Mean estimation: median-of-means tournaments

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based on joint work with
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estimating the mean

Given $X_1, \ldots, X_n$, a real i.i.d. sequence, estimate $\mu = \mathbb{E}X_1$. 

"Obvious" choice: empirical mean $\mu_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. 

By the central limit theorem, if $X$ has a finite variance $\sigma^2$, 

$$\lim_{n \to \infty} P\left\{ \left| \sqrt{n} (\mu_n - \mu) \right| > \sigma \sqrt{\frac{2 \log(2/\delta)}{\delta}} \right\} \leq \delta.$$ 

We would like non-asymptotic inequalities of a similar form. 

If the distribution is sub-Gaussian, $\mathbb{E} \exp(\lambda (X - \mu)) \leq \exp(\frac{\sigma^2 \lambda^2}{2})$, then with probability at least $1 - \delta$, 

$$|\mu_n - \mu| \leq \sigma \sqrt{\frac{2 \log(2/\delta)}{\delta} n}.$$
estimating the mean

Given $X_1, \ldots, X_n$, a real i.i.d. sequence, estimate $\mu = \mathbb{E}X_1$.

“Obvious” choice: empirical mean

$$\bar{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$
estimating the mean

Given \( \mathbf{X_1}, \ldots, \mathbf{X_n} \), a real i.i.d. sequence, estimate \( \mu = \mathbb{E} \mathbf{X_1} \).

“Obvious” choice: empirical mean

\[
\overline{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbf{X_i}
\]

By the central limit theorem, if \( \mathbf{X} \) has a finite variance \( \sigma^2 \),

\[
\lim_{n \to \infty} \mathbb{P} \left\{ \sqrt{n} \left| \overline{\mu}_n - \mu \right| > \sigma \sqrt{2 \log(2/\delta)} \right\} \leq \delta .
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We would like non-asymptotic inequalities of a similar form.

If the distribution is sub-Gaussian, $\mathbb{E} \exp(\lambda (X - \mu)) \leq \exp(\sigma^2 \lambda^2 / 2)$, then with probability at least $1 - \delta$,

$$|\bar{\mu}_n - \mu| \leq \sigma \sqrt{\frac{2 \log(2/\delta)}{n}}.$$
empirical mean–heavy tails

The empirical mean is computationally attractive.
Requires no a priori knowledge and automatically scales with $\sigma$.
If the distribution is not sub-Gaussian, we still have Chebyshev’s inequality: w.p. $\geq 1 - \delta$,

$$|\mu_n - \mu| \leq \sigma \sqrt{\frac{1}{n\delta}}.$$

Exponentially weaker bound. Especially hurts when many means are estimated simultaneously.
This is the best one can say. Catoni (2012) shows that for each $\delta$ there exists a distribution with variance $\sigma$ such that

$$\mathbb{P} \left\{ \left| \mu_n - \mu \right| \geq \sigma \sqrt{\frac{c}{n\delta}} \right\} \geq \delta.$$

\[
\hat{\mu}_{MM} \overset{\text{def}}{=} \text{median} \left( \frac{1}{m} \sum_{t=1}^{m} x_t, \ldots, \frac{1}{m} \sum_{t=(k-1)m+1}^{km} x_t \right)
\]

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**Lemma**

Let \( \delta \in (0, 1) \), \( k = 8 \log \delta^{-1} \) and \( m = \frac{n}{8 \log \delta^{-1}} \). Then with probability at least \( 1 - \delta \),

\[ |\hat{\mu}_{MM} - \mu| \leq \sigma \sqrt{32 \log(1/\delta) \over n} \]
proof

By Chebyshev, each mean is within distance $\sigma \sqrt{\frac{4}{m}}$ of $\mu$ with probability $\frac{3}{4}$.

