Adaptive High-Order Methods for Elliptic Problems: Convergence and Optimality

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Joint work with

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Foundations of Computational Mathematics
Barcelona, July 14 2017
Outline

Introduction

Adaptive Fourier methods

A framework for $hp$-Adaptivity

$hp$-Adaptive Approximation

Basic $hp$-Adaptive Algorithm

Realizations of the Algorithm

Conclusions
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Adaptive approximation of elliptic problems: the state of the art

- **Adaptivity for finite-order methods** [wavelets, $h$-type finite elements]: well-understood in terms of algorithms and theory (convergence, optimality)
  
Adaptive approximation of elliptic problems: the state of the art

- **Adaptivity for finite-order methods** [wavelets, \( h \)-type finite elements]: well-understood in terms of algorithms and theory (convergence, optimality)
  

- **Adaptivity for high-order methods** [spectral, \( h p \)-type finite elements]: heuristic algorithms, partial theory
  
  - **A posteriori error analysis:**
    
    [Gui and Babuška 1986, Oden, Demkowicz et al '89, Bernardi '96, Ainsworth and Senior '98, Schmidt and Siebert '00, Melenk and Wohlmuth '01, Heuvelin and Rannacher '03, Houston and Süli '05, Eibner and Melenk '07, Braess, Pillwein and Schöberl '08, Ern and Vohralík '14, ... ]

  - **Convergence and optimality:**
    
Challenges for high-order adaptivity

- A suitable combination of ‘\textit{h-refinement}’ and ‘\textit{p-enrichment}’ may yield a fast (e.g., exponential) decay of the approximation error, even for functions with poor global smoothness.
  
  - For instance, the function \( u(x) = x^\alpha \) with \( \alpha < 1 \) on \( I = [0, 1] \) can be approximated with an error of the form

    \[
    \text{approximation error} \sim C e^{-\beta \sqrt{N}} \quad \quad N = \# \text{degrees of freedom}
    \]

    on a graded mesh geometrically refined towards the origin, with polynomial degrees linearly growing away from the origin. [DeVore-Scherer '79, Babuška-Guo '86].
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  ▶ For instance, the function $u(x) = x^\alpha$ with $\alpha < 1$ on $I = [0, 1]$ can be approximated with an error of the form

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- Need of dealing with approximation classes of functions for which the (best) approximation error decays faster than algebraically (e.g., exponentially).

- The choice between ‘h-refinement’ and ‘p-enrichment’ is quite delicate. In an iterative adaptive algorithm, one of the two choices may appear preferable in an earlier stage, but eventually it may reveal itself short-sighted and non-optimal.

  One should incorporate the possibility of stepping back, and correcting early errors in the adaptive strategy.
Approximation classes

- **Best N-term approximation error**: Given \( v \in V \), define

\[
\sigma_N(v) = \inf_{V_N \subset V} \inf_{w \in V_N} \| v - w \|_V.
\]

- **Decay vs N identifies an approximation class**:

\[
\sigma_N(v) \lesssim \phi(N) \quad \text{with} \quad \phi \to 0 \quad \text{as} \quad N \to \infty.
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- **Algebraic class (finite-order methods):**

\[
v \in \mathcal{A}_B^s \quad \text{iff} \quad |v|_{A_B^s} := \sup_N \sigma_N(v) N^{s/d} < \infty.
\]

- **Exponential class (infinite-order methods):**

\[
v \in \mathcal{A}_G^{\eta,t} \quad \text{iff} \quad |v|_{A_G^{\eta,t}} := \sup_N \sigma_N(v) e^{\eta N^\tau} < \infty.
\]
Complexity

**Question:** What is the cost involved in reducing the best approximation error

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- **Algebraic decay:** Let \( E(v_k) \) decay algebraically

  \[ E(v_k) = AN_k^{-s} \]

  in terms of degrees of freedom \( N_k \). Then, a simple calculation yields

  \[ N_{k+1} = \rho^{-\frac{1}{s}} N_k \]

  The new number of degrees of freedom \( N_{k+1} \) is proportional to the current one \( N_k \). This is what the \( h \)-theory predicts.
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  The new number of degrees of freedom $N_{k+1}$ is proportional to the current one $N_k$. This is what the $h$-theory predicts.

- **Exponential decay:** Let $E(v_k)$ decay exponentially
  \[
  E(v_k) = Ae^{-\eta N_k}.
  \]
  Then, a simple calculation reveals that
  \[
  N_{k+1} - N_k = -\eta^{-1} \log \rho
  \]
  and the number of degrees of freedom must only grow by an additive constant. This property is very delicate to prove!
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Fourier methods

- **Periodic elliptic problem in** $\Omega = (0, 2\pi)^d$

  $$-\nabla \cdot (\nu \nabla u) + \sigma u = f \text{ in } \Omega, \quad u \text{ } (2\pi)^d\text{-periodic},$$

  formulated variationally in $V = H^1_{\text{per}}(\Omega)$ as

  $$u \in V : \quad a(u, v) = \langle f, v \rangle \quad \forall v \in V,$$

  and assumed to be continuous and coercive in $V$. 
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- **Fourier basis** $\{\phi_k : k \in \mathbb{Z}^d\}$, normalized in $V$

\[v = \sum_k \hat{v}_k \phi_k, \quad \text{with} \quad \|v\|_V^2 = \sum_k |\hat{v}_k|^2.\]
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$$v = \sum_k \hat{v}_k \phi_k, \quad \text{with} \quad \|v\|_V^2 = \sum_k |\hat{v}_k|^2.$$

- **Finite dimensional subspaces:** For arbitrary finite $\Lambda \subset \mathbb{Z}^d$, define

$$V_\Lambda = \text{span} \{\phi_k : k \in \Lambda\}.$$

and the orthogonal projection $P_\Lambda : V \rightarrow V_\Lambda$. 
Galerkin approximation and residual

- **Galerkin projection**

  \[ u_\Lambda \in V_\Lambda : \quad a(u_\Lambda, v_\Lambda) = \langle f, v_\Lambda \rangle \quad \forall v_\Lambda \in V_\Lambda. \]
Galerkin approximation and residual

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- **Residual** \( r_\Lambda = r(u_\Lambda) \in V' \) defined by

  \[ \langle r_\Lambda, v \rangle = \langle f, v_\Lambda \rangle - a(u_\Lambda, v) \quad \forall v \in V. \]

  It satisfies

  \[ \| r_\Lambda \|_{V'}^2 = \sum_{k \not\in \Lambda} |\hat{r}_k|^2, \quad \hat{r}_k = \langle r_\Lambda, \phi_k \rangle. \]
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  \[ \| r_\Lambda \|_{V'}^2 = \sum_{k \notin \Lambda} |\hat{r}_k|^2, \quad \hat{r}_k = \langle r_\Lambda, \phi_k \rangle. \]

