves a conjecture of Burton (2003) in the theory of normal surfaces.

*Joint work with Karim Adiprasito (Hebrew University of Jerusalem, Israel), Philip Brinkmann (Freie Universität Berlin, Germany), Pavel Paták (Hebrew University of Jerusalem, Israel), Zuzana Patáková (Hebrew University of Jerusalem, Israel), and Raman Sanyal (Goethe Universität Frankfurt, Germany).*

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B2 - July 14, 17:00 – 17:50

APPROXIMATION OF MIN CSPs

**Víctor Dalmau**

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An instance of the constraint satisfaction problem (CSP) is given by a family of constraints on overlapping sets of variables, and the goal is to assign values from a fixed domain to the variables so that all constraints are satisfied. In the optimization version, the goal is to maximize the number of satisfied constraints (MAX CSP) or, alternatively, to minimize the number of unsatisfied constraints (MIN CSP). This problem is usually parameterized by the set,  $\Gamma$ , of relations allowed in the constraints, usually called constraint language. It turns out that MAX CSP/MIN CSP is computationally hard for most constraint languages, which motivates the study of approximation algorithms. In this talk we will focus on the approximation of MIN CSPs. We shall start addressing the following question: which constraint languages give rise to a MIN CSPs that is constant-factor approximable? We shall also study some other weaker approximation notions such as polynomial loss and robust approximation.

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B2 - July 14, 18:00 – 18:25

#### ENUMERATION OF LABELLED 4-REGULAR PLANAR GRAPHS

**Juanjo Rué**

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The enumeration of planar graphs and related classes of graphs is currently an active area of research. For the case of 4-regular planar graphs there are known schemes for exhaustive generation starting from some basic graphs, such as the graph of the octahedron, but no counting results up to now. We present here a complete solution to the enumeration of 4-regular planar graphs.

To obtain our results we follow the classical technique introduced by Tutte: take a graph rooted at a directed edge and classify the possible configurations arising from the removal of the root edge. This produces several combinatorial classes that are further decomposed, typically in a recursive way. The decomposition translates into a system of polynomial equations for the associated generating functions. Along the way, we have to deal with the enumeration of quadrangulations and other classes of planar maps.

*Joint work with Marc Noy (Universitat Politècnica de Catalunya and BGSMATH, Spain) and Clément Requilé (Freie Universität Berlin and Berlin Mathematical School, Germany).*

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B2 - July 14, 18:30 – 18:55

#### A TUTTE POLYNOMIAL FOR GRAPHS EMBEDDED ON SURFACES

**Lluís Vena**

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In this talk, we present a graph polynomial for maps (graphs embedded in orientable surfaces) that contains the Las Vergnas polynomial, Bollobás-Riordan polynomial and Kruskal polynomial as specialisations.

The new polynomial invariant of maps is built following Tutte's construction of the dichromate of a graph (that is, the Tutte polynomial) as a unification of the chromatic polynomial

and the flow polynomial. In our case, we consider the analogues for maps of the chromatic polynomial (local tensions) and of the flow polynomial (local flows). Hence, by construction, the new polynomial includes among its evaluations the number of local tensions and local flows taking values in any given finite group. Other evaluations include the number of quasi-forests. An extension of the polynomial to graphs embedded on non-orientable surfaces is also discussed.

*Joint work with Andrew Goodall (Charles University Prague, Czech Republic), Thomas Krajewski (Aix-Marseille Université, France), Bart Litjens (University of Amsterdam, The Netherlands), and Guus Regts (University of Amsterdam, The Netherlands).*

B2 - July 15, 14:30 – 15:20

## GEOMETRIC REPRESENTATIONS OF GRAPHS: PARTIAL EXTENSIONS VERSUS SIMULTANEOUS EMBEDDINGS

**Jan Kratochvíl**

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Extending partial solutions is often provably more difficult than constructing a solution from scratch. Somewhat surprisingly for geometric representations of graphs, this does not seem to be the case. In most of the cases when the computational complexity of the extension problem is known, the complexity is the same as of the plain recognition problem. Another closely related problem are simultaneous representations of graphs. Closely related, but not always known to be of the same computational complexity. We will survey the known results, compare them, and comment on open problems. The classes of graphs we will be interested in are interval graphs, circle graphs, comparability graphs, and also several graph drawing types.

B2 - July 15, 15:30 – 16:20

## PLANAR ARCS

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Let  $\text{PG}_2(\mathbb{F}_q)$  denote the projective plane over  $\mathbb{F}_q$ . An arc (or planar arc) of  $\text{PG}_2(\mathbb{F}_q)$  is a set of points in which any 3 points span the whole plane. An arc is complete if it cannot be extended to a larger arc.

In 1967 Beniamino Segre proved that the set of tangents to a planar arc of size  $q + 2 - t$ , when viewed as a set of points in the dual plane, is contained in an algebraic curve of small degree  $d$ . Specifically, if  $q$  is even then  $d = t$  and if  $q$  is odd then  $d = 2t$ .

The main result to be presented here is the following theorem.

**Theorem 1.** *Let  $S$  be a planar arc of size  $q + 2 - t$  not contained in a conic. If  $q$  is odd then  $S$  is contained in the intersection of two curves, sharing no common component, each of degree at most  $t + p^{\lfloor \log_p t \rfloor}$ .*

This leads directly to the following theorem.

**Theorem 2.** *If  $q$  is odd then an arc of size at least  $q - \sqrt{q} + 3 + \max\{\sqrt{q}/p, \frac{1}{2}\}$  is contained in a conic.*

There are examples of complete arcs of size  $q + 1 - \sqrt{q}$  in  $\text{PG}_2(\mathbb{F}_q)$  when  $q$  is square, first discovered by Kestenband in 1981. These arcs are the intersection of two curves of degree  $t$ . It has long been conjectured that if  $q \neq 9$  is an odd square then any larger arc is contained in a conic.

*Joint work with Michel Lavrauw (Sabanci University, Turkey).*

B2 - July 15, 17:00 – 17:50

## SOME TOPOLOGICAL ALGORITHMS FOR GRAPHS ON SURFACES

**Éric Colin de Verdière**

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The aim of this talk is to survey various recent results for solving computational problems for graphs on surfaces, at the interface of topological graph theory and graph algorithms.

Given a topologically non-trivial surface (up to homeomorphism, a sphere with handles), how can we compute a shortest non-contractible closed curve (which cannot be continuously deformed to a point)? How can we cut the surface economically to make it planar (homeomorphic to a disk)? These are basic topological questions, which we revisit them from an algorithmic perspective.

Conversely, we will illustrate how topology can help devising more efficient graph algorithms in the case of graphs embeddable on a fixed surface, in the case of the minimum multicut problem.

B2 - Poster

## SCAFFOLDING SKELETONS USING SPHERICAL VORONOI DIAGRAMS

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Skeletons are used in 3D graphics for modeling and animating articulated shapes. The user can design a complex shape by sketching a simple and intuitive geometric object that is the input to a surface generating algorithm.

In this work, a skeleton is a finite set of line segments in space that do not intersect except at the endpoints. It can be seen as a graph with geometrical constraints (length of the edges, position of vertices). We study the problem of generating a mesh around a skeleton. The mesh we want to obtain should be as “coarse” as possible (in the sense of number of patches around each line segment), with mostly quad patches and must enclose the skeleton. Following [3] we call this mesh a scaffold.

This problem has been studied before in [2] and [1]. The main limitation in Ji & al., was the presence of irregularities in the final mesh. Furthermore the method proposed by Bærentzen & al., cannot handle closed skeletons, i.e., skeletons with cycles when seen as a graph. The following problems were also left as open : under what conditions the construction can be done; whether there is an optimal solution on the number of patches; or whether the mesh can be constructed such that each segment piece is enclosed by the same number of patches.

We present a method for constructing a scaffold for a skeleton using spherical Voronoi diagrams at the branch points. We prove that such a mesh can always be computed, even in the presence of cycles. We can compute a solution with minimal total number of patches, or, alternatively, a mesh with the same number of patches around each line segment. The main tools we use are: spherical Voronoi diagrams, Delaunay triangulations, Integer Linear Programming and a numerical characterization of inscribable graphs due to [4].

Our method has been implemented and is currently being used for developing a meshing algorithm for implicit surfaces defined around a skeleton [5]. We point out also that the problem is closely related to a perfect matching for graphs.

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*Joint work with Evelynne Hubert (INRIA Sophia Antipolis, France).*

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B2 - Poster

## COUNTING ARITHMETICAL STRUCTURES

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Let  $G$  be a finite, simple, connected graph. An arithmetical structure on  $G$  is a pair of positive integer vectors  $d, r$  such that  $(\text{diag}(d) - A)r = 0$ , where  $A$  is the adjacency matrix of  $G$ . Arithmetical graphs were introduced in the context of arithmetical geometry by Lorenzini in 1989 to model intersections of curves. We investigate the combinatorics of arithmetical structures on path and cycle graphs, as well as the associated critical groups (the cokernels of the matrices



( $\text{diag}(d) - A$ )). For paths, we prove that arithmetical structures are enumerated by the Catalan numbers, and we obtain refined enumeration results related to ballot sequences. For cycles, we prove that arithmetical structures are enumerated by the binomial coefficients  $\binom{2n-1}{n-1}$ , and we obtain refined enumeration results related to multisets. In addition, we determine the critical groups for all arithmetical structures on paths and cycles.

*Joint work with Benjamin Braun (University of Kentucky, USA), Hugo Corrales (CINVESTAV, México), Scott Corry (Lawrence University, USA), Darren Glass (Gettysburg College, USA), Nathan Kaplan (University of California, Irvine, USA), Jeremy L. Martin (University of Kansas, USA), Gregg Musiker (University of Minnesota, USA) and Carlos E. Valencia (CINVESTAV, México).*

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B2 - Poster

## NULL DECOMPOSITION OF TREES AND ITS CONSEQUENCES

**Daniel Alejandro Jaume**

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The purpose of this study is to determine which information about a tree could be obtained from the support of the null space its adjacency matrices. In order to do that, we introduce a new family of trees, the S-trees, which are based on the non-zero entries of vectors in null space. We show that every tree can be decomposed into a forest of S-trees and a forest of non-singular trees.

The support of a tree  $T$  is the set

$$\text{Supp}(T) := \{u \in V(T) : \exists x \in \mathcal{N}(T) \text{ such that } x_u \neq 0\}.$$

Where  $\mathcal{N}(T)$  is the null space of  $A(T)$ , the adjacency matrix of  $T$ . The cardinality of the support is denoted  $\text{supp}(T)$ .

**Definition.** A tree  $T$  is an S-tree if  $N[\text{Supp}(T)] = V(T)$ , where  $N[\text{Supp}(T)]$  is the closed neighborhood of  $\text{Supp}(T)$ .

**Definition.** Let  $T$  be an S-tree. The core of  $T$ , denoted by  $\text{Core}(T)$ , is defined to be the set of all non-supported vertices of  $T$ . We will denote by  $\text{core}(T)$  the cardinality of  $\text{Core}(T)$ .

A tree  $T$  is non-singular tree, N-tree for short, if its adjacency matrix  $A(T)$  is invertible. Thus  $T$  is a N-tree if and only if  $T$  has a perfect matching.

**Definition.** Let  $T$  be a tree. The S-forest of  $T$ , denoted by  $\mathcal{F}_S(T)$ , is defined to be the forest induced by the closed neighbor of  $\text{Supp}(T)$  in  $T$ :  $\mathcal{F}_S(T) := T \langle N[\text{Supp}(T)] \rangle$ . The N-forest of  $T$ , denoted by  $\mathcal{F}_N(T)$ , is defined to be the remain forest:  $\mathcal{F}_N(T) := T \setminus \mathcal{F}_S(T)$ . The pair of forests  $(\mathcal{F}_S(T), \mathcal{F}_N(T))$  is called the Null Decomposition of  $T$ .

**Theorem.** Let  $T$  be a tree. The S-forest  $\mathcal{F}_S(T)$  of  $T$  is a forest of S-trees, and the N-forest  $\mathcal{F}_N(T)$  of  $T$  is a forest of N-trees. Futhermore:

$$1) \nu(T) = \text{core}(T) + \frac{|V(\mathcal{F}_N(T))|}{2}, \text{ where } \nu(T) \text{ is the matching number of } T,$$

- 2)  $\alpha(T) = \text{supp}(T) + \frac{|V(\mathcal{F}_N(T))|}{2}$ , where  $\alpha(T)$  is the independence number of  $T$ ,
- 3)  $rk(T) = \text{core}(T) + |V(\mathcal{F}_N(T))|$ , where  $rk(T)$  is the rank of the adjacency matrix of  $T$ ,
- 4)  $\text{null}(T) = \text{supp}(T) - \text{core}(T)$ , where  $\text{null}(T)$  is the nullity of the adjacency matrix of  $T$ ,
- 5)  $m(T) = \prod_{S \in \mathcal{F}_N(T)} m(S)$ , where  $m(T)$  is the number of maximum matching of  $T$ , and
- 6)  $a(T) = \prod_{N \in \mathcal{F}_N(T)} a(N)$ , where  $a(T)$  is the number of maximum independent sets of  $T$ .

**Corollary.** *Let  $T$  be a tree. Then  $\nu(T) = \alpha(T) - \text{null}(T)$ .*

*Joint work with Gonzalo Molina (Universidad Nacional de San Luis, Argentina).*

B2 - Poster

## BOUNDS ON THE LETTER FREQUENCIES IN KOLAKOSKI SEQUENCES VIA CHVATAL'S SEQUENCE OF GRAPHS

**Bernd Sing**

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An infinite sequence  $z$  over the alphabet  $\{1, 2\}$  that equals its own run-length sequence, is called a Kolakoski-sequence (also see [2, 3]):

$$z = \underbrace{22}_2 \underbrace{11}_2 \underbrace{2}_1 \underbrace{1}_1 \underbrace{22}_2 \underbrace{1}_1 \underbrace{22}_2 \underbrace{11}_2 \underbrace{2}_1 \underbrace{11}_2 \dots = z.$$

Prepending a 1 to this sequence gives the other Kolakoski sequence over this alphabet. Although it is conjectured that the letter frequencies are equal in the infinite sequence (i.e., half the letters are 1s and half of them 2s), it is not even known if the frequencies actually exist.

Seeking to understand this mysterious sequences better, one can consider sequences that equal their own run-length sequence over other alphabets with two letters, see [4]: For alphabets consisting of two even numbers, one can easily establish that the limiting frequencies are  $1/2$ ; in the case of two odd numbers, the limiting frequencies are unequal (they are given by cubic irrationals); while in the case that of one even and one odd number, one arrives at the same conjecture about the letter frequencies as for the alphabet  $\{1, 2\}$ .

In order to find bounds on the letter frequencies of the Kolakoski sequence, Vasek Chvatal [1] considered a certain recursively defined sequence of digraphs. Using the 22nd term in this sequence, he showed that the letter frequencies of the Kolakoski sequence are confined to  $[0.499162, 0.500838]$ ; however, the number of vertices grows by a factor of 2 in each step (so the 22nd graph has  $2^{22}$  vertices).

In this paper we look at the structure of the digraphs one gets for different alphabets: One can again recursively define such a sequence of digraphs for any two letter alphabet. While the number of vertices always grows by a factor of 2, other properties like connectedness differ for different alphabets.

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B2 - Poster

**THE DEHN-SOMMERVILLE RELATIONS AND THE CATALAN MATROID****Nicole Yamzon**

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The  $f$ -vector of a  $d$ -dimensional polytope  $P$  stores the number of faces of each dimension. When  $P$  is simplicial the Dehn–Sommerville relations imply that to determine the  $f$ -vector of  $P$ , we only need to know approximately half of its entries. This raises the question: Which  $(\lceil \frac{d+1}{2} \rceil)$ -subsets of the  $f$ -vector of a general simplicial polytope are sufficient to determine the whole  $f$ -vector? We prove that the answer is given by the bases of the Catalan matroid.

*Joint work with Anastasia Chavez (University of California Berkeley, USA).*

## Workshop B3 Symbolic Analysis

Organizers: Bruno Salvy – Jacques-Arthur Weil – Irina Kogan

B3 - July 13, 14:30 – 14:55

### INVARIANTS OF TERNARY QUARTICS UNDER THE ORTHOGONAL GROUP

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Classical invariant theory has essentially addressed the action of the general linear group on homogeneous polynomials. Yet the orthogonal group arises in applications as the relevant group of transformations, especially in 3 dimensional space. Having a complete set of invariants for its action on quartics is, for instance, relevant in determining biomarkers for white matter from diffusion MRI [1]

We characterize a generating set of rational invariants of the orthogonal group by reducing the problem to the action of a finite subgroup on a slice. The invariants of the orthogonal group can then be obtained in an explicit way. But their numerical evaluation can be achieved more straightforwardly. These results can furthermore be generalised to sextics and higher even degree forms thanks to a novel and explicit construction of a basis of harmonic polynomials at each degree.

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*Joint work with Paul Görlach (INRIA, France & Max Planck Institute Leipzig, Germany), Théo Papadopoulos (INRIA, France).*

B3 - July 13, 15:00 – 15:25

### FORMAL RECURSION OPERATORS OF INTEGRABLE PDES OF THE FORM

$$q_{tt} = F(q, q_x, q_t, \dots)$$

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We will explain how the symmetry approach to integrability applies to partial differential equations of the form

$$q_{tt} = F(q, q_x, \dots, q_n, q_t, q_{tx}, \dots, q_{tm}).$$

Any such equation is integrable if it admits a formal recursion operator, i.e., a pseudodifferential operator  $\mathcal{R}$  of the form

$$\mathcal{R} := L + MD_t$$

where  $L$  and  $M$  are pseudodifferential operators in the derivation  $D_x$  satisfying the symmetry condition

$$\mathcal{F}(L + MD_t) = (\bar{L} + MD_t)\mathcal{F}.$$

Here  $\bar{L} := L + 2M_t + [M, V]$  and  $\mathcal{F} = D_t^2 - U - VD_t$  is the linearization operator of the PDE, i.e., the operator appearing in its variational equation, so

$$\begin{aligned} U &= u_n D^n + u_{n-1} D^{n-1} + \cdots + u_0 \\ V &= v_m D^m + v_{m-1} D^{m-1} + \cdots + v_0 \end{aligned}$$

with

$$u_i = \frac{\partial F}{\partial q_i}, \quad v_j = \frac{\partial F}{\partial q_{tj}}.$$

We are then confronted with solving an equation over pseudodifferential operators in two derivations, a rather nontrivial problem. The equation happens to have a somewhat triangular structure, making its resolution possible. But in the solving process there appear obstructions, written as conditions over the rhs  $F$  of the PDE, that are interpreted as integrability conditions.

The algebra of formal recursion operators has an interesting structure, and it has important relationships to algebras of commuting (pseudo)-differential operators in two derivations.

Some previous results and background were presented in [Rafael Hernández Heredero, V.V. Sokolov, and A.B. Shabat, “A new class of linearizable equations.” J. Phys. A: Math. Gen. 36 (2003), L605–L614].

*Joint work with Agustín Caparrós Quintero (Universidad Politécnica de Madrid, Spain).*

B3 - July 13, 15:30 – 15:55

## COMPUTING THE GALOIS–LIE ALGEBRA OF COMPLETELY REDUCIBLE DIFFERENTIAL SYSTEMS

**Jacques–Arthur Weil**

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This talk is related to the work presented by Thomas Dreyfus earlier in this session.

A linear differential system  $[A] : Y' = AY$  is called completely reducible when  $A$  is block-diagonal and each block is the matrix of an irreducible system. We show how to compute the Lie algebra  $\mathfrak{g}$  of the Galois group of  $[A]$ .

We will first explain how to represent a copy of  $\mathfrak{g}$  as a subsystem of a system constructed on  $[A]$ . We show how to guess this subsystem by using modular techniques or invariants. Validation of this guess is achieved through, first, computing an explicit conjugacy between Lie algebras and finally finding algebraic solutions of a (small) linear differential system.

*Joint work with Moulay Barkatou (Université de Limoges, France), Thomas Cluzeau (Université de Limoges, France), and Lucia Di Vizio (Université de Versailles, France).*

B3 - July 13, 16:00 – 16:25

# SINGULAR INITIAL VALUE PROBLEMS FOR QUASI-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

**Werner M. Seiler**

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We discuss the existence, (non-)uniqueness and regularity of solutions of initial value problems for quasi-linear ordinary differential equations where the initial data corresponds to a singularity of the equation. We show how the Vessiot distribution permits to reduce the question of existence and (non-)uniqueness to the study of the local solution behaviour of a vector field near a stationary point. For regularity results a similar analysis must be performed for different prolongations of the given equation.

*Joint work with Matthias Seiß (Universität Kassel, Germany).*

B3 - July 13, 17:00 – 17:25

# MULTISPACE AND VARIATIONAL METHODS

**Elizabeth Mansfield**

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Recently with Gloria Mari Beffa, a multi-dimensional version of multispace was constructed. This incorporates Lagrange and Hermite interpolations of functions but has the standard jet bundle embedded as a smooth submanifold. The construction allows for functional approximations and their smooth continuum limits to be studied simultaneously. Applications include invariants of Lie group actions via a moving frame on multispace, smooth and discrete integrable systems, and smooth and discrete variational calculus including Noether's theorem. Further, there is a multispace version of the preserved symplectic form for a variational system, allowing smooth and discrete Hamiltonian versions of Euler Lagrange equations to be studied simultaneously. The talk will include joint work with Gloria Mari Beffa and Peter Hydon, and perhaps others, depending on the particular application of the multispace variational calculus developed by the time of the conference.

*Joint work with Gloria Mari Beffa (UW, Madison) and Peter Hydon (University of Kent, UK).*

B3 - July 13, 17:30 – 17:55

# NON-INTEGRABILITY OF THE ARMBURSTER-GUCKENHEIMER-KIM QUARTIC HAMILTONIAN THROUGH MORALES-RAMIS THEORY

**Primitivo Acosta-Humánez**

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In this talk it will be presented the non-integrability of the three-parameter Armburster–Guckenheimer–Kim quartic Hamiltonian using Morales–Ramis theory, with the exception of the three already known integrable cases. Poincaré sections to illustrate the breakdown of regular motion for some parameter values. In particular we obtain conditions to get meromorphic non-integrability results through the Galoisian analysis of the variational equation of the AGK Hamiltonian which is a Legendre differential equation. On the other hand, the complete conditions for rational non-integrability results are presented through the Bostan–Combot–Safey El Din algorithm which is developed in maple.

*Joint work with Martha Álvarez-Ramírez (Universidad Autónoma Metropolitana Iztapalapa, México) and Teresa Stuchi (Universidade Federal do Rio de Janeiro, Brazil).*

B3 - July 13, 18:00 – 18:25

### COMPUTING THE DIFFERENTIAL GALOIS GROUP USING REDUCED FORMS

**Thomas Dreyfus**

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To a linear differential system one may associate a group, the differential Galois group, that measures the algebraic relations between the solutions of the system. The latter may be seen as an algebraic group and its computation is in general a difficult task. In this talk we explain how to transform the system so that the new system belong to the Lie algebra of the differential Galois group (such transformation always exists and the new system will be said to be on the Kolchin-Kovacic reduced form). Furthermore, as we will see, the latter reduction will be helpful for computing the differential Galois group.

*Joint work with Jacques-Arthur Weil (University of Limoges, France).*

B3 - July 14, 14:30 – 14:55

### FINITE AUTOMATA, AUTOMATIC SETS, AND DIFFERENCE EQUATIONS

**Michael Singer**

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A finite automaton is one of the simplest models of computation. Initially introduced by McCulloch and Pitts to model neural networks, they have been used to aid in software design as well as to characterize certain formal languages and number-theoretic properties of integers. A set of integers is said to be  $m$ -automatic if there is a finite automaton that decides if an integer is in this set given its base- $m$  representation. For example powers of 2 are 2-automatic but not 3-automatic. This latter result follows from a theorem of Cobham describing which sets of integers are  $m$ - and  $n$ -automatic for sufficiently distinct  $m$  and  $n$ . In recent work with Reinhard Schaefer, we gave a new proof of this result based on analytic results concerning normal forms of systems of difference equations. In this talk, I will describe this circle of ideas.

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B3 - July 14, 15:00 – 15:25

FACTORIZATION OF STATIONARY SCHRÖDINGER OPERATORS OVER KdV SPECTRAL CURVES

**Sonia Rueda**

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In 1928, J.L. Burchall and T.W. Chaundy established a correspondence between commuting differential operators and algebraic curves. With the discovery of solitons and the integrability of the Korteweg de Vries (KdV) equation, using the inverse spectral methods, their theory found applications to the study of partial differential equations called integrable (or with solitonic type solutions: Sine–Gordon, non linear Schrödinger, etc). Burchall and Chaundy had discovered the spectral curve, which was later computed by E. Previato, using differential resultants. The spectral curve allows an algebraic approach to handling the inverse spectral problem for the finite-gap operators, with the spectral data being encoded in the spectral curve and an associated line bundle.

In this work, we explore the benefits of using differential resultants to compute the Burchall and Chaundy polynomials. We review the definition of the differential resultant of two ordinary differential operators and its main properties. We revisit Enma Previato's result about the computation of the spectral curve of two commuting differential operators using differential resultants. We use these results to establish the appropriate fields where commuting operators have a common factor, which can be computed using differential subresultants.

These results will allow us to give new explanations to some well known results related with the celebrated KdV hierarchy. We will describe the centralizer of a Schrödinger operator  $L = -\partial^2 + u$ , for stationary potentials  $u$  subject to constraints given by the KdV hierarchy, using results of Goodearl. We present an algorithm to factor Schrödinger operators  $L_s - \lambda = -\partial^2 + u_s - \lambda$  with  $u_s$  satisfying the KdV<sub>s</sub> equation. Previous results, for hyperelliptic curves, construct factorizations as formulas using  $\theta$ -functions. The method we are presenting is effective and it points out the fact that closed formulas for factors of the Schrödinger operator over the curve can be obtained using: a global parametrization of the spectral curve (if it exists), and the subresultant formula operator. Furthermore, one can extend the coefficient field of  $L_s$  to the field of a spectral curve and then to the Liouvillian extension given by  $\Psi' = \phi_s \Psi$ , with  $\phi_s$  satisfying the Riccati equation  $\phi' + \phi^2 = u_s - \lambda$ . In this manner, we will describe the Picard–Vessiot fields of  $L_s - \lambda$ . A specialization process in  $\lambda$  can be adapted to our methods.

*Joint work with M.A. Zurro (Universidad Politécnica de Madrid, Spain) and J.J. Morales–Ruiz (Universidad Politécnica de Madrid, Spain).*

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B3 - July 14, 15:30 – 15:55

DESINGULARIZATION OF FIRST ORDER LINEAR DIFFERENCE SYSTEMS WITH RATIONAL FUNCTION COEFFICIENTS

**Moulay Barkatou**

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It is well known that for a first order system of linear difference equations with rational function coefficients, a solution that is holomorphic in some left half plane can be analytically continued to a meromorphic solution in the whole complex plane. The poles stem from the singularities of the rational function coefficients of the system. Just as for systems of differential equations, not all of these singularities necessarily lead to poles in a solution, as they might be what is called removable. In this talk, we show how to detect and remove these singularities and further study the connection between poles of solutions, removable singularities and the extension of numerical sequences at these points.

*Joint work with Maximilian Jaroschek (Johannes Kepler University Linz, Austria).*

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B3 - July 14, 16:00 – 16:25

#### SYMBOLIC COMPUTATION FOR PRESERVING CONSERVATION LAWS

**Gianluca Frasca-Caccia**

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The main goal of the field of investigation known as Geometric Integration is to reproduce, in the discrete setting, a number of geometric properties shared by the original continuous problem. Conservation laws are among a PDE's most fundamental geometric properties. In this talk a new strategy, which uses symbolic computation and is based on the fact that conservation laws are in the kernel of the Euler operator, is used to develop new methods that preserve multiple discrete conservation laws. The new schemes are numerically compared with some other schemes existing in literature.

*Joint work with Peter Hydon (University of Kent, UK).*

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B3 - July 14, 17:00 – 17:50

#### THE LINEAR MAHLER EQUATION: LINEAR AND ALGEBRAIC RELATIONS

**Boris Adamczewski**

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A Mahler function is a solution, analytic in some neighborhood of the origin, of a linear difference equation associated with the Mahler operator  $z \mapsto z^q$ , where  $q \geq 2$  is an integer. Understanding the nature of such functions at algebraic points of the complex open unit disc is an old number theoretical problem dating back to the pioneering works of Mahler in the late 1920s. In this talk, I will explain why it can be considered as totally solved now, after works of Ku. Nishioka, Philippon, Faverjon and the speaker.

*Joint work with Colin Faverjon (France).*

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B3 - July 14, 18:00 – 18:25

## A GENERALIZATION OF AN INTEGRABILITY THEOREM OF DARBOUX

**Irina Kogan**

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In Chapter I, Livre III of his monograph “Systèmes Orthogonaux” Darboux stated three integrability theorems. They provide local existence and uniqueness of solutions to systems of first order Partial Differential Equations of the type:

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha(x, u(x)), \quad i \in I_\alpha \subset \{1, \dots, n\}$$

where  $x_i$ ,  $i = 1, \dots, n$  are independent variables, and  $u_\alpha$ ,  $\alpha = 1, \dots, m$  are dependent variables and  $f_i^\alpha(x, u(x))$  are some given functions. For each dependent variable  $u^\alpha$ , the system prescribes partial derivatives in certain coordinate directions given by a subset  $I_\alpha$ . The data, near a given point  $\bar{x} \in \mathbb{R}^n$ , prescribe each of the unknown functions  $u_\alpha$  along the affine subspace spanned by the coordinate vectors complimentary to the coordinate vector defined by indices  $I_\alpha$ .

Darboux’s first theorem applies to determined systems, in which case  $|I_\alpha| = 1$  for all  $\alpha$ , while his second theorem is Frobenius’ Theorem for complete systems, in which case  $|I_\alpha| = n$  for all  $\alpha$ . The third theorem addresses the general situation where  $I_\alpha$  are arbitrary subsets varying with  $\alpha$ . Under the appropriate integrability conditions, Darboux proved his third theorem in the cases  $n = 2$  and  $n = 3$ . However, his argument does not appear to generalize in any easy manner to cases with more than three independent variables.

In the present work, we formulate and prove a theorem that generalizes Darboux’s third theorem to systems of the form

$$\mathbf{r}_i(u_\alpha)|_x = f_i^\alpha(x, u(x)), \quad i \in I_\alpha \subset \{1, \dots, n\}$$

where  $\{\mathbf{r}_i\}_{i=1}^n$  is an arbitrary local frame of vector-fields near  $\bar{x}$ . Furthermore, the data for  $u^\alpha$  can be prescribed along an arbitrary submanifold through  $\bar{x}$  transversal to the subset of vector-fields  $\{\mathbf{r}_i | i \in I_\alpha\}$ . Our proof applies to any number of independent variables and uses a nonstandard application of Picard iteration. The approach requires only  $C^1$  smoothness of the  $f_i^\alpha$  and the initial data. We note that, in the analytic case, this result can be proved using Cartan–Kähler theorem.

*Joint work with Michael Benfield (San Diego, USA) and Helge Kristian Jenssen (Pennsylvania State University, USA).*

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B3 - July 15, 14:30 – 15:20

## THE GOOD, THE BAD, AND THE UGLY: THE CARTAN ALGORITHM FOR OVERDETERMINED PDE SYSTEMS

**Jeanne Clelland**

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Cartan’s theory of exterior differential systems provides tools for analyzing spaces of local solutions to systems of PDEs that don’t fit nicely into any standard classification. In particular, the Cartan–Kahler Theorem — which generalizes the Cauchy–Kowalewski Theorem for determined systems — can often be used to compute the size of the local solution space, as long as the system itself and all initial data are assumed to be real analytic. This can be a powerful framework for analyzing the solution spaces to overdetermined PDE systems, including many that arise naturally in geometric contexts.

In this talk, we will illustrate the application of Cartan’s algorithm to the problem of finding strong Beltrami fields with nonconstant proportionality factor, i.e., vector fields  $\mathbf{u}$  on an open set  $U \subset \mathbb{R}^3$  with the property that

$$\nabla \times \mathbf{u} = f\mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0$$

for some nonconstant function  $f: U \rightarrow \mathbb{R}$ . This example illustrates both the power of the method and some significant challenges that may arise during its application.

*Joint work with Taylor Klotz (University of Colorado, Boulder, USA).*

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B3 - July 15, 15:30 – 15:55

#### DIFFERENTIAL INVARIANTS FOR TIME-LIKE CURVES AND THEIR APPLICATIONS TO A CONFORMALLY INVARIANT VARIATIONAL PROBLEM

**Emilio Musso**

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In this talk I will focus on local differential invariants of a time-like curve with respect to the group of the Lorentz conformal transformations. The invariant of lower-order is a differential form of degree four (the “conformal strain”) which, in turns, can be used to define natural parameterizations. When the conformal strain doesn’t possesses zeroes, one can integrate its fourth root along the curve. This provides the simplest conformally invariant variational problem for time-like curves. I will give some hint on how to find the critical curves and I will discuss the existence of closed critical curves.

#### Bibliography

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- [2] O. Eshkobilov & E. Musso, “A conformally invariant variational problem for time-like curves.” [ArXiv:1566997 \[mathDG22may2016\]](#).
- [3] E. Musso & L. Nicolodi, “Quantization of the conformal arclength functional on space curves.” *Communications in Analysis and Geometry*, to appear; [ArXiv:1501.04101 \[math.DG\]](#).

*Joint work with L. Nicolodi (University of Parma, Italy), A. Dzhililov (TTPU, Tashkent, Uzbekistan), and O. Eshkobilov (University of Torino, Italy).*

B3 - July 15, 16:00 – 16:25

# ON THE COMPUTATION OF SIMPLE FORMS AND REGULAR SOLUTIONS OF LINEAR DIFFERENCE SYSTEMS

**Thomas Cluzeau**

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In this talk, I will present a new algorithm for transforming any first-order linear difference system with factorial series coefficients into a simple system. Such an algorithm can be seen as a first step towards the computation of regular solutions since in a previous work we have developed an algorithm for computing regular solutions of simple linear difference system. Moreover, computing a simple form can also be used to characterize the nature of the singularity at infinity. If the singularity is regular, we are then reduced to a system of the first-kind. I will also present a direct algorithm for computing a formal fundamental matrix of regular solutions of such first-kind linear difference systems which yields an alternative to the algorithm for computing regular solutions of simple systems in the case of a regular singularity. Finally, the algorithms developed have been implemented in Maple thanks to our new package for handling factorial series. The talk will be illustrated by examples computed using our implementation.

*Joint work with Moulay Barkatou (University of Limoges; XLIM, France) and Carole El Bacha (Lebanese University, Faculty of Sciences II, Lebanon).*

B3 - July 15, 17:00 – 17:25

# ALGEBRAIC CERTIFICATES OF DISCONNECTEDNESS

**Didier Henrion**

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Given two disjoint semialgebraic subsets of a given compact semialgebraic set, we describe an algorithm that certifies that the two subsets are disconnected, i.e., that there is no continuous trajectory connecting them. The certificate of disconnectedness is a polynomial satisfying specific positivity conditions. It is computed by solving a hierarchy of convex moment-sum-of-squares semidefinite programs obtained by applying infinite-dimensional convex duality on a transport equation satisfied by measures attached to trajectories.

*Joint work with Mohab Safey El Din (Université Pierre et Marie Curie, Paris, France).*

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 B3 - July 15, 17:30 – 17:55

## DIFFERENTIAL GALOIS THEORY AND NON-INTEGRABILITY OF PLANAR POLYNOMIAL VECTOR FIELDS

**Juan J. Morales–Ruiz**

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We study a necessary condition for the integrability of the polynomials fields in the plane by means of the differential Galois theory. More concretely, as a corollary of a previous result with Ramis and Simó on Hamiltonian systems, it is proved that a necessary condition for the existence of a meromorphic first integral is that the identity component of the Galois group of the higher order variational equations around a particular solution must be abelian. We illustrate this theorem with several families of examples. A key point in these applications is to check whether a suitable primitive is or not elementary. Using a theorem by Liouville, the problem is equivalent to the existence of a rational solution of a certain first order linear equation, the Risch equation. This is a classical problem studied by Risch in 1969, and the solution is given by the “Risch algorithm”. In this way we will point out the connection of the non integrability of polynomial fields with some higher transcendent functions, like the error function.