The probability that the median is not within distance $\sigma \sqrt{\frac{4}{m}}$ of $\mu$ is at most $\mathbb{P}\{\text{Bin}(k, 1/4) > k/2\}$ which is exponentially small in $k$.  

median of means

- Sub-Gaussian deviations.
- Scales automatically with $\sigma$.
- Parameters depend on required confidence level $\delta$.
- Also works when the variance is infinite. If 
  $$
  \mathbb{E} \left[ |X - \mathbb{E}X|^{1+\alpha} \right] = M \text{ for some } \alpha \leq 1,
  $$
  then, with probability at least $1 - \delta$,

  $$
  |\hat{\mu}_{MM} - \mu| \leq \left( \frac{8(12M)^{1/\alpha} \ln(1/\delta)}{n} \right)^{\alpha/(1+\alpha)}
  $$

why sub-Gaussian?

Sub-Gaussian bounds are the best one can hope for when the variance is finite.

In fact, for any $M > 0$, $\alpha \in (0, 1]$, $\delta > 2e^{-n/4}$, and mean estimator $\hat{\mu}_n$, there exists a distribution $\mathbb{E} \left[ |X - \mathbb{E}X|^{1+\alpha} \right] = M$ such that

$$|\hat{\mu}_n - \mu| \geq \left( \frac{M^{1/\alpha} \ln(1/\delta)}{n} \right)^{\alpha/(1+\alpha)}.$$

Proof: The distributions $P_+(0) = 1 - p$, $P_+(c) = p$ and $P_-(0) = 1 - p$, $P_-(c) = p$ are indistinguishable if all $n$ samples are equal to 0.
why sub-Gaussian?

This shows optimality of the median-of-means estimator for all $\alpha$. It also shows that finite variance is necessary even for rate $n^{-1/2}$. One cannot hope to get anything better than sub-Gaussian tails. Catoni proved that sample mean is optimal for the class of Gaussian distributions.
Do there exist estimators that are sub-Gaussian simultaneously for all confidence levels?

An estimator is multiple-$\delta$-sub-Gaussian for a class of distributions $\mathcal{P}$ and $\delta_{\text{min}}$ if for all $\delta \in [\delta_{\text{min}}, 1)$, and all distributions in $\mathcal{P}$,

$$|\hat{\mu}_n - \mu| \leq L\sigma \sqrt{\frac{\log(2/\delta)}{n}}.$$
multiple-$\delta$ estimators

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An estimator is multiple-$\delta$ -sub-Gaussian for a class of distributions $\mathcal{P}$ and $\delta_{\min}$ if for all $\delta \in [\delta_{\min}, 1)$, and all distributions in $\mathcal{P}$,

$$|\hat{\mu}_n - \mu| \leq L\sigma \sqrt{\frac{\log(2/\delta)}{n}}.$$  

The picture is more complex than before.
known variance

Given $0 < \sigma_1 \leq \sigma_2 < \infty$, define the class

$$\mathcal{P}_2[\sigma_1^2, \sigma_2^2] = \{P : \sigma_1^2 \leq \sigma_P^2 \leq \sigma_2^2\}.$$

Let $R = \sigma_2 / \sigma_1$. 
known variance

Given $0 < \sigma_1 \leq \sigma_2 < \infty$, define the class

$$\mathcal{P}_2^{[\sigma_1^2, \sigma_2^2]} = \{ P : \sigma_1^2 \leq \sigma_P^2 \leq \sigma_2^2 \}. $$

Let $R = \sigma_2 / \sigma_1$.

- If $R$ is bounded then there exists a multiple-$\delta$-sub-Gaussian estimator with $\delta_{\min} = 4e^{1-n/2}$;
- If $R$ is unbounded then there is no multiple-$\delta$-sub-Gaussian estimate for any $L$ and $\delta_{\min} \to 0$.

A sharp distinction.

The exponentially small value of $\delta_{\min}$ is best possible.
construction of multiple-δ estimator

Reminiscent to Lepski’s method of adaptive estimation.