- **(Ideal) efficient and reliable error estimator**

  \[ \frac{1}{\alpha^*} \| r(u_\Lambda) \|_{V'} \leq \| u - u_\Lambda \|_V \leq \frac{1}{\alpha^*} \| r(u_\Lambda) \|_{V'}, \]
Dörfler marking

- **Active basis updating**: Fix any $\theta \in (0, 1)$. Given $\Lambda$, $u_\Lambda \in V_\Lambda$ and $r_\Lambda \in V'$, select
  
  $\Lambda_{\text{new}} = \Lambda \cup \partial \Lambda$

  by the condition

  $\|P_{\partial \Lambda} r_\Lambda\|_{V'} \geq \theta \|r_\Lambda\|_{V'}$,  
  i.e.,  
  \[ \sum_{k \in \partial \Lambda} |\hat{r}_k|^2 \geq \theta^2 \sum_{k \in \mathbb{Z}^d} |\hat{r}_k|^2. \]
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  \[ \sum_{k \in \partial \Lambda} |\hat{r}_k|^2 \geq \theta^2 \sum_{k \in \mathbb{Z}^d} |\hat{r}_k|^2 . \]

- **Minimality:** $\partial \Lambda$ may be chosen of minimal cardinality by a greedy approach, bases on the decreasing rearrangement of the moduli of the Fourier coefficients of $r_\Lambda$.

- **Feasibility:** Exploiting some more information on data, a feasible version exists, which requires exploring only a finite number of Fourier coefficients of $r_\Lambda$. 
An ideal adaptive algorithm

Algorithm ADFOUR($\theta$, $tol$)

Set $r_0 := f$, $\Lambda_0 := \emptyset$, $n = -1$

do

$n \leftarrow n + 1$

$\partial \Lambda_n := \text{D"ORFLER}(r_n, \theta)$

$\Lambda_{n+1} := \Lambda_n \cup \partial \Lambda_n$

$u_{n+1} := \text{GAL}(\Lambda_{n+1})$

$r_{n+1} := \text{RES}(u_{n+1})$

while $\|r_{n+1}\|_{V'} > tol$
An ideal adaptive algorithm

**Algorithm ADFOUR(θ, tol)**

Set $r_0 := f$, $Λ_0 := ∅$, $n = −1$

do

$n ← n + 1$

$∂Λ_n := \text{DÖRFLER}(r_n, θ)$

$Λ_{n+1} := Λ_n ∪ ∂Λ_n$

$u_{n+1} := \text{GAL}(Λ_{n+1})$

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**Theorem (contraction property of ADFOUR).** Let $θ ∈ (0, 1)$ and let $\{Λ_n, u_n\}_{n≥0}$ be the sequence generated by the adaptive algorithm above. Then,

$$\|u - u_{n+1}\| ≤ \sqrt{1 - \frac{α_0}{α_0^2} \theta^2} \|u - u_n\|$$

where $\|v\| = \sqrt{a(v, v)}$. 
A more aggressive version

If the coefficients of the equation are analytic, the Galerkin matrix is “quasi sparse”. Exploiting this property, one can slightly enrich the active set produced by Dörfler’s marking, and push the contraction constant towards 0.
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If the coefficients of the equation are analytic, the Galerkin matrix is “quasi sparse”. Exploiting this property, one can slightly enrich the active set produced by Dörfler’s marking, and push the contraction constant towards 0.

Algorithm A-ADFOUR(θ, tol)

Set $r_0 := f$, $\Lambda_0 := \emptyset$, $n = -1$

do

$n \leftarrow n + 1$

$\tilde{\partial} \Lambda_n := \text{DÖRFLER}(r_n, \theta)$

$\partial \Lambda_n := \text{ENRICH}(\tilde{\partial} \Lambda_n, \theta)$

$\Lambda_{n+1} := \Lambda_n \cup \partial \Lambda_n$

$u_{n+1} := \text{GAL}(\Lambda_{n+1})$

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Theorem (contraction property of A-ADFOUR). Let $\theta \in (0, 1)$ and let $
\{\Lambda_n, u_n\}_{n \geq 0}$ be the sequence generated by A-ADFOUR. Then,

$$\|u - u_{n+1}\| \leq 2\sqrt{\frac{\alpha^*}{\alpha^*}} \sqrt{1 - \theta^2} \|u - u_n\|.$$

Optimality issues

- **Target cardinality growth:** If the solution $u$ belongs to some exponential class $\mathcal{A}_{G}^{\eta,t}$, one should expect

$$
\#\Lambda_n \leq \left( \frac{1}{\eta} \log \frac{|u|_{\mathcal{A}_{G}^{\eta,t}}}{\|u - u_n\|} \right)^{1/\tau} + C, \quad n = 0, 1, 2, \ldots
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- **Residual obstruction:** Dörfler marking is based on the current residual $r_\Lambda$. In general, this belongs to a worse approximation class than the solution.

$$v \in A^{\eta,t}_G \quad \Rightarrow \quad r(v) \in A^{\bar{\eta},\bar{\tau}}_G \text{ for some } \bar{\eta} \leq \eta, \quad \bar{\tau} \leq \tau.$$
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- **Estimate on cardinality growth:**

$$\#\Lambda_{n} \leq \left( \frac{1}{\overline{\eta}} \log \frac{|u|_{\mathcal{A}_{G}^{\eta,t}}}{\|u - u_{n}\|} \right)^{1/\overline{\tau}} + C , \quad n = 0, 1, 2, \ldots$$

- **Remedies...**
Remedy I: incorporating a coarsening step

\[ \Lambda := \text{COARSE}(w, \varepsilon) \]

Given \( u \in A_G^{\eta, \tau} \) and a function \( w \in V \), which is known to satisfy

\[ \| u - w \| \leq \varepsilon , \]

the output \( \Lambda \) is a set of minimal cardinality such that

\[ \| w - P_{\Lambda} w \| \leq 2\varepsilon , \]

and

\[ \# \Lambda \leq \left( \frac{1}{\eta} \log \frac{|u|_{A_G^{\eta, \tau}}}{\varepsilon} \right)^{1/\tau} + 1. \]
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- The final sets \( \Lambda_n \) have quasi-optimal cardinality
  (while the intermediate sets have only suboptimal cardinality)

**Algorithm AC-ADFOUR(\( \theta, \; tol \))**

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\( n \leftarrow n + 1 \)
\( \hat{\partial} \Lambda_n := \text{DÖRFLER}(r_n, \theta) \)
\( \hat{\partial} \Lambda_n := \text{ENRICH}(\hat{\partial} \Lambda_n, \theta) \)
\( \Lambda_{n+1} := \Lambda_n \cup \hat{\partial} \Lambda_n \)
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while \( \| r_{n+1} \|_{V'} > tol \)
Remedy II: applying a super-aggressive Dörfler marking

- Dynamic choice of Dörfler parameter:

\[ \theta \rightarrow \theta_n \]  

such that  

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  This yields:

- **quadratic convergence of the algorithm:**

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  \| u - u_{n+1} \| \lesssim \| u - u_n \|^2;
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- **no need of coarsening.**
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Abstract Framework for $h\!p$-Adaptivity

- **Operator equation:** Consider a, possibly, parametric operator equation (eg., a PDE) in a domain $\Omega \subset \mathbb{R}^n$

  \[ A_\lambda u = g. \]

  - The forcing $g$ and the parameter $\lambda$ (representing, e.g., the coefficients of the operator) are taken from some spaces $G$ and $\Lambda$ of functions on $\Omega$.
  - For short, we will write $f = (g, \lambda) \in F = G \times \Lambda$.
  - We assume there exists a unique solution $u = u(f) \in V$, a space of functions on $\Omega$. We assume, for simplicity, that $V$ and $F$ are Hilbert spaces over $\mathbb{R}$. 
Abstract Framework for $h_p$-Adaptivity

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- **Binary trees:** From an initial partition of $\Omega$, we generate an infinite binary master tree $\mathcal{K}$ by recursively halving each element $K$ in two children $K'$ and $K''$.