*Joint work with Primitivo B. Acosta–Humánez (Universidad Simón Bolívar, Colombia), J. Tomás Lázaro (Universitat Politècnica de Catalunya, Spain), and Chara Pantazi (Universitat Politècnica de Catalunya, Spain).*

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 B3 - July 15, 18:00 – 18:25

 ALGORITHMIC PROOF FOR THE TRANSCENDENCE OF  $D$ -FINITE POWER SERIES

**Alin Bostan**

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Given a sequence represented by a linear recurrence with polynomial coefficients and sufficiently many initial terms, a natural question is whether the transcendence of its generating function can be decided algorithmically. The question is non trivial even for sequences satisfying a recurrence of first order. An algorithm due to Michael Singer is sufficient, in principle, to answer the general case. However, this algorithm suffers from too high a complexity to be effective in practice. We will present a recent method that we have used to treat a non-trivial combinatorial example. It reduces the question of transcendence to a (structured) linear algebra problem.

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B3 - Poster

ON STRONGLY CONSISTENT FINITE DIFFERENCE APPROXIMATIONS TO THE  
NAVIER-STOKES EQUATIONS

**Dmitry Lyakhov**

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The finite difference method is widely used for solving partial differential equations in the computational sciences. The decisive factor for its successful application is the quality of the underlying finite difference approximations. In this contribution, we present a computer algebra assisted approach to generate appropriate finite difference approximations to systems of polynomially nonlinear partial differential equations on regular Cartesian grids. The generated approximations satisfy the major quality criterion — strong consistency — which implies the preservation of fundamental algebraic properties of the system at the discrete level. This criterion admits a verification algorithm. We apply our approach to the Navier-Stokes equations and construct strongly consistent approximations. Moreover, we construct two approximations which are not only strongly consistent but also fully conservative.

*Joint work with Vladimir Gerdt (JINR, Russia) and Dominik Michels (King Abdullah University of Science and Technology, Saudi Arabia).*

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B3 - Poster

SYMBOLIC COMPUTATION FOR OPERATORS WITH MATRIX COEFFICIENTS

**Clemens Raab**

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In order to facilitate symbolic computations with systems of linear functional equations an algebraic framework for such systems is needed for effective computations in corresponding rings of operators. Normal forms of operators are a key ingredient for that.

We generalize the recently developed tensor approach from scalar equations to the matrix case, by allowing noncommutative coefficients. The tensor approach is flexible enough to cover many operators, like integral operators, that do not fit the well-established framework of skew-polynomials. Noncommutative coefficients even allow to handle systems of generic size. Normal forms rely on a confluent reduction system.

Based on our implementation of tensor reduction systems, we implemented the ring of integro-differential operators with time-delay and we worked out normal forms for those operators. We use this to partly automatize certain computations related to differential time-delay systems, e.g., Artstein's reduction of differential time-delay control systems.

*Joint work with Thomas Cluzeau (University of Limoges, France), Jamal Hossein Poor (RICAM, Austrian Academy of Sciences, Austria), Alban Quadrat (INRIA Lille – Nord Europe, France), and Georg Regensburger (Johannes-Kepler University Linz, Austria).*

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B3 - PosterDISCRETE MOVING FRAMES AND NOETHER'S FINITE DIFFERENCE CONSERVATION  
LAWS. EULER'S ELASTICA**Ana Rojo–Echeburúa**University of Kent, United Kingdom  
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Discrete moving frames have the potential to have a large spectrum of applications as well as leading to improvement regarding computations. Further, the theory of discrete moving frames provides an useful tool when solving variational problems or in the study of integrable systems. In this poster I will consider the discrete variational problem analogue to the minimisation of the integral of the curvature squared. The solutions of this equation are commonly known as Euler's elastica. I will show that the Euler–Lagrange difference equations and the Noether's difference conservation laws can be written in terms of the invariants of the action and a discrete moving frame and that the appearance of the moving frame in the expression for the conservation laws makes explicit the equivariance of the frame under the group action. I will exhibit how this formulation can allow one, to solve for the solutions in term of the original variables. By matching the use of the smooth and discrete frames, for the smooth and discrete problems respectively, I will show that it is possible to design approximations which have matching conservation laws.

*Joint work with Elizabeth Mansfield (University of Kent, UK).*

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## Workshop B4 Learning Theory

Organizers: Sébastien Bubeck – Lorenzo Rosasco – Alexandre Tsybakov

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B4 - July 13, 14:30 – 15:20

### OPTIMAL ALGORITHMS FOR SMOOTH AND STRONGLY CONVEX DISTRIBUTED OPTIMIZATION IN NETWORKS

**Francis Bach**

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In this work, we determine the optimal convergence rates for strongly convex and smooth distributed optimization in two settings: centralized and decentralized communications over a network. For centralized (ie master/slave) algorithms, we show that distributing Nesterov’s accelerated gradient descent is optimal and achieves a precision  $\varepsilon > 0$  in time  $O(\sqrt{\kappa_g}(1 + \Delta\tau) \ln(1/\varepsilon))$ , where  $\kappa_g$  is the condition number of the (global) function to optimize,  $\Delta$  is the diameter of the network, and  $\tau$  (resp. 1) is the time needed to communicate values between two neighbors (resp. perform local computations). For decentralized algorithms based on gossip, we provide the first optimal algorithm, called the multi-step dual accelerated (MSDA) method, that achieves a precision  $\varepsilon > 0$  in time  $O(\sqrt{\kappa_l}(1 + \frac{\tau}{\sqrt{\gamma}}) \ln(1/\varepsilon))$ , where  $\kappa_l$  is the condition number of the local functions and  $\gamma$  is the (normalized) eigengap of the gossip matrix used for communication between nodes. We then verify the efficiency of MSDA against state-of-the-art methods for two problems: least-squares regression and classification by logistic regression.

*Joint work with Kevin Scaman (INRIA, France), Sébastien Bubeck (Microsoft Research), Yin-Tat Lee (Microsoft Research), and Laurent Massoulié (INRIA, France).*

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B4 - July 13, 15:30 – 15:55

### SUBSPACE CLUSTERING VIA TANGENT CONES

**Rebecca Willett**

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In this talk, I will describe a novel approach to subspace clustering called “conic subspace clustering” (CSC), which offers new insight into the geometry of subspace clustering. Like many other subspace clustering methods, it operates by first estimating the affinity between pairs of points, where the affinity is one if two points are in the same subspace and zero otherwise, and then applying an off-the-self clustering method to the learned affinities. The main novelty of the proposed approach lies in the estimation of pairwise affinities. Specifically, CSC considers the convex hull of a collection of data points and the tangent cones at each sample  $x$ . The union of subspaces underlying the data imposes a specific geometry on the convex hull, so that the tangent cone at a point  $x$  has a strong association with the subspace containing  $x$ .



This insight leads to a tangent cone membership test that can be used to estimate the affinity between  $x$  and other points. In addition to describing this novel geometric perspective, this work provides deterministic and stochastic guarantees on the accuracy of the learned affinity matrix. More specifically, the test guarantees zero false positives and provides guarantees for any true positive rate. Furthermore, the true positive rate is potentially much larger than for competing methods that leverage sparsity regularization, which is particularly important in the presence of large numbers of samples from subspaces with complicated configurations. This increased true positive rate can be leveraged within an convex optimization formulation of the CSC affinity matrix estimation, leading to a computationally efficient algorithm.

*Joint work with Amin Jalali (University of Wisconsin-Madison, USA).*

B4 - July 13, 16:00 – 16:25

#### THE LOSS SURFACE OF DEEP AND WIDE NEURAL NETWORKS

**Matthias Hein**

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While the optimization problem behind deep neural networks is highly non-convex, it is frequently observed in practice that training deep networks seems possible without getting stuck in suboptimal points. It has been argued that this is the case as all local minima are close to being globally optimal. We show that this is (almost) true, in fact almost all local minima are globally optimal, for a fully connected network with squared loss and analytic activation function given that the number of hidden units of one layer of the network is larger than the number of training points and the network structure from this layer on is pyramidal.

*Joint work with Quynh Nguyen Ngoc (Saarland University, Germany).*

B4 - July 13, 17:00 – 17:25

#### LEARNING DETERMINANTAL POINT PROCESSES

**Philippe Rigollet**

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Determinantal Point Processes (DPPs) are a family of probabilistic models that have a repulsive behavior, and lend themselves naturally to many tasks in machine learning where returning a diverse set of objects is important. While there are fast algorithms for sampling, marginalization and conditioning, much less is known about learning the parameters of a DPP. In this talk, I will present recent results related to this problem, specifically

- (i) Rates of convergence for the maximum likelihood estimator: by studying the local and global geometry of the expected log-likelihood function we are able to establish rates of convergence for the MLE and give a complete characterization of the cases where these are parametric. We also give a partial description of the critical points for the expected log-likelihood.

- (ii) Optimal rates of convergence for this problem: these are achievable by the method of moments and are governed by a combinatorial parameter, which we call the cycle sparsity.
- (iii) Fast combinatorial algorithm to implement the method of moments efficiently.

The necessary background on DPPs will be given in the talk.

*Joint work with Victor-Emmanuel Brunel (Massachusetts Institute of Technology, USA), Ankur Moitra (Massachusetts Institute of Technology, USA), and John Urschel (Massachusetts Institute of Technology, USA).*

B4 - July 13, 17:30 – 17:55

#### ON THE EXPONENTIALLY WEIGHTED AGGREGATE WITH THE LAPLACE PRIOR

**Arnak Dalalyan**

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In this talk, we will present some results on the statistical behavior of the Exponentially Weighted Aggregate (EWA) in the problem of high-dimensional regression with fixed design. Under the assumption that the underlying regression vector is sparse, it is reasonable to use the Laplace distribution as a prior. The resulting estimator and, specifically, a particular instance of it referred to as the Bayesian lasso, was already used in the statistical literature because of its computational convenience, even though no thorough mathematical analysis of its statistical properties was carried out. We will present results that fill this gap by establishing sharp oracle inequalities for the EWA with the Laplace prior. These inequalities show that if the temperature parameter is small, the EWA with the Laplace prior satisfies the same type of oracle inequality as the lasso estimator does, as long as the quality of estimation is measured by the prediction loss. Extensions of the proposed methodology to the problem of prediction with low-rank matrices will be discussed as well.

*Joint work with Edwin Grappin (ENSAE/CREST, France) and Quentin Paris (HSE).*

B4 - July 13, 18:00 – 18:25

#### OPTIMAL GRAPHON ESTIMATION

**Olga Klopp**

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In the last decade, network analysis has become an important research field driven by applications. We study the twin problems of estimating the connection probability matrix of an inhomogeneous random graph and the graphon of a  $W$ -random graph. We consider classes of block constant matrices and step function graphons and establish the minimax estimation rates with respect to different distances:  $l_1$ ,  $l_2$  and the cut metric. We also extend our study to the case of unbounded graphons.

*Joint work with Nicolas Verzelen (INRIA, France).*

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 B4 - July 13, 18:30 – 18:55

## OVERLAPPING VARIABLE CLUSTERING WITH STATISTICAL GUARANTEES AND LOVE

**Florentina Bunea**

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Variable clustering is one of the most important unsupervised learning methods, ubiquitous in most research areas. In the statistics and computer science literature, most of the clustering methods lead to non-overlapping partitions of the variables. However, in many applications, some variables may belong to multiple groups, yielding clusters with overlap. It is still largely unknown how to perform overlapping variable clustering with statistical guarantees. To bridge this gap, we propose a novel Latent model-based OVERlapping clustering method (LOVE) to recover overlapping sub-groups of a potentially very large group of variables. In our model-based formulation, a cluster is given by variables associated with the same latent factor, and can be determined from an allocation matrix  $A$  that indexes our proposed latent model. We assume that some of the observed variables are pure, in that they are associated with only one latent factor, whereas the remaining majority has multiple allocations. We prove that the corresponding allocation matrix  $A$ , and the induced overlapping clusters, are identifiable, up to label switching. We estimate the clusters with LOVE, our newly developed algorithm, which consists in two steps. The first step estimates the set of pure variables, and the number of clusters. In the second step we estimate the allocation matrix  $A$  and determine the overlapping clusters. Under minimal signal strength conditions, our algorithm recovers the population level clusters consistently. Our theoretical results are fully supported by our empirical studies, which include extensive simulation studies that compare LOVE with other existing methods, and the analysis of a RNA-seq dataset.

*Joint work with Yang Ning (Cornell University, USA) and Marten Wegkamp (Cornell University, USA).*

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B4 - July 14, 14:30 – 14:55

## RANDOM MOMENTS FOR COMPRESSIVE STATISTICAL LEARNING

**Rémi Gribonval**

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We describe a general framework, *sketched statistical learning*, for memory-efficient large-scale learning. A low-dimensional *sketch* (a vector of random empirical generalized moments), that captures the information relevant to the considered learning task, is computed in one pass on the training collection. A near-minimizer of the risk is computed from the sketch using the generalized method of moments.

The framework is illustrated on sketched clustering, and sketched PCA, using empirical algorithms inspired by sparse recovery algorithms used in compressive sensing.

We provide theoretical guarantess to control the resulting generalization error, with sketch of size driven by an intrinsic measure of complexity of the learning task, which is independent of the volume of the training collection. Volume reductions of several orders of magnitude are demonstrated while preserving the overall learning quality.

*Joint work with Nicolas Keriven (Université de Rennes 1, France), Yann Traonmilin (INRIA – Rennes, France), and Gilles Blanchard (Universität Potsdam, Germany).*

B4 - July 14, 15:00 – 15:25

#### CONSISTENCY AND COMPARISON OF OBJECTIVE FUNCTIONALS IN SEMI-SUPERVISED LEARNING

**Dejan Slepcev**

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We consider a regression problem of semi-supervised learning: given real-valued labels on a small subset of data recover the function on the whole data set while taking into account the information provided by a large number of unlabeled data points. Objective functionals modeling this regression problem involve terms rewarding the regularity of the function estimate while enforcing agreement with the labels provided. We will discuss and prove which of these functionals make sense when the number of data points goes to infinity. Furthermore we will discuss distinct qualitative properties of function estimates that different regularizations lead to. In particular we will discuss regularizations motivated by p-Laplace equation and higher order fractional Laplacians.

*Joint work with Matthew Dunlop (Caltech, USA), Andrew Stuart (Caltech, USA), and Matthew Thorpe (CMU, USA).*

B4 - July 14, 15:30 – 15:55

#### PARALLELIZING SPECTRAL ALGORITHMS FOR KERNEL LEARNING

**Gilles Blanchard**

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We consider a distributed learning approach in supervised learning for a large class of spectral regularization methods in a reproducing kernel Hilbert space framework. The data set of size  $n$  is partitioned into  $m = O(n^\alpha)$  disjoint subsets which can be sent to different machines. On each subset, some spectral regularization method (belonging to a large class, including in particular Kernel Ridge Regression,  $L^2$ -boosting and spectral cut-off) is applied. The regression function  $f$  is then estimated via simple averaging. This leads to a substantial reduction in computation time with respect to using a single machine. We show that minimax optimal rates of convergence are preserved if  $m$  grows sufficiently slowly with  $n$ , the upper limit on the exponent  $\alpha$  depending on the smoothness assumptions on  $f$  and the intrinsic dimensionality.

*Joint work with Nicole Mücke (University of Potsdam, Germany).*

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B4 - July 14, 16:00 – 16:25

## MATRIX COMPLETION, SADDLEPOINTS, AND GRADIENT DESCENT

**Jason Lee**University of Southern California, USA  
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Matrix completion is a fundamental machine learning problem with wide applications in collaborative filtering and recommender systems. Typically, matrix completion are solved by non-convex optimization procedures, which are empirically extremely successful. We prove that the symmetric matrix completion problem has no spurious local minima, meaning all local minima are also global. Thus the matrix completion objective has only saddlepoints and a global minimum.

Next, we show that saddlepoints are easy to avoid for even Gradient Descent — arguably the simplest optimization procedure. We prove that with probability 1, randomly initialized Gradient Descent converges to a local minimizer. The same result holds for a large class of optimization algorithms including proximal point, mirror descent, and coordinate descent.

*Joint work with Michael Jordan (UC Berkeley), Benjamin Recht (UC Berkeley), Max Simchowitz (UC Berkeley), Rong Ge (Duke University, USA), and Tengyu Ma (Princeton University, USA).*

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B4 - July 14, 17:00 – 17:25

## DICTIONARY LEARNING: FROM LOCAL TO GLOBAL CONVERGENCE

**Karin Schnass**University of Innsbruck, Austria  
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In this talk we first give conditions that guarantee one iteration of an alternating dictionary learning scheme to contract an estimate for the desired generating dictionary towards this generating dictionary. Conversely we will provide examples of dictionaries not equal to the generating dictionary that are stable fixed points of the alternating scheme. Based on these characterisations we then propose a (cheap) replacement strategy for alternate dictionary learning to avoid local minima. Time permitting we will finally discuss how the replacement strategy can be used to automatically determine the dictionary size and sparsity level.

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B4 - July 14, 17:30 – 17:55

## PROJECTED GRADIENT DESCENT WITH NONCONVEX CONSTRAINTS

**Rina Barber**University of Chicago, USA  
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Nonconvex optimization arises in many applications of high-dimensional statistics and data analysis, where data models and regularization terms can both often exhibit nonconvexity. While

convex programs for structured signal recovery have been widely studied, comparatively little is known about the theoretical properties of nonconvex optimization methods. In this talk I will discuss the problem of projected gradient descent over nonconvex constraints, where the local geometry of the constraint set is closely tied to its convergence behavior. By measuring the local concavity of the constraint set, we can give concrete guarantees for convergence of projected gradient descent. Furthermore, by relaxing these geometric conditions, we can allow for approximate calculation of the projection step to speed up the algorithm.

*Joint work with Wooseok Ha (University of Chicago, USA).*

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B4 - July 15, 14:30 – 14:55

#### THE STATISTICAL FOUNDATIONS OF LEARNING TO CONTROL

**Benjamin Recht**

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Given the dramatic successes in machine learning and reinforcement learning over the past half decade, there has been a resurgence of interest in applying these techniques to continuous control problems in robotics, self-driving cars, and unmanned aerial vehicles. Though such applications appear to be straightforward generalizations of standard reinforcement learning, few fundamental baselines have been established prescribing how well one must know a system in order to control it. In this talk, I will discuss how one might merge techniques from statistical learning theory with robust control to derive such baselines for such continuous control. I will explore several examples that balance parameter identification against controller design and demonstrate finite sample tradeoffs between estimation fidelity and desired control performance. I will then describe how these simple baselines give us insights into shortcomings of existing reinforcement learning methodology. I will close by listing several exciting open problems that must be solved before we can build robust, safe learning systems that interact with an uncertain physical environment.

*Joint work with Ross Boczar (UC Berkeley, USA), Sarah Dean (UC Berkeley, USA), Moritz Hardt (Google Brain/UC Berkeley, USA), Tengyu Ma (Princeton University, USA), Horia Mania (UC Berkeley, USA), Andrew Packard (UC Berkeley, USA), and Stephen Tu (UC Berkeley, USA).*

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B4 - July 15, 15:00 – 15:25

#### LEARNING IN NATURE

**Nisheeth Vishnoi**

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In this talk, I will present two algorithmic connections between nature and humans that we recently discovered. The first is an equivalence of the dynamics of the slime mold and a popular algorithm in machine learning — the iteratively reweighted least squares (IRLS) method. The

second is between a distributed learning dynamics observed among social animals (including humans) and the multiplicative weights algorithm.

Not only have similar learning algorithms been independently chosen by human intelligence and nature (via evolution) to solve problems in entirely different context, these connections lead us to the first proof of correctness of the damped IRLS method and a novel distributed and low-memory implementation of the classic MWU method.

*Joint work with L. Elisa Celis (EPF Lausanne, Switzerland), Peter Krafft (MIT), and Damian Straszak (EPF Lausanne, Switzerland).*

B4 - July 15, 15:30 – 16:20

#### ESTIMATION OF SMOOTH FUNCTIONALS OF COVARIANCE OPERATORS

**Vladimir Koltchinskii**

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We discuss a problem of estimation of a smooth functional of unknown covariance operator of a mean zero Gaussian random variable in a Hilbert space based on a sample of its i.i.d. observations in a setting when either the dimension of the space, or so called effective rank of the covariance operator are allowed to be large (although much smaller than the sample size). The main example is the problem of estimation of a linear functional of unknown eigenvector of the true covariance that corresponds to its largest eigenvalue (the top principal component). In this case, in a joint work with Matthias Loeffler and Richard Nickl, we proved a minimax lower bound on the risk of an arbitrary estimator of a linear functional, developed an estimator for which this bound is attained (which is not true for a standard estimator based on the top principal component of sample covariance due to its large bias) and also proved a Berry–Esseen type bound on the accuracy of approximation of its distribution by normal. This estimator is based on a further development of a bias reduction method initially introduced in our joint work with Karim Lounici. We will also discuss a more general approach to bias reduction and efficient estimation of smooth functionals of the unknown covariance (developed in our joint work with Mayya Zhilova).

B4 - July 15, 17:00 – 17:25

#### A UNIFIED VIEW OF ENTROPY-REGULARIZED MARKOV DECISION PROCESSES

**Gergely Neu**

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We propose a general framework for entropy-regularized average-reward reinforcement learning in Markov decision processes (MDPs). Our approach is based on extending the linear-programming formulation of policy optimization in MDPs to accommodate convex regularization functions. Our key result is showing that using the conditional entropy of the joint state-action distributions as regularization yields a dual optimization problem closely resembling the Bellman optimality equations. This result enables us to formalize a number of state-of-the-art entropy-regularized reinforcement learning algorithms as approximate variants of Mirror Descent or Dual

Averaging, and thus to argue about the convergence properties of these methods. In particular, we show that the exact version of the TRPO algorithm of Schulman & al., (2015) actually converges to the optimal policy, while the entropy-regularized policy gradient methods of Mnih & al., (2016) may fail to converge to a fixed point. Finally, we illustrate empirically the effects of using various regularization techniques on learning performance in a simple reinforcement learning setup.

*Joint work with Anders Jonsson (Universitat Pompeu Fabra, Spain) and Vicenç Gómez (Universitat Pompeu Fabra, Spain).*



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**Workshop B5**  
**Random Matrices**

Organizers: Joel Tropp – Michel Ledoux – Sheehan Olver

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B5 - July 13, 14:30 – 15:20

SPECTRA OF RANDOM REGULAR AND QUASI-REGULAR GRAPHS

**Ioana Dumitriu**

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The spectra of random graphs and networks have been widely studied in the last two decades, both in theoretical contexts (relating to random matrices and universality) and practical ones (with applications to mixing, sampling, community detection). From a random matrix perspective, the “natural” random graphs have independent edges (Erdős–Renyi-like); however, regular graphs have also been shown to share many of the spectral properties of the independent-edge ones. Recently, some of these studies have been extended to quasi-regular random graphs. We will survey some of these results and potential applications to community detection.

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B5 - July 13, 15:30 – 15:55

RANDOM MATRICES, M-ESTIMATORS AND THE BOOTSTRAP

**Noureddine El Karoui**

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Random-matrices-inspired ideas and techniques can be used to understand a wide variety of problems in high-dimensional statistics, way beyond properties of correlation and covariance matrices. In this talk, I’ll discuss how RMT is a fundamental tool in understanding the properties of widely used methods such as M-estimators and the bootstrap in high-dimensional statistics. Interestingly, the corresponding analyses upend conventional statistical thinking: maximum likelihood methods are woefully inefficient (even among restricted classes of estimators) and the bootstrap typically fails to give statistically valid confidence statements, even for very simple problems. I’ll also discuss limitations of random matrix models for statistical applications.

*Joint work with Elizabeth Purdom (UC Berkeley, USA).*

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B5 - July 13, 16:00 – 16:25

STEIN’S METHOD FOR RANDOM MATRICES

**Elizabeth Meckes**

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Most of the random variables of interest which arise in random matrix theory come from random matrix ensembles in which dependence plays an important role. Stein’s method is a powerfully

robust tool for studying limiting behavior in such situations. I will describe a few applications of Stein's method in random matrix theory, for distributional results (including joint work with S. Chatterjee), and for obtaining concentration inequalities (work of Chatterjee, Tropp–Mackey, and others).

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B5 - July 13, 17:00 – 17:25

#### SOME RESULTS ON THE EIGENVECTORS OF RANDOM SYMMETRIC MATRICES

**Ke Wang**

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Eigenvectors of large matrices and graphs play an essential role in combinatorics and theoretical computer science. For instance, many properties of a graph can be deduced or estimated from its eigenvectors. It is conjectured that an eigenvector of a random symmetric matrix behaves like a random vector uniformly distributed on the unit sphere. I will talk about some recent partial results toward confirming this conjecture.

*Joint work with Sean O'Rourke (University of Colorado Boulder, USA) and Van Vu (Yale University, USA).*

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B5 - July 13, 17:30 – 17:55

#### OPTIMALITY OF THE JOHNSON-LINDENSTRAUSS LEMMA

**Jelani Nelson**

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One popular application of random matrices is the “random projections” tool offered by the Johnson-Lindenstrauss lemma. This lemma states that for any  $n, d > 1$  and any  $0 < \varepsilon < 1/2$ , and for any  $X \subset \ell_2^d$  of size  $n$ , there is a map  $f: X \rightarrow \ell_2^m$  with  $m = O(\varepsilon^{-2} \log n)$  with distortion  $1 + \varepsilon$ . All known proofs of this lemma operate by defining  $f(x) = \Pi x$  for some random matrix  $\Pi$ , with the distribution being used changing from proof to proof.

We show that this “random projection” method offered by the JL lemma is optimal for Euclidean dimensionality reduction. That is, for every  $n, d, \varepsilon$  as above where  $\varepsilon$  is not too small, there exists a set  $X \subset \ell_2^d$  with  $|X| = n$  such that any  $1 + \varepsilon$  distortion of embedding (even non-linear ones) of  $X$  into  $\ell_2^m$  must have  $m = \Omega(\varepsilon^{-2} \log n)$ .

*Joint work with Kasper Green Larhus (Aarhus University, Denmark).*

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B5 - July 14, 14:30 – 14:55

#### FINITE SIZE EFFECTS FOR SPACING DISTRIBUTIONS IN RANDOM MATRIX THEORY: CIRCULAR ENSEMBLES AND RIEMANN ZEROS

**Folkmar Bornemann**

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According to Dyson's three fold way, from the viewpoint of global time reversal symmetry there are three circular ensembles of unitary random matrices relevant to the study of chaotic spectra in quantum mechanics. These are the circular orthogonal, unitary and symplectic ensembles, denoted COE, CUE and CSE respectively. For each of these three ensembles and their thinned versions, whereby each eigenvalue is deleted independently with probability  $1 - \xi$ , we take up the problem of calculating the first two terms in the scaled large  $N$  expansion of the spacing distributions. It is well known that the leading term admits a characterisation in terms of both Fredholm determinants and Painlevé transcendents. We show that modifications of these characterisations also remain valid for the next to leading term, and that they provide schemes for high precision numerical computations. In the case of the CUE there is an application to the analysis of Odlyzko's data set for the Riemann zeros, and in that case some further statistics are similarly analyzed.

*Joint work with Peter J. Forrester (University of Melbourne, Australia) and Anthony Mays (University of Melbourne, Australia).*

B5 - July 14, 15:00 – 15:25

#### SELF-SIMILARITY IN THE SPECTRA OF RANDOM UNITARY MATRICES

**Mark Meckes**

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I will present a rigorous result roughly stating that, for a range of mesoscopic scales, the eigenvalues of an  $n \times n$  random unitary matrix are statistically indistinguishable from those of a  $2n \times 2n$  matrix, suitably rescaled. This result is inspired by a conjecture made by Coram and Diaconis in a statistical study of the relationship between eigenvalues of large random unitary matrices and zeroes of the Riemann zeta function.

*Joint work with Elizabeth Meckes (Case Western Reserve University, USA).*

B5 - July 14, 15:30 – 16:20

#### VARIATIONS ON PCA

**Amit Singer**

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Principal component analysis, or PCA, is a data analysis method used in nearly every scientific discipline. It allows the scientist to discover the main directions of variability in a dataset, known as the principal components. The principal components are used as factors to model the data, denoise and compress the data, and for visualization, clustering, or any other further analysis of the data.

Despite enormous progress in recent years on PCA in high dimensions, the theory and methods developed until now are mostly limited to simple applications involving the sample covariance matrix. Current methods for PCA are either statistically suboptimal or are not computationally

scalable for data that is corrupted by non-additive, non-Gaussian random effects, as is routinely encountered in problems involving missing data, image deconvolution, and Poissonian noise.

In this talk we will discuss two variants of PCA for which we recently developed new statistical and computational frameworks:

- 1) PCA for exponential family distributions, and
- 2) PCA from noisy linearly reduced measurements.

We will discuss applications to cryo-electron microscopy, X-ray free electron lasers, low-rank matrix completion, and noisy deconvolution that motivated this work.

*Joint work with Lydia Liu (Princeton University, USA), William Leeb (Princeton University, USA), Edgar Dobriban (Stanford University, USA), Joakim Anden (Princeton University, USA), Tejal Bhamre (Princeton University, USA), and Teng Zhang (University Central Florida, USA).*

B5 - July 14, 17:00 – 17:25

#### CONCENTRATION FOR COULOMB GASES AND COULOMB TRANSPORT INEQUALITIES

**Djalil Chafaï**

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This talk will present a recent joint work with Mylène Maïda and Adrien Hardy on the non-asymptotic behavior of Coulomb gases in dimension two and more. Such gases are modeled by an exchangeable Boltzmann–Gibbs measure with a singular two-body interaction. We obtain concentration of measure inequalities for the empirical distribution of such gases around their equilibrium measure, with respect to bounded Lipschitz and Wasserstein distances. This implies macroscopic as well as mesoscopic convergence in such distances. In particular, we improve the concentration inequalities known for the empirical spectral distribution of Ginibre random matrices. Our approach is remarkably simple and bypasses the use of renormalized energy. It crucially relies on new inequalities between probability metrics, including Coulomb transport inequalities which can be of independent interest.

*Joint work with Mylène Maïda (Université de Lille 1, France) and Adrien Hardy (Université de Lille 1, France).*

B5 - July 14, 17:30 – 17:55

#### EIGENVALUE ATTRACTION

**Ramis Movassagh**

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We prove that the complex conjugate (c.c.) eigenvalues of a smoothly varying real matrix attract. We offer a dynamical perspective on the motion and interaction of the eigenvalues in the complex plane, derive their governing equations and discuss applications. C.c. pairs closest to the real axis, or those that are ill-conditioned, attract most strongly and can collide to become

exactly real. We apply the results to various special cases including the Hatano–Nelson model and provide various numerical illustrations. It is our hope that the simple dynamical perspective herein might help better understanding of the aggregation and low density of the eigenvalues of real random matrices on and near the real line respectively.

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B5 - July 15, 14:30 – 14:55

THE SMALLEST SINGULAR VALUE OF ADJACENCY MATRICES OF RANDOM REGULAR DIGRAPHS

**Konstantin Tikhomirov**  
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Let  $A_n$  be the adjacency matrix of a random  $d$ -regular directed graph on  $n$  vertices, distributed uniformly on the set of all  $d$ -regular digraphs without multiple edges. By combining a specially developed chaining method with linear algebraic arguments and Littlewood–Offord-type anti-concentration inequalities, we derive a lower bound for the smallest singular value of matrices of the form  $A_n - z\text{Id}_n$ . Our result gives a bound  $s_{\min} \geq n^{-C}$  with probability at least  $1 - d^{-c}$ , for any  $d$  larger than an absolute constant and less than  $n/\log n$ . This result can be viewed as a step towards proving the circular law for uniform random digraphs in the “sparse” regime, complementing a recent result of Cook who established the circular law for  $d$  at least polylogarithmic in  $n$ .

*Joint work with A.E. Litvak (University of Alberta, Canada), A. Lytova (University of Alberta, Canada), N. Tomczak-Jaegermann (University of Alberta, Canada), and P. Youssef (Université de Paris – Diderot, France).*

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B5 - July 15, 15:00 – 15:25

FREE COMPONENT ANALYSIS

**Raj Rao**  
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We describe a method for unmixing mixtures of ‘freely’ independent random variables in a manner analogous to the independent component analysis (ICA) based method for unmixing independent random variables from their additive mixture. Random matrices play the role of free random variables in this context so the method we develop, which we call Free component analysis (FCA), unmixes matrices from an additive mixture of matrices. We describe the theory — the various ‘contrast functions’, computational methods and compare FCA to ICA on data derived from real-world experiments.

*Joint work with Hao Wu (University of Michigan, USA).*

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B5 - July 15, 15:30 – 15:55

UNIVERSALITY IN NUMERICAL COMPUTATIONS WITH RANDOM DATA

**Thomas Trogdon**

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This talk will concern recent progress on the statistical analysis of numerical algorithms with random initial data. In particular, with appropriate randomness, the fluctuations of the iteration count (halting time) of numerous numerical algorithms have been demonstrated to be universal, i.e., independent of the distribution on the initial data. This phenomenon has given new insights into random matrix theory. Furthermore, recent estimates from random matrix theory allow for fluctuation limit theorems for simple algorithms and halting time estimates for others.

*Joint work with Percy Deift (New York University, USA) and Govind Menon (Brown University, USA).*

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B5 - July 15, 16:00 – 16:25

NOVEL COMPUTATIONS WITH RANDOM MATRIX THEORY AND JULIA

**Alan Edelman**

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Over the many years of reading random matrix papers, it has become increasingly clear that the phenomena of random matrix theory can be difficult to understand in the absence of numerical codes to illustrate the phenomena. (We wish we could require that all random matrix papers that lend themselves to computing include a numerical simulation with publicly available code.) Of course mathematics exists without numerical experiments, and all too often a numerical experiment can be seen as an unnecessary bother. On a number of occasions, however, the numerical simulations themselves have an interesting twist of their own. This talk will illustrate a few of those simulations and illustrate why in particular the Julia computing language is just perfect for these simulations. Some topics we may discuss:

1. “Free” Dice.
2. Tracy Widom ( $O(n^{1/3})$ ).
3. Smallest Singular Value (High Parallel Monte Carlo).
4. Jacobians of Matrix Factorizations (Autodiff: Derivatives w/o Symbolic nor Finite Derivatives).

*Joint work with Bernie Wang (Amazon, USA).*

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B5 - PosterNEW EXPRESSIONS FOR THE EXTREME EIGENVALUES OF  $\beta$  RANDOM MATRICES**Plamen Koev**

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We present new identities for the hypergeometric function of a matrix argument of parameter  $\beta$ , which lead to new expressions for the extreme eigenvalues of the  $\beta$  random matrix ensembles — Laguerre, Wishart, and MANOVA.

The new expressions have lower computational complexity than the existing ones. All known expressions are in terms of the hypergeometric function of a matrix argument, an infinite series of Jack functions summed over all integer partitions. Using algebraic identities for the hypergeometric function and we are able to reduce the set of partitions over which the summation needs to take place and in certain cases (e.g., the trace of the  $\beta$  Wishart matrix) derive new expressions which require the summation over partitions of just one part.

These new expressions allow for substantial computational savings that are at least an order of magnitude faster and more than that for “spiked” covariances.

*Joint work with Raymond Kan (University of Toronto, Canada).*

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**Workshop B6****Multiresolution and Adaptivity in Numerical PDEs**Organizers: Pedro Morin – Rob Stevenson – Christian Kreuzer

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B6 - July 13, 14:30 – 14:55

**LOCAL ESTIMATES FOR THE DISCRETE  $(p)$ -HARMONIC FUNCTIONS FOR FULLY ADAPTIVE MESHES****Lars Diening**

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It is well known that harmonic functions and  $p$ -harmonic functions have higher interior regularity. In 1957 De Giorgi introduced a new technique that allows for example to estimate the maximum of the solution on a ball by an mean integral of the solution on an enlarged ball. A similar result holds for  $p$ -harmonic functions. The proof is based on subtle Cacciopoli estimates using truncation operators. In this talk we present similar estimates for discretely harmonic and  $p$ -harmonic functions. Our solutions can be scalar valued as well as vector valued, which makes a big difference for  $p$ -harmonic functions.

Various results already exist in this direction for harmonic functions (e.g., Schatz–Wahlbin '95). However, the main obstacle in this direction even in the linear case is adaptivity. All of the results obtained so far, require that the mesh size does not vary too much locally.

Our approach differs in such that we allow for arbitrary highly graded meshes (still shape regular). However, our approach uses certain properties of the Lagrange basis functions. This restrict our approach at the moment to acute meshes and linear elements. The proof of our result is based on a discretized version of the De Giorgi technique.

*Joint work with Toni Scharle (University of Oxford, UK).*

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B6 - July 13, 15:00 – 15:25

**IMPROVED REGULARITY ESTIMATES FOR SOLUTIONS TO THE  $p$ -POISSON EQUATION****Markus Weimar**

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It is well-known that the rate of convergence of numerical schemes which aim to approximate solutions to operator equations is closely related to the maximal regularity of these solutions in certain scales of smoothness spaces of Sobolev and Besov type. For linear elliptic PDEs on Lipschitz domains, a lot of results in this direction already exist. In contrast, it seems that not too much is known for nonlinear problems.