For \( k = 1, \ldots, K = \log_2(1/\delta_{\text{min}}) \), use the median-of-means estimator to construct confidence intervals \( I_k \) such that

\[
\mathbb{P}\{\mu \notin l_k\} \leq 2^{-k}.
\]

(This is where knowledge of \( \sigma_2 \) and boundedness of \( R \) is used.) Define

\[
\hat{k} = \min \left\{ k : \bigcap_{j=k}^{\infty} l_j \neq \emptyset \right\}.
\]

Finally, let

\[
\hat{\mu}_n = \text{mid point of } \bigcap_{j=\hat{k}}^{\infty} l_j.
\]
proof

For any \( k = 1, \ldots, K \),

\[
\mathbb{P}\{|\hat{\mu}_n - \mu| > |l_k|\} \leq \mathbb{P}\{\exists j \geq k : \mu \notin I_j\}
\]

because if \( \mu \in \bigcap_{j=k}^{K} I_j \), then \( \bigcap_{j=k}^{K} I_j \) is non-empty and therefore \( \hat{\mu}_n \in \bigcap_{j=k}^{K} I_j \).

But

\[
\mathbb{P}\{\exists j \geq k : \mu \notin I_j\} \leq \sum_{j=k}^{K} \mathbb{P}\{\mu \notin I_j\} \leq 2^{1-k}
\]
For $\eta \geq 1$ and $\alpha \in (2, 3]$, define

$$\mathcal{P}_{\alpha, \eta} = \{ P : \mathbb{E}|X - \mu|^{\alpha} \leq (\eta \sigma)^{\alpha} \}.$$ 

Then for some $C = C(\alpha, \eta)$ there exists a multiple-$\delta$ estimator with a constant $L$ and $\delta_{\text{min}} = e^{-n/C}$ for all sufficiently large $n$. 
This follows from a more general result: Define

\[
p_-(j) = \mathbb{P}\left\{ \sum_{i=1}^{j} X_i \leq j \mu \right\} \quad \text{and} \quad p_+(j) = \mathbb{P}\left\{ \sum_{i=1}^{j} X_i \geq j \mu \right\}.
\]

A distribution is \( k \)-regular if

\[
\forall j \geq k, \min(p_+(j), p_-(j)) \geq 1/3.
\]

For this class there exists a multiple-\( \delta \) estimator with a constant \( L \) and \( \delta_{\min} = e^{-n/k} \) for all \( n \).
Let $X$ be a random vector taking values in $\mathbb{R}^d$ with mean $\mu = \mathbb{E}X$ and covariance matrix $\Sigma = \mathbb{E}(X - \mu)(X - \mu)^T$.

Given an i.i.d. sample $X_1, \ldots, X_n$, we want to estimate $\mu$ that has sub-Gaussian performance.
Let $X$ be a random vector taking values in $\mathbb{R}^d$ with mean $\mu = \mathbb{E}X$ and covariance matrix $\Sigma = \mathbb{E}(X - \mu)(X - \mu)^T$.

Given an i.i.d. sample $X_1, \ldots, X_n$, we want to estimate $\mu$ that has sub-Gaussian performance.

What is sub-Gaussian?

If $X$ has a multivariate Gaussian distribution, the sample mean $\overline{\mu}_n = (1/n) \sum_{i=1}^n X_1$ satisfies, with probability at least $1 - \delta$,

$$
\|\overline{\mu}_n - \mu\| \leq \sqrt{\frac{\text{Tr}(\Sigma)}{n}} + \sqrt{\frac{2\lambda_{\text{max}} \log(1/\delta)}{n}},
$$

Can one construct mean estimators with similar performance for a large class of distributions?
Coordinate-wise median of means yields the bound:

\[ \| \hat{\mu}_{MM} - \mu \| \leq K \sqrt{ \frac{\text{Tr}(\Sigma) \log(d/\delta)}{n} } . \]

We can do better.
multivariate median of means


Minsker proposes an analogous estimate that uses the multivariate median

\[
\text{Med}(x_1, \ldots, x_N) = \arg\min_{y \in \mathbb{R}^d} \sum_{i=1}^{N} \|y - x_i\|.
\]

For this estimate, with probability at least \(1 - \delta\),

\[
\|\hat{\mu}_{\text{MM}} - \mu\| \leq K \sqrt{\frac{\text{Tr}(\Sigma) \log(1/\delta)}{n}}.
\]

No further assumption or knowledge of the distribution is required. Computationally feasible. Almost sub-Gaussian but not quite. Dimension free.
We propose a new estimator with a purely sub-Gaussian performance, without further conditions.

The mean $\mu$ is the minimizer of $f(x) = \mathbb{E}\|X - \mu\|^2$.