- **$h$-partitions:** A finite subtree of $\mathcal{K}$ defines an essentially disjoint $h$-partition $\mathcal{K}$ of $\Omega$, by collecting all the leaves of the subtree. The set of all $h$-partitions is denoted by $\mathcal{K}$. 
• **$hp$-partitions:** A $hp$-element is a pair $D = (K, d) \in \mathfrak{K} \times \mathbb{N}$, i.e., a geometric element $K$ together with a dimension $d$.

Given a $h$-partition $\mathcal{K}$, an associated $hp$-partition of $\Omega$ is a collection

$$\mathcal{D} = \{D = (K_D, d_D) : K_D \in \mathcal{K}\}.$$ 

The set of all $hp$-partitions is denoted by $\mathbb{D}$. 
Local Spaces for $h_p$-Adaptivity

- **Local spaces:** For all $K \in \mathcal{K}$, let $V_K$ and $F_K$ be (infinite dimensional) spaces of functions on $K$, such that for any $K \in \mathcal{K}$ we have

$$V \subseteq \prod_{K \in \mathcal{K}} V_K, \quad F \subseteq \prod_{K \in \mathcal{K}} F_K.$$
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- **Discrete local spaces**: For all $hp$-elements $D = (K, d) \in \mathcal{K} \times \mathbb{N}$, given $Z \in \{V, F\}$ we let $Z_{K,d} \subset Z_K$ be finite dimensional spaces of functions on $K$ such that

$$Z_{K,d} \subseteq Z_{K,d+1}, \quad Z_{K,d} \subset Z_{K',d} \times Z_{K'',d}.$$

We write $Z_D = Z_{K,d}$ and observe that any $Z_D$ will be a polynomial space of dimension $\approx d$. 

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$$V \subseteq \prod_{K \in \mathcal{K}} V_K, \quad F \subseteq \prod_{K \in \mathcal{K}} F_K.$$

- **Discrete local spaces**: For all $hp$-elements $D = (K, d) \in \mathcal{K} \times \mathbb{N}$, given $Z \in \{V, F\}$ we let $Z_{K,d} \subset Z_K$ be finite dimensional spaces of functions on $K$ such that

$$Z_{K,d} \subseteq Z_{K,d+1}, \quad Z_{K,d} \subset Z_{K',d} \times Z_{K'',d}.$$

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- **Discrete local spaces:** For all \(hp\)-elements \(D = (K, d) \in \mathcal{K} \times \mathbb{N}\), given \(Z \in \{V, F\}\) we let \(Z_{K,d} \subset Z_K\) be finite dimensional spaces of functions on \(K\) such that

\[
Z_{K,d} \subseteq Z_{K,d+1}, \quad Z_{K,d} \subset Z_{K',d} \times Z_{K'',d}.
\]

We write \(Z_D = Z_{K,d}\) and observe that any \(Z_D\) will be a polynomial space of dimension \(\approx d\).

- **Example:** When \(K\) is an \(n\)-simplex, \(V_{K,d}\) may be chosen as \(\mathbb{P}_p(K)\), where the associated polynomial degree \(p = p(d)\) can be defined as the largest value in \(\mathbb{N}\) such that \(\dim \mathbb{P}_{p-1}(K) = \binom{n+p-1}{p-1} \leq d\).
  - This definition normalizes the starting value \(p(1) = 1\) for all \(n \in \mathbb{N}\).
  - Only for \(n = 1\), it holds that \(p(d) = d\) for all \(d \in \mathbb{N}\).
Local Error Functional and Monotonicity

- **Local error functional**: This is a quantity

\[ e_D = e_D(v, f) \geq 0, \]

defined for all \((v, f) \in V \times F\), which measures the (squared) distance between \((v|_{K_D}, f|_{K_D})\) and its local approximation \((v_D, f_D)\).
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- **Local monotonicity**: We assume that \(e_D = e_D(v, f)\) is non-increasing under both ‘h-refinements’ and ‘p-enrichments’, in the sense that

  - **h-refinement**

\[ e_{D'} + e_{D''} \leq e_D \quad \text{if } K_{D'}, K_{D''} \text{ are children of } K_D \text{ and } d_{D'} = d_{D''} = d_D; \]

  - **p-enrichment**

\[ e_{D'} \leq e_D \quad \text{if } K_{D'} = K_D \text{ and } d_{D'} \geq d_D. \]
Global Error Functional and Monotonicity

- **Global error functional**: For an \( hp \)-partition \( \mathcal{D} = \{ D = (K_D, d_D) \} \) of \( \Omega \), the *global error functional*

  \[
  E_D(v, f) := \sum_{D \in \mathcal{D}} e_D(v, f),
  \]

  measures the (squared) distance between \((v, f)\) and its projection onto \( V_D \times F_D \), where \( Z_D = (Z_D)_{D \in \mathcal{D}} \).

- **Global monotonicity**:

  \[
  E_{\tilde{D}}(v, f) \leq E_D(v, f) \quad \text{if} \quad \mathcal{D} \leq \tilde{\mathcal{D}}.
  \]
Global Error Functional and Monotonicity

- **Global error functional**: For an $hp$-partition $\mathcal{D} = \{D = (K_D, d_D)\}$ of $\Omega$, the *global error functional*

$$E_{\mathcal{D}}(v, f) := \sum_{D \in \mathcal{D}} e_D(v, f),$$

measures the (squared) distance between $(v, f)$ and its projection onto $V_{\mathcal{D}} \times F_{\mathcal{D}}$, where $Z_{\mathcal{D}} = (Z_D)_{D \in \mathcal{D}}$.

- **Global monotonicity**:

$$E_{\tilde{\mathcal{D}}}(v, f) \leq E_{\mathcal{D}}(v, f) \quad \text{if} \quad \mathcal{D} \leq \tilde{\mathcal{D}}.$$

- **Notation**:
  
  - The cardinality of $\mathcal{D}$ is defined as $\#\mathcal{D} := \sum_{D \in \mathcal{D}} d_D$ ($d_D$ local dimension).
  