In this talk, we are concerned with the  $p$ -Laplace operator which has a similar model character for quasi-linear equations as the ordinary Laplacian for linear problems. It finds applications in models, e.g., for turbulent flows of a gas in porous media, radiation of heat, as well as in



non-Newtonian fluid theory. We discuss a couple of local regularity estimates for the gradient of the unknown solutions. These assertions are then used to derive global smoothness properties by means of wavelet-based proof techniques. Finally, the presented results imply that adaptive algorithms are able to outperform (at least asymptotically) their commonly used counterparts based on uniform refinement.

The material extends assertions which were obtained earlier in joint work with S. Dahlke, L. Diening, C. Hartmann, and B. Scharf, *Nonlinear Anal.* 130 (2016), 298–329.

B6 - July 13, 15:30 – 16:20

#### TRANSPORT AND ADAPTIVITY

**Wolfgang Dahmen**

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Linear transport equations arise as limits of convection diffusion equations and form core constituents of kinetic models of Boltzmann type. We begin with indicating the importance of stable variational formulations of transport equations in such contexts. We discuss, in particular, the relevance of efficient and reliable a posteriori error bounds that have not been available so far. We then present some core ingredients of deriving such error bounds. We highlight essential distinctions from elliptic type problems which are mainly due to the anisotropic structure of the relevant function spaces arising in connection with transport problems. In particular, due to the fact that the a posteriori error indicators do not depend explicitly on mesh size parameters, it becomes much more involved to show that refinements, based on typical bulk-criteria, result in a fixed error reduction rate. We indicate some of the key ingredients and conclude with a short outlook on further consequences.

*Joint work with Felix Gruber (RWTH Aachen, Germany), Olga Mula (Paris Dauphine University), and Rob Stevenson (University of Amsterdam).*

B6 - July 13, 17:00 – 17:25

#### ERROR ANALYSIS OF A VARIATIONAL MULTISCALE STABILIZATION FOR CONVECTION-DOMINATED DIFFUSION EQUATIONS IN 2D

**Mira Schedensack**

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The talk formulates a stabilized quasi-optimal Petrov–Galerkin method for singularly perturbed convection-diffusion problems based on the variational multiscale method. The stabilization is of Petrov–Galerkin type with a standard finite element trial space and a problem-dependent test space based on precomputed fine-scale correctors. The exponential decay of these correctors and their localisation to local patch problems, which depend on the direction of the velocity field and the singular perturbation parameter, is rigorously justified. Under moderate assumptions, this stabilization guarantees stability and quasi-optimal rate of convergence for arbitrary mesh Peclet numbers on fairly coarse meshes at the cost of additional inter-element communication.

*Joint work with Guanglian Li (Universität Bonn, Germany) and Daniel Peterseim (Universität Bonn, Germany).*

B6 - July 13, 17:30 – 17:55

#### ADAPTIVE FINITE ELEMENT METHODS FOR UNRESOLVED DIFFUSION COEFFICIENTS

**Andrea Bonito**

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Elliptic partial differential equations (PDEs) with discontinuous diffusion coefficients occur in application domains such as diffusions through porous media, electro-magnetic field propagation on heterogeneous media, and diffusion processes on rough surfaces.

In most applications, the discontinuities do not lie on the boundaries of the cells in the initial triangulation. Rather, the discontinuities occur on curves, surfaces, or manifolds, and could even be unknown beforehand. One of the obstacles to treating such discontinuity problems is that the usual perturbation theory for elliptic PDEs assumes bounds for the distortion of the coefficients in the  $L_\infty$  norm and this in turn requires that the discontinuities are matched exactly when the coefficients are approximated.

We present a new approach based on distortion of the coefficients in an  $L_q$  norm with  $q < \infty$ , which therefore does not require the exact matching of the discontinuities. We then use this new distortion theory to formulate new adaptive finite element methods (AFEMs) for such discontinuity problems. We show that such AFEMs are optimal in the sense of distortion versus number of computations, and report insightful numerical results supporting our analysis.

If time permits, we will proceed further and discuss how this perturbation theory can be used in turn to derive stability estimates for parameter recovery processes in diffusion problems.

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*Joint work with Ronald A. DeVore (Texas A&M University, USA) and Ricardo H. Nochetto (University of Maryland, USA).*

B6 - July 13, 18:00 – 18:25

#### COMPUTATION OF LOCAL AND QUASI-LOCAL EFFECTIVE DIFFUSION TENSORS IN ELLIPTIC HOMOGENIZATION

**Daniel Peterseim**

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The talk discusses a re-interpretation of existing multiscale methods by means of a discrete integral operator acting on standard finite element spaces. The exponential decay of the involved integral kernel motivates the use of a diagonal approximation and, hence, a localized piecewise constant effective coefficient. This local model turns out to be appropriate when the localized coefficient satisfies a certain homogenization criterion, which can be verified a posteriori. An a priori error analysis of the local model is presented and illustrated in numerical experiments.

## Bibliography

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*Joint work with Dietmar Gallistl (Karlsruhe Institute of Technology, Germany).*

B6 - July 13, 18:30 – 18:55

### $H^1$ -STABILITY OF THE $L^2$ -PROJECTION AND APPLICATIONS TO ADAPTIVE METHODS

**Fernando Gaspoz**

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The  $L^2$ -projection onto discrete spaces plays an essential role in the analysis of finite element discretizations. On uniform grids  $H^1$ -stability of the  $L^2$ -projection can easily be deduced by an inverse estimate. This simple proof hinges on the fact that the minimal mesh-size is comparable to the maximal mesh-size. We provide a technique that sidestep this restriction and prove the stability in  $H^1$  of the  $L^2$ -projection for piecewise continuous Finite Element Spaces for a class of adaptive meshes. We also present some new applications of this estimate.

*Joint work with Claus-Justus Heine (University of Stuttgart, Germany) and Kunibert Siebert (University of Stuttgart, Germany).*

B6 - July 14, 14:30 – 14:55

### GUARANTEED AND ROBUST A POSTERIORI BOUNDS FOR LAPLACE EIGENVALUES AND EIGENVECTORS

**Martin Vohralik**

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We derive a posteriori error estimates for numerical approximation of the Laplace eigenvalue problem with a homogeneous Dirichlet boundary condition. In particular, upper and lower bounds for an arbitrary simple eigenvalue are given. These bounds are guaranteed, fully computable, and converge with optimal speed to the given exact eigenvalue, under a separation condition from the surrounding smaller and larger exact eigenvalues that we can check in practice. Guaranteed, fully computable, optimally convergent, and polynomial-degree robust bounds

on the energy error in the approximation of the associated eigenvector are derived as well, under the same hypotheses. Remarkably, there appears no unknown (solution-, regularity-, or polynomial-degree-dependent) constant in our theory. Inexact algebraic solvers are taken into account, so that the estimates are valid on each iteration and can serve for design of adaptive stopping criteria for eigenvalue solvers. The framework can be applied to conforming, nonconforming, discontinuous Galerkin, and mixed finite element approximations of arbitrary polynomial degree. We illustrate it numerically on a set of test problems.

*Joint work with Eric Cances (École des Ponts, France), Genevieve Dusson (University Paris VI, France), Yvon Maday (University Paris VI, France), and Benjamin Stamm (RWTH Aachen, Germany).*

B6 - July 14, 15:00 – 15:25

## RELIABLE COMPUTATIONS AND SELF ADAPTED METHODS IN NONLINEAR PHENOMENA

**Charalambos Makridakis**

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The computation of singular phenomena (shocks, defects, dislocations, interfaces, cracks) arises in many complex systems. For computing such phenomena, it is natural to seek methods that are able to detect them and to devote the necessary computational resources to their accurate resolution. At the same time, we would like to have mathematical guarantees that our computational methods approximate physically relevant solutions. Our purpose in this talk is to review results and discuss related computational challenges for such nonlinear problems modelled either by PDEs or are a result of Micro / Macro adaptive modelling methods.

B6 - July 14, 15:30 – 15:55

## ON THE ERROR CONTROL FOR FULLY DISCRETE APPROXIMATIONS OF THE TIME-DEPENDENT STOKES EQUATION

**Fotini Karakatsani**

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We consider fully discrete finite element approximations to the time-dependent Stokes system. The space discretization is based on popular stable spaces, including Crouzeix-Raviart and Taylor-Hood finite element methods. Implicit Euler is applied for the time discretization. The finite element spaces are allowed to change with time steps and the projection steps include alternatives that are hoped to cope with possible numerical artifices and the loss of the discrete incompressibility of the schemes. The final estimates are of optimal order in  $L^\infty(L^2)$  for the velocity error.

*Joint work with Eberhard Bänsch (University of Erlangen, Germany) and Charalambos Makridakis (University of Sussex, UK).*

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B6 - July 14, 16:00 – 16:25

APOSTERIORI ERROR ANALYSIS FOR  $hp$ -DISCONTINUOUS GALERKIN DISCRETIZATIONS OF PARABOLIC PROBLEMS

**Omar Lakkis**

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We consider fully discrete space–time approximations of abstract linear parabolic partial differential equations (PDEs) consisting of an  $hp$ -version discontinuous Galerkin time stepping scheme in conjunction with standard (conforming) Galerkin discretizations in space. We derive computable a posteriori error bounds in the  $L_\infty(L_2)$ - and  $L_2(H_0^1)$ -type norms based on a novel space-time reconstruction approach.

*Joint work with Emmanuil H. Georgoulis (University of Leicester, GB, and National Technical University of Athens, GR) and Thomas P. Wihler (Universität Bern, CH).*

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B6 - July 14, 17:00 – 17:25

A POSTERIORI ERROR ESTIMATES IN  $L^2(H^1) \cap H^1(H^{-1})$  AND  $L^2(H^1)$ -NORMS FOR HIGH-ORDER DISCRETIZATIONS OF PARABOLIC PROBLEMS

**Iain Smears**

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We present a posteriori error estimates in parabolic energy norms for space-time discretisations based on arbitrary-order conforming FEM in space and arbitrary-order discontinuous Galerkin methods in time. Using the heat equation as a model problem, we first consider a posteriori error estimates in a norm of  $L^2(H^1) \cap H^1(H^{-1})$ -type that is extended to the nonconforming discrete space. We construct a flux equilibration by solving at each time-step some local mixed finite element problems posed on the patches of the mesh, which yields estimators with guaranteed upper bounds on the error, and locally space-time efficient lower bounds with respect to this extended norm. Furthermore, the efficiency constants are robust with respect to the discretisation parameters, including the polynomial degrees in both space and time, and also with respect to refinement and coarsening between time-steps, thereby removing the need for the transition conditions required in earlier works. In the last part of the talk, we consider the same flux equilibration in the context of  $L^2(H^1)$ -norm a posteriori error estimation, where we show that in the practically relevant situation where the time-step size  $\tau \gtrsim h^2$  the mesh-size, then the spatial part of the estimators have the additional feature of being locally efficient with respect to the  $L^2(H^1)$ -norm of the error plus the temporal jumps. Our analysis of efficiency in  $L^2(H^1)$ -norms thus removes the more severe time-step and mesh size conditions required previously in the literature.

*Joint work with Alexandre Ern (ENPC, France) and Martin Vohralik (INRIA Paris, France).*

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 B6 - July 14, 17:30 – 17:55

## NEAR-BEST APPROXIMATION ON ADAPTIVE PARTITIONS

**Peter Binev**

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We consider adaptive approximation of a function on a given domain  $\Omega$  using piecewise polynomial functions. To study this approximation, the partitioning of  $\Omega$  is related to building a binary tree  $T$  whose leaves represent the elements of the partition. The problem of approximating a function is presented then as the process of identifying a tree  $T$  and the related approximating polynomials  $P(\Delta)$  for each of its leaves  $\Delta$ . We consider two cases of adaptivity: the h-adaptivity in which the polynomials  $P(\Delta)$  are of the same order and the hp-adaptivity in which the orders of  $P(\Delta)$  may vary for different  $\Delta$ -s. We use error functionals to drive the adaptive process and prove that the presented algorithms provide a near-best approximation in both cases.

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 B6 - July 14, 18:00 – 18:25

## QUARKLET FRAMES IN ADAPTIVE NUMERICAL SCHEMES

**Stephan Dahlke**

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This talk is concerned with the design of adaptive numerical schemes for operator equations that are based on subatomic (quarkonial) decompositions. Besides the usual space refinement, in quarkonial decompositions also a polynomial enrichment is included. We construct compactly supported, piecewise polynomial functions whose dilates and translates (quarklets) generate frames for Sobolev spaces. All frame elements except those on the coarsest level have vanishing moment properties. As a consequence, the matrix representation of elliptic operator equations in quarklet frame coordinates is compressible, which is a first important step towards the design of adaptive algorithms.

*Joint work with Philipp Keding (Philipps-University of Marburg) and Thorsten Raasch (Johannes Gutenberg-University of Mainz).*

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 B6 - July 15, 14:30 – 14:55

## ADAPTIVE HIERARCHICAL B-SPLINES: CONVERGENCE AND OPTIMALITY

**Ricardo H. Nochetto**

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We present and analyze an adaptive hierarchical B-splines method, of any (fixed) order and maximal regularity, for linear elliptic equations. We first set the framework for approximation

with hierarchical B-splines, and next develop an a posteriori error estimator based on solutions of discrete local problems on stars. We next propose a novel refinement strategy to increase the local resolution of the spaces and derive a contraction property for Dörfler's marking which implies linear convergence. We also study the complexity of our refinement strategy in terms of marked cells, as well as the optimal relation of energy error vs degrees of freedom for a suitable nonlinear class of functions.

*Joint work with Pedro Morin (Universidad Nacional del Litoral, Argentina) and M. Sebastian Pauletti (Universidad Nacional del Litoral, Argentina).*

B6 - July 15, 15:00 – 15:25

# OPTIMAL CONVERGENCE RATES FOR ADAPTIVE FEM FOR COMPACTLY PERTURBED ELLIPTIC PROBLEMS

**Dirk Praetorius**

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We consider adaptive FEM for problems, where the corresponding bilinear form is symmetric and elliptic up to some compact perturbation. We prove that adaptive mesh-refinement is capable of overcoming the preasymptotic behavior and eventually leads to convergence with optimal algebraic rates. As an important consequence of our analysis, one does not have to deal with the a priori assumption that the underlying meshes are sufficiently fine. In particular, our analysis covers adaptive mesh-refinement for the Helmholtz equation from where our interest originated.

*Joint work with Alex Bespalov (University of Birmingham, UK) and Alexander Haberl (TU Wien, Austria).*

B6 - July 15, 15:30 – 15:55

# MIXED ADAPTIVE FINITE ELEMENT DISCRETIZATION OF LINEAR ELLIPTIC EQUATIONS IN NONDIVERGENCE FORM

**Dietmar Gallistl**

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This talk discusses formulations of second-order elliptic partial differential equations in nondivergence form with Cordes coefficients on convex domains as equivalent variational problems. These formulations enable the use of standard finite element techniques for variational problems in subspaces of  $H^2$  as well as mixed finite element methods from the context of fluid computations. Besides the immediate quasi-optimal a priori error bounds, the variational setting allows for a posteriori error control with explicit constants and adaptive mesh-refinement. The convergence of an adaptive algorithm is proved. Numerical results on uniform and adaptive meshes are included.

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 B6 - July 15, 16:00 – 16:25

 ADAPTIVE FINITE ELEMENT METHODS FOR  $C^2$  SURFACES

**Alan Demlow**

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FEM for elliptic PDE on smooth surfaces involve approximating both the surface itself (leading to a “geometric” consistency error) and the PDE (leading to a standard Galerkin error). The behavior of the geometric error depends heavily on the choice of surface representation. Parametric representations based on mappings from Euclidean domains are more flexible and generally easier to implement, but lead to a lower-order (larger) geometric error. Implicit representations based on a closest-point projection are less flexible because they require a  $C^2$  surface and because the closest-point projection is not usually analytically computable. However, the geometric error derived from such representations is of higher order due to special properties of the closest-point projection. Because existing estimators based on the closest-point projection require actually computing it, non-computability of the closest-point projection becomes critical when carrying out adaptive computations. In this talk we merge these two perspectives by constructing estimators which preserve many of the best properties of closest-point and parametric estimators and discuss behavior of AFEM based on our new estimators.

*Joint work with Andrea Bonito (Texas A&M University, USA).*

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B6 - July 15, 17:00 – 17:50

## OSCILLATION IN A POSTERIORI ERROR ESTIMATION

**Andreas Veese**

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In a posteriori error estimation, one devises computationally accessible estimators for the error of an approximate PDE solution. Such estimators allow evaluating the approximation quality and are also used to guide adaptive mesh refinement.

In the available approaches to a posteriori error estimation, the estimators are equivalent to the approximation error, up to so-called oscillation terms. These terms vanish for data resolved by the underlying mesh, but they can be arbitrarily bigger than the error. Oscillation was believed to be exclusively the price of (a better) computability and to converge at least as fast as the error. A remarkable example of Cohen, DeVore and Nochetto (2012) shows that both beliefs are wrong for the oscillation terms in the available approaches.

This talk presents a new approach to a posteriori error estimation and applications. In contrast to preceding ones, the arising, new oscillation terms are bounded by the error. The new twist is an interpolation operator that splits the residual into an approximate residual with oscillation-free data and second part concerning the not yet resolved data.

*Joint work with Christian Kreuzer (Ruhr-Universität Bochum, Germany).*



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B6 - July 15, 18:00 – 18:25

## RATE OPTIMAL ADAPTIVITY FOR NON-SYMMETRIC FEM/BEM COUPLING

**Michael Feischl**

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We develop a framework which allows us to prove the essential general quasi-orthogonality for the non-symmetric Johnson–Nédélec finite element/boundary element coupling. General quasi-orthogonality was first proposed in [Carstensen, Feischl, Page, Praetorius 2014] as a necessary ingredient of optimality proofs and is the major difficulty on the way to prove rate optimal convergence of adaptive algorithms for many strongly non-symmetric problems. The proof exploits a new connection between the general quasi-orthogonality and  $LU$ -factorization of infinite matrices. We then derive that a standard adaptive algorithm for the Johnson–Nédélec coupling converges with optimal rates. The developed techniques are fairly general and can most likely be applied to other problems like Stokes equation.

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**Workshop B7****Numerical Linear Algebra**Organizers: Froilan Dopico – Alex Townsend – Volker Mehrmann

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B7 - July 13, 14:30 – 15:00

**NUMERICAL OPTIMIZATION STRATEGIES FOR LARGE-SCALE (CONSTRAINED,  
COUPLED) MATRIX/TENSOR FACTORIZATION****Lieven De Lathauwer**

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We give an overview of recent developments in numerical optimization-based computation of tensor decompositions. By careful exploitation of tensor product structure in methods such as quasi-Newton and nonlinear least squares, good convergence is combined with fast computation. A modular approach extends the computation to coupled factorizations and structured factors. Compact representations (polyadic, Tucker, ...) of large data sets may be obtained by stochastic optimization, randomization, compressed sensing, ... Careful exploitation of the representation structure allows us to scale the numerical algorithms to large problem size. The discussion is illustrated with application examples.

**Bibliography**

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- [2] N. Vervliet, O. Debals, L. Sorber, M. Van Barel, & L. De Lathauwer, “Tensorlab 3.0.” Available online, (2016) <http://www.tensorlab.net/>

*Joint work with Nico Vervliet (KU Leuven, Belgium), Otto Debals (KU Leuven, Belgium), Laurent Sorber (KU Leuven, Belgium) (now Forespell, Belgium), and Marc Van Barel (KU Leuven, Belgium).*

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B7 - July 13, 15:00 – 15:30

**ON THE COMPLEXITY OF SOLVING ELLIPTIC PDES USING LOW-RANK TENSOR  
APPROXIMATIONS****Markus Bachmayr**

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In the construction of solvers for high-dimensional problems based on hierarchical tensor representations, one faces a trade-off between ensuring convergence and retaining small ranks of arising intermediate quantities. When approximately solving PDEs, in addition, low-rank approximation errors need to be balanced with discretization errors. We give an overview of available results on the complexity of solvers for hierarchical tensor representations, as well as of some open questions. We also consider what can or cannot be gained by tensor methods applied to different classes of high-dimensional problems.

*Joint work with Albert Cohen (UPMC – Paris VI, France), Wolfgang Dahmen (RWTH Aachen, Germany), and Reinhold Schneider (TU Berlin, Germany).*

B7 - July 13, 15:30 – 16:00

#### ORTHOGONAL TENSORS AND THEIR EXTREMAL SPECTRAL PROPERTIES

**André Uschmajew**

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Let  $\|X\|_2$ ,  $\|X\|_F$  and  $\|X\|_*$  denote the spectral, Frobenius and nuclear norm of an  $n_1 \times \cdots \times n_d$  tensor  $X$ . The sharp upper bounds for  $\|X\|_F/\|X\|_2$  and  $\|X\|_*/\|X\|_F$  seem to be unknown for many configurations of dimensions  $(n_1, \dots, n_d)$  with  $d \geq 3$ . Assuming  $n_1 \leq \cdots \leq n_d$ , a trivial upper bound for both ratios is  $\sqrt{n_1 \cdots n_d}$ . It is sharp for matrices ( $d = 2$ ) and achieved if and only if  $X$  is a multiple of a matrix with pairwise orthonormal rows. Using a natural definition of orthogonal higher-order tensors, we can generalize this fact in the sense that orthogonality is necessary (up to scaling) and sufficient to achieve the trivial upper bound for both ratios. However, as it turns out, orthogonal tensors do not necessarily exist for every configuration  $(n_1, \dots, n_d)$ . When  $d = 3$ , the question of existence of real orthogonal tensors admits an equivalent algebraic formulation known as Hurwitz' problem on composition formulas for bilinear forms. A classical result (due to Hurwitz) then implies that real  $n \times n \times n$  orthogonal tensors only exist for  $n = 1, 2, 4, 8$ . Surprisingly, the situation is different in complex spaces: unitary  $n \times n \times n$  do not exist for any  $n \geq 2$ .

*Joint work with Zhening Li (University of Portsmouth), Yuji Nakatsukasa (University of Oxford, UK), and Tasuku Soma (University of Tokyo).*

B7 - July 13, 16:00 – 16:30

#### POLYNOMIAL ROOT-FINDING USING COMPANION MATRICES

**Fernando De Terán**

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The use of companion matrices for computing the roots of scalar polynomials is a standard approach (it is, for instance, the one followed by the command `roots` in MATLAB). It consists in computing the roots of a scalar polynomial as the eigenvalues of a companion matrix. In this talk, I will review several numerical and theoretical issues on this topic. I will pay special attention to the backward stability of solving the polynomial root-finding problem using companion matrices.

More precisely, the question is the following: Even if the computed eigenvalues of the companion matrix are the eigenvalues of a nearby matrix, does this guarantee that they are the roots of a nearby polynomial? Usually, the companion matrix approach focuses on monic polynomials, since one can always divide by the leading coefficient, if necessary. But, is it enough for the backward stability issue to focus on monic polynomials? I will also pay attention to some other (more theoretical) questions like: How many companion matrices are there and what do they look like?

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B7 - July 13, 17:00 – 17:30

#### AVERAGING POSITIVE DEFINITE MATRICES

**Pierre-Antoine Absil**

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This talk concerns the problem of averaging a collection of symmetric positive-definite (SPD) matrices. Generally speaking, averaging methods are required notably to aggregate several noisy measurements of the same object, or to compute the mean of clusters in k-means clustering algorithms, or as a subtask in higher-level tasks such as curve fitting. The problem of averaging SPD matrices arises for example in medical imaging (denoising and segmentation tasks in Diffusion Tensor Imaging), mechanics (elasticity tensor computation), and in video tracking and radar detection tasks. Among several possible definitions for the mean of SPD matrices (including the straightforward arithmetic mean), the “Karcher mean” (specifically the least-squares mean in the sense of the so-called affine-invariant metric) is of widespread interest in the research literature, since it possesses several pleasant properties while being challenging to compute. In this talk, we will review recent advances in iterative methods that converge to the Karcher mean, and in methods that approach it using limited resources.

*This work is in collaboration with Xinru Yuan and Kyle Gallivan (Florida State University, USA), Wen Huang (Rice University, USA), and Estelle Massart and Julien Hendrickx (Université Catholique de Louvain, Belgium).*

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B7 - July 13, 17:30 – 18:00

#### EIGENVALUE OPTIMIZATION WITH APPLICATIONS TO SEMIDEFINITE PROGRAMS AND RECTANGULAR EIGENVALUE PROBLEMS

**Emre Mengi**

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The majority of interest into eigenvalue optimization has originated from its intimate connection with convex semidefinite programs. This interest has diminished to a degree after the invention of interior point methods, which made numerical solutions of small convex semidefinite programs possible. Nevertheless to this day the convex semidefinite programs with large matrix variables or with large number of constraints stand as a challenge. On the other side, nonconvex

eigenvalue optimization problems have been explored especially after late 1980s, the research in this direction has concentrated on particular problems for instance motivated by control theory and depending on only a few optimization variables.

This talk starts with a brief review of a unified algorithm for nonconvex eigenvalue optimization problems over a few variables. The algorithm is based on global piece-wise quadratic models for the eigenvalue functions. It works very well in practice provided bounds on second derivatives of eigenvalue functions are available analytically or heuristically.

The second part features a greedy subspace framework for eigenvalue optimization problems involving large matrices. At every iteration the subspace framework solves a projected eigenvalue optimization problem onto a small subspace, and expands the subspace with the addition of the eigenvectors at the optimal points of the small problem. The proposed framework converges superlinearly both in theory and in practice.

The third part concerns the adaption of the subspace framework for convex semidefinite programs with large matrix variables. Here we benefit from the eigenvalue optimization characterization of the dual of a semidefinite program. The small problems are solved by interior point methods. One difficulty is to start with a subspace that leads to a feasible semidefinite program, we describe an approach to overcome this difficulty.

Finally, we focus on large generalized rectangular eigenvalue problems whose spectra consist of a few imaginary eigenvalues. Such problems arise for instance from robust stability considerations of dissipative Hamiltonian systems. Extraction of their imaginary eigenvalues can be viewed as a nonconvex eigenvalue optimization problem depending on one variable, on which we employ the subspace framework. The small projected problems are nonconvex and solved by the unified algorithm.

*Joint work with Nicat Aliyev (Koc University, Turkey), Fatih Kangal (Koc University, Turkey), Karl Meerbergen (KU Leuven, Belgium), Wim Michiels (KU Leuven, Belgium), and Emre Alper Yildirim (Koc University, Turkey).*

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B7 - July 14, 14:30 – 15:00

## REVISITING THE PERFECT SHIFT STRATEGY

**Paul Van Dooren**

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In this paper we revisit the problem of performing a QR-step on an unreduced Hessenberg matrix when we know an exact eigenvalue of this matrix. Under exact arithmetic, this eigenvalue will appear on diagonal of the transformed Hessenberg matrix and will be decoupled from the remaining part of the Hessenberg matrix, thus resulting in a deflation. But it is well known that in finite precision arithmetic the so-called perfect shift gets blurred and the eigenvalue can not be deflated and/or is perturbed significantly. In this paper, we develop a new strategy for computing such a QR step so that the deflation is always successful.

*Joint work with Nicola Mastronardi (CNR Bari, Italy).*

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 B7 - July 14, 15:00 – 15:30

## DYNAMICS VIA NONLINEAR PSEUDOSPECTRA

**David Bindel**

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Nonlinear eigenvalue problems arise naturally from transform analysis of linear differential and functional differential equations involving effects such as damping, delays, and radiation. But just as linear eigenvalue analysis tells us about asymptotic stability without necessarily providing a full picture of the intermediate transient behavior, so to does nonlinear eigenvalue analysis give an incomplete picture of the dynamics of the associated system. In this talk, we describe how generalizations of pseudospectra and related inclusion regions to the case of nonlinear eigenvalue problems, and show how they can provide information about the non-asymptotic dynamics of systems with radiation and delay.

*Joint work with Amanda Hood (Cornell University, USA).*

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B7 - July 14, 15:30 – 16:00

## VECTOR SPACES OF LINEARIZATIONS FOR RECTANGULAR MATRIX POLYNOMIALS

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The seminal work [4] introduced vector spaces of matrix pencils, with the property that almost all the pencils in the vector space are strong linearizations of a given square regular matrix polynomial. This work was subsequently extended to include the case of square singular matrix polynomials in [1]. Extending these ideas, we construct similar vector spaces of rectangular matrix pencils such that almost every matrix pencil of the space is a strong linearization of a given rectangular matrix polynomial  $P$  in a generalized sense. Moreover, the minimal indices of  $P$  can be recovered from those of the matrix pencil.

We further show that such pencils can be ‘trimmed’ to form smaller pencils that are unimodular equivalent to  $P$ . The resulting pencils are almost always strong linearizations of  $P$ . Moreover they are easier to construct and are often smaller than the Fiedler linearizations of  $P$  introduced in [2]. Further, the backward error analysis carried out in [3] when applied to these trimmed linearizations shows that under suitable conditions, the computed eigenstructure of the linearizations obtained from some backward stable algorithm yield the exact eigenstructure of a slightly perturbed matrix polynomial.

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*Joint work with Biswajit Das (Indian Institute of Technology Guwahati).*

B7 - July 14, 16:00 – 16:30

#### THE COMPUTATIONAL COMPLEXITY OF DUALITY

**Shmuel Friedland**

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We show that for any given norm ball or proper cone, weak membership in its dual ball or dual cone is polynomial-time reducible to weak membership in the given ball or cone. A consequence is that the weak membership or membership problem for a ball or cone is NP-hard if and only if the corresponding problem for the dual ball or cone is NP-hard. In a similar vein, we show that computation of the dual norm of a given norm is polynomial-time reducible to computation of the given norm. This extends to convex functions satisfying a polynomial growth condition: for such a given function, computation of its Fenchel dual/conjugate is polynomial-time reducible to computation of the given function. Hence the computation of a norm or a convex function of polynomial-growth is NP-hard if and only if the computation of its dual norm or Fenchel dual is NP-hard. We discuss implications of these results on the weak membership problem for a symmetric convex body and its polar dual, the polynomial approximability of Mahler volume, and the weak membership problem for the epigraph of a convex function with polynomial growth and that of its Fenchel dual.

*Joint work with Lek-Heng Lim (University of Chicago, USA).*

B7 - July 14, 17:00 – 17:30

#### NUMERICAL METHODS FOR LYAPUNOV MATRIX EQUATIONS WITH BANDED SYMMETRIC DATA

**Valeria Simoncini**

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We are interested in the numerical solution of the large-scale Lyapunov equation

$$AX + XA^T = C,$$

where  $A, C \in \mathcal{R}^{n \times n}$  are both large and banded matrices. We suppose that  $A$  is symmetric and positive definite and  $C$  is symmetric and positive semidefinite. While the case of low rank  $C$  has been successfully addressed in the literature, the more general banded setting has not received much attention, in spite of its possible occurrence in applications. In this talk we aim to fill this gap.

It has been recently shown that if  $A$  is well conditioned, the entries of the solution matrix  $\mathbf{X}$  decay in absolute value as their indexes move away from the sparsity pattern of  $C$ . This property can be used in a memory-saving matrix-oriented Conjugate Gradient method to obtain a banded approximate solution.

For  $A$  not well conditioned, the entries of  $\mathbf{X}$  do not sufficiently decay to derive a good banded approximation. Nonetheless, we show that it is possible to split  $\mathbf{X}$  as  $\mathbf{X} = \mathbf{Z}_b + \mathbf{Z}_r$ , where  $\mathbf{Z}_b$  is banded and  $\mathbf{Z}_r$  is numerically low rank. We thus propose a novel strategy that efficiently approximates both  $\mathbf{Z}_b$  and  $\mathbf{Z}_r$  with acceptable memory requirements.

Numerical experiments are reported to illustrate the potential of the discussed methods.

*Joint work with Davide Palitta (Università di Bologna, Italy).*

B7 - July 14, 17:30 – 18:00

# ON THE SOLUTION OF QUADRATIC MATRIX EQUATIONS WITH INFINITE-DIMENSIONAL STRUCTURED COEFFICIENTS

**Beatrice Meini**

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Matrix equations of the kind  $A_{-1} + A_0X + A_1X^2 = X$ , where both the matrix coefficients and the unknown are semi-infinite matrices belonging to a Banach algebra, are considered. These equations, where coefficients are nonnegative quasi-Toeplitz matrices, are encountered in certain Quasi-Birth-Death (QBD) stochastic processes, as the tandem Jackson queue or the reflecting random walk in the quarter plane. We provide a numerical framework for approximating the minimal nonnegative solution of these equations which relies on semi-infinite quasi-Toeplitz matrix arithmetic. This arithmetic does not perform any finite size truncation, instead it approximates the infinite matrices through the sum of a banded Toeplitz matrix and a compact correction. In particular, we show that the algorithm of Cyclic Reduction can be effectively applied and can approximate the infinite dimensional solutions with quadratic convergence at a cost which is comparable to that of the finite case. This way, we may compute a finite approximation of the sought solution, as well as of the invariant probability measure of the associated QBD process, within a given accuracy. Numerical experiments, performed on a collection of benchmarks, confirm the theoretical analysis.

*Joint work with D.A. Bini (University of Pisa), S. Massey (Scuola Normale Superiore, Pisa), and L. Robol (CNR, Pisa).*

B7 - July 14, 18:00 – 18:30

# RECENT ADVANCES IN LOW-RANK APPROXIMATIONS

**Ivan Oseledets**

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In this talk, I will give a brief overview of our recent results in theory, algorithms, and applications of low-rank approximations of matrices and tensors. These include: 1) Jacobi-Davidson



method for the optimization over low-rank matrix manifolds 2) Desingularization approach for optimization over low-rank matrix sets 3) Several new applications of tensors in chemistry, physics, machine learning.

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B7 - July 15, 14:30 – 15:00

### LOW-RANK UPDATES OF MATRIX FUNCTIONS

**Daniel Kressner**

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This talk is concerned with the development and analysis of fast algorithms for updating a matrix function  $f(A)$  if  $A$  undergoes a low-rank change  $A + L$ . For example, when  $A$  is the Laplacian of an undirected graph then removing one edge or vertex of the graph corresponds to a rank-one change of  $A$ . The evaluation and update of such matrix functions (or parts thereof) is of relevance in the analysis of network community structure of networks and signal graph processing. Our algorithms are based on the tensorization of polynomial or rational Krylov subspaces involving  $A$  and  $A^T$ . The choice of a suitable element from such a tensorized subspace for approximating  $f(A + L) - f(A)$  is straightforward in the symmetric case but turns out to be more intricate in the nonsymmetric case. We show that the usual convergence results for Krylov subspace methods for matrix functions can be extended to our algorithms. If time permits, we will also touch upon the closely related problem of applying the Fréchet derivative of a matrix function to a low-rank matrix.

*Joint work with Bernhard Beckermann (Université de Lille 1, France) and Marcel Schweitzer (EPF Lausanne, Switzerland).*

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B7 - July 15, 15:00 – 15:30

### SUBSPACE METHODS FOR COMPUTING THE CRAWFORD NUMBER AND THE REAL PSEUDOSPECTRAL ABSCISSA

**Bart Vandereycken**

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Certain eigenvalue or singular value optimization algorithms require repeatedly calculating the spectrum of a smoothly varying matrix. Two examples are the computation of the Crawford number and the real pseudospectral abscissa. In this talk, I will show how the computed eigenvectors and singular vectors can be used to construct subspace methods that approximate the original problems increasingly well. For the Crawford number, the subspace method is equivalent to one in [Kangal & al., “A subspace method for large scale eigenvalue optimization,” 2017] but our new convergence analysis predicts the numerical order of convergence very well. For the real pseudospectral abscissa, our algorithm is new and we prove its superlinear convergence.

*Joint work with Ding Lu (University of Geneva, Switzerland) and Daniel Kressner (EPF Lausanne, Switzerland).*

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 B7 - July 15, 15:30 – 16:00

## SEPARABLE NONNEGATIVE MATRIX FACTORIZATION

**Nicolas Gillis**

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Nonnegative Matrix Factorization (NMF) is a linear dimensionality reduction technique for nonnegative data. NMF has become a very popular technique in data mining and machine learning because it automatically extracts meaningful features through a sparse and part-based representation. NMF consists in approximating a nonnegative data matrix with the product of two nonnegative matrices. In this talk, we first introduce NMF and illustrate its usefulness with some application examples. We then focus on the separability assumption that allows to provably solve the NMF problem (although it is NP-hard in general). We present several recent algorithms, including geometric algorithms and algorithms based on convex optimization. We illustrate the results with some numerical experiments.