  - The set $\mathbb{D}$ of all $hp$-partitions contains the subset $\mathbb{D}^c$ of the ‘conforming’ partitions. We assume that for any $\mathcal{D} \in \mathbb{D}$ there exists a conforming partition $\mathcal{C}(\mathcal{D})$ such that $\mathcal{D} \leq \mathcal{C}(\mathcal{D})$. 
Example of Choice of Error Functional $e_D$

- Consider the model elliptic problem in $\Omega$

$$-\Delta u = f, \quad u = 0 \quad \text{in} \quad \partial\Omega,$$

- Define as a local error functional

$$e_D(v, f) := |v - P_{p_D}^1 v|_{H^1(K_D)}^2 + \frac{1}{\kappa} \|p_{p_D}^{-1} h_D(f - P_{p_D}^0 f)\|_{L^2(K_D)}^2 \quad \forall D \in \mathcal{D},$$

where

- $P_{p_D}^1, P_{p_D}^0$ resp. are orthogonal projectors on $\mathbb{P}_p(K_D)$ in the inner products of $L^2(K_D), H^1_0(K_D)$, resp.

- $\kappa$ is a parameter to be chosen later on.
Outline

Introduction
Adaptive Fourier methods
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$h_p$-Adaptive Approximation
Basic $h_p$-Adaptive Algorithm
Realizations of the Algorithm
Conclusions
• **Goal:** Given two functions \((v, f) \in V \times F\) and a target accuracy \(\varepsilon > 0\), find a “near optimal” \(h p\)-partition \(D\) such that
\[
E_D(v, f) \leq \varepsilon.
\]

• The task will be realized in two stages...
**h-Adaptive Tree Approximation**

- **Admissible binary tree:** Given $\mathcal{K} \in \mathbb{K}$, an admissible binary tree $T$ is the set of all $K \in \mathcal{K}$ and their ancestors. We note that $T \subset \mathbb{K}$ is finite and denote by $\mathcal{L}(T)$ the leaves of $T$, i.e., elements without successors.

- **Local $h$-error functional:** This is a subadditive quantity $e_K$

  $$e_{K'} + e_{K''} \leq e_K \quad \forall K \in \mathbb{K},$$

  where $K'$ and $K''$ denote the children of $K$. Given a function $v \in L^2(\Omega)$, $e_K$ is simply the square of the best $L^2$-error in $K$.

- **Global $h$-error functional:** $E_{\mathcal{K}} = \sum_{K \in \mathcal{K}} e_K \quad \forall \mathcal{K} \in \mathbb{K}$.

- **Best $h$-approximation:** Given $N \in \mathbb{N}$, let

  $$\sigma_N := \inf_{\#\mathcal{K} \leq N} E_{\mathcal{K}}.$$

  For functions in $L^2(\Omega)$ this gives the best $L^2$-error but computing a tree that realizes the min has exponential complexity.
Near-Best $h$-Adaptive Tree Approximation (Binev-DeVore)

- **Modified local error functional:** $\tilde{e}_K$ for all $K \in \mathcal{K}$
  
  - $\tilde{e}_K := e_K$ if $K$ is a root;
  
  - $\frac{1}{\tilde{e}_K} := \frac{1}{e_K} + \frac{1}{\tilde{e}_{K^*}}$ where $K^*$ is the parent of $K$ and $e_K \neq 0$; else $\tilde{e}_K = 0$. 
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- **Greedy algorithm on $\{\tilde{e}_K\}_{K \in \mathcal{K}}$:** Given a tree $\mathcal{K}_N$, with $\#\mathcal{K}_N = N$, construct $\mathcal{K}_{N+1}$ by bisecting the leaf $K \in \mathcal{L}(\mathcal{K})$ with largest $\tilde{e}_K$. 

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- **Instance optimality:** The sequence of trees $\{\mathcal{K}_N\}$ given by the greedy algorithm on $\{\tilde{e}_K\}_{K \in \mathcal{K}}$ provides a near-best $h$-adaptive approximation in the sense
  
  $$E_{\mathcal{K}_N} \leq \frac{N}{N - n + 1} \sigma_n$$

  for any integer $n \leq N$. The complexity for obtaining $\mathcal{K}_N$ is $\mathcal{O}(N)$. 
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  $\tilde{e}_K$ is defined as:
  
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  $$E_{\mathcal{K}_N} \leq \frac{N}{N - n + 1} \sigma_n$$

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- **Interpretation:** Given $N$ let $n = \lceil \frac{N}{2} \rceil$. Then $N - n + 1 \geq N/2$ and

  $$E_{\mathcal{K}_N} \leq 2\sigma_{\lceil \frac{N}{2} \rceil}.$$
**hp-adaptivity: the Ghost h-Tree...**

Ghost $h$-tree $\mathcal{T}$ (left) with 10 leaves ($\#\mathcal{L}(\mathcal{T}) = 10$); the root $K$ of $\mathcal{T}$ thus has an admissible dimension (polynomial degree for $n = 1$) $d(K, \mathcal{T}) = 10$. 
hp-adaptivity: the Ghost $h$-Tree... and Subordinate $hp$-Tree (Binev)

Ghost $h$-tree $\mathcal{T}$ (left) with 10 leaves ($\# \mathcal{L}(\mathcal{T}) = 10$); the root $K$ of $\mathcal{T}$ thus has an admissible dimension (polynomial degree for $n = 1$) $d(K, \mathcal{T}) = 10$.

The subordinate $hp$-tree $\mathcal{P}$ (right) results from $\mathcal{T}$ upon trimming 3 subtrees and raising the polynomial degrees of the interior nodes of $\mathcal{T}$, now leaves of $\mathcal{P}$, to $d(K, \mathcal{T}) = 2, 3, 2$ respectively.
Adaptive Strategy for $hp$-Refinements: $hp$-NEARBEST(Binev)

- **Ghost $h$-tree $\mathcal{T}$**: This is the previous $h$-tree associated with $v \in L^2(\Omega)$.

- **Admissible dimension**: Given $K \in \mathcal{T}$, the dimension $d(K, \mathcal{T})$ is

$$d(K, \mathcal{T}) = \#\mathcal{L}(\mathcal{T}(K)),$$

where $\mathcal{T}(K)$ is the subtree of $\mathcal{T}$ emanating from $K$. This quantity depends on both $\mathcal{T}$ and the underlying tree $\mathcal{T}$. 
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where $\mathcal{T}(K)$ is the subtree of $\mathcal{T}$ emanating from $K$. This quantity depends on both $\mathcal{T}$ and the underlying tree $\mathcal{T}$.

- **Local $hp$-error functionals**: Let $e_{K,d}$ be the local error functional on $K \in \mathcal{T}$ with polynomial dimension $d$. The modified local $hp$-error functional $e_K(\mathcal{T})$ reads

  - $e_K(\mathcal{T}) := e_{K,1}$ provided $K \in \mathcal{L}(\mathcal{T})$ is a leaf;

  - $e_K(\mathcal{T}) := \min \{ e_{K'}(\mathcal{T}) + e_{K''}(\mathcal{T}), e_{K,d(K,\mathcal{T})} \}$ otherwise.

- **Subordinate $hp$-tree $\mathcal{P}$**: This tree is obtained from the $h$-tree upon eliminating the subtree $\mathcal{T}(K)$ whenever increasing the polynomial dimension in $K$ from 1 to $d(K, \mathcal{T})$ reduces the error, i.e.

$$e_K(\mathcal{T}) = e_{K,d(K,\mathcal{T})}.$$
Instance Optimality of $hp$-NEARBEST

- **Theorem (Binev):** The subordinate $hp$-tree $\mathcal{P}_N$ with cardinality
  
  \[
  \#\mathcal{P}_N = \sum_{K \in \mathcal{L}(\mathcal{P}_N)} d(K, \mathcal{T}_N) = \#\mathcal{L}(\mathcal{T}_N) = N
  \]

  gives a $hp$ partition $\mathcal{D}_N$ with $\#\mathcal{D}_N = N$ and near-best $hp$-approximation over $\mathcal{D}_N$ in the sense that the global error functional satisfies

  \[
  E_{\mathcal{D}_N}(v, f) \leq \frac{2N}{N - n + 1} \sigma_n(v, f) \quad \forall N \geq n,
  \]

  where $\sigma_n$ is the best $hp$-error for $(v, f)$ with $n$ total degrees of freedom.
Instance Optimality of hp-NEARBEST

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  where $\sigma_n$ is the best $hp$-error for $(v, f)$ with $n$ total degrees of freedom.

  - The cost for constructing $\mathcal{D}_N$ is bounded by $O\left(\sum_{K \in \mathcal{T}_N} d(K, \mathcal{T}_N)\right)$, and
  
  varies from $O(N \log N)$ for well balanced trees to $O(N^2)$ for highly unbalanced trees.
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- **Interpretation:** Choosing $B > 1$, $n = \frac{N}{B}$ and $b = \frac{1}{2}(1 - \frac{1}{B}) < 1$ implies

  - $E_{\mathcal{D}_N}(v, f) \leq \varepsilon$
  - $\#\mathcal{D}_N \leq B \#\mathcal{D}$ for all $\mathcal{D} \in \mathcal{D}$ such that $E_{\mathcal{D}}(v, f) \leq b\varepsilon$. 
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Basic Modules

We assume availability of the following routines, which realize the two fundamental steps of the algorithm.

- \([\mathcal{D}, f_\mathcal{D}] := \text{hp-NEARBEST}(\varepsilon, v, f)\)

  The routine \text{hp-NEARBEST} takes as input \(\varepsilon > 0\) and \((v, f) \in V \times F\), and outputs \(\mathcal{D} \in \mathcal{D}\) as well as \(f_\mathcal{D}\), such that

  \(\begin{align*}
  & (i) \quad E_\mathcal{D}(v, f)^{\frac{1}{2}} \leq \varepsilon; \\
  & (ii) \quad \#\mathcal{D} \leq B \#\hat{\mathcal{D}} \text{ for any } \hat{\mathcal{D}} \in \mathcal{D} \text{ with } E_{\hat{\mathcal{D}}}(v, f)^{\frac{1}{2}} \leq b\varepsilon, \text{ for some constants } 0 < b \leq 1 \leq B.
  \end{align*}\)

  This routine may be implemented via Binev’s algorithm.

- \([\bar{\mathcal{D}}, \bar{u}] := \text{PDE}(\varepsilon, \mathcal{D}, f_\mathcal{D})\)

  The routine \text{PDE} takes as input \(\varepsilon > 0\), \(\mathcal{D} \in \mathcal{D}^c\), and data \(f_\mathcal{D} \in F_\mathcal{D}\). It outputs \(\bar{\mathcal{D}} \in \mathcal{D}^c\) with \(\mathcal{D} \leq \bar{\mathcal{D}}\) and \(\bar{u} \in V^c_\mathcal{D}\) such that \(\|u(f_\mathcal{D}) - \bar{u}\|_V \leq \varepsilon.\)
Assumptions on Global Error Functional

We assume the existence of constants $C_1, C_2 > 0$ with

$$C_1 C_2 < b,$$

such that the following properties hold:

- **Continuity of the solution upon data:**

  $$\| u(f) - u(f_D) \|_V \leq C_1 \inf_{w \in V} E_D(w, f)^{\frac{1}{2}} \quad \forall D \in \mathcal{D}, \forall f \in F,$$

- **Lipschitz continuity of $E_D$ upon state:**

  $$\sup_{f \in F} \left\| E_D(w, f)^{\frac{1}{2}} - E_D(v, f)^{\frac{1}{2}} \right\| \leq C_2 \| w - v \|_V \quad \forall D \in \mathcal{D}, \forall v, w \in V.$$
Verifying the Assumptions on Global Error Functional

**Error and Oscillation:** Let

\[ e_D(v, f) := |v - P_{pD}^1 v|_{H^1(K_D)}^2 + \frac{1}{\kappa} \text{osc}_D(f)^2 \quad \forall D \in \mathcal{D}, \]

with

\[ \text{osc}_D(f)^2 = \|p_D^{-1}h_D(f - P_{pD-1}^0 f)\|^2_{L^2(K_D)} \]

- **Continuity of the solution upon data:**

\[ \|u(f) - u(f_D)\|_{H^1(\Omega)} \leq C \text{osc}_D(f) \leq C\kappa^{\frac{1}{2}} E_D(w, f)^{\frac{1}{2}} = C_1 \quad \forall w \in V. \]

- **Lipschitz continuity of \( E_D \) upon state:**

\[ \left| E_D(w, f)^{\frac{1}{2}} - E_D(v, f)^{\frac{1}{2}} \right| \leq \|w - v\|_{H^1(\Omega)} \quad \Rightarrow \quad C_2 = 1. \]
Verifying the Assumptions on Global Error Functional

Error and Oscillation: Let

\[ e_D(v, f) := |v - P_{P_D}^1 v|_{H^1(K_D)}^2 + \frac{1}{\kappa} \text{osc}_D(f)^2 \quad \forall D \in D, \]

with

\[ \text{osc}_D(f)^2 = \|p_D^{-1} h_D(f - P_{P_D}^0 f)\|^2_{L^2(K_D)} \]

- Continuity of the solution upon data:

\[ \|u(f) - u(f_D)\|_{H^1(\Omega)} \leq C \text{osc}_D(f) \leq C\kappa \frac{1}{2} \mathcal{E}_D(w, f)^{\frac{1}{2}} = C_1 \quad \forall w \in V. \]

- Lipschitz continuity of \( \mathcal{E}_D \) upon state:

\[ \left| \mathcal{E}_D(w, f)^{\frac{1}{2}} - \mathcal{E}_D(v, f)^{\frac{1}{2}} \right| \leq \|w - v\|_{H^1(\Omega)} \quad \Rightarrow \quad C_2 = 1. \]

- Bound on constants: \( C_1 C_2 = C\kappa \frac{1}{2} C_2 < b \) for \( \kappa \) sufficiently small.
**Basic $hp$-AFEM**

- **$hp$-AFEM:**

  \[
  hp\text{-AFEM}(\bar{u}_0, f, \varepsilon_0)
  \]

  % Input: \((\bar{u}_0, f) \in V \times F, \varepsilon_0 > 0\) with \(\|u(f) - \bar{u}_0\|_V \leq \varepsilon_0\).