*Joint work with Robert Luce (EPF Lausanne, Switzerland) and Stephen Vavasis (University of Waterloo, Canada).*

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 B7 - July 15, 16:00 – 16:30

## GEOMETRY OF MATRIX POLYNOMIAL SPACES

**Andrii Dmytryshyn**

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We study how small perturbations of matrix polynomials may change their elementary divisors and minimal indices by constructing the closure hierarchy graphs (stratifications) of orbits and bundles of matrix polynomial Fiedler linearizations. We show that the stratification graphs do not depend on the choice of Fiedler linearization which means that all the spaces of the matrix polynomial Fiedler linearizations have the same geometry (topology). We also develop the theory for structure preserving stratification of skew-symmetric matrix polynomials. The results are illustrated by examples using the software tool Stratigraph.

*Joint work with Stefan Johansson (Umeå University), Bo Kågström (Umeå University), Paul Van Dooren (Université Catholique de Louvain, Belgium).*

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 B7 - July 15, 17:00 – 17:30

## PARAMETER-DEPENDENT RANK-ONE PERTURBATIONS OF SINGULAR HERMITIAN PENCILS

**Christian Mehl**

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We investigate the effect of perturbations of singular Hermitian pencils  $\lambda E + A$  that are (1) of rank one, (2) structure-preserving, (3) generic, (4) parameter-dependent, and (5) regularizing, i.e., generic perturbations that have the form

$$\lambda(E + \tau e u u^T) + A + \tau a u u^T,$$

where the perturbed pencil is regular.

It is known that in this situation the eigenvalues of the perturbed pencil are constant in the parameter  $\tau$ . One would now expect that for a generic regularizing rank-one perturbation all eigenvalues of the perturbed pencil are simple. While this is indeed the case for pencils without symmetry structure, surprisingly this is different for real symmetric pencils where it happens that generically double eigenvalues will occur. While these eigenvalues will be constant in  $\tau$  as expected, this will not be the case for their geometric multiplicities.

In this talk, we will explain why this peculiar behavior occurs and investigate what happens when the geometric multiplicity of eigenvalues changes.

*Joint work with Volker Mehrmann (TU Berlin, Germany) and Michal Wojtylak (University of Krakow).*

B7 - July 15, 17:30 – 18:00

# NUMERICAL INSTABILITY OF ALGEBRAIC-GEOMETRIC METHODS TO SOLVE POLYNOMIAL EQUATIONS

**Vanni Noferini**

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Among the existing algorithms to solve systems of polynomial equations, those based on algebraic geometry have particular relevance. In practice, they are often implemented either in a mixed symbolic-numeric environment, or purely in floating point arithmetic.

We focus on two important classes of methods, those based on resultant matrices and those based on Groebner bases. The common paradigm is to convert first the polynomial root-finding problem into a related eigenvalue problem, and then to find approximate solutions to the latter with standard, and reliable, numerical linear algebra algorithms. One can then recover an approximate solution to the original problem from the computed eigenvalues and eigenvectors. However, even if the conversion step is performed symbolically, the eigenvalue problem is potentially worse conditioned than the original problem. As a result, the polynomial system solver may be numerically unstable, even if the eigenvalue problem is formed without any approximation error and even it is then solved in a numerically stable manner.

This issue has been long known to practitioners. Here, we provide recent theoretical developments that quantitatively describe a worst-case spectacular increase of the condition number. We also comment on the difference between the various variants of these “numerical algebraic geometry” algorithms, discussing whether and how this effect can be mitigated when implementing these methods.

*Joint work with Sujit Rao (Cornell University, USA) and Alex Townsend (Cornell University, USA).*

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B7 - Poster

# FAST NUMERICAL SOLVER FOR HIGHER ORDER LOCAL PLATFORMS BASED ON ERROR CORRECTION METHODS

**Sunyoung Bu**

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Error correction method (ECM) [1, 2] is based on the construction of a local approximation to the solution on each time step. In this study, a higher-order continuous local platform is constructed to develop higher-order semi-explicit one-step ECM due to the excellent convergence order  $O(h^{2p+2})$  of ECM, provided the local approximation has a local residual error  $O(h^p)$ . Unfortunately, the construction generates inevitably a huge size of matrix. To overcome this complexity, a fast numerical matrix solver is also provided. Special choices of parameters for the local platform can lead to the improvement of the well-known explicit fourth and fifth order Runge-Kutta methods. Numerical experiments demonstrate the theoretical results.

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*Joint work with Philsu Kim (Kyungpook National University, South Korea).*

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B7 - Poster

# PERRON-FROBENIUS THEOREM FOR NONNEGATIVE TENSORS

**Antoine Gautier**

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Recently, the Perron-Frobenius theory has been extended to various class of nonlinear spectral problems involving nonnegative tensors. Relevant examples include the  $\ell^{p,q}$ -singular vectors of matrices [1], the  $\ell^p$ -eigenvectors of a square tensor [2], the  $\ell^{p,q}$ -singular vectors of rectangular tensors [3] and the  $\ell^{p_1, \dots, p_d}$ -singular vectors of tensors [4, 5].

We present the results of [6] where a unified Perron-Frobenius theorem based on the concept of multi-homogeneous mappings is proposed. This general result can be applied to a variety of problems which includes all mentioned spectral problems for nonnegative tensors and allows to identify their common structure. It implies and improves many existing results for each of these nonnegative tensor spectral problems. In particular, when applied to these special cases, our theorem provide new characterizations of the spectral radius as well as novel convergence guarantees of the power method.

More precisely, we discuss a weak Perron-Frobenius theorem for multi-homogeneous maps which implies the existence of a nonnegative eigenvector corresponding to the spectral radius as well as a Collatz-Wielandt characterization and a number of Gelfand type formulas for the spectral radius. Furthermore, we present a strong Perron-Frobenius theorem where conditions for the existence, uniqueness and maximality of a positive eigenvector are given. We also discuss the convergence of the power-method towards the unique positive eigenvector together with a linear convergence rate. When applied to the special case of a multi-homogeneous map associated to a tensor spectral problem, our theorem allows equivalent or wider choices for  $p, q$  and  $p_1, \dots, p_d$  than existing results. Moreover, the generality of our formulation allows to unify the definitions of irreducibility, weak irreducibility and weak primitivity for tensors which have been introduced separately by various authors.

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*Joint work with F. Tudisco (University of Padua, Italy) and M. Hein (Saarland University, Germany).*

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B7 - Poster

## NUMERICAL METHODS FOR SOLVING NONLINEAR MATRIX EQUATIONS

**Hyun–Min Kim**

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We consider the existence of solutions to several types of nonlinear matrix equation, for example the quadratic matrix equations and the matrix polynomials, etc. The existences of elementwise positive solution, elementwise negative solution and positive definite solution are introduced. Also, we apply the some numerical methods to find such solutions which are Newton’s method with and without incorporating exact line searches and relaxation method and functional iterations. Finally we show some numerical results.

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*Joint work with Jie Meng (Pusan National University).*

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B7 - Poster

## STABLE ALS APPROXIMATION IN THE TT-FORMAT FOR RANK-ADAPTIVE TENSOR COMPLETION

**Sebastian Kraemer**

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Low rank tensor completion is a highly ill-posed inverse problem, particularly when the data model is not accurate, and some sort of regularization is required in order to solve it. In this article we focus on the calibration of the data model. For alternating optimization, we observe that existing rank adaption methods do not enable a continuous transition between manifolds of different ranks. We denote this flaw as *instability (under truncation)*. As a consequence of this flaw, arbitrarily small changes in the singular values of an iterate can have arbitrarily large influence on the further reconstruction. We therefore introduce a singular value based regularization to the standard alternating least squares (ALS), which is motivated by averaging in micro-steps. We prove its *stability* and derive a natural semi-implicit rank adaption strategy. We further prove that the standard ALS micro-steps are only stable on manifolds of fixed ranks, and only around points that have what we define as *internal tensor restricted isometry property iTRIP*. Finally, we provide numerical examples that show improvements of the reconstruction quality up

to orders of magnitude in the new Stable ALS Approximation (SALSA) compared to standard ALS.

*Joint work with Lars Grasedyck (IGPM at RWTH Aachen University, Germany).*

B7 - Poster

# AN UPPER $J$ - HESSENBERG REDUCTION OF A MATRIX THROUGH SYMPLECTIC HOUSEHOLDER TRANSFORMATIONS

**Ahmed Salam**

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In this work, we introduce a reduction of a matrix to a condensed form, the upper  $J$ - Hessenberg form, via elementary symplectic Householder transformations, which are rank-one modification of the identity. This reduction is the crucial step for constructing an efficient SR-algorithm. The method is the analog of the reduction of a matrix to Hessenberg form, via Householder transformations, when instead of an Euclidean linear space, one takes a symplectic one. Features of the reduction are highlighted. Two variants numerically more stables are then derived. Some numerical experiments are given, showing the efficiency of these variants.

*Joint work with Haithem Ben Kahla (Université de Lille Nord de France, France).*

B7 - Poster

# A NEW FORMULATION FOR THE NEAREST STABLE MATRIX PROBLEM

**Punit Sharma**

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A square matrix is stable if all its eigenvalues are in the closed left half of the complex plane and those on the imaginary axis are semisimple. In this talk, we consider the problem of computing the nearest stable matrix to an unstable one: given a unstable matrix  $A \in \mathbb{R}^{n,n}$ , minimize the Frobenius norm of  $\Delta_A \in \mathbb{R}^{n,n}$  such that  $A + \Delta_A$  is a stable matrix. This is the converse of the stability radius problem, where a stable matrix is given and one looks for the smallest perturbation that moves an eigenvalue outside the stability region. This problem occurs for example in system identification where one needs to identify a stable system from observations.

We propose new algorithms to solve this problem based on a reformulation using linear dissipative Hamiltonian systems: we show that a matrix  $A$  is stable if and only if it can be written as  $A = (J - R)Q$ , where  $J$  is skew-symmetric,  $R$  is positive semidefinite and  $Q$  is positive definite. This reformulation results in an equivalent optimization problem with a simple convex feasible set. We propose three strategies to solve the problem in variables  $(J, R, Q)$ :

- (i) a block coordinate descent method,
- (ii) a projected gradient descent method, and
- (iii) a fast gradient method inspired from smooth convex optimization.

These methods require  $\mathcal{O}(n^3)$  operations per iteration. We show the effectiveness of the fast gradient method compared to the other approaches and to several state-of-the-art algorithms.

The paper is under review and available from <https://arxiv.org/abs/1611.00595>.

*Joint work with Nicolas Gillis (University of Mons, Belgium).*

B7 - Poster

## THE GSVD AND CSD: MATRIX TRIGONOMETRY OR WHERE ARE THE ELLIPSES?

**Yuyang Wang**

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The SVD ellipse picture for a matrix  $A$  is a very familiar visual for the action of  $A$  on the unit ball. We are not aware of any ellipse pictures in the literature nor even a notion that a natural ellipse picture exists for the generalized SVD (GSVD) of two matrices  $A$  and  $B$  or the closely related CSD (CS Decomposition) of an orthogonal matrix. We believe that the lack of a geometric view of the GSVD is part of the reason that the GSVD is not as widely understood or as widely used as it should be. In this paper, we reveal the trigonometry, the ellipse picture, and further geometry of the GSVD. In particular, we show how the GSVD reveals whether  $B$  is “small” relative to  $A$ , in the same way that  $B/A$  is the slope of a vector in 2d. In higher dimensions,  $B$  may be small relative to  $A$  in some directions, and large in others. We further characterize the connections between the generalized singular values of  $(A, B)$ , which we call the cotangent form of the gsvd, and the singular values of  $AB^\dagger$  (e.g.,  $A$  times the pseudoinverse of  $B$ ).

*Joint work with Alan Edelman (MIT, USA).*



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**Workshop C1****Computational Harmonic Analysis and Compressive Sensing**Organizers: Holger Rauhut – Karlheinz Gröchenig – Thomas Strohmer

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C1 - July 17, 14:30 – 15:20

**ENHANCING RESOLUTION IN UNDERSAMPLED PHYSICAL IMAGING****Robert Calderbank**

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Compressed sensing MRI was first proposed by Lustig and Donoho, and it is now starting to appear in commercial systems. Candes, Tao and Romberg used Incoherence between measurements and representations of signals to develop performance guarantees for compressed sensing. We demonstrate here that fidelity and resolution in physical imaging systems can be improved by taking advantage of shared asymptotic characteristics of signals and measurement devices. We present new experimental results, explaining why and how improvements are possible.

*Joint work with Bogdan Roman (University of Cambridge, UK), Ben Adcock (Simon Fraser University, Canada), Daniel Nietlispach (University of Cambridge, UK), Mark Bostock (University of Cambridge, UK), Irene Calvo-Almazan (University of Cambridge, UK), Martin Graves (University of Cambridge, UK), and Anders Hansen (University of Cambridge, UK).*

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C1 - July 17, 15:30 – 15:55

**THE SAMPLE COMPLEXITY OF MULTI-REFERENCE ALIGNMENT****Afonso Bandeira**

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Many problems in signal and image processing amount to estimating a signal, or image, from corrupted measurements. A particularly challenging form of measurement corruption are latent transformations of the underlying signal to be recovered.

A fundamental example of this problem is the multi-reference alignment problem: in its simplest form the goal is to estimate a signal from noisy cyclically shifted copies. We will show that the number of observations needed has a surprising dependency on the signal-to-noise ratio (SNR). Computationally efficient methods that achieve optimal sample complexity will also be described.

We will also discuss the sample complexity of the heterogeneous multi-reference alignment problem where the samples come from a mixture of signals, and provide the first known procedure that provably achieves signal recovery in the low SNR regime. A related problem is heterogenous reconstruction in Cryo-Electron Microscopy (Cryo-EM) imaging where projections of a mixture of molecule densities are taken from unknown rotations. Our work provides a first step towards a complete statistical theory of heterogenous Cryo-EM.

*Joint work with Jonathan Weed (MIT, USA), Amelia Perry (MIT, USA), Philippe Rigollet (MIT, USA), and Amit Singer (Princeton University, USA).*

C1 - July 17, 16:00 – 16:25

## ROBUSTNESS TO UNKNOWN ERROR IN SPARSE REGULARIZATION

**Ben Adcock**

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Sparse regularization techniques have become increasingly popular in the last decade with the advent of techniques such as compressed sensing and matrix completion. Most theoretical guarantees for sparse regularization assume that an upper bound for the noise is known in advance. However, such a bound is absent in most, if not all, practical scenarios. While estimation of the noise level may be possible, e.g., via cross validation, there are few theoretical results which explain the effect of unknown or estimated noise levels on the overall reconstruction. In this talk I will present new results on the performance of several popular sparse regularization techniques when subject to unknown noise. These results cover both Gaussian random matrices, and large classes of structured random matrices with heavy-tailed rows. Time permitting, I will give several applications of this work, including high-dimensional function approximation, infinite-dimensional sparse regularization for inverse problems, and fast algorithms for non-Cartesian Magnetic Resonance Imaging.

*Joint work with Simone Brugiapaglia (SFU, Canada).*

C1 - July 17, 17:00 – 17:25

## ENERGY PROPAGATION IN DEEP CONVOLUTIONAL NEURAL NETWORKS

**Philipp Grohs**

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Many practical machine learning tasks employ very deep convolutional neural networks. Such large depths pose formidable computational challenges in training and operating the network. It is therefore important to understand how many layers are actually needed to have most of the input signal's features be contained in the feature vector generated by the network. This question can be formalized by asking how quickly the energy contained in the feature maps decays across layers. In addition, it is desirable that none of the input signal's features be "lost" in the feature extraction network or, more formally, we want energy conservation in the sense of the energy contained in the feature vector being proportional to that of the corresponding input signal. This paper establishes conditions for energy conservation for a wide class of deep convolutional neural networks and characterizes corresponding feature map energy decay rates. Specifically, we consider general scattering networks, and find that under mild analyticity and high-pass conditions on the filters (which encompass, inter alia, various constructions of Weyl–Heisenberg filters, wavelets, ridgelets,  $(\alpha)$ -curvelets, and shearlets) the feature map energy decays at least polynomially fast. For broad families of wavelets and Weyl–Heisenberg filters, the guaranteed

decay rate is shown to be exponential. Our results yield handy estimates of the number of layers needed to have at least  $((1 - \varepsilon) \cdot 100)\%$  of the input signal energy be contained in the feature vector.

*Joint work with Helmut Bölcskei (ETH Zürich, Switzerland) and Thomas Wiatowski (ETH Zürich, Switzerland).*

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C1 - July 17, 17:30 – 17:55

#### ADAPTIVE REGRESSION ON LOW-DIMENSIONAL SETS IN HIGH-DIMENSIONS

**Mauro Maggioni**

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We discuss the problem of regressing a function of unknown regularity on an unknown low-dimensional set (e.g., a manifold) embedded in high-dimensional space, given samples on the set and corresponding noisy function values. We introduce a multiscale approximation scheme that adapts to the unknown smoothness, has strong theoretical guarantees, and is implemented by fast algorithms.

*Joint work with Wenjing Liao (Johns Hopkins University, USA) and Stefano Vigogna (Johns Hopkins University, USA).*

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C1 - July 17, 18:00 – 18:25

#### A POLYNOMIAL-TIME RELAXATION OF THE GROMOV–HAUSDORFF DISTANCE

**Soledad Villar**

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The Gromov–Hausdorff distance provides a metric on the set of isometry classes of compact metric spaces. Unfortunately, computing this metric directly is believed to be computationally intractable. Motivated by applications in shape matching and point-cloud comparison, we study a semidefinite programming relaxation of the Gromov–Hausdorff metric. This relaxation can be computed in polynomial time, and somewhat surprisingly is itself a pseudometric. We describe the induced topology on the set of compact metric spaces. Finally, we demonstrate the numerical performance of various algorithms for computing the relaxed distance and apply these algorithms to several relevant data sets. In particular we propose a greedy algorithm for finding the best correspondence between finite metric spaces that can handle hundreds of points.

*Joint work with Afonso Bandeira (New York University, USA), Andrew Blumberg (The University of Texas at Austin, USA), and Rachel Ward (The University of Texas at Austin, USA).*

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 C1 - July 17, 18:30 – 18:55

TBA

**Gilad Lerman**
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 C1 - July 18, 14:30 – 14:55

## LOW-RANK MATRIX RECOVERY FROM CLIFFORD ORBITS

**David Gross**
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We prove that low-rank matrices can be recovered efficiently from a small number of measurements that are sampled from orbits of a certain matrix group. We focus in particular on the phase retrieval problem — a special case of low-rank recovery for rank-1 matrices. Variants of the group in question have appeared under different names in many areas of mathematics. In coding theory and quantum information, it is the complex Clifford group; in time-frequency analysis the oscillator group; and in mathematical physics the metaplectic group. It affords one particularly small and highly structured orbit that includes and generalizes the discrete Fourier basis: While the Fourier vectors have coefficients of constant modulus and phases that depend linearly on their index, the vectors in said orbit have phases with a quadratic dependence. In quantum information, the orbit is used extensively and is known as the set of stabilizer states. We argue that due to their rich geometric structure and their near-optimal recovery properties, stabilizer states form an ideal model for structured measurements for phase retrieval.

*Joint work with R. Kueng (University of Cologne, Germany) and H. Zhu (University of Cologne, Germany).*

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 C1 - July 18, 15:00 – 15:25

## COMPUTATIONAL HARMONIC ANALYSIS MEETS DEEP LEARNING

**Gitta Kutyniok**
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Deep learning has shown impressive results in real-world applications. However, most of the research on deep neural networks is based on heuristics, lacking a fundamental mathematical theory. In this talk, we will focus on the relation between the complexity of a neural network in terms of sparse connectivity and its approximation properties. We will derive a fundamental lower bound on this sparsity for a prescribed approximation accuracy. Using methods from computational harmonic analysis, in particular,  $\alpha$ -shearlets, we then prove that this bound is in fact optimal. Finally, our numerical experiments do not only show that this bound is achieved when training a deep neural network by the standard backpropagation algorithm, but

by using a carefully chosen architecture, the trained network surprisingly even exhibits elements of representation systems from computational harmonic analysis.

*Joint work with Helmut Bölcskei (ETH Zürich, Switzerland), Philipp Grohs (Universität Wien, Austria), and Philipp Petersen (Technische Universität Berlin, Germany).*

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C1 - July 18, 15:30 – 15:55

#### STATISTICS, COMPUTATION AND LEARNING WITH GRAPH NEURAL NETWORKS

**Joan Bruna**

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Deep Learning, thanks mostly to Convolutional architectures, has recently transformed computer vision and speech recognition. Their ability to encode geometric stability priors, while offering enough expressive power, is at the core of their success. In such settings, geometric stability is expressed in terms of local deformations, and it is enforced thanks to localized convolutional operators that separate the estimation into scales.

Many problems across applied sciences, from particle physics to recommender systems, are formulated in terms of signals defined over non-Euclidean geometries, and also come with strong geometric stability priors. In this talk, I will present techniques that exploit geometric stability in general geometries with appropriate graph neural network architectures. We will show that these techniques can all be framed in terms of local graph generators such as the graph Laplacian. We will present some stability certificates, as well as applications to computer graphics, particle physics and graph estimation problems. In particular, we will describe how graph neural networks can be used to reach statistical detection thresholds in community detection, and attack hard combinatorial optimization problems, such as the Quadratic Assignment Problem.

*Joint work with Xiang Li (UC Berkeley, USA), Soledad Villar (NYU, USA), Alex Nowak (NYU, USA), and Afonso Bandeira (NYU, USA).*

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C1 - July 18, 16:00 – 16:25

#### WHAT CAN DEEP LEARNING LEARN FROM COMPRESSIVE SENSING?

**Benjamin Recht**

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Recent successes in neural networks have demonstrated that models with an excessive numbers parameters can achieve tremendous improvements in pattern recognition. Moreover, empirical evidence demonstrates that such performance is achievable without any obvious forms of regularization or capacity control. These findings at first glance suggest that traditional learning theory fails to explain why large neural networks generalize. In this talk, I will discuss possible mechanisms to explain generalization in such large models, appealing to insights from linear predictors. I will discuss how many of the observations can be understood by direct analogies to the linear case. I will close by proposing some possible directions of future research that connect a decade of work in compressive sensing with the contemporary phenomenology in large-scale machine learning.

*Joint work with Samy Bengio (Google Brain), Moritz Hardt (Google Brain/University of California Berkeley), Oriol Vinyals (DeepMind), Chiyuan Zhang (Massachusetts Institute of Technology).*

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C1 - July 18, 17:00 – 17:25

## VANDERMONDE MATRICES AND THE LARGE SIEVE

**Helmut Bölcskei**

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Vandermonde matrices arise in many fields of applied mathematics and engineering, e.g., subspace methods — such as MUSIC and ESPRIT — for the estimation of cisoid parameters, super-resolution, line spectral estimation, compressed sensing, interpolation and approximation theory, sampling theory, differential equations, and control theory. In this talk, we establish a systematic connection between Vandermonde matrices and the large sieve, a set of inequalities developed in analytic number theory by Linnik, Rényi, Roth, and Bombieri. Based on this relationship, we present new bounds on the extremal singular values and the condition number of Vandermonde matrices with nodes in the unit disk. We then build on these bounds to develop a deterministic, finite-SNR, finite sample-size performance analysis of MUSIC, ESPRIT, and the matrix pencil method. These results demonstrate that polynomial root finding (e.g., Berlekamp–Massey and Peterson–Gorenstein–Zierler) algorithms over the reals as used widely in the sparse signal recovery literature do not exhibit an intrinsic lack of numerical stability.

*Joint work with Céline Aubel (ETH Zürich, Switzerland).*

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C1 - July 18, 17:30 – 17:55

## PACKINGS IN REAL PROJECTIVE SPACES

**Dustin Mixon**

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This talk discusses techniques from algebraic and differential geometry to determine how to best pack points in real projective spaces. We present a computer-assisted proof of the optimality of a particular 6-packing in  $\mathbb{RP}^3$ , we introduce a linear-time constant-factor approximation algorithm for packing in the so-called Gerzon range, and we provide local optimality certificates for two infinite families of packings. Finally, we present perfected versions of various putatively optimal packings from Sloane’s online database, along with a handful of infinite families they suggest, and we prove that these packings enjoy a certain weak notion of optimality.

*Joint work with Matthew Fickus (Air Force Institute of Technology, USA) and John Jasper (University of Cincinnati, USA).*

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C1 - July 18, 18:00 – 18:25

MULTIREFERENCE ALIGNMENT AND CLUSTERING

**Amit Singer**

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Multi-reference alignment consists of estimating a signal from its many noisy shifted copies, where the shifts are unknown. The number of observations needed for accurate estimation, efficient recovery algorithms, extension to recovery of multiple signals, and application to cryo-electron microscopy will be discussed.

*Joint work with Tamir Bendory (Princeton University, USA), Nicolas Boumal (Princeton University, USA), Roy Lederman (Princeton University, USA), William Leeb (Princeton University, USA), Nir Sharon (Princeton University, USA), Emmanuel Abbe (Princeton University, USA), Joao Pereira (Princeton University, USA), and Afonso Bandeira (NYU, USA).*

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C1 - July 19, 14:30 – 14:55

EXTRAPOLATION FROM SPARSITY

**Laurent Demanet**

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This talk considers the basic question of frequency extrapolation of sparse signals observed over some frequency band, such as scattered bandlimited waves. How far, and how stably can we extend? I will review recent progress on the mathematical aspects of this question, which are tied to the notion of super-resolution. I will also discuss the robust “phase tracking” algorithmic approach, which is suitable for seismic imaging where the bandlimiting model is far from accurately known.

*Joint work with Nam Nguyen and Yunyue Elita Li.*

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C1 - July 19, 15:00 – 15:25

THE USEFULNESS OF A MODIFIED RESTRICTED ISOMETRY PROPERTY

**Simon Foucart**

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The restricted isometry property is arguably the most prominent tool in the theory of compressive sensing. In its classical version, it features  $\ell_2$ -norms as inner and outer norms. The modified version considered in this talk features the  $\ell_1$ -norm as the inner norm, while the outer norm depends a priori on the distribution of the random entries populating the measurement matrix. The modified version holds for a wider class of random matrices and still accounts for the success of sparse recovery via basis pursuit and via iterative hard thresholding. In the special case

of Gaussian matrices, the outer norm actually reduces to an  $\ell_2$ -norm. This fact allows one to retrieve results from the theory of one-bit compressive sensing in a very simple way. Extensions to one-bit matrix recovery are then straightforward.

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C1 - July 19, 15:30 – 15:55

## JOINT BLIND DECONVOLUTION AND BLIND DEMIXING VIA NONCONVEX OPTIMIZATION

**Shuyang Ling**

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We study the question of extracting a sequence of functions  $\{f_i, g_i\}_{i=1}^s$  from observing only the sum of their convolutions, i.e., from  $y = \sum_{i=1}^s f_i * g_i$ . While convex optimization techniques are able to solve this joint blind deconvolution-demixing problem provably and robustly under certain conditions, for medium-size or large-size problems we need computationally faster methods without sacrificing the benefits of mathematical rigor that come with convex methods. We present a non-convex algorithm which guarantees exact recovery under conditions that are competitive with convex optimization methods, with the additional advantage of being computationally much more efficient. Our two-step algorithm converges to the global minimum linearly and is also robust in the presence of additive noise. While the derived performance bounds are suboptimal in terms of the information-theoretic limit, numerical simulations show remarkable performance even if the number of measurements is close to the number of degrees of freedom. We will discuss blind deconvolution as a special case and an application of the proposed framework in wireless communications.

*Joint work with Thomas Strohmer (University of California Davis, USA), Xiaodong Li (University of California Davis, USA), and Ke Wei (University of California Davis, USA).*

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C1 - July 19, 16:00 – 16:25

## MULTI-TAPER ESTIMATORS WITH GEOMETRIC CONSTRAINTS

**Jose Luis Romero**

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We consider the problem of estimating a stationary point process from samples taken on a set with possibly general geometry. The multi-taper approach consists in weighting the data with adequate masks to then perform an average that reduces variance. We describe the performance of such estimators, and the computational challenges in their implementation. We consider workarounds for the main computational obstacles and quantify their effectiveness.

*Joint work with Luis Daniel Abreu (Acoustics Research Institute, Austrian Academy of Sciences).*



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C1 - July 19, 17:00 – 17:50

TBA

**Ingrid Daubechies**

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C1 - Poster

THE BEURLING-SELBERG BOX MINORANT PROBLEM

**Jacob Carruth**

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Let  $F$  be a function on  $\mathbb{R}^d$  which is bandlimited to the unit cube and which minorizes the characteristic function of the unit cube. How large can the integral of  $F$  be? We show that for a sufficiently large dimension  $d^*$  the answer to this question is zero for  $d \geq d^*$ . Furthermore, we discuss a numerical method which gives an explicit upper bound on  $d^*$ . We also show that the integral of  $F$  can be larger than zero when  $d \leq 5$  by explicitly constructing non-trivial minorants. Selberg first asked this question to solve a problem in number theory but it has since found many applications including to signal recovery, crystallography, and sphere packing. We highlight some of the applications to signal recovery.

*Joint work with Noam Elkies (Harvard University, USA), Felipe Goncalves (University of Alberta, Canada) and Michael Kelly (Center for Communications Research).*

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C1 - Poster

EXACT SUPPORT RECOVERY IN UNMIXING PROBLEMS BY AN ALTERED LASSO-PATH ALGORITHM FOR MULTI-PENALTY FUNCTIONALS

**Timo Klock**

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Inverse problems of unmixing type arise in many real-life applications such as audio processing or medical image analysis. In such problems additive noise directly affects a sparse signal before being measured through a sampling matrix. Consequently, the noise in the measurement is amplified through the sampling process and the so-called noise folding phenomenon occurs. This amplification worsens the results on support identification by means of “classical” sparse recovery techniques based on the  $\ell_1$ -penalised Lasso functional. Several recent works suggest to apply a multi-penalty framework for a correct modeling and separation of the original signal in such type of problems. Theoretical results and numerical experiments with oracle-given regularisation parameters have shown that this approach allows to recover the correct support in a larger number of problems than its single-penalty counterpart. Admittedly, the parameter choice in such multi-penalty functionals becomes more involved compared to the Lasso or other single-penalty methods.

In this poster, we introduce the multi-penalty regularisation for the unmixing problem. It has been shown that the resulting functional can be rewritten as a parameterised Lasso such that we can solve the multi-penalty minimisation with known techniques for the Lasso. In this spirit, we provide an extension of the Lasso-path algorithm for an efficient calculation of large parts of the multi-penalty solution space without performing extensive grid-searches over regularisation parameters. Finally, by using a naive support selection heuristic based on signal-to-noise ratios, we identify a unique support that is compared to results from conventional sparse recovery techniques. Such experiments confirm improved results of the multi-penalty function, especially when compared to single-penalty counterparts.

*Joint work with Valeriya Naumova (Simula Research Laboratory) and Markus Grasmair (Norwegian University of Science and Technology).*

C1 - Poster

## HARMONIC MEAN ITERATIVELY REWEIGHTED LEAST SQUARES FOR LOW-RANK MATRIX RECOVERY

**Christian Kümmerle**

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We propose a new iteratively reweighted least squares (IRLS) algorithm for the recovery of a matrix  $X \in \mathbb{C}^{d_1 \times d_2}$  of rank  $r \ll \min(d_1, d_2)$  from incomplete linear observations, solving a sequence of low complexity linear problems. The easily implementable algorithm, which we call harmonic mean iteratively reweighted least squares (HM-IRLS), optimizes a non-convex Schatten- $p$  quasi-norm penalization to promote low-rankness and carries three major strengths, in particular for the matrix completion setting. First, the algorithm converges globally to the low-rank matrix for relevant, interesting cases, for which any other (non-)convex state-of-the-art optimization approach fails the recovery. Secondly, HM-IRLS exhibits an empirical recovery probability close to 1 even for a number of measurements very close to the theoretical lower bound  $r(d_1 + d_2 - r)$ , i.e., already for significantly fewer linear observations than any other tractable approach in the literature. Thirdly, HM-IRLS exhibits a locally superlinear rate of convergence (of order  $2 - p$ ) if the linear observations fulfill a suitable null space property. While for the first two properties we have so far only strong empirical evidence, we prove the third property as our main theoretical result.

*Joint work with Juliane Sigl (Technische Universität München, Germany).*

C1 - Poster

## NOVEL ASPECTS OF APPROXIMATING HILBERT SCHMIDT OPERATORS VIA GABOR MULTIPLIERS AND SPLINE-TYPE SPACES

**Darian Onchis**

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In this work, we introduce two new aspects to improve the approximation of Hilbert Schmidt operators over locally compact Abelian groups via generalized Gabor multipliers (GGM). One aspect is to formulate both as particular examples of pseudodifferential operators and to consider the approximation of the symbol of an Hilbert-Schmidt operator in the spline-type space associated to a Gabor multiplier. This gives the possibility to employ a selection procedure of the analysis and synthesis function, interpreted as time-frequency lag; hence, with the related algorithm it is possible to handle both underspread and overspread operators. In the numerical section, we exploit the case of approximating overspread operators having compact spreading function and time-varying systems. For the latter, the approximation of discontinuities in the symbol is not straightforward achievable in the GGM setting. For this reason, another aspect is to further process the symbol through a Hough transform, to detect discontinuities and to smooth them using a new class of approximants. This procedure creates a bridge between features detections techniques and harmonic analysis methods and in specific cases it almost doubles the accuracy of approximation.

*Joint work with Simone Zappala (University of Vienna, Austria).*

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C1 - Poster

#### ESTIMATION OF BLOCK SPARSITY IN COMPRESSIVE SENSING

**Zhiyong Zhou**

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Explicitly using the block structure of the unknown signal can achieve better recovery performance in compressive sensing. An unknown signal with block structure can be accurately recovered from underdetermined linear measurements provided that it is sufficiently block sparse. However, in practice, the block sparsity level is typically unknown. In this paper, we consider a soft measure of block sparsity,  $k_\alpha(\mathbf{x}) = (\|\mathbf{x}\|_{2,\alpha}/\|\mathbf{x}\|_{2,1})^{\frac{\alpha}{1-\alpha}}$ ,  $\alpha \in [0, \infty]$  and propose a procedure to estimate it by using multivariate isotropic symmetric  $\alpha$ -stable random projections without sparsity or block sparsity assumptions. The limiting distribution of the estimator is given. Some simulations are conducted to illustrate our theoretical results.

*Joint work with Jun Yu (Umeå University, Sweden).*

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**Workshop C2**
**Computational Mathematical Biology with emphasis on the Genome**

 Organizers: Steve Smale – Mike Shub – Indika Rajapakse
 

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C2 - July 17, 14:30 – 15:20

## BIOLOGICAL CLOCKS IN CELL DIVISION AND INFECTIOUS DISEASE

**John Harer**

 Duke University, Geometric Data Analytics and Mimetics Biosciences, United States  
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Gene regulatory networks (GRNs) can drive cyclic and temporally ordered processes in biological systems. One of the best-known GRNs of this type is the circadian clock, which drives rhythmic behaviors with a period of approximately 24hrs. The circadian clock network exerts its control, in part, by regulating a dynamic program of gene expression where substantial fractions of the genome are expressed during distinct phases of the circadian cycle. We have proposed that a different GRN controls the cyclic program of gene expression that is observed during the cell division cycle. In fact, we have observed similar gene expression programs across time scales from hours to days, and across organisms that are evolutionarily diverged by millions of years. These observations suggest that a class of GRNs may serve as central mechanisms that drive temporal gene expression programs in biological systems.

In this talk we will discuss a pipeline of mathematical methods designed to infer GRNs that control various cyclic and temporally ordered processes, including the Yeast Cell Cycle and Parasite Clock Networks.

*Joint work with Steve Haase (Duke University and Mimetics Biosciences, USA), Anastasia Deckard (Geometric Data Analytics, USA), Francis Motta (Duke University, USA), Tina Keliher (Duke University, USA), Kevin McGoff (UNC Charlotte, USA), and Zin Guo (The Hong Kong Polytechnic University).*

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C2 - July 17, 15:30 – 16:00

## REGULATION NETWORKS FOR STEM CELLS

**Natalia Komarova**

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Design principles of biological networks have been studied extensively in the context of protein-protein interaction networks, metabolic networks, and regulatory (transcriptional) networks. Here we focus on less studied regulation networks that occur on larger scales, namely, the cell-to-cell signaling networks that connect groups of cells in multicellular organisms. These are the feedback loops that orchestrate the complex dynamics of cell fate decisions and are necessary for the maintenance of homeostasis in stem cell lineages. We focus on “minimal”

networks, that is those that have the smallest possible numbers of controls. Using the formalism of digraphs, we show that in two-compartment lineages, reducible systems must contain two 1-cycles, and irreducible systems one 1-cycle and one 2-cycle; stability follows from the signs of the controls and does not require magnitude restrictions. In three-compartment systems, irreducible digraphs have a tree structure or have one 3-cycle and at least two more shorter cycles, at least one of which is a 1-cycle. Our results may serve as a first step toward an understanding of ways in which these networks fail in cancer.