  % Parameters: \(\mu \in (0, 1)\) such that \(C_1C_2 < b(1 - \mu)\), and \(\omega \in \left(\frac{C_2}{b}, \frac{1-\mu}{C_1}\right)\).

  for \(i = 1, 2, \ldots\) do
    \[
    [D_i, f_{D_i}] := hp\text{-NEARBEST}(\omega \varepsilon_{i-1}, \bar{u}_{i-1}, f)
    \]
    \[
    [\bar{D}_i, \bar{u}_i] := PDE(\mu \varepsilon_{i-1}, C(D_i), f_{D_i})
    \]
    \[
    \varepsilon_i := (\mu + C_1 \omega) \varepsilon_{i-1}
    \]
  end do

- **Error Reduction:** Note that \(\varepsilon_i = (\mu + C_1 \omega)^i \varepsilon_0\) where \(\mu + C_1 \omega < 1\).
Basic $h p$-AFEM

- **hp-AFEM:**

  $hp$-AFEM($\bar{u}_0, f, \varepsilon_0$)

  % Input: ($\bar{u}_0, f$) $\in V \times F$, $\varepsilon_0 > 0$ with $\|u(f) - \bar{u}_0\|_V \leq \varepsilon_0$.
  % Parameters: $\mu \in (0, 1)$ such that $C_1 C_2 < b(1 - \mu)$, and $\omega \in \left(\frac{C_2}{b} , \frac{1-\mu}{C_1}\right)$.

  for $i = 1, 2, \ldots$ do
  [${\mathcal D}_i, f_{{\mathcal D}_i}] := hp$-NEARBEST($\omega \varepsilon_{i-1}, \bar{u}_{i-1}, f$)
  [${\mathcal D}_i, \bar{u}_i] := PDE(\mu \varepsilon_{i-1}, C({\mathcal D}_i), f_{{\mathcal D}_i})$
  $\varepsilon_i := (\mu + C_1 \omega)\varepsilon_{i-1}$
  end do

- **Error Reduction:** Note that $\varepsilon_i = (\mu + C_1 \omega)^i \varepsilon_0$ where $\mu + C_1 \omega < 1$.

- **Coarsening:** Tolerance for $\bar{u}_i$ within PDE is $\tau_i := \mu \varepsilon_{i-1}$ and the subsequent input tolerance of $hp$-NEARBEST is $\omega \varepsilon_i = \omega(\mu + C_1 \omega)\varepsilon_{i-1} > \omega \tau_i$. Since in our applications $C_2 = 1$ and $\omega > C_2/b \geq 1$, we see that $\omega \varepsilon_i > \tau_i = \mu \varepsilon_{i-1}$. 
**Convergence and Instance Optimality**

**Theorem.** Let the previous assumptions on the global error functional $E_D$ be satisfied. Then, for the sequences $(\bar{u}_i)$, $(D_i)$ produced in $hp$-AFEM, it holds that

$$
\|u - \bar{u}_i\|_V \leq \varepsilon_i \quad \forall i \geq 0, \quad E_{D_i}(u, f)^{1/2} \leq \frac{\omega + C_2}{\mu + C_1\omega} \varepsilon_i \quad \forall i \geq 1,
$$

and

$$
\#D_i \leq B\#D \quad \text{for any } D \in \mathbb{D} \quad \text{with } E_D(u, f)^{1/2} \leq \frac{b\omega - C_2}{\mu + C_1\omega} \varepsilon_i
$$

where $u = u(f)$.
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A practical \( hp \)-adaptive algorithm

- Recall PDE:
  
  \[ [\bar{D}, \bar{u}] := \text{PDE}(\varepsilon, D, f_D) \]

  The routine \text{PDE} takes as input \( \varepsilon > 0, \ D \in \mathbb{D}^c \), and data \( f_D \in F_D \). It outputs \( \bar{D} \in \mathbb{D}^c \) with \( D \leq \bar{D} \) and \( \bar{u} \in V^c_D \) such that \( \| u(f_D) - \bar{u} \|_V \leq \varepsilon \).
A practical $h p$-adaptive algorithm

- **Recall PDE:**
  
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- **Error reduction:** For efficiency, $\text{PDE}$ should exploit the work already carried out within $hp$-AFEM. Precisely, for any desired error reduction factor $\varrho \in (0, 1)$, it should give

  $$\|u(f_D) - \bar{u}\|_V \leq \varrho \inf_{v \in V_{\bar{D}}} \|u(f_D) - v\|_V.$$
A practical \(hp\)-adaptive algorithm

- **Recall PDE:**
  
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  \[\bar{\mathcal{D}}, \bar{u}\] := \text{PDE}(\varepsilon, \mathcal{D}, f_D)
  \]
  The routine \(\text{PDE}\) takes as input \(\varepsilon > 0, \mathcal{D} \in \mathcal{D}^c\), and data \(f_D \in F_D\). It outputs \(\bar{\mathcal{D}} \in \mathcal{D}^c\) with \(\mathcal{D} \leq \bar{\mathcal{D}}\) and \(\bar{u} \in V_\mathcal{D}^c\) such that \(\|u(f_D) - \bar{u}\|_V \leq \varepsilon\).

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- Indeed, at each call of \(\text{PDE}\) within \(hp\)-\text{AFEM}, we will be already guaranteed to have
  \[
  \inf_{v \in V_\mathcal{D}^c} \|u(f_D) - v\|_V \leq C\varepsilon,
  \]
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  \]
A practical $h p$-adaptive algorithm

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  \[ \bar{D}, \bar{u} := \text{PDE}(\varepsilon, D, f_D) \]

  The routine \text{PDE} takes as input $\varepsilon > 0$, $D \in \mathbb{D}^c$, and data $f_D \in F_D$. It outputs $\bar{D} \in \mathbb{D}^c$ with $D \leq \bar{D}$ and $\bar{u} \in V_D$ such that $\|u(f_D) - \bar{u}\|_V \leq \varepsilon$.

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  whence a suitable choice of $\varrho$ will yield
  
  \[ \|u(f_D) - \bar{u}\|_V \leq \varepsilon. \]

- **Remark:** The input data $f_D$ is piecewise polynomial on the input partition $D$, hence no data oscillation appears.
• **PDE:** This module may be implemented by the usual loop

\[
\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{GROW}
\]

where

\textbf{GROW} may be either an \( h \)-refinement or a \( p \)-enrichment.
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• **Features:**

  ▶ **ESTIMATE:** should be based on a reliable and efficient *a posteriori* error indicator, with constants independent of the current \( hp \)-partition \( D \) (in particular, “\( p \)-robust”).
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\[
\|u - u_{\mathcal{D}}\|_{V} \lesssim \|u_{\mathcal{D}_{\text{new}}} - u_{\mathcal{D}}\|_{V}
\]

(all constants independent of \( \mathcal{D} \)).
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  ▶ **GROW:** should yield a new \( hp \) partition \( \mathcal{D}_{\text{new}} \) with \( \#D_{\text{new}} \lesssim \#D \) and guaranteed *saturation property*

\[
\| u - u_{\mathcal{D}} \|_V \lesssim \| u_{\mathcal{D}_{\text{new}}} - u_{\mathcal{D}} \|_V
\]

(all constants independent of \( \mathcal{D} \)).

▶ **MARK:** might be skipped. Indeed, since the task of producing a near-best \( hp \)-partition is assigned to \( \text{hp-NEARBEST} \), in principle even a *uniform* refinement/enrichment is allowed.
Applications to elliptic self-adjoint problems

We detail three possible realizations of PDE, based on:

- a residual estimator, in dimension 1,
- a residual estimator, in dimension 2
- an equilibrated flux estimator, in dimension 2.
Applications to elliptic self-adjoint problems

We detail three possible realizations of PDE, based on:

- a residual estimator, in dimension 1,
- a residual estimator, in dimension 2
- an equilibrated flux estimator, in dimension 2.

- **In a 1D domain**, for the general elliptic self-adjoint problem

  \[ u \in H^1_0(\Omega) : -(\mu u_x)_x + \sigma u = f + g_x \quad \text{in} \quad H^{-1}(\Omega), \]

  - the residual-based error estimator \( \eta_D(u_D, f_D) \) defined by

    \[ \eta_D^2(u_D, f_D) = \sum_{D \in \mathcal{D}} \| r_D \|^2_{H^{-1}(K_D)} \]

    is \( p \)-robust and easily computable element-wise;

  - the saturation property is guaranteed by raising the polynomial degree from \( p_D \) to some \( \hat{p}_D \leq 2p_D + 3 \) in each marked element.

  - Thus, \( hp \)-AFEM is fully optimal.
Residual-based estimators

- In a 2D domain (a polygon), for the model problem

\[ u \in H^1_0(\Omega) : -\Delta u = f \text{ in } \Omega, \]

> the Melenk-Wohlmuth residual-based error estimator \( \eta_D(u_D, f_D) \) defined element-wise by

\[
\eta^2_D(u_D, f_D) : = \frac{|K_D|}{p^2_D} \| f_D + \Delta u_D \|_{L^2(K_D)}^2 \\
+ \sum_{\{e \in \mathcal{E}(D) : e \subset \partial K_D \cap \Omega\}} \frac{|e|}{2 p_e, D} \| [\nabla u_D \cdot n_e] \|_{L^2(e)}^2.
\]

induces a factor \( \simeq \| p_D \|^{-2-2\varepsilon} \) in the efficiency estimate.

Consequently, the number \( M \) of iterations in each call of \( \text{PDE} \) for reducing the error by a factor \( \varrho \) scales like \( M \approx \log \varrho^{-1} \| p_D \|_{\infty}^{2+\varepsilon} \), leading to a convergence analysis that is not \( p \)-robust.

> The saturation property is guaranteed if each marked element is replaced by its four grandchildren, while preserving the polynomial degree.
Equilibrated Flux Estimators

- **p-robust convergence:** This can be achieved for \texttt{hp-AFEM} in 2D upon resorting to equilibrated flux estimators.

- **Equilibrated flux estimator:** We introduce the following standard notation:

  - Given a partition \( \mathcal{D} \) made of triangles \( K \), with vertices \( a \in \mathcal{A}_D \), denote by \( \omega_a \) the star (or patch) of elements containing \( a \).

  - For any such vertex, define the local energy space

    \[
    H_1^*(\omega_a) := \begin{cases} 
    \{ v \in H^1(\omega_a) : \langle v, 1 \rangle_{\omega_a} = 0 \} & a \in \mathcal{A}_D^{\text{int}}, \\
    \{ v \in H^1(\omega_a) : v = 0 \text{ on } \partial\omega_a \cap \partial\Omega \} & a \in \mathcal{A}_D^{\text{bdry}}. 
    \end{cases}
    \]

  - Define the global and local residuals for the Galerkin solution \( u_\mathcal{D} \in \mathcal{V}_\mathcal{D} \)

    \[
    r(v) := \langle f, v \rangle_\Omega - \langle \nabla u_\mathcal{D}, \nabla v \rangle_\Omega, \quad r_a(v) = r(\phi_a v).
    \]

    where \( \phi_a \) is the piecewise linear hat function centered at \( a \).
\textbf{\textit{p}-Robust A Posteriori Estimates}

- **Upper and lower bounds:**

\[
\|\nabla (u-u_D)\|^2_{\Omega} \leq 3 \sum_{a \in A_D} \|r_a\|^2_{H^1_*(\omega_a)}, \quad \|r_a\|_{H^1_*(\omega_a)} \lesssim \|\nabla (u-u_D)\|_{\omega_a} \quad \forall a \in A_D.
\]
p-Robust A Posteriori Estimates

- **Upper and lower bounds:**
  \[
  \| \nabla (u - u_D) \|^2_\Omega \leq 3 \sum_{a \in A_D} \| r_a \|^2_{H^1_*(\omega_a)'}, \quad \| r_a \|_{H^1_*(\omega_a)'} \lesssim \| \nabla (u - u_D) \|_{\omega_a} \quad \forall a \in A_D.
  \]

- **p-robust equivalence:**
  \[
  \| r_a \|_{H^1_*(\omega_a)'} \simeq \| \sigma_a \|_{\omega_a}
  \]

where \( \sigma_a \in \mathcal{RT}(D_a) \) is a suitable equilibrated flux for \( u_D \) (i.e., it satisfies \( \langle \nabla \cdot \sigma_a, 1 \rangle_T = \langle f, 1 \rangle_T \) for all \( T \subset \omega_a \)),

[Braess, Pillwein and Schöberl (2009)]
\textbf{p-Robust A Posteriori Estimates}

- Upper and lower bounds:

\[
\|\nabla (u-u_D)\|_\Omega^2 \leq 3 \sum_{a \in A_D} \| r_a \|^2_{H^1_{\ast}(\omega_a)}, \quad \| r_a \|^1_{H^1_{\ast}(\omega_a)} \lesssim \|\nabla (u-u_D)\|_{\omega_a} \quad \forall a \in A_D.
\]

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[Braess, Pillwein and Schöberl (2009)]

- \textbf{Computability}. A particular equilibrated flux \( \sigma_a \) can be efficiently computed by solving a discrete saddle-point problem in the space \( \mathcal{RT}(D_a) \).