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C2 - July 17, 16:00 – 16:30

#### AN APPROACH TO CONTROL PARTIALLY KNOWN NETWORKS

**Gemunu Gunaratne**

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Coupled networks are needed to represent a wide-range of problems including bio-molecular processes in cells, interactions within social groups, and ecosystems. One of the major goals of network analyses is to design methods to control such systems to a pre-specified target state. For example, in cellular processes, we may inquire if consequences of genetic mutations or chromosomal rearrangements can be mitigated through external intervention. The obvious approach is to start from a realistic model of the underlying network. Unfortunately, this is an extremely difficult task. Precise quantitative forms of interactions between biomolecules are currently unavailable, and are unlikely to be available in the near future. The model-independent approach proposed here relies on representing the “state” of a cell through its gene expression profile; i.e., the levels of mRNA within a cell. The data can be extracted using techniques such as RNASeq. Under the assumption, key ingredients needed for control are “response surfaces,” each of which expresses how the gene expression profile responds to a specific external perturbation. The number of control nodes, i.e., nodes whose levels are to be externally controlled, can be systematically increased in order to reach the target state. Importantly, the most appropriate control node to be added at a given level is determined computationally from prior data. Analyses of synthetic models and (experiments on) nonlinear electrical circuits show that the target state can be typically reached with a few (3 or 4) control genes. This observation is consistent with seminal work on reprogramming mature cells to stem cells using the transcription factors Myc, Oct3/4, Sox2, and Klf4 [Takahashi and Yamanaka, Cell 126, 663, 2006] and on reprogramming fibroblasts to cardiomyocytes using three transcription factors Gata4, Mef2c and Tbx5 [Ieda & al., Cell 142, 375, 2010]. The work outlined is a model-free approach to select, systematically, a sequence of genes whose levels need to be subjected to external control in order to approach the pre-specified final state. We propose experimental validations of the approach in the sleep-deprivation network and in an addiction network in *Drosophila*.

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C2 - July 17, 17:00 – 17:30

## GENOME AND TRANSCRIPTOME DYNAMICS IN CANCER DEVELOPMENT

**Thomas Ried**

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The development of carcinomas, e.g., cervical or colorectal tumors is defined by the sequential acquisition of chromosomal copy number changes, also referred to as aneuploidy. Importantly, these changes can be invariably characterized as follows: (1) they occur in premalignant lesions before the transition to invasive disease; (2) they are surprisingly tissue specific. In cervical cancers we observe copy number gains of chromosome 3, whereas colon cancers carry extra copies of chromosomes 7, 13 and 20, along with losses of 18; (3) these chromosomal aneuploidies are maintained in metastases and in cell lines, despite ongoing, apparently random, chromosome mis-segregation. Our observations indicate that they are drivers of cancer development and its evolutionary bottleneck. We have conducted extensive analyses to show that genome copy number changes directly affect the expression of genes that reside on the affected chromosomes, both in primary tumors and in experimental models, and therefore result in a complex deregulation of the cancer transcriptome. In addition to chromosomal aberrations, specific genetic signaling pathways are activated by mutations. We are presently analyzing how these cancer specific genetic alterations affect structure/function relationships in the cell nucleus. This is achieved by correlating high-resolution maps of nuclear chromosomal architecture with global gene expression analyses. We realize that such changes cannot be understood in a meaningful way without considering the time aspect in which they occur. We propose applying data guided mathematical methods to understand the dynamical aspect of tumor development, with the ultimate goal of controlling the cancer genome, and reversing cancer cells to the differentiated state they were derived from.

*Joint work with Indika Rajapakse (University of Michigan, USA).*

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C2 - July 17, 17:30 – 18:00

## CONSIDERING QUANTUM INFORMATION AND OTHER NON-SEQUENCE INFORMATION IN UNDERSTANDING THE OPERATING PARAMETERS OF THE GENOME

**Christian Macedonia**

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The discovery of biological information encoding in nucleotide sequences was one of the most profound scientific breakthroughs of the 20th century. Scientists learned that the information necessary to regenerate living things resided in the nucleus but more pointedly found that the basic code for the generation of proteins was reducible to a remarkably simple series of three nucleotide combinations. This discovery was the result of an inspiring collaboration between biologists, chemists, and mathematicians. Unfortunately, it also set forth a “central dogma” that has readily been disproved but has had lingering negative effects on the full comprehension of genomic information storage and processing. For instance, biologists continue to use

the term “non-coding regions” to describe areas of the human genome that clearly have information encoded but not encoded to produce proteins. Biological information is also manifest within chromosomal material in non-sequence motifs such as through epigenetics, histone coiling, and other geometric modifications of DNA and RNA crystal structure. A single diploid human zygote contains roughly 1.5 GB of raw data of which only about 30 MB is exomic and yet something as extraordinarily complex as a newborn baby is produced from such sparse information. Either nature uses compression algorithms heretofore unimaginable to the world’s best cryptographers or there are layers of information encoding and processing yet to be discovered. While the application of particle physics and quantum information theory are used ever more frequently in the world’s top data encryption laboratories, there appears to be no serious effort in applying these disciplines to understanding information encoding and processing in the genome. While it is known that DNA is held together with quantum entangled electrons along the entirety of the DNA backbone, we have yet to explore the possibility that this or other non-sequence physical property has any contribution to information storage and management in the genome. Genomics as an information sciences discipline needs to expand beyond simply understanding nucleotide sequence. Without this, a full understanding of genomic contributions to health and disease can never be completely realized.

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C2 - July 17, 18:00 – 18:30

THE LOCAL RDGE MACHINE: INFERENCE OF DYNAMIC MODELS OF GENE  
REGULATION

**Xin Guo**

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We present a novel approach, the Local Edge Machine, for the inference of regulatory interactions directly from time-series gene expression data. We demonstrate its performance, robustness, and scalability on in silico datasets with varying behaviors, sizes, and degrees of complexity. Moreover, we demonstrate its ability to incorporate biological prior information and make informative predictions on a well-characterized in vivo system using data from budding yeast that have been synchronized in the cell cycle. Finally, we use an atlas of transcription data in a mammalian circadian system to illustrate how the method can be used for discovery in the context of large complex networks.

*Joint work with Kevin A. McGoff (UNC Charlotte, USA), Anastasia Deckard (Duke University, USA), Christina M. Kelliher (Duke University, USA), Adam R. Leman (Duke University, USA), Lauren J. Francey (Duke University, USA), John B. Hogenesch (University of Cincinnati, USA), Steven B. Haase (Duke University, USA), and John L. Harer (Duke University, USA).*

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C2 - July 18, 14:30 – 15:20

TOPOLOGICAL DETERMINATION OF H-BONDS ROTATIONS IN PROTEINS

**Jørgen Andersen**

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In this talk we will first recall our H-bond rotation descriptor of the local geometry in a proteins around an H-bond. We will then report on recent experimental work in progress on how to determine this descriptor by combinatorial topological methods. Finally we will discuss possible strategies for determining combinatorial topological data from Primary sequence.

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C2 - July 18, 15:30 – 16:00

EXPLORATION OF CANCER DETECTION AND TREATMENT WITH DATA SCIENCE

**Xuan Michael**

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There is a lot of attention and exploration in the detection and treatment of certain types of cancer with the help of big data and machine learning methods.

We would like to report some of development and exploration in this field in UniData Technology

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C2 - July 18, 16:00 – 16:30

CENTRALITY IN BIOLOGICAL NETWORKS

**Alfred Hero**

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Vertex centrality measures arise in topological network analysis, in particular for community detection, clustering, and vertex nomination. This talk will discuss centrality in the context of biological networks. We will then introduce a new variational representation for betweenness centrality in large Euclidean graphs that can be used to characterize influential vertices in gene expression networks.

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C2 - July 18, 17:00 – 17:30

DSGRN: GLOBAL GUIDE TO REGULATORY NETWORK DYNAMICS I: APPLICATIONS

**Tomas Gedeon**

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Dynamic Signatures Generated by Regulatory Networks (DSGRN) provides a queryable description of global dynamics over the entire parameter space. DSGRN is based on a new approach to dynamical systems, which moves the focus away from trajectories and invariant sets, and toward robust, scalable and computable description of dynamics in terms of lattices and posets. On the level of software, DSGRN takes as input a regulatory network and outputs a queryable SQL database that provides information about the structure of global dynamics over all of the associated parameter space.

In this first lecture of the series of two, we apply DSGRN to challenging problems in gene regulation. Experimental data on gene regulation is mostly qualitative, and quantitative data



is subject to large uncertainty. There are no first principle models for gene regulatory networks, and no canonical choices of nonlinearities. Given these realities, it is very difficult to make reliable predictions of dynamical behavior of these networks using mathematical models. The current approach of choosing reasonable nonlinearities, reasonable parameter values, a few initial conditions and run simulations, is severely subsampling both the parameter and phase space, and does not provide provable predictions about the dynamics. The coarse description of dynamics over entire parameter space obtained by our method provably captures several important aspects of the dynamics, and addresses the limitations imposed by sub-cellular biology.

In this talk we describe work on E2F-Rb network, that controls mammalian cell cycle restriction point and has been implicated in many cancers. We show that a large portion of the parameters support either the proliferative state, quiescent state, or hysteresis between these two states. We sample perturbations of this network and study robustness of this dynamics in the network space.

*Joint work with Bree Cummins (Montana State University, USA), Shawn Harker (Rutgers University, USA), and Konstantin Mischaikow (Rutgers University, USA).*

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C2 - July 18, 17:30 – 18:00

## GLOBAL GUIDE TO REGULATORY NETWORK DYNAMICS II: THEORY

**Konstantin Mischaikow**

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Dynamic Signatures Generated by Regulatory Networks (DSGRN) provides a queryable description of global dynamics over the entire parameter space. DSGRN is based on a new approach to dynamical systems, which moves the focus away from trajectories and invariant sets, and toward robust, scalable and computable description of dynamics in terms of lattices and posets. On the level of software, DSGRN takes as input a regulatory network and outputs a queryable SQL database that provides information about the structure of global dynamics over all of the associated parameter space.

In this second lecture of the series of two, we discuss the theory and algorithms that form the basis for the DSGRN computations. In particular, we will describe tasks that DSGRN performs including:

1. the decomposes parameter space into a finite collection of regions with explicit expressions as semi-algebraic sets,
2. for each element of parameter space representation of the dynamics in the form of a state transition graph with the property that the state transition graph is constant over the above mentioned regions of the parameter space,
3. the reduction of information of the state transition graph into a poset, called a Morse graph, for which the nodes represent recurrent dynamics and the edges indicate the direction of gradient-like dynamics.

Time permitting we will discuss current questions including the following:

1. theorems that relate the DSGRN information to more classical concepts from dynamical systems such as attractors and Morse decompositions,
2. introducing algebraic topological methods for analyzing the structure of the nonlinear dynamics,
3. refinement of the computations,
4. potential methods for compressing the information in the database.

*Joint work with Tomas Gedeon (Montana State University, USA), Shaun Harker (Rutgers University, USA), and Bree Cummins (Montana State University, USA).*

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C2 - July 18, 18:00 – 18:30

#### DYNAMICS IN LARGE GENETIC NETWORKS

**Leon Glass**

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Genetic activity is partially regulated by a complicated network of proteins, coded by genes, called transcription factors. I will describe a mathematical framework dating to the 1970s to relate the structure of a network to qualitative features of the dynamics. The underlying idea is to first capture the topology and logic of the network interactions by a Boolean network, and to then embed the Boolean network into a continuous piecewise linear differential equation. A symbolic representation of any given network and of its dynamics is given by a directed graph on a Boolean  $N$ -cube, where  $N$  is the number of genes. Assuming no self-input, each different network of  $N$  genes generates a different directed  $N$ -cube in which each edge is uniquely directed. An attracting cycle on a directed  $N$ -cube is a cycle for which each vertex on the cycle is connected to  $N - 2$  edges not on the cycle that are directed towards it. In any dimension, an attracting cycle on the hypercube implies a stable unique stable limit cycle in an associated piecewise linear differential equation. A simple example is the 3-dimensional repressilator. More complicated examples in 4 and higher dimensions can have chaotic dynamics. I pose the question: What can we tell about the dynamics in a given piecewise linear equation from its associated directed  $N$ -cube as  $N$  becomes large? If this model is a reasonable way to think about real genetic networks, then there must be strong restrictions on the class of possible logical structures in order to have well controlled dynamics. I have benefited from collaborations with many colleagues, especially S.A. Kauffman and J. Pasternack in the 1970s and R. Edwards more recently.

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C2 - July 18, 18:30 – 19:00

#### AN ALGORITHM FOR CELLULAR REPROGRAMMING

**Rajapakse Indika**

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In 2007, a remarkable discovery was made that with just four external inputs (transcription factors), it was possible to change differentiated cells into embryonic-like cells. This type of cellular reprogramming changes the fundamental nature of a cell. It invites the possibility of building a universal template for transcription factor-guided reprogramming. I will present our work on an algorithm for cellular reprogramming that uses advanced genomics technologies + mathematics.

*Joint work with Anthony Bloch (University of Michigan, USA) and Roger Brockett (Harvard University, USA).*

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C2 - July 19, 14:30 – 15:00

CAN A MAXIMUM ENTROPY MODEL EXPLAIN CODON AND AMINO ACID ABUNDANCES  
IN GENOMES AND PROTEOMES?

**Ignacio Enrique Sánchez**

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Living cells share some nearly universal features, such as the genetic code and the monomers that constitute proteins and nucleic acids. Our hypothesis is that the relative abundances of codons in genomes and amino acids in proteomes can be explained from a reduced number of nearly universal principles. In previous work, we were able to explain the relative abundances of amino acids in over 100 proteomes by postulating that all organisms minimize the energy flux through amino acid metabolism. We now present a two-layer maximum entropy model that can simultaneously explain the relative abundances of amino acids in proteomes and codons in genomes by adding coding constraints to the analysis.

*Joint work with Michael Shub (City College of the City University of New York, USA) and Teresa Krick (Universidad de Buenos Aires & CONICET, Argentina).*

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C2 - July 19, 15:00 – 15:30

LARGE SCALE 3D CHROMATIN RECONSTRUCTION FROM CHROMOSOMAL CONTACTS

**Shuaicheng Li**

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Recent advances in genome analysis techniques have established that chromatin has preferred three-dimensional (3D) conformations. Spatial folding can bring two distant genes along the nucleotide sequence into contact. Identifying these contacts is important to understand activities of genes. This has motivated the proposal of methods like Hi-C to detect long-range interactions in the past decade. One can further gain insights on the contacts by applying distance geometry techniques to infer the chromosomal 3D structures from Hi-C data, as has been demonstrated by algorithms such as ChromSDE and ShRec3D. These matrix-based algorithms, however, are space- and time- consuming on very large datasets. A human genome of 100 kilobase resolution would involve 30,000 loci, requiring gigabytes in merely storing the matrices. In this work, we

propose a succinct representation of the distance matrix which tremendously reduces this space requirement. We give a complete solution, called SuperRec, for the inference of chromosomal structures from Hi-C data, through solving the large-scale weighted multidimensional scaling problem. SuperRec runs faster than earlier systems without compromise on result accuracy. Using SuperRec, we were able to reconstruct a structure of 30,000 loci more than 400 times faster than existing methods — SuperRec solved the structure in 43 minutes whereas the state of the art method ShRec3D took 13 days.

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C2 - July 19, 15:30 – 16:00

#### CLUSTERING WITH SPARSE DATA: THE CASE OF SINGLE-CELL GENOMIC PROFILES

**Jun Li**

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Sequencing-based genomic profiling of individual cells, such as single-cell RNA-sequencing (scRNA-seq), produces low-count integer data for the number of transcripts of each gene measured in each cell. These data are called “sparse”, because of high fractions of missing values as well as the issue of low-rank approximation. My group, along with other members of the Michigan Center for Single-Cell Genomic Data Analytics, have been wrestling with the problem of ab initio class discovery using such sparse data. Many specialized methods have appeared in recent years to address noise modeling, normalization, batch effect correction, rare cell type identification, network inference, and integration across sparse ‘omics datatypes. However, several challenges remain in making statistically principled inference. Foremost among these is the need for community standards in declaring and objectively assessing clustering solutions. This is challenging as the true “structure” of the data is not known beforehand, and the full range of generative models include not only manifolds of all possible shapes, but also discrete vs. continuous clusters, simple partitioning vs. hierarchical, full membership vs partial membership, or combinations of these scenarios. Further, distribution density in the high-dimensional space could vary greatly from one region of the manifold to another, calling for adaptive learning of “local structure”. In practice, however, development of new methods and their application in real datasets have often considered only a subset of these models; and the underlying assumptions are often unacknowledged. With this backdrop, our Center has begun designing best practices that systematically evaluate diverse generative models, and providing guidance on how to identify the most appropriate ones. We also assess how the underlying assumptions impact the choice of data normalization and clustering algorithms. Addressing these challenges are relevant in other data science areas, including learning from electronic health records, consumer purchasing and rating data, location and usage pattern of mobile devices, connectivity in social networks, and medical imaging data. Similarly, the task of manifold alignment across omics data types represents a special case of integration over multiple networks inferred with imperfect information.

*Joint work with Sue Hammoud, Xiang Zhou, and Anna Gilbert (University of Michigan, USA).*

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C2 - July 19, 16:00 – 16:30

MATHEMATICS OF TRANSCRIPTION: DECODING CIS-REGULATION FROM THE  
DROSOPHILADAЕ TO THE SEPSIDAE AND BACK AGAIN

**John Reinitz**

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The syncytial organization of the blastoderm stage of *Drosophila* development affords unique advantages for the mathematical modeling of transcriptional control. It is possible to use the entire embryo as a spatially resolved microarray in which the response of reporters to quantitatively assayed transcription factors can be monitored at cellular resolution. This provides an opportunity to construct quantitative and predictive models of transcriptional control that are not limited to single enhancers. I will discuss progress in this effort, with emphasis on a case where understanding conservation of function in the absence of conservation of sequence leads to the identification of functional clusters of binding sites that are arguably interpretable as cis-regulatory codons.

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C2 - July 19, 17:00 – 17:30

A DIFFERENTIAL INCLUSION APPROACH TO VARIABLE SELECTION AND APPLICATIONS

**Yuan Yao**

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A differential inclusion approach is presented for variable selection in high dimensional statistics. The differential inclusions arise from the limits of the Linearized Bregman Iterations, or equivalently the Mirror Descent algorithm associated with sparsity-induced Bregman distances. A path consistency theory is presented to explain its observed superb performance in comparison to the (generalized) Lasso estimate. Application examples are given, including a case study in Alzheimer Disease detection.

*Joint work with this talk contains joint work with: Chendi Huang (Peking University, China), Lingjing Hu (Chinese Capital Medical University, China), Stan Osher (UCLA, USA), Feng Ruan (Stanford University, USA), Xinwei Sun (Peking University, China), Yizhou Wang (Peking University, China) and Jiechao Xiong (Peking University and Tencent AI Lab, China).*

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## Workshop C3 Continuous Optimization

Organizers: Javier Peña – Coralia Cartis – Etienne de Klerk

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C3 - July 17, 14:30 – 15:00

### TRANSLATIVE PACKINGS IN THREE DIMENSIONS

**Frank Vallentin**

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Finding optimal packings of spheres and more generally of nonspherical objects is a central problem in discrete geometry which gives rise to challenging optimization problems. In this talk I will show new lower and new upper bounds for the optimal density of translative packings of superballs (unit balls of  $l_p$ -norms) in three dimensions. For finding new lower bounds we use Minkowski's classical approach based on nonlinear optimization. For finding new upper bounds we use the Cohn-Elkies bound based on linear optimization.

*Joint work with M. Dostert, C. Guzmán, and F.M. de Oliveira Filho.*

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C3 - July 17, 15:00 – 15:30

### NEW UPPER BOUNDS FOR THE KISSING NUMBER VIA COPOSITIVE OPTIMIZATION

**Juan Vera**

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We build a hierarchy of upper bounds on the kissing number using copositive optimization. Recently, it has been shown that the kissing number can be reformulated as an infinite dimensional optimization problem over the set of copositive kernels on a sphere.

To obtain a hierarchy for the kissing number, we extend existing sdp-based hierarchies for the finite dimensional copositive cone to the infinite dimensional one; obtaining a hierarchy based on (infinite dimensional) sdp-kernels. Then we exploit the symmetry of the sphere to obtain a finite dimensional SDP formulation.

*Joint work with Olga Kuryatnikova (Tilburg University, The Netherlands).*

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C3 - July 17, 15:30 – 16:00

### SEMIDEFINITE PROGRAMS WITH A DASH OF SMOOTHNESS: WHY AND WHEN THE LOW-RANK APPROACH WORKS

**Nicolas Boumal**

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Semidefinite programs (SDPs) can be solved in polynomial time by interior point methods, but scalability can be an issue. To address this shortcoming, over a decade ago, Burer and Monteiro proposed to solve SDPs with few equality constraints via low-rank, non-convex surrogates. Remarkably, for some applications, local optimization methods seem to converge to global optima of these non-convex surrogates reliably.

In this presentation, we show that the Burer-Monteiro formulation of SDPs in a certain class almost never has any spurious local optima, that is: the non-convexity of the low-rank formulation is benign (even saddles are strict). This class of SDPs covers applications such as max-cut, community detection in the stochastic block model, robust PCA, phase retrieval and synchronization of rotations.

The crucial assumption we make is that the low-rank problem lives on a manifold. Then, theory and algorithms from optimization on manifolds can be used.

Optimization on manifolds is about minimizing a cost function over a smooth manifold, such as spheres, low-rank matrices, orthonormal frames, rotations, etc. We will present the basic framework as well as parts of the more general convergence theory, including recent complexity results. (Toolbox: <http://www.manopt.org>).

*Joint work with P.-A. Absil (Université Catholique de Louvain, Belgium), A. Bandeira (Courant Institute, NYU), C. Cartis (University of Oxford, UK), and V. Voroninski (formerly MIT).*

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C3 - July 17, 16:00 – 16:30

#### ROBUST-TO-DYNAMICS LINEAR PROGRAMMING

**Amir Ali Ahmadi**

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We consider a class of robust optimization problems that we call “robust-to-dynamics optimization” (RDO). The input to an RDO problem is twofold: (i) a mathematical program (e.g., an LP, SDP, IP, etc.), and (ii) a dynamical system (e.g., a linear, nonlinear, discrete, or continuous dynamics). The objective is to maximize over the set of initial conditions that forever remain feasible under the dynamics. The focus of this presentation is on the case where the optimization problem is a linear program and the dynamics are linear or switched linear. We establish some structural properties of the feasible set and prove that if the linear system is asymptotically stable, then under mild conditions the RDO problem can be solved in polynomial time. We also present a semidefinite programming based algorithm for providing upper bounds on robust-to-dynamics linear programs.

*Joint work with Oktay Gunluk (IBM Research, USA).*

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C3 - July 17, 17:00 – 17:30

PIECEWISE PARAMETRIC STRUCTURE IN THE POOLING PROBLEM — FROM SPARSE,  
STRONGLY-POLYNOMIAL AOLUTIONS TO NP-HARDNESS

**Ruth Misener**

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Standard pooling is an NP-hard, non-convex blending problem that arises in process networks applications such as crude oil scheduling and more widely as a general problem pattern [Ceccon & al., AIChE J, 2016]. Unfortunately, state-of-the-art general solvers do not scale well to large-scale instances of the problem which are commercially-relevant.

In this talk we introduce an algorithm that solves specialized pooling problem instances to global optimality and integrate it within a Branch and Bound framework for more generic instances. The approach parameterizes the optimization problem with respect to the pool concentration variables and uncovers embedded sparsity and polyhedral/topological properties for a variety of instances. We generalize and extend recent work analyzing the computational complexity of the pooling problem [Haugland, J. Glob Optim, 2016; Boland & al., J. Glob Optim, 2016; Haugland and Hendrix, J. Optim Theory Appl., 2016]. Our analysis also integrates source-to-output streams and both upper and lower bounds on the network parameters.

*Joint work with Radu Baltean-Lugoian.*

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C3 - July 17, 17:30 – 18:00

MANIFOLD SAMPLING FOR PIECEWISE LINEAR COMPOSITE NONCONVEX  
OPTIMIZATION

**Stefan Wild**

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We address minimization of a function  $\psi(x) + h(F(x))$ , where  $\psi$  and  $F$  are smooth functions,  $h$  is piecewise linear, and the Jacobian of  $F$  is unavailable. We are motivated by cases where  $F$  represents the outputs of an expensive blackbox simulation and  $h$  represents a nonsmooth loss function.

We employ a derivative-free trust-region framework, whereby smooth models are built to approximate  $F$  using zero-order information. Under minimal assumptions, we show that our resulting “master model” generates descent directions and that the algorithm’s cluster points are Clarke stationary. Numerical experiments demonstrate the benefit of the proposed algorithm as well as current limitations to achieving better performance in practice.

*Joint work with Jeff Larson (Argonne National Laboratory) and Kamil Khan (McMaster University).*



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C3 - July 17, 18:00 – 18:30

DIRECT SEARCH BASED ON PROBABILISTIC FEASIBLE DESCENT FOR BOUND AND  
LINEARLY CONSTRAINED PROBLEMS

**Luis Nunes Vicente**

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Direct search is a methodology for derivative-free optimization whose iterations are characterized by evaluating the objective function using a set of polling directions. In deterministic direct search applied to smooth objectives, these directions must somehow conform to the geometry of the feasible region and typically consist of positive generators of approximate tangent cones (which then renders the corresponding methods globally convergent in the linearly constrained case). One knows however from the unconstrained case that randomly generating the polling directions leads to better complexity bounds as well as to gains in numerical efficiency, and it becomes then natural to consider random generation also in the presence of constraints.

In this paper, we study a class of direct search based on sufficient decrease for solving smooth linearly constrained problems where the polling directions are randomly generated (in approximate tangent cones). The random polling directions must satisfy probabilistic feasible descent, a concept which reduces to probabilistic descent in the absence of constraints. Such a property is instrumental in establishing almost-sure global convergence and worst-case complexity bounds with overwhelming probability. Preliminary numerical results show that the randomization of polling directions compares favorably to the classical deterministic approach.

*Joint work with Serge Gratton (Toulouse, France), Clément W. Royer (Madison, USA), and Zaikun Zhang (Hong Kong, Hong Kong).*

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C3 - July 17, 18:30 – 19:00

FASTER ALGORITHMS FOR COMPUTING THE STATIONARY DISTRIBUTION

**Aaron Sidford**

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Computing the stationary distribution of a Markov Chain is one of the most fundamental problems in optimization. It lies at the heart of numerous computational tasks including computing personalized PageRank vectors, evaluating the utility of policies in Markov decision process, and solving asymmetric diagonally dominant linear systems.

Despite the ubiquity of these problems, until recently the fastest known running times for computing the stationary distribution either depended polynomially on the mixing time or appealed to generic linear system solving machinery, and therefore ran in super-quadratic time. In this talk I will present recent results showing that the stationary distribution and related quantities can all be computed in almost linear time. Moreover, I will discuss connections between this work and recent developments in solving Laplacian systems and optimizing over directed graphs.

*Joint work with Michael B. Cohen (MIT, USA), Jonathan Kelner (MIT, USA), John Peebles (MIT, USA), Richard Peng (Georgia Tech, USA), and Adrian Vladu (MIT, USA).*

C3 - July 17, 19:00 – 19:30

### SOLVING CHANCE-CONSTRAINED PROBLEMS USING A KERNEL VAR ESTIMATOR

**Andreas Waechter**

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We present a reformulation of nonlinear optimization problems with chance constraints that replaces a probabilistic constraint by a nonparametric value-at-risk estimator. The estimator smooths the historical VaR via a kernel, resulting in a better approximation of the quantile and reducing the nonconvexity of the feasible region. An optimization algorithm is proposed that permits the efficient treatment of joint chance constraints. Theoretical convergence properties and numerical experiments are presented.

*Joint work with Alejandra Pena-Ordieres (Northwestern University, USA).*

C3 - July 18, 14:30 – 15:00

### EFFICIENCY OF MINIMIZING COMPOSITIONS OF CONVEX FUNCTIONS AND SMOOTH MAPS

**Dmitriy Drusvyatskiy**

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Numerous optimization tasks can be posed as minimization of a finite convex function composed with a smooth map. I will discuss various aspects of this problem class, including efficiency of first-order methods, stochastic algorithms for large finite sums, fast local rates of convergence, and termination criteria.

*Joint work with Alexander D. Ioffe (Technion, Israel), Adrian S. Lewis (Cornell University, USA), and Courtney Paquette (University of Washington, USA).*

C3 - July 18, 15:00 – 15:30

### SHARPNESS, RESTART AND ACCELERATION

**Alexandre d'Aspremont**

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The Lojasiewicz inequality shows that sharpness bounds on the minimum of convex optimization problems hold almost generically. Here, we show that sharpness directly controls the performance of restart schemes. The constants quantifying sharpness are of course unobservable, but optimal restart strategies are fairly robust, hence searching for the best scheme only increases the complexity by a logarithmic factor versus the optimal bound. Overall, this shows that restart schemes generically accelerate accelerated methods.

*Joint work with Vincent Roulet (D.I., École Normale Supérieure, France).*

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C3 - July 18, 15:30 – 16:20

## ONLINE FIRST-ORDER FRAMEWORK FOR ROBUST CONVEX OPTIMIZATION

**Fatma Kilinc-Karzan**

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Robust optimization (RO) has emerged as one of the leading paradigms to efficiently model parameter uncertainty. The recent connections between RO and problems in statistics and machine learning domains demand for solving RO problems in ever more larger scale. However, the traditional approaches for solving RO formulations based on building and solving robust counterparts or the iterative approaches utilizing nominal feasibility oracles can be prohibitively expensive and thus significantly hinder the scalability of RO paradigm.

We present a general and flexible iterative framework to solve robust convex optimization problems that is built on a fully online first-order paradigm. In contrast to the existing literature, a key distinguishing feature of our approach is that it requires access to only cheap first-order oracles for the original problem that are remarkably cheaper than pessimization or nominal feasibility oracles, while maintaining the same convergence rates. This, in particular, makes our approach much more scalable and hence preferable in large-scale applications. In the case of strongly convex functions, we also establish a new improved iteration complexity addressing an open question in the literature. We demonstrate our approach on robust quadratic programs with ellipsoidal uncertainty.

*Joint work with Nam Ho-Nguyen (Carnegie Mellon University, USA).*

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C3 - July 18, 17:00 – 17:30

## PERFORMANCE ESTIMATION OF FIRST-ORDER METHODS FOR COMPOSITE CONVEX OPTIMIZATION

**François Glineur**

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Composite convex optimization consists in the minimization of a convex objective function equal to the sum of several convex functions with different properties, for example a smooth term and a nonsmooth term. We consider a large class of first-order methods designed to solve such composite problems, which rely on specific oracles for each term. We show that the worst-case performance of each of those methods can be computed exactly by solving a semidefinite optimization problem, which also produces an explicit problem instance for which this worst-case is attained.

The performance estimation methodology was born in the original work of Drori and Teboulle in 2014, which introduced a semidefinite relaxation to study the behaviour of first-order optimization algorithms for smooth unconstrained convex optimization. In this talk, we present a framework that produces exact convergence bounds for fixed-step linear first-order methods

applied to general composite convex optimization. These methods include classical and accelerated gradient methods (including constrained and proximal variants), conditional gradient and subgradient methods, and also allow inexact gradient computations.

In particular, our approach allows us to derive exact convergence rates for the proximal gradient method in function value, gradient residual norm and distance to the solution, backed by independently-checkable proofs. We also use numerical computations to obtain worst-case rates for several well-known methods including accelerated gradient and the conditional gradient method, improving on the best published rates. We conclude with a quick overview of a MATLAB toolbox implementing our framework, called PESTO (Performance Estimation Toolbox), available from the URL below:

<https://github.com/AdrienTaylor/Performance-Estimation-Toolbox>.

*Joint work with Adrien B. Taylor (Université catholique de Louvain, Belgium) and Julien M. Hendrickx (Université catholique de Louvain, Belgium).*

C3 - July 18, 17:30 – 18:00

## HUGE-SCALE CONVEX OPTIMIZATION, AND INCREMENTAL SUBGRADIENT METHODS

**James Renegar**

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Five years ago, Nesterov considered “a new class of huge-scale problems, the problems with sparse subgradients.” He showed, for example, that for the problem of minimizing a piecewise-linear convex function  $x \mapsto \max\{\alpha_i^T x + \beta_i \mid i = 1, \dots, m\}$ , a full iteration of a subgradient method could be achieved with cost only logarithmic in  $m$ , assuming the vectors  $\alpha_i$  to be very sparse, and assuming for each  $j$  that the coefficient  $\alpha_{i,j}$  is nonzero for only a modest number of the vectors  $\alpha_i$ .

Under analogous assumptions, we show how an iteration cost of  $O(\log m)$  can be achieved for the problem of minimizing  $c^T x$  subject to  $x \in \cap_{i=1}^m Q_i$ , where the  $Q_i$  are closed, convex sets. Our development happens to also lead naturally to a deterministic (as opposed to stochastic) incremental subgradient method for minimizing summations  $\sum_{i=1}^m f_i(x)$ , assuming the functions  $f_i$  to satisfy certain conditions including being convex and lower semicontinuous. The algorithm here is interesting particularly in that unlike previous incremental methods, over time it does not treat the individual summands  $f_i$  equally, but instead discovers some of the summands  $f_i$  as being more crucial to minimizing the overall summation.

(This is work in progress.)

C3 - July 18, 18:00 – 18:30

## SPARSE INTERPOLATION VIA SUPER-RESOLUTION AND COMPRESSED SENSING

**Jean B. Lasserre**

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We consider the sparse interpolation polynomial problem, that is, given an unknown black-box polynomial with very few coefficients, retrieve its coefficients and exponents from a few evaluations only. We show that this problem can be formulated as a Super-Resolution problem on the Torus for which exact recovery is guaranteed under some spacing assumption on the support. It also implies that exact recovery is guaranteed via  $L_1$ -norm minimization, i.e., solving a compressed sensing LP, even though the RIP property is not satisfied. We also compare this approach (with & without noise) with the well-known Prony method on some numerical examples.

*Joint work with Cédric Jozs (LAAS-CNRS, Toulouse, France) and Bernard Mourrain (INRIA, Sophia-Antipolis, France).*

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C3 - July 18, 18:30 – 19:00

#### A NEW VARIANCE REDUCTION METHOD FOR FINITE SUMS WITH RECURSIVE STOCHASTIC GRADIENT

**Katya Scheinberg**

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We propose a StochAstic Recursive grAdient algorithM (SARAH) for finite-sum minimization problems. Similarly to SVRG, SARAH admits a simple recursive framework for updating stochastic gradient estimates. Linear convergence rate of SARAH is shown under strong convexity assumption, in addition we show a sublinear convergence rate (in the strongly convex case) for an inner loop of SARAH, the property that SVRG does not have. We also discuss the convergence rates for the nonconvex case and demonstrate the efficiency of the algorithm via numerical experiments.

*Joint work with Lam Ngyuen (Lehigh University, USA), Jie Liu (Lehigh University, USA), and Martin Takáč (Lehigh University, USA).*

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C3 - July 19, 14:30 – 15:20

#### ACCELERATING THE UNIVERSAL NEWTON METHODS

**Yurii Nesterov**

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In this talk we present new results related to the universal second-order methods, which can automatically adjust the level of smoothness of the objective function. Our methods converge in accordance to the best rates allowed by a Holder condition introduced for the Hessians. As compared with the usual Newton method, the reason for acceleration of our schemes consists in accumulation of global information on the behavior of the objective, represented by a linear model. Our methods solve the convex optimization problems in composite form.

*Joint work with Geovani Grapiglia (Federal University of Parana, Brazil).*

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C3 - July 19, 15:30 – 16:00

TOWARDS UNDERSTANDING THE CONVERGENCE BEHAVIOR OF NEWTON-TYPE  
METHODS FOR STRUCTURED OPTIMIZATION PROBLEMS

**Anthony Man–Cho So**

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Recently, there has been a growing interest in applying Newton-type methods to solve structured optimization problems that arise in machine learning and statistics. A major obstacle to the design and analysis of such methods is that many problems of interest are neither strongly convex nor smooth. In this talk, we will present some design techniques for overcoming such obstacle and report some recent progress on analyzing the convergence rates of the resulting Newton-type methods using error bounds. We will also discuss some directions for further study.

*Joint work with Man-Chung Yue (The Chinese University of Hong Kong) and Zirui Zhou (Simon Fraser University).*

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C3 - July 19, 16:00 – 16:30

COMPLEXITY OF NONCONVEX OPTIMIZATION: THE STORY SO FAR

**Coralia Cartis**

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We give an overview of results regarding complexity of nonconvex optimization (without constraints), focusing on two antipodal algorithmic cases, namely, high-order regularisation methods with improved complexity and universal properties, and zero-order regularization methods with probabilistic models. A story of robust methods, with continuously-varying and often-sharp complexity bounds, emerges.

*Joint work with Nick Gould (Rutherford Appleton Laboratory, UK), Philippe Toint (University of Namur, Belgium), and Katya Scheinberg (Lehigh University, USA).*

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C3 - July 19, 17:00 – 17:30

ON SOLVING THE QUADRATIC SHORTEST PATH PROBLEM

**Renata Sotirov**

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The quadratic shortest path problem (QSPP) is the problem of finding a path in a digraph such that the sum of weights of arcs and the sum of interaction costs over all pairs of arcs on the path is minimized. The QSPP is NP-hard. In this talk we consider first linearizable QSPP instances whose optimal solution can be obtained by solving the corresponding instance of the shortest path problem. We also present a polynomial-time algorithm that verifies whether a QSPP instance on the directed grid graph is linearizable. Further, we present a semidefinite

programming relaxation for the QSPP that provides strong bounds for non-linearizable problem instances of moderate size.

*Joint work with Hao Hu (Tilburg University, The Netherlands).*

C3 - July 19, 17:30 – 18:00

### SINGULAR VALUE DECOMPOSITIONS AND OPTIMIZATION OF SYMMETRIC FUNCTIONS

**Raphael Hauser**

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In this talk we will derive a novel characterization of the leading part SVD of real matrices in terms of optimization models over symmetric functions. The new characterization frees the singular value decomposition from its inherent dependency on the Frobenius distance and the orthogonality of factors, leading to new dimensionality-reduction models and algorithms in data science.

*Joint work with Armin Eftekhari (Alan Turing Institute, London, UK).*

C3 - July 19, 18:00 – 18:30

### A POLYNOMIAL ALGORITHM FOR LINEAR FEASIBILITY PROBLEMS AND ITS APPLICATIONS IN INFINITE-DIMENSIONAL OPTIMIZATION

**Sergei Chubanov**

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In this talk, we will consider an algorithm for solving systems of linear inequalities given by separation oracles. Without loss of generality, we assume that the system in question is homogeneous, i.e., that it defines a cone. We are looking for a nonzero solution of this system. The algorithm is based on a procedure for finding sufficiently short convex combinations of the rows of the coefficient matrix. Whenever such a combination is found, the algorithm performs a linear transformation of the space which depends on this convex combination. The running time of the algorithm is bounded by a polynomial in the number of variables and in the maximum binary size of an entry of the coefficient matrix, provided that the separation oracle is a polynomial algorithm.

In many situations, like for instance in the case of the minimum-cost flow over time problem (which is NP-hard) we are looking for a function optimizing a linear functional subject to an infinite number of linear constraints, feasible solutions being square-integrable functions. Identifying functions which are equal up to a set of measure zero, we come to linear optimization problems in Hilbert spaces. Similarly to the finite-dimensional case, we can reduce such problems to a sequence of conic feasibility problems. The algorithm mentioned above can be modified to solve such systems with a given accuracy. The algorithm does not use discretization and works in the space of the original problem.

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C3 - July 19, 18:30 – 19:00

COMPARISON OF LASSERRE’S MEASURE-BASED BOUNDS FOR POLYNOMIAL  
OPTIMIZATION TO BOUNDS OBTAINED BY SIMULATED ANNEALING

**Etienne de Klerk**

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We consider the problem of minimizing a continuous function  $f$  over a compact set  $K$ . We compare the hierarchy of upper bounds proposed by Lasserre in [SIAM J. Optim. 21 (2011), no. 3, 864–885] to bounds that may be obtained from simulated annealing. We show that, when  $f$  is a polynomial and  $K$  a convex body, this comparison yields a faster rate of convergence of the Lasserre hierarchy than what was previously known in the literature.

*Joint work with Monique Laurent (CWI, Amsterdam, and Tilburg University, The Netherlands).*

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C3 - Poster

GRADIENT METHOD WITH INEXACT ORACLE FOR COMPOSITE NON-CONVEX  
OPTIMIZATION

**Pavel Dvurechensky**

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We develop new first-order method for composite non-convex minimization problems with simple constraints and inexact oracle information. The objective function is given as a sum of “hard”, possibly non-convex part, and “simple” convex part. Informally speaking, oracle inexactness means that, for the “hard” part, at any point we can approximately calculate the value of the function and construct a quadratic function, which approximately bounds the whole function from above. We give several examples of such inexactness: smooth non-convex functions with inexact Hölder-continuous gradient, functions given by auxiliary uniformly concave maximization problem, which can be solved only approximately. For the introduced class of problems, we propose a gradient-type method, which allows to use different proximal setup to adapt to geometry of the feasible set, adaptively chooses controlled oracle error, allows for inexact proximal mapping. We provide convergence rate for our method in terms of the norm of generalized gradient mapping and show that, in the case of inexact Hölder-continuous gradient, our method is universal with respect to Hölder parameters of the problem. Details can be found in <https://arxiv.org/abs/1703.09180>.

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C3 - Poster

NONLINEAR SPECTRAL METHODS FOR NONCONVEX OPTIMIZATION WITH GLOBAL  
OPTIMALITY GUARANTEES

**Antoine Gautier**

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We present the Nonlinear Spectral Method, introduced in [1, 2], which can be used to solve a class of nonconvex optimization problems over the product of nonnegative  $\ell^p$  spheres with global optimality guarantees and linear convergence rate. Advantages of this method are that the global optimality assumptions as well as the convergence rate can be easily checked before running the method by computing the spectral radius of a small nonnegative matrix which depends on the objective function and the norm constraints. The method is inspired by the theory of (sub)-homogeneous nonlinear eigenproblems on convex cones which has its origin in the Perron-Frobenius theory for nonnegative matrices. In fact this work is motivated by the closely related Perron-Frobenius theory for multi-homogeneous problems developed in [3].

Following [1], we show that this method can be applied for the training of a class of generalized polynomial neural networks on real-world datasets. This is the first practically feasible method for training a class of nonlinear neural networks with global convergence guarantees. We also discuss how the method can be used to compute projective norms of nonnegative tensors.

## Bibliography

- [1] A. Gautier, Q. Nguyen, & M. Hein, “Globally optimal training of generalized polynomial neural networks with nonlinear spectral methods.” NIPS, 2016.
- [2] Q. Nguyen, A. Gautier, & M. Hein, “Nonlinear spectral methods for nonconvex optimization with global optimality.” Workshop on Optimization for Machine Learning, 2016.
- [3] A. Gautier, F. Tudisco, & M. Hein, “The Perron-Frobenius theorem for multi-homogeneous maps.” (2017), [ArXiv:1702.03230](#).

*Joint work with Q. Nguyen (Saarland University, Germany) and M. Hein (Saarland University, Germany).*

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C3 - Poster

## DC DECOMPOSITION OF NONCONVEX POLYNOMIALS WITH ALGEBRAIC TECHNIQUES

**Georgina Hall**

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We consider the problem of decomposing a multivariate polynomial as the difference of two convex polynomials. We introduce algebraic techniques which reduce this task to linear, second order cone, and semidefinite programming. This allows us to optimize over subsets of valid difference of convex decompositions (dcds) and find ones that speed up the convex-concave procedure (CCP). We prove, however, that optimizing over the entire set of dcds is NP-hard.

*Joint work with Amir Ali Ahmadi (Princeton University, USA).*

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C3 - Poster

ON THE CONJECTURE BY DEMYANOV-RYABOVA IN CONVERTING FINITE EXHAUSTERS

**Tian Sang**

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The Demyanov-Ryabova conjecture is a geometric problem originating from duality relations between nonconvex objects. Given a finite collection of polytopes, one obtains its dual collection as convex hulls of the maximal faces of sets in the original collection, for each direction in the space (thus constructing upper convex representations of positively homogeneous functions from lower ones and vice versa, via Minkowski duality). It is conjectured that an iterative application of this conversion procedure to finite families of polytopes results in a cycle of length at most two.

We prove a special case of the conjecture assuming an affine independence condition on the vertices of polytopes in the collection. We also obtain a purely combinatorial reformulation of the conjecture.

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## Workshop C4

### Foundations of Numerical PDEs

Organizers: Ricardo Nochetto – Annalisa Buffa – Endre Suli

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C4 - July 17, 14:30 – 14:55

#### COUPLING BULK AND SURFACE PDEs: NUMERICAL APPROXIMATION OF DYNAMIC BIOMEMBRANES

**Robert Nürnberg**

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A parametric finite element approximation of a fluidic membrane, whose evolution is governed by a surface Navier–Stokes equation coupled to bulk Navier–Stokes equations, is presented. The elastic properties of the membrane are modelled with the help of curvature energies of Willmore and Helfrich type. Forces stemming from these energies act on the surface fluid, together with a forcing from the bulk fluid.

We introduce a stable parametric finite element method to solve this complex free boundary problem. Local inextensibility of the membrane is ensured by solving a tangential Navier–Stokes equations, taking surface viscosity effects of Boussinesq–Scriven type into account. In our approach the bulk and surface degrees of freedom are discretized independently, which leads to an unfitted finite element approximation of the underlying free boundary problem. Bending elastic forces resulting from an elastic membrane energy are discretized using an approximation introduced by Dziuk. The obtained numerical scheme can be shown to be stable and to have good mesh properties.

*Joint work with John W. Barrett (Imperial College London, UK) and Harald Garcke (Regensburg University, Germany).*

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C4 - July 17, 15:00 – 15:25

#### DYNAMICAL LOW RANK APPROXIMATION OF RANDOM TIME DEPENDENT PDEs

**Fabio Nobile**

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We investigate dynamical low rank approximations of time dependent PDEs with random data where at each time instant, the approximate solution is sought in the manifold of functions in separable form (separating space and stochastic variables) of fixed rank, and is obtained by Galerkin projection of the governing equations onto the tangent space to the manifold. Suitable equations are then derived for the evolution of the deterministic and stochastic modes. We discuss error estimates and implementation issues. In particular, we investigate and present numerical results on parabolic equations, Navier–Stokes equations for incompressible fluids at low Reynolds number, with emphasis on the case of stochastic boundary conditions, as well as second order wave equations.

## Bibliography

- [1] E. Musharbash & F. Nobile, “Dual dynamically orthogonal approximation of incompressible Navier–Stokes equations with random boundary conditions.” MATHICSE REPORT no. 3 (2017), EPFL, 2017.
- [2] E. Musharbash, F. Nobile, & T. Zhou, “Error analysis of the dynamically orthogonal approximation of time dependent random PDEs.” Siam Journal on Scientific Computing 37 (2015), no. 2, A776–A810.

*Joint work with Eleonora Musharbash (École Polytechnique Fédérale de Lausanne, Switzerland).*

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C4 - July 17, 15:30 – 15:55

### CAUSAL DISCRETIZATIONS OF ANISOTROPIC EIKONAL EQUATIONS

**Jean–Marie Mirebeau**

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Motivated by applications to path planning and image segmentation, we develop numerical methods for finding globally optimal paths subject to non-holonomic constraints. This encompasses various forms of path curvature penalization, such as the Reeds–Shepp car, Euler elastica, or Dubins car models. For that purpose we discretize eikonal equations, or first order static Hamilton–Jacobi–Bellman PDEs, with respect to strongly anisotropic riemannian or finlerian metrics. Our approach relies on tools from discrete geometry, such as Voronoi’s reductions of quadratic forms, to design numerical schemes obeying a property of causality, which are then efficiently solved via the single pass fast marching algorithm.

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C4 - July 17, 16:00 – 16:25

### CONVERGENCE OF ADAPTIVE DISCONTINUOUS GALERKIN METHODS

**Christian Kreuzer**

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Consulting the unprecedented developments of convergence and optimality theory of conforming adaptive finite element methods during the last two decades, the strict Dörfler marking and the resulting reduction of some error quantity appears to be fundamental for most of the results. A straight forward generalisation to discontinuous Galerkin (dG) methods is difficult due to the fact that the penalty term is not necessarily monotone. However, for large enough penalty parameters Karakashian & Pascal (2007) proved error reduction and Bonito and Nochetto (2010) proved optimality for the adaptive symmetric interior penalty method with Dörfler marking.

In contrast to these results, in this talk, we generalise the basic convergence analysis by Morin, Siebert, & Veiser (2008) for adaptive discontinuous Galerkin methods. On the one hand, this provides convergence without rates. On the other hand the theory covers different

dG schemes as well as all practically relevant marking strategies. Another key feature of the presented result is, that it does not require an extra enlargement of the penalty parameter. The analysis is based on a quasi interpolation into a newly developed limit space of the adaptively created non-conforming discrete spaces.

*Joint work with Emmanuil Gourgoulis (University of Leicester, UK and National Technical University, GR).*

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C4 - July 17, 17:00 – 17:50

#### NUMERICAL SOLUTION OF NONSMOOTH PROBLEMS AND APPLICATION TO DAMAGE EVOLUTION AND OPTIMAL INSULATION

**Soeren Bartels**

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Nonsmooth minimization problems and singular partial differential equations arise in the description of inelastic material behavior, image processing, and modeling of non-Newtonian fluids. The numerical discretization and iterative solution is often based on regularizing or stabilizing terms. In this talk we address the influence of such modifications on error estimates and the robustness of iterative solution methods. In particular, we present an unconditional stability result for semi-implicit discretizations of a class of singular flows and devise a variant of the alternating direction of multipliers method with variable step sizes. The results and methods are illustrated by numerical experiments for a damage evolution model and a problem of optimal insulation leading to a break of symmetry.

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C4 - July 18, 14:30 – 14:55

#### CONCURRENT VERSUS SEQUENTIAL MULTI-SCALE METHODS FOR ATOMISTIC MATERIALS MODELLING

**Christoph Ortner**

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A common atomic-scale materials modelling task is to characterise the lattice defects that control material response. Due to their multi-scale nature (complex defect cores coupled to long-range elastic far-fields) **concurrent** multi-scale methods, including e.g., the popular Quasicontinuum and QM/MM methods, are a popular tool for their computational investigation.

A difficult challenge in the development of concurrent multi-scale schemes is the interface condition coupling models valid at different scales (e.g., quantum, atomistic and continuum). The emerging mathematical theory of concurrent multi-scale methods precisely quantifies the effect of the coupling and coarse-graining mechanisms on the numerical error, leading to optimised schemes with rigorous convergence rates.

A new realisation that has come out of this mathematical theory is that — with some care — **sequential** multi-scale methods, providing a static boundary condition to the atomistic simulation rather than a concurrent coupling, may provide comparable accuracy at significantly

reduced model complexity. Indeed, as I will show in this talk, one can even surpass the accuracy of concurrent schemes.

*Joint work with Julian Braun (University of Warwick, UK), Huajie Chen (Peking Normal, China), Maciej Buze (University of Warwick, UK), and Tom Hudson (University of Warwick, UK).*

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C4 - July 18, 15:00 – 15:25

## A PDE APPROACH TO REGULARIZATION IN DEEP LEARNING

**Adam Oberman**

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Deep neural networks have achieved significant success in a number of challenging engineering problems. There is consensus in the community that some form of smoothing of the loss function is needed, and there have been hundreds of papers and many conferences in the past three years on this topic. However, so far there has been little analysis by mathematicians.

The fundamental tool in training deep neural networks is Stochastic Gradient Descent (SGD) applied to the loss function,  $f(x)$ , which is high dimensional and nonconvex.

$$dx_t = -\nabla f(x_t)dt + dW_t \quad (\text{SDG})$$

There is a consensus in the field that some form of regularization of the loss function is needed, but so far there has been little progress. This may be in part because smoothing techniques, such as a convolution, which are useful in low dimensions, are computationally intractable in the high dimensional setting.

Two recent algorithms have shown promise in this direction. The first uses a mean field approach to perform SGD in parallel. The second replaced  $f$  in (SDG) with  $f_\gamma(x)$ , the *local entropy* of  $f$ , which is defined using notions from statistical physics.

We interpret both algorithms as replacing  $f$  with  $f_\gamma$ , where  $f_\gamma = u(x, \gamma)$  and  $u$  is the solution of the viscous Hamilton-Jacobi PDE

$$u_t(x, t) = -\frac{1}{2}|\nabla u(x, t)|^2 + \beta^{-1}\Delta u(x, t)$$

along with  $u(x, 0) = f(x)$ . This interpretation leads to theoretical validation for empirical results.

However, what is needed for (SDG) is  $\nabla f_\gamma(x)$ . Remarkably, for short times, this vector can be computed efficiently by solving an auxiliary convex optimization problem, which has much better convergence properties than the original non-convex problem. Tools from optimal transportation are used to justify the fast convergence of the solution of the auxiliary problem.

In practice, this algorithm has significantly improved the training time (speed of convergence) for Deep Networks in high dimensions. The algorithm can also be applied to nonconvex minimization problems where (SDG) is used.

*Joint work with Pratik Chaudhari (UCLA, USA), Stanley Osher (UCLA, USA), Stefano Soatto (UCLA, USA), and Guillaume Carlier (UCLA, USA).*

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C4 - July 18, 15:30 – 15:55

WHAT ARE WE COMPUTING WITH NUMERICAL METHODS FOR HYPERBOLIC SYSTEMS  
OF CONSERVATION LAWS?

**Siddhartha Mishra**

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Efficient numerical methods for approximating hyperbolic systems of conservation laws have been in existence for the past three decades. However, rigorous convergence results to entropy solutions are only available in the case of scalar conservation laws. We present numerical evidence that demonstrates the lack of convergence of state of art numerical methods to entropy solutions of multi-dimensional systems. On the other hand, an ensemble averaged version of these numerical methods is shown to converge to entropy measure-valued solutions. However, these solutions are not unique. We impose additional admissibility criteria by requiring propagation of information on all multi-point correlations. This results in the concept of statistical solutions or time-parametrized probability measures on integrable functions, as a solution framework. We derive sufficient conditions for convergence of ensemble-averaged numerical methods to statistical solutions and present numerical experiments illustrating these solutions.

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C4 - July 18, 16:00 – 16:25

A FIRST APPROACH TO VIRTUAL ELEMENTS FOR MAGNETOSTATIC PROBLEMS

**Lourenco Beirao da Veiga**

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The Virtual Element Method (VEM) is a recent technology for the discretization of partial differential equations that follows a similar paradigm to standard Finite Elements, but allows to make use of general polytopal meshes, including non-convex elements and hanging nodes. In addition, the flexibility of the VEM construction allows in many problems to develop new appealing schemes also for simplex and quad/hexa-type meshes.

In the present talk we investigate the development of some Virtual Element families for a simple (classical) magnetostatic problem. We start by introducing a first set of Virtual Spaces that constitute a complex and detail its application for the discretization of the problem. Afterwards, we present a modified set of spaces that are more efficient in terms of degrees of freedom (in particular in three dimensions) and still constitute a complex. We support the theoretical analysis of the scheme with a set of numerical tests.

*Joint work with Franco Brezzi (IMATI-CNR, Pavia, Italy), Franco Dassi (Università di Milano-Bicocca, Italy), Donatella Marini (Università di Pavia, Italy), and Alessandro Russo (Università di Milano-Bicocca, Italy).*

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C4 - July 18, 17:00 – 17:50

ADAPTIVE WAVELET METHODS FOR SPACE-TIME VARIATIONAL FORMULATIONS OF  
EVOLUTIONARY PDES

**Rob Stevenson**

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Space-time discretization methods require a well-posed space-time variational formulation. Such formulations are well-known for parabolic problems. The (Navier)-Stokes equations can be viewed as a parabolic problem for the divergence-free velocities. Yet to avoid the cumbersome construction of divergence-free trial spaces, we present well-posed variational formulations for the saddle-point problem involving the pair of velocities and pressure.

We discuss adaptive wavelet methods for the optimal adaptive solution of simultaneous space-time variational formulations of evolutionary PDEs. Thanks to use of tensor products of temporal and spatial wavelets, the whole time evolution problem can be solved at a complexity of solving one instance of the corresponding stationary problem.

*Joint work with Christoph Schwab (ETH, Zürich).*

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C4 - July 19, 14:30 – 14:55

DISCRETE THEORIES FOR ELLIPTIC PROBLEMS IN NON-DIVERGENCE FORM

**Michael Neilan**

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In this talk, two discrete theories for elliptic problems in non-divergence form are presented. The first, which is applicable to problems with continuous coefficients and is motivated by the strong solution concept, is based on discrete Calderon-Zygmund-type estimates. The second theory relies on discrete Miranda-Talenti estimates for elliptic problems with discontinuous coefficients satisfying the Cordes condition. Both theories lead to simple, efficient, and convergent finite element methods. We provide numerical experiments which confirm the theoretical results, and we discuss possible extensions to fully nonlinear second order PDEs.

*Joint work with Xiaobing Feng (The University of Tennessee, USA) and Mohan Wu (University of Pittsburgh, USA).*

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C4 - July 19, 15:00 – 15:25

WEIGHTS AND APPLICATIONS IN NUMERICS

**Abner J. Salgado**

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The use of weights and weighted norm inequalities has a rich history in harmonic analysis and the study of regularity properties to solutions of partial differential equations (PDE). Starting from classical results, we will present an overview of the application of some of these ideas to the numerical analysis of PDEs. Our main attention will be on some recent results concerning the use of weights in fractional diffusion, problems with singular data and PDE constrained optimization problems. Although these seem as disparate and unrelated applications, it is remarkable that the only structural assumption on the weight is that it belongs to a so-called Muckenhoupt  $A_p$  class, which has been thoroughly studied in harmonic analysis since the 1970's.

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C4 - July 19, 15:30 – 15:55

DISCRETE MAXIMAL PARABOLIC REGULARITY FOR GALERKIN FINITE ELEMENT  
SOLUTIONS AND THEIR APPLICATIONS

**Dmitriy Leykekhman**

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Maximal parabolic regularity is an important analytical tool and has a number of applications, especially to nonlinear problems and/or optimal control problems when sharp regularity results are required. Recently, there have been a lot of interest in establishing similar results for various time discretization methods. In my talk, I will describe our results for discontinuous Galerkin time schemes and show how such results can be used, for example, in establishing pointwise best approximation estimates for fully discrete Galerkin solutions without any coupling between the spatial mesh size  $h$  and the time steps  $k$ .

*Joint work with Boris Vexler (TU Munich, Germany).*

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C4 - July 19, 16:00 – 16:25

THE KURATOWSKI–RYLL–NARDZEWSKI THEOREM AND SEMISMOOTH NEWTON  
METHODS FOR HAMILTON–JACOBI–BELLMAN EQUATIONS

**Iain Smears**

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The numerical solution of fully nonlinear Hamilton–Jacobi–Bellman (HJB) partial differential equations (PDE) leads to the task of solving large systems of equations, with a nonlinearity that is often not differentiable in a classical sense. Nevertheless, a well-established iterative method, commonly called policy iteration or Howard's algorithm, often converges rapidly in practice, as well as constituting an essential ingredient for constructive proofs of the existence of discrete numerical solutions for monotone discretization schemes. More recently, it has become apparent that the notions of semismoothness and of semismooth Newton methods provide a suitable framework for establishing the superlinear convergence of the algorithm. In this talk, we present the proof of semismoothness of HJB differential operators posed on Sobolev spaces, with a possibly infinite control set that is merely assumed to be a compact metric space. In particular, we will show how the measurable selection theorem of Kuratowski and Ryll–Nardzewski plays a

central role in guaranteeing the existence of a generalized differential for the HJB operator. We illustrate the theory with numerical experiments showing the performance of the semismooth Newton method in applications to high-order discretizations of HJB equations with Cordes coefficients.

*Joint work with Endre Süli (University of Oxford, UK).*

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C4 - July 19, 17:00 – 17:50

## $C^1$ ISOGEOMETRIC SPACES ON MULTIPATCH DOMAINS

**Giancarlo Sangalli**

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One key feature of isogeometric analysis is that it allows smooth shape functions. Indeed, when isogeometric spaces are constructed from  $p$ -degree splines (and extensions, such as NURBS), they enjoy up to  $C^{p-1}$  continuity within each patch. However, global continuity beyond  $C^0$  on so-called multi-patch geometries poses some significant difficulties. We consider planar multi-patch domains that have a parametrization which is only  $C^0$  at the patch interface. On such domains we study the  $h$ -refinement of  $C^1$ -continuous isogeometric spaces. These spaces in general do not have optimal approximation properties. The reason is that the  $C^1$ -continuity condition easily over-constrains the solution which is, in the worst cases, fully locked to linears at the patch interface. However, recent studies have given numerical evidence that optimal convergence occurs for bilinear two-patch geometries and cubic (or higher degree)  $C^1$  splines. This is the starting point of our study. We introduce the class of analysis-suitable  $G^1$  geometry parametrizations, which includes piecewise bilinear parametrizations. We then analyze the structure of  $C^1$  isogeometric spaces over analysis-suitable  $G^1$  parametrizations and, by theoretical results and numerical testing, discuss their approximation properties. We also consider examples of geometry parametrizations that are not analysis-suitable, showing that in this case optimal convergence of  $C^1$  isogeometric spaces is prevented.

*Joint work with Pablo Antolin (EPF Lausanne, Switzerland), Annabelle Collin (Institut Polytechnique de Bordeaux), Mario Kapl (RICAM Linz), and Thomas Takacs (Johannes-Kepler University Linz, Austria).*

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**Workshop C5**  
**Information-Based Complexity**

Organizers: Tino Ullrich – Frances Kuo – Erich Novak

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C5 - July 17, 14:30 – 15:20

OPTIMAL ALGORITHMS FOR NUMERICAL INTEGRATION: A SURVEY

**Mario Ullrich**

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I will outline some recent (and several classical) results on the complexity of numerical integration using cubature rules with specific point sets. The point sets under consideration will be (admissible) lattices and digital nets, as they are the only presently known constructions that lead to optimal algorithms for the spaces considered here.

The aim of the talk is to present the state of the art in this area of research, including results in the deterministic and randomized setting, results on tractability, and numerical experiments. Additionally, we provide pointers to other relevant research areas and related open problems.

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C5 - July 17, 15:30 – 15:55

OPTIMALITY AND LOWER BOUNDS IN APPROXIMATION THEORY

**Jan Vybiral**

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We discuss optimality of approximation algorithms and lower bounds on their complexity using Carl's inequality, which states that for any two Banach spaces  $X$  and  $Y$  and any  $\alpha > 0$  there exists a constant  $\gamma_\alpha > 0$  such that

$$\sup_{1 \leq k \leq n} k^\alpha e_k(T) \leq \gamma_\alpha \sup_{1 \leq k \leq n} k^\alpha c_k(T)$$

holds for all linear and bounded operators  $T: X \rightarrow Y$ . Here  $e_k(T)$  is the  $k$ -th entropy number of  $T$  and  $c_k(T)$  is the  $k$ -th Gelfand number of  $T$ . This inequality has been used to obtain lower bounds on Gelfand numbers, which in turn are a useful tool in the study of optimality of algorithms.

We extend the Carl's inequality to quasi-Banach spaces, which allows to extend this approach also to the area of sparse recovery, where  $\ell_p^N$  spaces play an important role also for  $p < 1$ . Further, we give estimates of entropy numbers of embeddings of Schatten classes of matrices. We close by an application of these results to the optimality of algorithms of low-rank matrix recovery and matrix completion.

*Joint work with A. Hinrichs (Johannes-Kepler University Linz, Austria), A. Kolleck (TU Berlin, Germany), and J. Prochno (University of Hull, UK).*

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C5 - July 17, 16:00 – 16:25

## GENERAL COST BOUNDS FOR RANDOMIZED SMOLYAK ALGORITHM

**Marcin Wnuk**

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We are considering randomized sparse grid (Smolyak) algorithms  $A(q, d)$  on  $d$ -dimensional tensor product spaces  $F_d = \bigotimes_{j=1}^d F^{(j)}$ , where  $F^{(j)}, j = 1, \dots, d$ , are separable Hilbert spaces of functions. We measure the quality of algorithm meant to approximate a given linear tensor product operator  $S_d: F_d \rightarrow G_d$  ( $G_d = \bigotimes_{j=1}^d G^{(j)}$ , where  $G^{(j)}, j = 1, \dots, d$ , are separable Hilbert spaces) via the randomized error:

$$e^{ran}(A(q, d)) = \left( \sup_{\substack{f \in F_d, \\ \|f\|_{F_d} \leq 1}} \mathbb{E} \| (S_d - A(q, d))f \|_{G_d}^2 \right)^{\frac{1}{2}}$$

and the worst case error:

$$e^{wce}(A(q, d)) = \left( \mathbb{E} \sup_{\substack{f \in F_d, \\ \|f\|_{F_d} \leq 1}} \| (S_d - A(q, d))f \|_{G_d}^2 \right)^{\frac{1}{2}}.$$

We prove upper bound on the approximation error of  $A(q, d)$  with explicit dependence on the number of variables and the number  $N$  of information used.

As an application we rigorously prove a claim from [1] that the results obtained there hold also for randomized algorithms. Moreover, our findings prove a substantial generalization of the main results from the aforementioned paper.

Under additional conditions on  $F_d$  our upper bounds may be to some extent improved upon. We demonstrate this for a multivariate integration problem on a family of Haar wavelet spaces. On those spaces we perform a rigorous comparison of pure QMC quadratures based on scrambled  $(t, m, s)$ -nets and the Smolyak algorithm.

## Bibliography

- [1] L. Plaskota & G. Waskowski, “Tractability of infinite-dimensional integration in the worst case and randomized settings.” *Journal of Complexity* 27 (2011), 505–518.

*Joint work with Michael Gnewuch (Kiel University, Germany).*

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C5 - July 17, 17:00 – 17:25

THE MARCINKIEWICZ-TYPE DISCRETIZATION THEOREMS FOR THE HYPERBOLIC  
CROSS POLYNOMIALS

**Vladimir Temlyakov**

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The talk is devoted to discretization of integral norms of functions from a given finite dimensional subspace — the hyperbolic cross polynomials. This problem is important in applications but there is no systematic study of it. We present here a new technique, which works well for discretization of the integral norm. It is a combination of probabilistic technique, based on chaining, with results on the entropy numbers in the uniform norm.

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C5 - July 17, 17:30 – 17:55

TRUNCATION DIMENSION FOR LINEAR PROBLEMS ON MULTIVARIATE FUNCTION  
SPACES

**Peter Kritzer**

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We discuss linear problems on weighted anchored and ANOVA spaces of high-dimensional functions. The main questions addressed are: When is it possible to approximate a continuous linear operator applied to multivariate functions with all but the first  $k$  variables set to zero, so that the corresponding error is small? What is the truncation dimension, i.e., the smallest number  $k = k(\varepsilon)$  such that the corresponding error is bounded by a given error demand  $\varepsilon$ ? As it turns out,  $k(\varepsilon)$  could be very small for sufficiently fast decaying weights.

*Joint work with Aicke Hinrichs (Johannes-Kepler University Linz, Austria), Friedrich Pillichshammer (Johannes-Kepler University Linz, Austria), and Greg W. Wasilkowski (University of Kentucky, USA).*

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C5 - July 17, 18:00 – 18:25

BOUNDEDNESS AND APPROXIMATE RECOVERY OF PSEUDO-DIFFERENTIAL  
OPERATORS ON  $m$ -DIMENSIONAL TORUS

**Daurenbek Bazarkhanov**

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In the talk we will discuss several recent results on  $L_p - L_q$ -boundedness of pseudo-differential operators on  $m$ -dimensional torus with symbols from Hörmander's classes or rough symbols. Furthermore, we will consider a problem of approximate recovery of those pseudo-differential operators on some function classes

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 C5 - July 17, 18:30 – 18:55

COMPUTING THE NON-COMPUTABLE – ON FUNDAMENTAL BARRIERS IN  
COMPUTATIONAL MATHEMATICS

**Anders Hansen**

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In the beginning of the 20th century Hilbert asked about the existence of algorithms for decision problems, which sparked the birth of modern theoretical computer science. Similarly, in the 1980s, Smale asked about the existence of algorithms for problems in scientific computing, such as polynomial root finding. This led to a comprehensive program on the foundations of computational mathematics. Questions à la Hilbert and Smale, regarding the existence of algorithms, can be asked in any area of computational mathematics. As we show in this talk, possibly surprisingly, the answer is often negative in many contemporary fields such as compressed sensing, statistical estimation, image processing, machine learning, deep neural networks, computer aided proofs, computational quantum mechanics etc. The non-existence of algorithms for many of these problems may seem like a great paradox given that they are used successfully in so many applications. In this talk we will discuss how these paradoxes yield a very rich classification theory in the foundations of computational mathematics. This is done through the Solvability Complexity Index (SCI) hierarchy. By using the SCI hierarchy one can show how many core problems are formally non-computable yet explain why they are computed efficiently on a daily basis. Moreover, this allows for a complexity theory for the many non-computable problems in the applications mentioned above. In particular, the new results show that theories such as Information Based Complexity (IBC) can be applied in a much wider context than traditionally used.

*Joint work with Alex Bastounis (University of Cambridge, UK), Matt Colbrooke (University of Cambridge, UK), and Verner Vlacic (University of Cambridge, UK).*

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 C5 - July 18, 14:30 – 14:55

ON THE GENERATION OF RANDOM FIELDS

**Ian Sloan**

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The generation of Gaussian random fields is a challenging problem in computational mathematics, especially when the correlation length is short and the field is rough. The traditional approach is to make use of a truncated Karhunen–Loève (KL) expansion, but the generation of even a single realisation of the field may then be effectively beyond reach (especially for 3-dimensional domains) if the need is to obtain an expected  $L_2$  error of say 5%, because of the potentially very slow convergence of the KL expansion. In this talk a completely different approach is used, in which the field is initially generated at a regular grid on a rectangle that contains the physical domain, and then possibly interpolated to obtain the field

at other points. In that case there is no need for any truncation, rather the main problem becomes the factorisation of a large dense matrix. For this we use circulant embedding and FFT ideas.

We note that the use of IBC ideas to study the random field generation problem, for example to find the standard information cost of obtaining an expected  $L_2$  error, has as far as we know not been studied. It might be interesting to do so, but with the reservation that in the grid-based approach, as above, the main computational issues concern not the information cost but rather the factorisation of large matrices.

*Joint work with Ivan Graham (University of Bath), Frances Kuo (University of New South Wales), Dirk Nuyens (KU Leuven, Belgium), and Robert Scheichl (University of Bath).*

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C5 - July 18, 15:00 – 15:25

ON ARBITRARILY SLOW CONVERGENCE RATES FOR STRONG NUMERICAL  
APPROXIMATIONS OF COX-INGERSOLL-ROSS PROCESSES AND SQUARED BESSEL  
PROCESSES

**Mario Hefter**

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Cox-Ingersoll-Ross (CIR) processes are extensively used in state-of-the-art models for the approximative pricing of financial derivatives. In particular, CIR processes are day after day employed to model instantaneous variances (squared volatilities) of foreign exchange rates and stock prices in Heston-type models and they are also intensively used to model short-rate interest rates. The prices of the financial derivatives in the above mentioned models are very often approximately computed by means of explicit or implicit Euler- or Milstein-type discretization methods based on equidistant evaluations of the driving noise processes. In this article we study the strong convergence speeds of all such discretization methods. More specifically, the main result of this article reveals that each such discretization method achieves at most a strong convergence order of  $\delta/2$ , where  $0 < \delta < 2$  is the dimension of the squared Bessel process associated to the considered CIR process. In particular, we thereby reveal that discretization methods currently employed in the financial industry may converge with arbitrarily slow strong convergence rates to the solution of the considered CIR process. This article thus discovers the alarming situation that discretization methods currently employed in the financial engineering industry are thus not capable to solve CIR processes in the strong sense in a reasonable computational time. We thereby lay open the need of the development of other more sophisticated approximation methods which are capable to solve CIR processes in the strong sense in a reasonable computational time and which thus can not belong to the class of algorithms which use equidistant evaluations of the driving noise processes.

*Joint work with Arnulf Jentzen (ETH Zürich, Switzerland).*

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C5 - July 18, 15:30 – 16:20

# ON SUB-POLYNOMIAL LOWER ERROR BOUNDS FOR STRONG APPROXIMATION OF SDEs

**Larisa Yaroslavtseva**

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We consider the problem of strong approximation of the solution of a stochastic differential equation (SDE) at the final time based on finitely many evaluations of the driving Brownian motion  $W$ . While the majority of results for this problem deals with equations that have globally Lipschitz continuous coefficients, such assumptions are typically not met for real world applications. In recent years a number of positive results for this problem has been established under substantially weaker assumptions on the coefficients such as global monotonicity conditions: new types of algorithms have been constructed that are easy to implement and still achieve a polynomial rate of convergence.