[Ern and Vohralík (2015)]
- **Upper and lower bound for discrete functions:** If $\widetilde{D} \geq D$ yields $V_D \subset V_{\widetilde{D}}$, then
  \[
  \| \nabla (u_{\widetilde{D}} - u_D) \|^2_\Omega \leq 3 \sum_{a \in A_D} \| r_a \|^2_{(H^1_*(\omega_a) \cap V_{\widetilde{D}}(\omega_a))'},
  \]
  and
  \[
  \| r_a \|_{(H^1_*(\omega_a) \cap V_{\widetilde{D}}(\omega_a))'} \lesssim \| \nabla (u_{\widetilde{D}} - u_D) \|_{\omega_a} \quad \forall a \in A_D.
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• **Upper and lower bound for discrete functions:** If $\tilde{D} \geq D$ yields $V_D \subset V_{\tilde{D}}$, then

$$\|\nabla(u_{\tilde{D}} - u_D)\|_\Omega^2 \leq 3 \sum_{a \in A_D} \|r_a\|^2_{(H^1_*(\omega_a) \cap V_{\tilde{D}}(\omega_a))'},$$

and

$$\|r_a\|_{(H^1_*(\omega_a) \cap V_{\tilde{D}}(\omega_a))'} \lesssim \|\nabla(u_{\tilde{D}} - u_D)\|_{\omega_a} \quad \forall a \in A_D.$$

• **Marking:** Suppose we apply a *star-based Dörfler marking*, and that for any marked star we can find a local space $V_{\tilde{D}}(\omega_a) \supset V_D(\omega_a)$ for which it holds

$$\|r_a\|_{H^1_*(\omega_a)'} \lesssim \|r_a\|_{(H^1_*(\omega_a) \cap V_{\tilde{D}}(\omega_a))'} \quad \text{uniformly in } p.$$
• **Upper and lower bound for discrete functions:** If $\tilde{D} \geq D$ yields $V_D \subset V_{\tilde{D}}$, then
  \[
  \|\nabla (u_{\tilde{D}} - u_D)\|_\Omega^2 \leq 3 \sum_{a \in A_D} \|r_a\|_2^2 (H_1^1(\omega_a) \cap V_{\tilde{D}}(\omega_a))',
  \]
  and
  \[
  \|r_a\| (H_1^1(\omega_a) \cap V_{\tilde{D}}(\omega_a))' \lesssim \|\nabla (u_{\tilde{D}} - u_D)\|_{\omega_a} \quad \forall a \in A_D.
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• **Marking:** Suppose we apply a *star-based Dörfler marking*, and that for any marked star we can find a local space $V_{\tilde{D}}(\omega_a) \supset V_D(\omega_a)$ for which it holds
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  \|r_a\|_{H^1_*(\omega_a) \cap V_{\tilde{D}}(\omega_a))'} \lesssim \|\nabla (u_{\tilde{D}} - u_D)\|_{\omega_a} \quad \forall a \in A_D.
  \]

• **Saturation property:** Then, we immediately obtain the *$p$-robust saturation property*
  \[
  \|\nabla (u - u_D)\|_\Omega^2 \lesssim \|\nabla (u_{\tilde{D}} - u_D)\|_\Omega^2
  \]

• **Contraction property:** This implies the following bound with $\rho < 1$
  \[
  \|\nabla (u - u_{\tilde{D}})\|_\Omega \leq \rho \|\nabla (u - u_D)\|_\Omega
  \]
Checking the saturation condition

- **Reduction to a reference domain.** The problem of verifying

\[ \| r_a \|_{H^1_* (\omega_a)'} \lesssim \| r_a \|_{(H^1_* (\omega_a) \cap V_D (\omega_a))'} \]

for a suitable \( V_D (\omega_a) \supset V_D (\omega_a) \) can be reduced to the problem of establishing, in a reference domain, norm equivalences between the exact and the Galerkin solution of certain elliptic problems with polynomial data \( p \), assuming that the Galerkin solution is a polynomial of suitable degree \( q > p \).
Checking the saturation condition

- **Reduction to a reference domain.** The problem of verifying

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for a suitable \( V_D(\omega_a) \supset V_D(\omega_a) \) can be reduced to the problem of establishing, in a reference domain, norm equivalences between the exact and the Galerkin solution of certain elliptic problems with polynomial data \( p \), assuming that the Galerkin solution is a polynomial of suitable degree \( q > p \).

- **A prototypical problem is as follows:**
  
  Let \( \hat{E} \) be a reference triangle or square. For any given \( g \in \mathbb{P}_p(\hat{E}) \), let \( u = u(g) \in \hat{V} := H^1_0(\hat{E}) \) be the solution of

  \[ \int_{\hat{E}} \nabla u \cdot \nabla v = \int_{\hat{E}} g v \quad \forall v \in H^1_0(\hat{E}), \]

  and let \( u_q = u_q(g) \in \hat{V}_q := H^1_0(\hat{E}) \cap \mathbb{P}_q(\hat{E}) \) be the solution of

  \[ \int_{\hat{E}} \nabla u_q \cdot \nabla v = \int_{\hat{E}} g v \quad \forall v \in H^1_0(\hat{E}) \cap \mathbb{P}_q(\hat{E}). \]
Prototypal problem (cont’d) One seeks a function

\[ q = q(p) > p \]

and a constant \( C > 0 \) independent of \( g \) and \( p \) such that

\[ \| \nabla u \|_{0, \hat{E}} \leq C \| \nabla u_{q(p)} \|_{0, \hat{E}}. \]
• **Prototypal problem (cont’d)** One seeks a function

\[ q = q(p) > p \]

and a constant \( C > 0 \) independent of \( g \) and \( p \) such that

\[ \| \nabla u \|_{0,\hat{E}} \leq C \| \nabla u_{q(p)} \|_{0,\hat{E}}. \]

• On the reference square \( \hat{R} = (0, 1)^2 \), the result is proven to be true with \( q(p) = p + c \) for a suitable constant \( c > 0 \).

• On the reference simplex \( \hat{T} \), there is clear numerical evidence of a similar result (and the proof is under construction).
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- We have considered several adaptive spectral methods, with guaranteed linear or quadratic convergence; we have discussed their optimality properties in terms of cardinality of activated degrees of freedom.

- We have introduced an abstract framework for $hp$-adaptivity.

- We have presented an algorithm for $hp$-adaptive approximation, with instance optimality.

- We have considered a general, convergent and nearly-optimal $hp$-adaptive finite element method, and we have discussed several specific realizations.

- Various extension are waiting:
  - Discontinuous Galerkin (underway)
  - Stokes system
  - anisotropic adaptivity
  - non-symmetric operators
  - ...

Some references


- C. Canuto, V. Simoncini and M. Verani, *On the decay of the inverse of matrices that are sum of Kronecker products*, Linear Algebra Appl., 452 (2014), 21–39