In our talk we present negative results for this problem. First we show that there exist SDEs with bounded smooth coefficients such that their solutions can not be approximated by means of any kind of adaptive method with a polynomial rate of convergence. Even worse, we show that for any sequence  $(a_n)_{n \in \mathbb{N}} \subset (0, \infty)$ , which may converge to zero arbitrarily slowly, there exists an SDE with bounded smooth coefficients such that no approximation method based on  $n$  adaptively chosen evaluations of  $W$  on average can achieve a smaller absolute mean error than the given number  $a_n$ .

While the diffusion coefficients of these pathological SDEs are globally Lipschitz continuous, the first order partial derivatives of the drift coefficients are, essentially, of exponential growth. In the second part of the talk we show that sub-polynomial rates of convergence may happen even when the first order partial derivatives of the coefficients have at most polynomial growth, which is one of the typical assumptions in the literature on numerical approximation of SDEs with globally monotone coefficients.

*Joint work with Arnulf Jentzen (ETH Zürich, Switzerland) and Thomas Müller-Gronbach (University of Passau, Germany).*

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C5 - July 18, 17:00 – 17:25

# APPROXIMATION OF ANISOTROPIC PERIODIC SOBOLEV FUNCTIONS

**Thomas Kühn**

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I will report about recent progress on approximation of periodic functions belonging to anisotropic Sobolev spaces on the  $d$ -torus, for any dimension  $d$ . The error will be measured in the  $L_2$ -norm and expressed by approximation numbers of embedding operators.

In particular, I will present new asymptotic and preasymptotic estimates, with special emphasis on the  $d$ -dependence of the hidden constants. Then I will give an interpretation of these results in the language of IBC, in terms of tractability.



*Joint work with Winfried Sickel (Friedrich-Schiller-Universität Jena, Germany) and Tino Ullrich (Hausdorff Center for Mathematics, Bonn, Germany).*

C5 - July 18, 17:30 – 17:55

#### AN UPPER BOUND OF THE MINIMAL DISPERSION VIA DELTA COVERS

**Daniel Rudolf**

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For a point set of  $n$  elements in the  $d$ -dimensional unit cube and a class of test sets we are interested in the largest volume of a test set which does not contain any point. For all natural numbers  $n, d$  and under the assumption of a  $\delta$ -cover with cardinality  $|\Gamma_\delta|$  we prove that there is a point set, such that the largest volume of such a test set without any point is bounded by  $\frac{\log |\Gamma_\delta|}{n} + \delta$ . For axis-parallel boxes on the unit cube this leads to a volume of at most  $\frac{4d}{n} \log(\frac{9n}{d})$  and on the torus to  $\frac{4d}{n} \log(2n)$ .

C5 - July 18, 18:00 – 18:25

#### THE COMPLEXITY OF HIGH AND INFINITE DIMENSIONAL INTEGRATION

**Aicke Hinrichs**

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We present recent results on the complexity of integration. We focus on high and infinite dimensional settings. In particular, we explain how embedding theorems between certain scales of function spaces can be used to transfer complexity results between different settings. This approach is equally useful for approximation problems.

*Joint work with Michael Gnewuch (CAU Kiel, Germany), Mario Hefter (Technische Universität Kaiserslautern, Germany), Klaus Ritter (Technische Universität Kaiserslautern, Germany), Greg Wasilkowski (University of Kentucky, USA).*

C5 - July 18, 18:30 – 18:55

#### FROLOV'S CUBATURE FORMULA IN LOW DIMENSIONS

**Christopher Kacwin**

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Frolov's cubature formula has returned to academic focus recently, because it achieves optimal convergence rates for many function classes, including Sobolev and Besov functions with dominating mixed smoothness. It can be stated as follows: for a given integration domain  $\Omega$ , choose an admissible lattice  $\Gamma$  and set

$$\Phi_\Gamma(f) = \det \Gamma \sum_{x \in \Gamma \cap \Omega} f(x) \approx \int_\Omega f(x) dx.$$

This allows us a certain degree of freedom in the choice of the lattice  $\Gamma$ , and gives rise to the question as to which lattices give optimal numerical results. In this talk, we present a construction method for admissible lattices and provide a list for dimensions  $d = 2, \dots, 10$  of what we believe are nearly optimal choices. We back up our findings with numerical results, comparing Frolov's cubature formula for different admissible lattices as well as with other numerical integration methods.

*Joint work with Jens Oettershagen (University of Bonn, Germany), Mario Ullrich (Johannes-Kepler University Linz, Austria), and Tino Ullrich (University of Bonn, Germany).*

C5 - July 19, 14:30 – 14:55

## ON THE COMPLEXITY OF COMPUTING THE $L_q$ NORM

**Stefan Heinrich**

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We study the complexity of computing

$$S(f) = \left( \int_Q f(x)^q dx \right)^{1/q},$$

that is, the  $L_q$  norm of a function. We assume that  $f$  is from the unit ball of the Sobolev space  $W_p^r(Q)$ , where  $r \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $1 \leq q < \infty$ ,  $r/d > 1/p - 1/q$ ,  $Q = [0, 1]^d$ . Information is standard, that is, consists of function values.

A general result of G.W. Wasilkowski [“Some nonlinear problems are as easy as the approximation problem,” *Comp. & Maths. with Appls.* 10 (1984), 351–363] states that in the deterministic setting the  $n$ -th minimal errors  $e_n^{\det}(S)$  are of the same order as  $e_n^{\det}(J)$ , where  $J$  is the embedding  $J: W_p^r(Q) \rightarrow L_q(Q)$ , thus we can derive the order of  $e_n^{\det}(S)$  directly from known results on approximation. In the randomized setting such a general result no longer holds.

We present and analyze a randomized algorithm for computing  $S(f)$  and provide lower bounds, which allow to determine the order of the randomized  $n$ -th minimal errors  $e_n^{\text{ran}}(S)$ . We also provide comparisons to  $e_n^{\det}(S)$  as well as to  $e_n^{\text{ran}}(J)$  and discuss some extensions.

C5 - July 19, 15:00 – 15:25

## RECOVERY FROM BINARY MEASUREMENTS USING $\ell_1$ -SVMs

**Anton Kollock**

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In this talk we discuss the recovery of sparse signals  $x \in \mathbb{R}^d$  from quantized linear measurements in the most extreme case, i.e., we want to recover  $x$  from  $y = \text{sign}(Ax)$ .

We follow the idea of support vector machines and show that we only need about  $s \log(d)$  measurements to recover an  $s$ -sparse signal  $x$  from  $y$ , even in the case when the measurement process is disturbed by random bitflips.

*Joint work with Jan Vybiral (Charles University, Czech Republic).*

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C5 - July 19, 15:30 – 15:55

ON MULTIVARIATE INTEGRATION WITH RESPECT TO THE GAUSSIAN MEASURE

**Christian Irrgeher**

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We consider the Gaussian integration problem, i.e., we want to approximate multivariate integrals with respect to the Gaussian measure in an efficient way. For that, we study the problem in terms of convergence rate and dependence on the dimension. The efficient approximation of such integrals is of great interest in many applications where Gaussian models are used (e.g., finance, uncertainty quantification,...).

In this talk we give an overview of the state of research and present current results regarding the Gaussian integration problem.

*Joint work with Josef Dick (UNSW Sydney, Australia), Gunther Leobacher (KFU Graz, Austria) and Friedrich Pillichshammer (Johannes-Kepler University Linz, Austria).*

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C5 - July 19, 16:00 – 16:25

MONTE CARLO METHODS FOR THE  $L_\infty$ -APPROXIMATION ON SOBOLEV SPACES WITH BOUNDED MIXED DERIVATIVES

**Van Kien Nguyen**

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In this talk we give tight lower and upper bounds for the order of convergence of linear Monte Carlo approximation of functions from periodic Sobolev spaces with bounded mixed derivatives in the norm of  $L_\infty$ . Especially, in the case of two dimensions we obtain the sharp estimates. In addition, we shall compare this result with the already known behaviour of worst-case approximation errors of linear deterministic methods.

*Joint work with Glenn Byrenheid (Hausdorff-Center for Mathematics, Bonn, Germany) and Robert J. Kunsch (Friedrich-Schiller-University Jena, Germany).*

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C5 - July 19, 17:00 – 17:25

HIGHER-ORDER CUBATURE AND THE MDM FOR INTEGRALS ON  $\mathbb{R}^N$

**Dirk Nuyens**

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We study the application of higher-order cubature rules to approximate an infinite-variate integral over  $\mathbb{R}^N$ . The error consists of a dimension-truncation error and the cubature rule error. We look for conditions in the form of weights to achieve a polynomially decaying error bound.

*Joint work with Dong T.P. Nguyen (KU Leuven, Belgium).*

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C5 - July 19, 17:30 – 17:55

FAST ESTIMATION OF BOUNDS FOR GLOBAL SENSITIVITY INDICES AND APPLICATIONS  
TO SEISMIC HAZARD ANALYSIS

**Hernan Leovey**  
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Global sensitivity indices are usually the method of choice for parameter identification in model calibration. Nevertheless, when the parameter dimensionality is huge, direct estimation of all Sobol' sensitivity indices for ranking becomes intractable, as is usually the case in many applications in Geo-sciences. We show a method that considers Poincaré-type inequalities combined with techniques of algorithmic differentiation to obtain a fast estimation of upper bounds on Sobol' sensitivity indices, when the input parameters of the model are assumed to have independent distributions. We discuss complexity of the estimation method, applications in seismic hazard analysis, and motivation for construction and error analysis of multi-objective quadrature rules. This talk is based in a recent publication in the Bulletin of the Seismological Society of America (doi:10.1785/0120160185).

*Joint work with Christian Molkenhain (Potsdam University, Germany), Frank Scherbaum (Potsdam University, Germany), Andreas Griewank (Yachaytech, Ecuador), Sergei Kucherenko (Imperial College London, UK), and Fabrice Cotton (Potsdam University, Germany).*

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C5 - July 19, 18:00 – 18:25

MINIMAL ASYMPTOTIC ERRORS FOR GLOBAL APPROXIMATION OF JUMP-DIFFUSION  
SDEs

**Pawel Przybylowicz**  
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We study strong global approximation of stochastic differential equations (SDEs) of the following form

$$\begin{cases} dX(t) = a(t, X(t))dt + b(t, X(t))dW(t) + c(t, X(t-))dN(t), & t \in [0, T], \\ X(0) = x_0, \end{cases}$$

where  $x_0 \in \mathbf{R}$ ,  $a, b, c: [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  satisfy certain regularity conditions,  $W = \{W(t)\}_{t \in [0, T]}$  is a one-dimensional Wiener process and  $N = \{N(t)\}_{t \in [0, T]}$  is a non-homogeneous Poisson process with an intensity function  $\lambda = \lambda(t) > 0$ . We assume that the jump and diffusion coefficients of the underlying SDE satisfy jump commutativity condition.

We present the exact convergence rate of the minimal errors that can be achieved by arbitrary algorithms based on a finite number of observations of the Wiener and Poisson process. We consider two classes of methods, that use equidistant and nonequidistant sampling for  $W$  and  $N$ . We define optimal schemes, that are based on the classical Milstein scheme, which asymptotically

attain established minimal errors. We discuss method based on an adaptive stepsize control. It turns out that methods based on nonequidistant mesh are more efficient than those based on the equidistant sampling.

*Joint work with Andrzej Kaluza (AGH University of Science and Technology, Poland).*

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C5 - July 19, 18:30 – 18:55

MULTIVARIATE DECOMPOSITION GALERKIN FINITE ELEMENT METHOD FOR ELLIPTIC  
PDES WITH RANDOM DIFFUSION COEFFICIENTS

**Dong Nguyen**

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We introduce *multivariate decomposition Galerkin finite element method* for solving elliptic PDEs with random diffusion coefficients. The proposed method combines multivariate decomposition method to compute infinite dimensional integration with Galerkin finite element method. Based on a priori error bounds, parameters of multivariate decomposition Galerkin finite element method are chosen in order to achieve a prescribe accuracy under minimal computational work.

*Joint work with Dirk Nuyens (KU Leuven, Belgium).*

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## Workshop C6 Real-Number Complexity

Organizers: Carlos Beltrán – Saugata Basu – Mark Braverman

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C6 - July 17, 14:30 – 14:55

### TRIANGULATIONS OF MONOTONE FAMILIES

**Nicolai Vorobjov**

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Let  $K \subset \mathbb{R}^n$  be a compact definable set in an o-minimal structure over the reals, e.g., a semi-algebraic or a subanalytic set. A definable family  $\{S_\delta \mid 0 < \delta \in \mathbb{R}\}$  of compact subsets of  $K$ , is called a monotone family if  $S_\delta \subset S_\eta$  for all sufficiently small  $\delta > \eta > 0$ . The main result of the talk is that when  $\dim K \leq 2$  there exists a definable triangulation of  $K$  such that for each (open) simplex  $\Lambda$  of the triangulation and each small enough  $\delta > 0$ , the intersection  $S_\delta \cap \Lambda$  is equivalent to one of the five standard families in the standard simplex (the equivalence relation and a standard family will be formally defined). The set of standard families is in a natural bijective correspondence with the set of all five lex-monotone Boolean functions in two variables. As a consequence, we prove the two-dimensional case of the topological conjecture of Gabrielov and Vorobjov on approximation of definable sets by compact families.

*Joint work with Saugata Basu (Purdue University, USA) and Andrei Gabrielov (Purdue University, USA).*

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C6 - July 17, 15:00 – 15:25

### TIGHT SPACE-NOISE TRADEOFFS IN COMPUTING THE ERGODIC MEASURE

**Jon Schneider**

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In this work we obtain tight bounds on the space-complexity of computing the ergodic measure of a low-dimensional discrete-time dynamical system affected by Gaussian noise. If the scale of the noise is  $\varepsilon$ , and the function describing the evolution of the system is not by itself a source of computational complexity, then the density function of the ergodic measure can be approximated within precision  $\delta$  in space polynomial in  $\log 1/\varepsilon + \log \log 1/\delta$ . We also show that this bound is tight up to polynomial factors.

In the course of showing the above, we prove a result of independent interest in space-bounded computation: that it is possible to exponentiate an  $n$  by  $n$  matrix to an exponentially large power in space polylogarithmic in  $n$ .

*Joint work with Mark Braverman (Princeton University, USA) and Cristóbal Rojas (Universidad Andres Bello, Chile).*

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C6 - July 17, 15:30 – 16:20

DESCRIPTIVE MATHEMATICS AND COMPUTER SCIENCE WITH POLYNOMIAL ORDINARY  
DIFFERENTIAL EQUATIONS

**Olivier Bournez**

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The purpose of the talk is to present the following fact: If you know what 0, 1, -1 are, as well as an addition and a multiplication, and if you remember what an ordinary differential equation is, then you can (re-)define and program many concepts from Mathematics and Computer Science.

In particular we will present/rediscover descriptive complexity, computability and complexity using polynomial ordinary differential equations only.

A title for this talk could also be “Programming with Ordinary Differential Equations,” as these questions also relate to analog models of computations, and in particular to the 1941 General Purpose Analog Computer of Claude Shannon, and the forgotten art of their programming.

*Joint work with Daniel Graça (Universidade do Algarve and Instituto de Telecomunicações) and Amaury Pouly (École Polytechnique and University of Oxford and Max Planck Institute for Software Systems).*

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C6 - July 17, 17:00 – 17:25

INTERACTIVE PROOFS OVER THE REALS

**Klaus Meer**

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Interactive proofs provide a powerful tool in complexity theory to verify statements. An almighty prover interacts in several rounds with a verifier to convince the latter of a statement. The verifier usually is a randomized polynomial time machine and should detect erroneous proofs with high probability. Shamir’s famous result characterizes the class IP of problems that can be accepted by such interactive proofs as PSPACE or, equivalently, by the class PAR of problems decidable in parallel polynomial time.

We study interactive proofs in the framework of real number complexity theory as introduced by Blum, Shub, and Smale. Since space resources alone are known not to make much sense in real number computations the question arises whether a real version of IP can be similarly characterized by real number complexity classes. We shall derive both an upper and a lower bound for the power of real interactive proofs. In particular, we show that Shamir like techniques can be applied to give interactive protocols for problems that can be solved in real parallel polynomial time.

*Joint work with Martijn Baartse (until 2016: Brandenburg University of Technology).*

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C6 - July 17, 17:30 – 17:55

PROBABILISTIC SCHUBERT CALCULUS

**Antonio Lerario**

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Classical enumerative geometry deals with questions like: “How many lines intersect four lines in general position in three-dimensional space?” The answer to this type of questions is provided by a beautiful, sophisticated machinery, which goes under the name of Schubert calculus. Over the real numbers this classical approach fails — there is no generic number of real solutions, which can already be seen in the basic problem of counting the real solutions to a polynomial equation. Understanding even the possible outcomes for higher dimensional versions of the problem becomes increasingly complicated. In this talk I will discuss how enumerative problems over the reals can be studied in a probabilistic way, by answering questions like: “On average, how many real lines intersect four random lines in three-dimensional space?”.

*Joint work with Peter Bürgisser (TU Berlin).*

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C6 - July 17, 18:00 – 18:25

ASYMPTOTIC VARIANCE OF THE NUMBER OF REAL ROOTS OF RANDOM POLYNOMIAL SYSTEMS

**Diego Armentano**

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In this talk we study the asymptotic variance, as the degree goes to infinity, of the normalized number of real roots of a square Kostlan-Shub-Smale random polynomial system of any size. Our main tools are the Kac-Rice formula for the second factorial moment of the number of roots and a Hermite expansion of this random variable.

*Joint work with J.M. Azaïs (Université de Toulouse, France), F. Dalmao (Universidad de la República, Uruguay), and J.R. León (Universidad Central de Venezuela, Venezuela).*

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C6 - July 18, 14:30 – 14:55

WEAK AVERAGE-CASE COMPLEXITY

**Martin Lotz**

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It is a well-established fact that, in practical settings, iterative numerical algorithms typically outperform their theoretical convergence guarantees substantially; examples in optimization and numerical analysis abound. This gap does not disappear in the setting of average-case or



smoothed analysis. We present the concept of weak average-case complexity, which is based on the observation that removing an extremely small, “numerically invisible” set from the inputs, is enough to get complexity bounds that are well in line with practical experience. A few applications, interesting relations to random matrix theory, and some probabilistic challenges are presented.

*Joint work with Dennis Amelunxen (City University of Hong Kong, Hong Kong).*

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C6 - July 18, 15:00 – 15:25

### COMPUTING THE HOMOLOGY OF REAL PROJECTIVE SETS

**Teresa Krick**

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We describe and analyze a numerical algorithm for computing the homology of real projective varieties. Its cost depends on the condition of the input as well as on its size: it is singly exponential in the number of variables (the dimension of the ambient space) and polynomial in the condition and the degrees of the defining polynomials. The algorithm relies on a new connection between Smale’s  $\gamma$  quantity and the injectivity radius of the normal bundle of the variety restricted to the sphere.

*Joint work with Felipe Cucker (City University of Hong Kong) and Mike Shub (City University of New York, USA).*

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C6 - July 18, 15:30 – 15:55

### ON THE COMPUTATION OF THE HOMOLOGY OF SEMIALGEBRAIC SETS

**Felipe Cucker**

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The computation of homology groups is one of the problems for which the dawn of the critical points method has not resulted in single exponential time algorithms. Instead, the double exponential time of computing them with CAD is still the best known upper bound. But recent work has found single exponential time algorithms for smooth complex projective varieties and weak exponential time for real projective varieties. This last complexity bound amount to say that, if we endow the space of inputs with a Gaussian probability measure, out of a negligible (i.e., exponentially small) set, the algorithm takes exponential time.

In the talk we describe an extension of the last result to basic semialgebraic sets.

*Joint work with Peter Buerger (TU Berlin, Germany) and Pierre Lairez (INRIA Saclay, France).*

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C6 - July 18, 16:00 – 16:25

ON THE COMPLEXITY OF SPARSE POLYNOMIAL SOLVING

**Gregorio Malajovich**

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I shall report on the algorithmic cost of solving sparse polynomial systems by homotopy algorithms. This cost depends on a sparse condition number, which measures how the solutions change in terms of the coefficients. For this analysis to capture the main features of sparseness, the roots have to be considered as elements of a certain toric variety.

Smale's 17-th problem asked about the existence of a polynomial time algorithm for finding one root of a random dense system. One of the main tools in the solution of this problem was invariance through the unitary group action. This tool is not available any more, so other group actions had to be introduced.

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C6 - July 18, 17:00 – 17:25

IMPROVED BOUNDS FOR EQUIVARIANT BETTI NUMBERS OF SYMMETRIC  
SEMI-ALGEBRAIC SETS

**Cordian Riener**

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Let  $R$  be a real closed field and  $S \subset R^k$  be a semi-algebraic set defined by symmetric polynomials whose degree is  $d$ . The quantitative study of the equivariant rational cohomology groups of such symmetric semi-algebraic sets was initiated recently and it was shown that in sharp contrast to the ordinary Betti numbers, the equivariant Betti numbers can be bounded by quantity that is polynomial in the number of variables  $k$ . These results were obtained using arguments from equivariant Morse-theory. In this talk we present a different method which improves these previous results. In particular, we show that the equivariant cohomology groups vanish in dimension  $d$  and larger and give new bounds on the equivariant Betti numbers for this setup. More importantly, this new approach yields an algorithm with polynomially bounded (in  $k$ ) complexity for computing these equivariant Betti numbers. This is in sharp contrast to the situation of a general semi-algebraic set, where not even an algorithm with a single exponential complexity is not known for computing all the Betti numbers.

*Joint work with Saugata Basu (Purdue University, USA).*

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C6 - July 18, 17:30 – 17:55

NUMERICAL APPROXIMATION OF MULTIPLE ISOLATED ROOTS OF ANALYTIC SYSTEMS

**Jean-Claude Yakoubsohn**

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It is classical that the convergence of the Newton's method at the neighborhood of a singular isolated root of a system of equations is no longer quadratic. It may even diverge. To fix this problem we propose a new operator, named singular Newton operator, generalizing the classical Newton operator defined in the regular case.

To do so we construct a finite sequence of equivalent systems named *deflation sequence*, where the multiplicity of the root drops strictly between two successive elements of the sequence. Hence the root is a regular root for the last system. Then there exists a regular square system extracted from this one named *deflated system*. The singular Newton operator is defined as the classical Newton operator associated to this deflated system.

The main idea is the following: since the Jacobian matrix is rank deficient at the root, there exists relation between the lines (respectively columns) of this Jacobian matrix. These relations are given by the Schur complement of the Jacobian matrix. When the elements of the Schur complement are added to the initial system, we obtain an equivalent system where the multiplicity of the root has dropped. We call this operation *kerneling*. In this way, a sequence of equivalent system can be defined iteratively.

Finally we perform a local  $\alpha$ -theory of Smale of this singular Newton operator, which achieves the numerical study.

*Joint work with Marc Giusti (École Polytechnique, France).*

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C6 - July 18, 18:00 – 18:25

## QUANTITATIVE EQUIDISTRIBUTION OF GALOIS ORBITS OF POINTS OF SMALL HEIGHT ON THE $N$ -DIMENSIONAL ALGEBRAIC TORUS

**Marta Narváez–Clauss**

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Bilu's equidistribution theorem establishes that, given a strict sequence of points on the  $N$ -dimensional algebraic torus whose Weil height tends to zero, the Galois orbits of the points are equidistributed with respect to the Haar probability measure of the compact subtorus,  $(S^1)^N$ . For the case of dimension one, quantitative versions of this result were independently obtained by Petsche and by Favre and Rivera–Letelier.

We present a quantitative version of Bilu's result for the case of any dimension. Our result gives, for a given point, an explicit bound for the discrepancy between its Galois orbit and the uniform distribution on the compact subtorus, in terms of the height and the generalized degree of the point.

As a corollary of our quantitative version for the case of dimension  $N$ , we give an estimate for the distribution of the solutions of a system of Laurent polynomials with rational coefficients in terms of the size of the coefficients of the polynomials and their degrees.

*Joint work with Carlos D'Andrea (Universitat de Barcelona, Spain) and Martín Sombra (Universitat de Barcelona, Spain).*

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 C6 - July 19, 14:30 – 14:55

## THE CONDITION NUMBER OF JOIN DECOMPOSITIONS

**Paul Breiding**

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Joins of real manifolds appear in a wide area of application and often one is interested in decomposing a point on a join into its factors. Naturally, one may expect that the performance of any numerical algorithm that solves the join decomposition problem is governed by the condition number of the problem. In this talk I will define the condition number of the join decomposition problem and describe it as the inverse distance to some ill-posed set in a product of Grassmannians. Moreover, I will model the join decomposition problem as an optimization problem, provide a Newton-like algorithm for it and explain how the condition number influences the performance of this algorithm. Finally, I will show you the behaviour of the condition number close to boundary points of the join.

*Joint work with Nick Vannieuwenhoven (KU Leuven, Belgium).*

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 C6 - July 19, 15:00 – 15:25

## SMALL DESIGNS ON COMPACT ALGEBRAIC MANIFOLDS

**Jordi Marzo**

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Let  $M$  be a compact smooth real algebraic manifold

$$M = \{x \in \mathbb{R}^n : P_1(x) = \dots = P_k(x) = 0\},$$

defined by polynomials  $P_1(x), \dots, P_k(x) \in \mathbb{R}[x]$ , and let  $\mu$  be the Lebesgue measure. A collection of points  $x_1, \dots, x_N \in M$  is a  $t$ -design (called also averaging set) if

$$\frac{1}{N} \sum_{i=1}^N P(x_i) = \frac{1}{\mu(M)} \int_M P(x) d\mu(x),$$

for all polynomials  $P(x)$  of total degree at most  $t$ .

We show the existence of  $t$ -designs with a number of points comparable to the dimension of the space of polynomials of degree at most  $t$  restricted to  $M$ . This generalizes results on the sphere by Bondarenko, Radchenko and Viazovska (2013). Of particular interest is the case of the Grassmannians where our results improve the bounds that had been proved previously.

*Joint work with Ujué Etayo (Universidad de Cantabria, Spain) and Joaquim Ortega-Cerdà (Universitat de Barcelona, Spain).*

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C6 - July 19, 15:30 – 16:20

AMORTIZED COMPLEXITY BOUNDS FOR POLYNOMIAL WITH ALGEBRAIC  
COEFFICIENTS AND APPLICATION TO CURVE TOPOLOGY

**Marie–Francoise Roy**

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Let  $P \in \mathbb{Z}[X, Y]$  be a square-free polynomial of total degree  $d$  and coefficients of bitsize  $\tau$ , and  $\mathcal{C}(P) = \{(\alpha, \beta) \in \mathbb{R}^2, P(\alpha, \beta) = 0\}$  be the real algebraic curve defined by  $P$ . One of our main results is an algorithm for the computation of the local topology in a neighbourhood of each of the singular points and critical points of the projection wrt the  $X$ -axis in  $\tilde{O}(d^5\tau + d^6)$  bit operations where  $\tilde{O}$  means that we ignore logarithmic factors in  $d$  and  $\tau$ . Compared to state of the art sub-algorithms used for computing a Cylindrical Algebraic Decomposition, this result avoids entirely a generic shear. It gives a deterministic algorithm for the computation of the topology of  $\mathcal{C}(P)$  i.e a straight-line planar graph isotopic to  $\mathcal{C}(P)$  inside  $\mathbb{R}^2$  in  $\tilde{O}(d^5\tau + d^6)$  bit operations.

This result can be seen as an application of new amortized complexity bounds for polynomial with algebraic coefficients that we prove in a first part of our work.

*Joint work with Daouda Niang Diatta (Université de Ziguinchor, Senegal), Seny Diatta (Université de Ziguinchor, Senegal), Fabrice Rouillier (INRIA, France), and Michael Sagraloff (Max-Planck-Institut für Informatik, Germany).*

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C6 - July 19, 17:00 – 17:25

A TRULY UNIVERSAL POLYNOMIAL DIFFERENTIAL EQUATION

**Amaury Pouly**

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Lee A. Rubel proved in 1981 that there exists a universal fourth-order algebraic differential equation and provided an explicit example:

$$3y'^4 y'' y'''' - 4y'^4 y'' y''' + 6y'^3 y'' y'' y''' + 24y'^2 y'' y'' y'''' - 12y'^3 y'' y'''' - 29y'^2 y'' y''' y'''' + 12y''''^7 = 0. \quad (1)$$

It is universal in the sense that for any continuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$  and any continuous positive function  $\varepsilon: \mathbb{R} \rightarrow (0; +\infty)$ , there exists a  $C^\infty$  solution  $y: \mathbb{R} \rightarrow \mathbb{R}$  to (1) such that  $|f(t) - y(t)| < \varepsilon(t)$  for all  $t \in \mathbb{R}$ . In other words, there is always a solution of (1) that is  $\varepsilon$ -close to  $f$  everywhere. However this result suffers from a big shortcoming, identified as an open question by Rubel in his paper:

“It is open whether we can require that the solution that approximates  $f$  be the unique solution for its initial data.”

Indeed, the solution to (1) is not unique when one adds the condition that  $y(0) = y_0$  for example. In fact, one can add a countable number of such conditions and still not get uniqueness of the solution. This is not surprising because the construction of Rubel fundamentally relies on the non-uniqueness of the solution to work.

Our main result is to answer Rubel's question positively. More precisely, we show that there is a polynomial  $P$  of degree  $k$  such that for any continuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$  and any continuous positive function  $\varepsilon: \mathbb{R} \rightarrow (0; +\infty)$ , there exists  $y_0, y'_0, \dots, y_0^{(k-1)} \in \mathbb{R}$  such that the unique analytic solution  $y$  to  $P(y, y', \dots, y^{(k)}) = 0$  and  $y(0) = y_0, y'(0) = y'_0, \dots, y^{(k-1)}(0) = y_0^{(k-1)}$  is such that  $|f(t) - y(t)| < \varepsilon(t)$  for all  $t \in \mathbb{R}$ . The major difference with Rubel's result is that we get unicity by requiring that the solution be analytic. This requires a completely different and more involved construction. A by-product of the proof is that  $y_0$  is constructive and in fact computable from  $f$  and  $\varepsilon$ .

*Joint work with Olivier Bournez (LIX, École Polytechnique, France).*

C6 - July 19, 17:30 – 17:55

## AN UPDATE ON THE $fg + 1$ PROBLEM FOR NEWTON POLYGONS

**Pascal Koiran**

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Let  $f, g$  be bivariate polynomials with at most  $t$  monomials each. The product  $fg$  can have up to  $t^2$  monomials, but only  $2t$  of them can appear as vertices of the Newton polygon of  $fg$ . If we add a constant (say, the constant 1) to this product, the determination of the maximum number of vertices of the corresponding Newton polygon becomes quite difficult. This is due to possible cancellations with the constant term of  $fg$  (which may expose some monomials from the interior of the Newton polygon of  $fg$ ).

This problem is motivated by a connection to an algebraic version of the P versus NP problem, as explained in my paper on the “tau-conjecture for Newton polygons” (joint work with Natacha Portier, Sébastien Tavenas and Stéphan Thomassé).

In this talk I will present some connections with problems from combinatorial geometry, and some results obtained by William Aufort for his Master's degree. His internship report is available at: <http://perso.ens-lyon.fr/pascal.koiran/Publis/aufort.pdf>.

C6 - Poster

## GEODESICS IN THE CONDITION METRIC AND CURVATURE

**Juan G. Criado del Rey**

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Given a Riemannian manifold  $(\mathcal{M}, g)$  and a smooth submanifold  $\mathcal{N}$ , the condition metric is the metric in  $\mathcal{M} \setminus \mathcal{N}$  given by  $g_\kappa(x) = d(x, \mathcal{N})^{-2}g(x)$ , where  $d(\cdot, \mathcal{N})$  is the distance function to  $\mathcal{N}$ . A question about the behaviour of the geodesics in the condition metric arises from the works of Beltrami, Dedieu, Malajovich and Shub: is it true that for every geodesic segment in the condition metric the closest point to  $\mathcal{N}$  is one of its endpoints? Previous works show

that, under some smoothness hypotheses, the answer to this question is positive when  $\mathcal{M}$  is the Euclidean space  $\mathbb{R}^n$ . We extend this result by proving that the answer is also positive when  $\mathcal{M}$  has non-negative sectional curvatures. We also prove that this property fails if there is some point in  $\mathcal{M}$  with negative sectional curvatures.

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C6 - Poster

ON THE NUMBER OF TANGENTS TO HYPERSURFACES IN  $\mathbb{R}P^n$  IN RANDOM POSITION

**Khazhgali Kozhasov**

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Given  $2n - 2$  smooth hypersurfaces  $X_1, \dots, X_{2n-2}$  in  $\mathbb{R}P^n$  we are interested in the average number  $T(X_1, \dots, X_{2n-2})$  of projective lines simultaneously tangent to  $g_1 X_1, \dots, g_{2n-2} X_{2n-2}$ , where  $g_1, \dots, g_{2n-2}$  are independent and uniformly distributed  $O(n + 1)$ -transformations.

We express  $T(X_1, \dots, X_{2n-2})$  in terms of the expected degree of the grassmanian  $Gr(2, n + 1)$  (introduced recently by P. Burgisser and A. Lerario) and some curvature integrals of  $X_1, \dots, X_{2n-2}$ .

For example, when  $n = 3$  and all  $X_1, \dots, X_4$  are spheres (in the metric of  $\mathbb{R}P^3$ ) of radius  $r \in (0, \frac{\pi}{2})$  we have

$$T(X_1, \dots, X_4) \approx 1.72 \left( \frac{8}{\pi} \sin r \cos r \right)^4.$$

*Joint work with Antonio Lerario (SISSA, Italy).*

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**Workshop C7**  
**Special Functions and Orthogonal Polynomials**

Organizers: Francisco Marcellán – Kerstin Jordaan – Andrei Martinez-Finkelshtein

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C7 - July 17, 14:30 – 15:00

ORTHOGONALITY AND APPROXIMATION IN SOBOLEV SPACES

**Yuan Xu**

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The best approximation polynomials in a  $L^2$  space are the partial sums of the Fourier orthogonal expansions in the same space. This can be extended to a Sobolev space, for which the orthogonality is defined with respect to an inner product that contains derivatives and approximation holds for functions and their derivatives simultaneously. We explain recent results in this talk, starting with approximation via Sobolev orthogonal polynomials on an interval with the Jacobi weight and continuing to results on the unit ball and on a triangle.

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C7 - July 17, 15:00 – 15:30

MATRIX-VALUED ORTHOGONAL POLYNOMIALS IN SEVERAL VARIABLES RELATED TO  
 $SU(n+1) \times SU(n+1)$

**Pablo Román**

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We study matrix-valued spherical functions for the symmetric pair  $G = SU(n+1) \times SU(n+1)$  and  $K = SU(n+1)$  diagonally embedded. Under certain assumptions these functions give rise to a family of matrix-valued polynomials in several variables. These polynomials are orthogonal with respect to a matrix weight which is described explicitly and is irreducible, i.e., it does not have non-trivial invariant subspaces. From the group theoretic interpretation, we obtain two commuting matrix-valued differential operators having the matrix-valued orthogonal polynomials as eigenfunctions. Remarkably one of these differential operators is of order one. In the case  $n = 2$ , we obtain polynomials in two-variables. The weight matrix in this case is supported on the interior of the Steiner hypocycloid.

*Joint work with Erik Koelink (Radboud University Nijmegen, The Netherlands) and Maarten van Pruijssen (Universität Paderborn, Germany).*



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C7 - July 17, 15:30 – 16:00

POSITIVE AND NEGATIVE RESULTS ABOUT INTEGRAL REPRESENTATIONS FOR  
MULTIVARIABLE BESSEL FUNCTIONS

**Margit Rösler**

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There exist various interesting classes of multivariable Bessel functions, such as Bessel functions of matrix argument which are important in multivariate statistics, or the Bessel functions associated with root systems in Dunkl theory.

In this talk, we shall focus on integral representations for such multivariable Bessel functions which generalize the classical Sonine integral for the one-variable Bessel function  $j_\alpha(z) = {}_0F_1(\alpha + 1; -z^2/4)$ ,

$$j_{\alpha+\beta}(z) = \frac{2\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + 1)\Gamma(\beta)} \int_0^1 j_\alpha(zt) t^{2\alpha+1} (1 - t^2)^{\beta-1} dt \quad (\alpha > -1, \beta > 0).$$

For Bessel functions of matrix argument there are analogous representations, going back already to Herz, and for certain Bessel functions associated with root systems of type  $B$  there are similar results by Macdonald. In these known cases, however, the range of indices is restricted. Similar to the theory of Gindikin for Riesz measures, we shall extend these integral representations to larger index ranges in a distributional sense, and study under which conditions the representing distributions are actually given by positive measures. There turn out to be gaps in the admissible range of indices which are determined by the so-called Wallach set.

As a consequence, we shall obtain examples where the Dunkl intertwining operator between Dunkl operators associated with multiplicities  $k \geq 0$  and  $k' \geq k$  is not positive, which disproves a long-standing conjecture.

*Joint work with Michael Voit (TU Dortmund, Germany).*

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C7 - July 17, 16:00 – 16:30

A SYMBOLIC APPROACH TO PERTURBED SECOND DEGREE FORMS

**Zélia da Rocha**

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The so-called second degree forms constitute an important set of regular forms that include as particular cases the four Chebyshev forms [1, 2]. The second degree character of a form is preserved by several transformations among which is the perturbation that corresponds to a finite modification of the coefficients that appear in the recurrence relation of order two satisfied by the sequence of polynomials orthogonal with respect to that form. Thus, perturbed second degree forms are of second degree; furthermore, they are semi-classical. In this work, we present a symbolic algorithm [1] that allows to explicit several semi-classical properties of perturbed second degree forms, namely: the functional equation, the class of

the form, the Stieltjes equation, a closed formula for the Stieltjes function, a structure relation and the second order linear differential equation. We apply the algorithm to the Chebyshev form of second kind and we explicit these properties for perturbations of several orders [1, 2]. From these results, we can easily derive similar ones for the other three forms of Chebyshev [2].

*Keywords:* Perturbed orthogonal polynomials; second-degree forms; semi-classical forms; Chebyshev forms; differential equations; symbolic computations.

## Bibliography

- [1] Z. da Rocha, “A general method for deriving some semi-classical properties of perturbed second degree forms: the case of the Chebyshev form of second kind.” J. Comput. Appl. Math. 296 (2016), 677–689.
- [2] Z. da Rocha, “On the second order differential equation satisfied by perturbed Chebyshev polynomials.” J. Math. Anal. 7(2016), no. 1, 53–69.

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C7 - July 17, 17:00 – 17:30

### ASYMPTOTICS OF CHEBYSHEV POLYNOMIALS

**Jacob Stordal Christiansen**

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Given an infinite compact set  $E \subset \mathbb{R}$ , the  $n$ th Chebyshev polynomial,  $T_n(z)$ , is the unique monic polynomial of degree  $n$  that minimizes the sup-norm  $\|T_n\|_E =: t_n$  on  $E$ . While the lower bound  $t_n \geq C(E)^n$  is classical, I shall briefly discuss upper bounds of the form

$$t_n \leq K \cdot C(E)^n$$

for some  $K > 0$ . Here,  $C(E)$  is the logarithmic capacity of  $E$ . The main focus of the talk will be on asymptotics of  $T_n(z)$  (and  $t_n$ ). I'll explain how to solve a 45+ year old conjecture of Widom for finite gap subsets of  $\mathbb{R}$  and then discuss how one can go far beyond this simple class of subsets.

*Joint work with Barry Simon (Caltech, Pasadena, USA), Peter Yuditskii (Johannes-Kepler University Linz, Austria) and Maxim Zinchenko (UNM, Albuquerque, USA).*

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C7 - July 17, 17:30 – 18:00

### RECURRENCE COEFFICIENTS OF ORTHOGONAL POLYNOMIALS AND THE PAINLEVÉ EQUATIONS

**Galina Filipuk**

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In this talk I shall review recent results on the connection of recurrence coefficients of orthogonal polynomials to the solutions of the Painlevé equations. In particular, for certain weights the coefficients in the three term recurrence relation satisfy some forms of discrete Painlevé equations, namely, dPI and dPIV. Moreover, I shall explain how to derive differential equations with respect to certain parameters.

The talk is based on the following two papers.

## Bibliography

- [1] G. Filipuk & M.N. Rebocho, “Differential equations for families of semi-classical orthogonal polynomials of class one.” Submitted.
- [1] G. Filipuk & M.N. Rebocho, “Discrete Painlevé equations for recurrence coefficients of Laguerre-Hahn orthogonal polynomials of class one.” *Integral Transforms and Special Functions* 27 (2016), 548–565.

*Joint work with M.N. Rebocho (UBI, Portugal).*

C7 - July 17, 18:00 – 18:30

## ON THREE-FOLD SYMMETRIC POLYNOMIALS WITH A CLASSICAL BEHAVIOUR

**Ana F. Loureiro**

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I will discuss sequences of polynomials of a single variable that are orthogonal with respect to a vector of weights defined in the complex plane. Such polynomial sequences satisfy a recurrence relation of finite (and fixed) order higher than 2. The main focus will be on polynomial sequences possessing a three-fold symmetry and whose multiple orthogonality is preserved under the action of the derivative operator.

C7 - July 17, 18:30 – 19:00

## BERGMAN POLYNOMIALS AND TORSIONAL RIGIDTY

**Brian Simanek**

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The torsional rigidity of a simply connected domain is a constant indicative of the resistance to twisting of a cylindrical beam with the given cross-section. We will show how one can use Bergman polynomials to calculate or estimate torsional rigidity in very general cases. A special emphasis will be placed on the simplicity of the necessary calculations and several examples that display the efficiency of our algorithm.

*Joint work with Matthew Fleeman (Baylor University, USA).*

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C7 - July 18, 14:30 – 15:00

## ZEROS OF WRONSKIANS OF HERMITE POLYNOMIALS

**Walter van Assche**

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Wronskians of Hermite polynomials appear as exceptional Hermite polynomials but also in rational solutions of Painlevé IV, where the rational solutions are in terms of generalized Hermite polynomials and generalized Okamoto polynomials. For exceptional Hermite polynomials it is important to know the zeros of such Wronskians, because real zeros are not allowed in the construction of exceptional Hermite polynomials. For the rational solutions of Painlevé IV the zeros give information of the poles of Painlevé transcendents. We will discuss recent results of Felder, Hemery and Veselov (2012) and Buckingham (2017) about the location of the zeros of these Wronskians.

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C7 - July 18, 15:00 – 15:30

## THE KONTSEVICH MATRIX INTEGRAL AND PAINLEVÉ HIERARCHY

**Marco Bertola**

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The Kontsevich integral is a matrix integral (aka “Matrix Airy function”) whose logarithm, in the appropriate formal limit, generates the intersection numbers on  $\mathcal{M}_{g,n}$ . In the same formal limit it is also a particular tau function of the KdV hierarchy; truncation of the times yields thus tau functions of the first Painlevé hierarchy. This, however is a purely formal manipulation that pays no attention to issues of convergence.

The talk will try to address two issues:

1. Issue 1: how to make an analytic sense of the convergence of the Kontsevich integral to a tau function for a member of the Painlevé I hierarchy? Which particular solution(s) does it converge to? Where (for which range of the parameters)?
2. Issue 2: it is known that (in fact for any  $\beta$ ) the correlation functions of  $K$  points in the  $GUE_\beta$  ensemble of size  $N$  are dual to the correlation functions of  $N$  points in the  $GUE_{4/\beta}$  of size  $K$ . For  $\beta = 2$  they are self-dual.

Consider  $\beta = 2$ : this duality is lost if the matrix model is not Gaussian; however we show that the duality resurfaces in the scaling limit near the edge (soft and hard) of the spectrum.

In particular we want to show that the correlation functions of  $K$  points near the edge of the spectrum converge to the Kontsevich integral of size  $K$  as  $N \rightarrow \infty$ .

This line of reasoning was used by Okounkov in the  $GUE_2$  for his “edge of the spectrum model”. This is based on joint work with Mattia Cafasso (Angers).

Time permitting I will discuss a work-in-progress with G. Ruzza (SISSA) extending these results to the generating function of open intersection numbers.

*Joint work with Mattia Cafasso (University of Angers, France) and Giulio Ruzza (SISSA, Italy).*

C7 - July 18, 15:30 – 16:00

#### SIX-VERTEX MODEL, YANG—BAXTER EQUATIONS, AND ORTHOGONAL POLYNOMIALS

**Pavel Bleher**

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We will discuss an exact solution of the six-vertex model with domain wall and half-turn boundary conditions in each of the phase regions, disordered, ferroelectric, and anti-ferroelectric. The solution is based on the Yang–Baxter equations, the Izergin–Korepin–Kuperberg determinantal formulae, the Zinn–Justin transform, and on the Riemann–Hilbert approach to the asymptotic analysis of orthogonal polynomials, of both continuous and discrete type.

*Joint work with Karl Liechty (DePaul University, Chicago, USA).*

C7 - July 18, 16:00 – 16:30

#### THE NORMAL MATRIX MODEL, MULTIPLE ORTHOGONAL POLYNOMIALS AND THE MOTHER BODY PROBLEM

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The normal matrix model with algebraic potential has gained a lot of attention recently, partially in virtue of its connection to several other topics as quadrature domains, inverse potential problems and the Laplacian growth.

In this presentation we consider the normal matrix model with cubic plus linear potential. Following previous works of Bleher & Kuijlaars and Kuijlaars & López, we introduce multiple orthogonal polynomials that should, in the large degree limit, coincide with the average characteristic polynomial of the normal matrix model. Developing the Deift–Zhou nonlinear steepest descent method to the associated Riemann–Hilbert problem we obtain asymptotics for these polynomials, in particular determining their limiting zero distribution  $\mu_*$ , which will be the main topic of our talk.

We show that the measure  $\mu_*$  can be characterized through the mother body problem associated with the eigenvalues of the normal matrix model. Interestingly, we also find a phase transition for the measure  $\mu_*$ , showing that its support undergoes a “one-cut to three-cut” phase transition.

At the technical level, the construction of  $\mu_*$  involves deformation techniques for quadratic differentials that we also plan to present (in case time permits).

*Joint work with Pavel Bleher (IUPUI, USA).*

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C7 - July 18, 17:00 – 17:50

PAINLEVÉ EQUATIONS AND UNIVERSALITY RESULTS IN HAMILTONIAN PDES

**Tamara Grava**

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We describe the critical behaviour of solutions to weakly dispersive Hamiltonian systems considered as a perturbation of elliptic or hyperbolic systems of hydrodynamic type with two components. Near the points of singularity formation for the unperturbed system, the solutions of the perturbed systems are described by particular solutions to the Painlevé equations. Painlevé equations play in this case the nonlinear analogue of the Airy function and Pearcey integral that are used to describe the semiclassical limit of the linear Schrödinger equation near transition regimes. The important feature of this result is that generic solutions to Hamiltonian perturbations of hyperbolic or elliptic systems behave, near critical points, as solutions to an integrable equation.

*Joint work with B. Dubrovin (SISSA, Italy), C. Klein (Dijon), and A. Moro (New Castle).*

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C7 - July 18, 18:00 – 18:30

ON DIFFERENCE EQUATIONS FOR ORTHOGONAL POLYNOMIALS ON NONUNIFORM LATTICES

**Mama Foupouagnigni**

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Classical orthogonal polynomials on nonuniform lattices  $x(s)$  are known to satisfy a second-order divided difference equation

$$\phi(x(s)) D_x^2 P_n(x(s)) + \psi(x(s)) S_x D_x P_n(x(s)) + \lambda_n P_n(x(s)) = 0,$$

with the divided-difference operators  $D_x$  and  $S_x$  defined by

$$D_x f(x(s)) = \frac{f(x(s + \frac{1}{2})) - f(x(s - \frac{1}{2}))}{x(s + \frac{1}{2}) - x(s - \frac{1}{2})},$$

$$S_x f(x(s)) = \frac{f(x(s + \frac{1}{2})) + f(x(s - \frac{1}{2}))}{2},$$

where  $\phi$  and  $\psi$  are polynomials of degree maximum 2 and 1 respectively and  $\lambda_n$  is a constant term with respect to  $x(s)$ . The lattice  $x(s)$  is defined as

$$x(s) = \begin{cases} c_1 q^s + c_2 q^{-s} + c_3 & \text{if } q \neq 1 \\ c_4 s^2 + c_5 s + c_6 & \text{if } q = 1. \end{cases}$$

We first derive an appropriate polynomial basis for the operators  $D_x$  and  $S_x$ , denoted  $(F_n)_n$  and fulfilling

$$D_x F_n = a_n F_{n-1}, \quad S_x F_n = b_n F_n + c_n F_{n-1},$$

where  $a_n$ ,  $b_n$ , and  $c_n$  are given constants.

Secondly, we use this basis combined with properties of the operators  $D_x$  and  $S_x$  to derive many properties of classical orthogonal polynomials on nonuniform lattice including: 1) Proof of existence of polynomial solution of the above mentioned second-order divided difference equation; 2) providing algorithmic method to find explicit solution to holonomic divided-difference equation and 3) the derivation of some structure relations for classical orthogonal polynomials on nonuniform lattices such the determination of the coefficients of the connection and the linearisation problems involving classical orthogonal polynomials on nonuniform lattices.

We end up by showing how this basis could be used, together with the operators  $D_x$  and  $S_x$  to derive a fourth-order partial divided-difference equations for bivariate classical orthogonal polynomials on the nonuniform lattice.

*Joint work with Wolfram Koepf (University of Kassel, Germany), Salifou Mboutngam (University of Maroua, Cameroon), Maurice Kenfack-Nangho (University of Dschang, Cameroon), Daniel Duviol Tcheutia (University of Kassel, Germany), and Patrick Njionou Sadjang (University of Douala, Cameroon).*

C7 - July 18, 18:30 – 19:00

#### TRIDIAGONALIZATION OF THE HEUN EQUATION

**Luc Vinet**

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It will be explained that the tridiagonalization of the hypergeometric operator  $L$  yields the generic Heun operator  $M$ . The algebra generated by the operators  $L$ ,  $M$  and  $Z = [L, M]$  will be seen to be quadratic and to form a one-parameter generalization of the Racah algebra. Racah-Heun orthogonal polynomials will be introduced as overlap coefficients between the eigenfunctions of the operators  $L$  and  $M$ . An interpretation in terms of the Racah problem for the  $su(1,1)$  algebra and separation of variables in a superintegrable system will be discussed.

*Joint work with F. Alberto Grünbaum (UC Berkeley, USA) and Alexei Zhedanov (Renmin University, China).*

C7 - July 19, 14:30 – 15:20

#### AHLFORS PROBLEM FOR POLYNOMIALS

**Peter Yuditskii**

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We raise a conjecture that asymptotics for Chebyshev polynomials in a complex domain can be given in terms of the reproducing kernels of a suitable Hilbert space of analytic functions in this domain. It is based on two classical results due to Garabedian and Widom. To support this conjecture we study asymptotics for Ahlfors extremal polynomials in the complement to a system of intervals on  $\mathbb{R}$ , arcs on  $\mathbb{T}$ , and its continuous counterpart.

*Joint work with Benjamin Eichinger (Johannes-Kepler University Linz, Austria).*

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C7 - July 19, 15:30 – 16:00

CONSTRUCTION OF OPTIMAL SET OF TWO QUADRATURE RULES IN THE SENSE OF BORGES

**Marija Stanić**

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In this paper we investigate the numerical method for construction of optimal set of quadrature rules in the sense of Borges [Numer. Math. 67 (1994), 271–288] for two definite integrals with the same integrand and interval of integration, but with different weight functions, related to an arbitrary multi-index. Presented method is illustrated by numerical example.

*Joint work with Tatjana Tomović (University of Kragujevac, Serbia).*

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C7 - July 19, 16:00 – 16:30

OUTLIER DETECTION THROUGH CHRISTOFFEL FUNCTIONS

**Bernhard Beckermann**

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We discuss how recent results on asymptotics for the Christoffel kernel built with orthogonal polynomials of one or several complex variables can help to detect outliers in a cloud of points.

*Joint work with Mihai Putinar (University of California at Santa Barbara, USA), Edward B. Saff (Vanderbilt University, USA) and Nikos Stylianopoulos (University of Cyprus, Cyprus).*

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C7 - July 19, 17:00 – 17:30

SELF-INTERLACING AND HURWITZ STABLE POLYNOMIALS IN APPLICATIONS TO ORTHOGONAL POLYNOMIALS AND INTEGRABLE SYSTEMS

**Mikhail Tyaglov**

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In the talk, we describe some interrelations between theory of orthogonal polynomials, moment theory, and the root distributions of polynomials. The recently discovered self-interlacing polynomials and their dual Hurwitz stable polynomials play an important role in these interrelations. Possible applications of Hurwitz stable and self-interlacing orthogonal polynomials to integrable systems and quantum transportation are discussed.



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C7 - July 19, 17:30 – 18:00

A CONNECTION BETWEEN ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE AND  
THE REAL LINE, VIA CMV MATRICES

**María-José Cantero**

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It is very well known the connection between orthogonal polynomials on the unit circle and orthogonal polynomials on the real line, given by Szegő. This connection, based on a map which transforms the unit circle onto certain interval of the real line, induces a one-to-one correspondence between symmetric measures on the unit circle and measures on the mentioned interval.

A new relation between these two kind of polynomials has been recently discovered by Derevyagin, Vinet and Zhedanov (DVZ). The DVZ connection starts from a factorization of real CMV matrices  $\mathcal{C}$  (unitary analogue of Jacobi matrices) into two tridiagonal factors, whose sum yields a Jacobi matrix  $\mathcal{K}$ . The authors provide closed formulas for the orthogonality measure and the orthogonal polynomials related to  $\mathcal{K}$  in terms of those corresponding to  $\mathcal{C}$ . DVZ consider even a more ambitious problem in which the Jacobi matrix  $\mathcal{K}$  is a linear combination of the referred CMV factors (generalized DVZ connection). Regarding this generalization, their main result refers to the Jacobi matrix  $\mathcal{K}$  built out of the Jacobi polynomials on the unit circle. They obtain a connection between the orthogonal polynomials of  $\mathcal{K}$  and the so called big  $-1$  Jacobi polynomials. However, for generalized DVZ, the relation between the orthogonal polynomials and orthogonality measures of  $\mathcal{K}$  and  $\mathcal{C}$  is missing.

We will present a different approach to this connection which allows us to go further than DVZ. We start by using CMV tools to obtain directly the orthogonal polynomials associated with  $\mathcal{K}$  in terms of the basis related to  $\mathcal{C}$ , and then we use this to discover the relation between the corresponding orthogonality measures. The advantages of our approach are more evident for the generalized DVZ connection, where we obtain explicit formulas for the relation between the orthogonal polynomials and orthogonality measures associated with  $\mathcal{K}$  and  $\mathcal{C}$ . The utility of these results will be illustrated with some examples providing new families of orthogonal polynomials on the real line.

*Joint work with Francisco Marcellán (University Carlos III of Madrid, Spain), Leandro Moral (University of Zaragoza, Spain), and Luis Velázquez (University of Zaragoza, Spain).*

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C7 - July 19, 18:00 – 18:30

CHRISTOFFEL FORMULA FOR CHRISTOFFEL-DARBOUX KERNELS ON THE UNIT  
CIRCLE

**Alagacone Sri Ranga**

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Given a nontrivial positive measure  $\mu$  on the unit circle, the associated Christoffel-Darboux kernels can be considered as polynomials in  $z$  given by  $K_n(z, w; \mu) = \sum_{k=0}^n \overline{\varphi_k(w; \mu)} \varphi_k(z; \mu)$ ,  $n \geq 0$ , where  $\varphi_k(\cdot; \mu)$  are the orthonormal polynomials with respect to the measure  $\mu$ . Let the positive measure  $\nu$  on the unit circle be given by  $d\nu(z) = z^{-m} G_{2m}(z) d\mu(z)$ , where  $G_{2m}$  is a conjugate reciprocal polynomial of exact degree  $2m$  such that  $z^{-m} G_{2m}(z)$  is non-negative within the support of the  $\mu$ . We consider the determinantal formula expressing  $\{K_n(z, w; \nu)\}_{n \geq 0}$  directly in terms of  $\{K_n(z, w; \mu)\}_{n \geq 0}$ . Special attention is given to the case  $w = 1$ . Example are also given.

*Joint work with Cleonice F. Bracciali (Universidade Estadual Paulista, Brazil), Andrei Martinez-Finkelshtein (Universidad de Almeria, Spain), and Daniel O. Veronese (Universidade Federal do Triangulo Mineiro, Brazil).*

C7 - July 19, 18:30 – 19:00

# THE PROJECTIVE ENSEMBLE AND DISTRIBUTION OF POINTS IN ODD-DIMENSIONAL SPHERES

**Ujué Etayo**

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We define a determinantal point process on the complex projective space that reduces to the so-called spherical ensemble for complex dimension 1 under identification of the 2-sphere with the Riemann sphere.

Through this determinantal point process we propose a point process in odd-dimensional spheres that produces fairly well-distributed points, in the sense that the expected value of the Riesz 2-energy for these collections of points is smaller than all previously known bounds.

*Joint work with Carlos Beltrán (Universidad de Cantabria, Spain).*

C7 - Poster

# GALOISIAN APPROACH TO ORTHOGONAL POLYNOMIALS

**Primitivo Acosta-Humánez**

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In this poster I will present some links between Classical and Differential Galois Theory with the Theory of Orthogonal Polynomials. The starting point corresponds to some general results about the Galois groups of orthogonal polynomials and differential equations with orthogonal polynomials as solutions. In particular, we compute the Galois group of some families of classical polynomials with base field the rational numbers as well the differential Galois groups of some families of differential equations, with base differential field the rational functions, involving orthogonal polynomials in their solutions and spectra.

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C7 - Poster

COLLOCATION SPLINE INTERPOLATION FOR NUMERICAL SOLUTION OF HIGH-ORDER  
LIDSTONE-TYPE BOUNDARY VALUE PROBLEMS

**Maria Italia Gualtieri, Anna Napoli**

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In this paper we consider Lidstone and Complementary Lidstone expansions for a sufficiently smooth function [1]. The corresponding interpolating polynomials have been used in [2, 3, 4] in the field of interpolation theory and its application, particularly, for the numerical solution of boundary value problems.

Here we present a new collocation method for solving nonlinear boundary value problems of high-order  $m$  with Lidstone and Complementary Lidstone boundary conditions. The exact solution is approximated by spline interpolation functions of order  $m + 2$  which are written in the basis of Lidstone and Complementary Lidstone polynomials, respectively, for even and odd-order problems. The use of piecewise polynomials and spline-type functions allows to overcome the drawback of polynomial interpolation which may produce widely oscillatory approximations.

Some results on local and global errors are given. The cases of third and fourth order boundary value problems have been analyzed in detail and some numerical tests are given in order to confirm the validity of the presented method.

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*Joint work with Francesco Aldo Costabile (University of Calabria, Italy).*

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C7 - Poster

ON SUPERSYMMETRIC EIGENVECTORS OF THE  $5D$  DISCRETE FOURIER TRANSFORM

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An explicit form of a discrete analogue of the quantum number operator, constructed in terms of the difference lowering and raising operators that govern eigenvectors of the 5D discrete (finite) Fourier transform  $\Phi^{(5)}$ , has been explored. This discrete number operator  $\mathcal{N}^{(5)}$  has distinct eigenvalues, which are employed to systematically classify eigenvectors of the  $\Phi^{(5)}$ , thus avoiding the ambiguity caused by the well-known degeneracy of the eigenvalues of the latter operator. In addition, we show that the hidden symmetry of the discrete number operator  $\mathcal{N}^{(5)}$  manifests itself in the form of the unitary Lie superalgebra  $psl(5|5)$ .

*Joint work with Mesuma Atakishiyeva (Universidad Autónoma del Estado de Morelos, México).*

C7 - Poster

#### ON PERTURBATIONS OF $g$ -FRACTIONS AND RELATED CONSEQUENCES

**Kiran Kumar Behera**

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The purpose of the work is to investigate some structural and qualitative aspects of two different perturbations of the parameters of  $g$ -fractions. In this context, the concept of gap- $g$ -fractions is introduced. While tail sequences of continued fractions play a significant role in the first perturbation, Schur fractions are used in the second perturbation of the  $g$ -parameters that are considered. Illustrations are provided using Gaussian hypergeometric functions.

*Joint work with A. Swaminathan (Indian Institute of Technology Roorkee, India).*

C7 - Poster

#### HANKEL DETERMINANTS AND SPECIAL FUNCTION SOLUTIONS OF PAINLEVÉ IV

**Alfredo Deaño**

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We consider  $n \times n$  Hankel determinants involving moments of discontinuous Hermite weights defined on the real line, and we present asymptotic results as  $n \rightarrow \infty$ . These Hankel determinants are related to Laguerre and Hermite semiclassical polynomials, to special function solutions of the Painlevé IV equation, which are constructed using Wronskians of parabolic cylinder functions, and also to thinning processes of Gaussian ensembles of random matrices.

*Joint work with Christophe Charlier (Université Catholique de Louvain, Belgium).*

C7 - Poster

#### FOURTH ORDER PARTIAL DIFFERENTIAL EQUATIONS FOR KRALL-TYPE ORTHOGONAL POLYNOMIALS ON THE TRIANGLE

**Lidia Fernández**

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We construct bivariate polynomials orthogonal with respect to a Krall-type inner product on the triangle defined by adding Krall terms over the border and the vertexes to the classical inner product. We prove that these Krall-type orthogonal polynomials satisfy fourth order partial differential equations with polynomial coefficients, as an extension of the classical theory introduced by H.L. Krall in the 40's. Also, a connection formula between classical orthogonal polynomials on the bidimensional simplex and orthogonal polynomials associated with Krall-type inner product is deduced.

*Joint work with Antonia M. Delgado and Teresa E. Pérez (Universidad de Granada, Spain).*

C7 - Poster

### TOEPLITZ MINORS FOR SZEGÖ AND FISHER-HARTWIG SYMBOLS

**David García-García**

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We study minors of Toeplitz matrices of both finite and large dimension, comprising also the case of symbols with Fisher-Hartwig singularities. We express the minors in terms of specializations of symmetric polynomials. Several implications of this formulation are presented, including explicit formulas for a Selberg-Morris integral with two Schur polynomials and for the specialization of skew-Schur polynomials. For the latter result, the inverse of a Toeplitz matrix with a pure Fisher-Hartwig singularity is computed, using both our results on minors and the Duduchava-Roch formula.

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*Joint work with Miguel Tierz (Universidade de Lisboa, Portugal).*

C7 - Poster

### GENERALIZED FREUD POLYNOMIALS AND THE PAINLEVÉ EQUATIONS

**Kerstin Jordaan**

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We investigate certain properties of monic polynomials orthogonal with respect to a semi-classical generalized Freud weight

$$w_\lambda(x; t) = |x|^{2\lambda+1} \exp(-x^4 + tx^2),$$

where  $x \in \mathbb{R}$ ,  $\lambda > 0$  and  $t \in \mathbb{R}$ . Generalized Freud polynomials arise from a symmetrization of semi-classical Laguerre polynomials. We prove that the coefficients in the three-term recurrence relation associated with the generalized Freud weight  $w_\lambda(x; t)$  can be expressed in terms of Wronskians of parabolic cylinder functions that appear in the description of special function solutions of the fourth Painlevé equation. This closed form expression for the recurrence coefficients allows the investigation of certain properties of the generalized Freud polynomials. We obtain an explicit formulation for the generalized Freud polynomials in terms of the recurrence coefficients, investigate the higher order moments as well as the Pearson equation satisfied by the generalized Freud weight  $w_\lambda(x; t)$ . We also derive a differential-difference equation using a method due to Shohat and second-order linear ordinary differential equation satisfied by the polynomials. Further, we provide an extension of Freud's conjecture for the recurrence coefficient  $\beta_n(t; \lambda)$  associated with the generalized Freud weight.

*Joint work with Peter Clarkson (University of Kent, UK) and Abey Kelil (University of Pretoria, South Africa).*

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C7 - Poster

# EIGENVALUES OF A DIFFERENTIAL OPERATOR FOR A FAMILY OF SOBOLEV ORTHOGONAL POLYNOMIALS

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We consider the Sobolev inner product

$$(f, g)_S := \int_{-1}^1 f(x)g(x)(1-x^2)^\alpha dx + M[f^{(j)}(-1)g^{(j)}(-1) + f^{(j)}(1)g^{(j)}(1)],$$

where  $\alpha > -1$ ,  $j \geq 0$  and  $M > 0$ .

We denote by  $\{B_n^{(\alpha, j)}\}_{n \geq 0}$  the sequence of orthonormal polynomials with respect to this inner product. These polynomials are called Gegenbauer-Sobolev orthogonal polynomials since involved the classical Gegenbauer weight.

In [1] the authors give conditions to assure that there is a operator differential equation,  $\mathbf{L}$ , such that the polynomials  $B_n^{(\alpha, j)}$  are eigenfunctions of

$$\mathbf{L}B_n^{(\alpha, j)}(x) = \lambda_n B_n^{(\alpha, j)}(x),$$

where  $\lambda_n$  are the corresponding eigenvalues of  $\mathbf{L}$ .

For certain applications we are interested in properties of the related polynomial kernel scaled by the eigenvalues:

$$K^r(x, y) = \sum_{i=0}^{\infty} \lambda_i^{-r} \frac{B_i^{(\alpha, j)}(x)B_i^{(\alpha, j)}(y)}{(B_i^{(\alpha, j)}, B_i^{(\alpha, j)})_S}$$

If the series converges for some  $r_0$ , then  $K^r$  is a reproducing kernel for all  $r \geq r_0$ . Therefore the *minimal*  $r_0 > 0$  for which the series converges for all  $r > r_0$  is a number of interest.

Our main result is to determinate this  $r_0$ , it is shown in [2]

$$\lim_{n \rightarrow +\infty} \frac{\log(\max_{x \in [-1,1]} |B_n^{(\alpha,j)}(x)|)}{\log(\lambda_n)} = r_0.$$

Finally, we study the Mehler-Heine type asymptotics for the polynomials  $B_n^{(\alpha,j)}$ .

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*Joint work with Lance L. Littlejohn (Baylor University, USA), Juan J. Moreno-Balcázar (Universidad de Almería, Spain), and Richard Wellman (Westminster College, USA).*

C7 - Poster

### ZEROS OF SOME FAMILIES OF SOBOLEV ORTHOGONAL POLYNOMIALS

**Juan José Moreno-Balcázar**

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The computation of the zeros of the Sobolev orthogonal polynomials (SOP) can be rather complicated. Thus, the study of the asymptotic behavior of these zeros can be useful as an approximation of them. In the literature the use of Mehler–Heine type formulae has permitted to obtain asymptotic approximations to the zeros of the SOP in terms of the zeros of Bessel functions or combination of these functions. From a general analytical result for discrete SOP, we will show some examples focusing our attention on the numerical experiments.

*Joint work with Juan F. Mañas-Mañas (Universidad de Almería, Spain).*

C7 - Poster

### FAST AND ACCURATE ALGORITHM FOR THE GENERALIZED EXPONENTIAL INTEGRAL $E_\nu(x)$ FOR POSITIVE REAL ORDER

**Guillermo Navas-Palencia**

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We describe an algorithm for the numerical evaluation of the generalized exponential integral  $E_\nu(x)$  for positive values of  $\nu$  and  $x$ . The generalized exponential integral appears in many fields of physics and engineering and plays an important role in some exponentially-improved asymptotic expansions for the confluent hypergeometric function. The performance and accuracy of

the resulting algorithm is analysed and compared with open-source software packages. This analysis shows that our implementation is competitive and more robust than other state-of-the-art codes. The described algorithm is part of the author's library Chypergeo.

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C7 - Poster

## ZEROS OF BIVARIATE CLASSICAL ORTHOGONAL POLYNOMIALS ON THE UNIT DISK

**Teresa E. Perez**

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The behaviour of the zeros of orthogonal polynomials in one variable has been studied extensively for years. In the positive-definite case, it is well known that the  $n$ th polynomial has exactly  $n$  distinct zeros, that have important applications in quadrature formulae. In its classical sense, a zero of a bivariate orthogonal polynomial is an algebraic curve, and depends on the chosen basis in every case.

As far as we know, in two variables, the problem was tackled by Charles Hermite in 1865 for the biorthogonal basis of classical orthogonal polynomials on the unit disk. This work is devoted to show some old and new results, difficulties and open problems about this subject.

*Joint work with Antonia M. Delgado (Universidad de Granada, Spain), Lidia Fernández (Universidad de Granada, Spain), and Miguel A. Piñar (Universidad de Granada, Spain).*

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C7 - Poster

## ENTROPIC FUNCTIONALS OF LAGUERRE AND GEGENBAUER POLYNOMIALS WITH LARGE PARAMETERS

**David Puertas Centeno**

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The determination of the physical entropies (Rényi, Shannon, Tsallis) of high-dimensional quantum systems subject to a central potential requires the knowledge of the asymptotics of some power and logarithmic integral functionals of the hypergeometric orthogonal polynomials which control the wavefunctions of the stationary states. For the D-dimensional hydrogenic and oscillator-like systems, the wavefunctions of the corresponding bound states are controlled by the Laguerre ( $\mathcal{L}_m^{(\alpha)}(x)$ ) and Gegenbauer ( $\mathcal{C}_m^{(\alpha)}(x)$ ) polynomials in both position and momentum spaces, where the parameter  $\alpha$  linearly depends on D. In this work we study the asymptotic behavior as  $\alpha \rightarrow \infty$  of the associated entropy-like integral functionals of these two families of hypergeometric polynomials. Finally, further extension of these results are applied to hyperspherical harmonics together with some quantum applications.

*Joint work with Nico M. Temme (IAA, The Netherlands), Irene V. Toranzo (Universidad de Granada, Spain), and Jesús S. Dehesa (Universidad de Granada, Spain).*



## C7 - Poster

A NEW FRAMEWORK FOR NUMERICAL ANALYSIS OF NONLINEAR SYSTEMS: THE  
SIGNIFICANCE OF THE STAHL'S THEORY AND ANALYTIC CONTINUATION VIA PADÉ  
APPROXIMANTS

**Sina Sadeghi Baghsorkhi**

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An appropriate embedding of polynomial systems of equations in a Riemann surface renders the variables as functions of a single complex variable. The relatively recent developments in the theory of approximation of multivalued functions in the extended complex plane provides a set of powerful tools to solve and analyze this class of problems. The underlying concepts, namely the algebraic curves and the Stahl's theory are presented along with a critical application in the voltage collapse study of electricity networks.

*Joint work with Nikolay Ikonov (Institute of Mathematics and Informatics, Bulgarian Academy of Sciences) and Sergey Suetin (Steklov Mathematical Institute, Russian Academy of Sciences).*

## C7 - Poster

ON COOLEY-TUKEY-TYPE-ALGORITHMS BASED ON GENERALIZED CHEBYSHEV  
POLYNOMIALS

**Bastian Seifert**

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Weyl groups associated to semisimple, compact Lie groups give rise to generalized Chebyshev polynomials. In recent years those of second kind have found a wide variety of applications, ranging from discretization of partial differential equations to cubature formulas on special domains. In this presentation we illustrate how one can use generalized Chebyshev polynomials of the first kind to develop Cooley-Tukey-type-algorithms in the setting of algebraic signal processing theory for the lattices they constitute. As an illustrating example, we show how one can use this technique to process voxel data on the face centered lattice.

*Joint work with Knut Hüper (University of Würzburg, Germany) and Christian Uhl (Ansbach University of Applied Sciences, Germany).*

## C7 - Poster

LINEARIZATION AND KREIN-LIKE FUNCTIONALS OF HYPERGEOMETRIC ORTHOGONAL  
POLYNOMIALS BY MEANS OF LAURICELLA FUNCTIONS

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The Krein-like functionals of the hypergeometric orthogonal polynomials  $\{p_n(x)\}$  with kernel of the form  $x^s[\omega(x)]^\beta p_{m_1}(x) \dots p_{m_r}(x)$ , being  $\omega(x)$  the weight function on the interval  $\Delta \in \mathbb{R}$ , are considered and analytically computed by means of the Srivastava linearization method. The particular ( $r = 2$ )-functionals of the form  $x^s[\omega(x)]^\beta p_n(x)p_m(x)$ , which are particularly relevant in quantum physics, are explicitly given in terms of the degrees and the characteristic parameters of the polynomials. They include the well-known power moments  $\langle x^s \rangle_n$  and the novel Krein-like moments  $\langle [\omega(x)]^k \rangle_n$ , where  $\langle f(x) \rangle_n := \int_\Delta f(x) \rho_n(x) dx$  and  $\rho_n(x) = \omega(x)p_n^2(x)$  is the Rakhmanov density of the polynomials. Moreover, various related types of exponential and logarithmic functionals of the form  $\langle x^k e^{-\alpha x} \rangle_n$ ,  $\langle (\log x)^k \rangle_n$  and  $\langle [\omega(x)]^k \log \omega(x) \rangle_n$ , are also investigated.

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