For any pair $a, b \in \mathbb{R}^d$, we try to guess whether $f(a) < f(b)$ and set up a “tournament”.

Partition the data points into $k$ blocks of size $m = n/k$.

We say that $a$ defeats $b$ if

$$\frac{1}{m} \sum_{i \in B_j} \|X_i - a\|^2 < \frac{1}{m} \sum_{i \in B_j} \|X_i - b\|^2$$

on more than $k/2$ blocks $B_j$.
median-of-means tournament

Within each block compute

$$Y_j = \frac{1}{m} \sum_{i \in B_j} X_i.$$  

Then \(a\) defeats \(b\) if

$$\|Y_j - a\| < \|Y_j - b\|$$

on more than \(k/2\) blocks \(B_j\).

Lemma. Let \(k = \lceil 200 \log(2/\delta) \rceil\). With probability at least \(1 - \delta\), \(\mu\) defeats all \(b \in \mathbb{R}^d\) such that \(\|b - \mu\| \geq r\), where

$$r = \max \left(800 \left(\sqrt{\frac{\text{Tr}(\Sigma)}{n}}, 240 \sqrt{\frac{\lambda_{\max} \log(2/\delta)}{n}}\right)\right).$$
sub-gaussian estimate

For each \( \mathbf{a} \in \mathbb{R}^d \), define the set

\[
S_a = \left\{ \mathbf{x} \in \mathbb{R}^d : \text{such that } \mathbf{x} \text{ defeats } \mathbf{a} \right\}
\]

Now define the mean estimator as

\[
\hat{\mu}_N \in \arg\min_{\mathbf{a} \in \mathbb{R}^d} \text{radius}(S_a).
\]

By the lemma, w.p. \( \geq 1 - \delta \),

\[
\text{radius}(S_{\hat{\mu}_N}) \leq \text{radius}(S_{\mu}) \leq r
\]

and therefore

\[
\|\hat{\mu}_n - \mu\| \leq r.
\]
sub-gaussian performance

Theorem. Let $k = \lceil 200 \log(2/\delta) \rceil$. Then, with probability at least $1 - \delta$,

$$\|\hat{\mu}_n - \mu\| \leq r$$

where

$$r = \max \left( 800 \left( \sqrt{\frac{\text{Tr}(\Sigma)}{n}}, 240 \sqrt{\frac{\lambda_{\text{max}} \log(2/\delta)}{n}} \right) \right).$$

- No other condition other than existence of $\Sigma$.
- “Infinite-dimensional” inequality: the same holds in Hilbert spaces.
- The constants are explicit but sub-optimal.
proof of lemma: sketch

Let $\overline{X} = X - \mu$ and $v = b - \mu$. Then $\mu$ defeats $b$ if

$$- \frac{1}{m} \sum_{i \in B_j} \langle \overline{X}_i, v \rangle + \|v\|^2 > 0$$

on the majority of blocks $B_j$. We need to prove that this holds for all $v$ with $\|v\| = r$.

**Step 1:** For a fixed $v$, by Chebyshev, with probability at least $9/10$,

$$\left| \frac{1}{m} \sum_{i \in B_j} \langle \overline{X}_i, v \rangle \right| \leq \sqrt{10} \|v\| \sqrt{\frac{\lambda_{\text{max}}}{m}} \leq r^2 / 2$$

So by a binomial tail estimate, with probability at least $1 - \exp(-k/50)$, this holds on at least $8/10$ of the blocks $B_j$. 
proof sketch

**Step 2:** Now we take a minimal $\epsilon$ cover the set $r \cdot S^{d-1}$ with respect to the norm $\langle v, \Sigma v \rangle^{1/2}$.

This set has $< e^{k/100}$ points if

$$\epsilon = 5r \left( \frac{1}{k} \text{Tr}(\Sigma) \right)^{1/2},$$

so we can use the union bound over this $\epsilon$-net.

**Step 3:** To extend to all points in $r \cdot S^{d-1}$, we need that, with probability at least $1 - \exp(-k/200)$,

$$\sup_{x \in r \cdot S^{d-1}} \frac{1}{k} \sum_{j=1}^{k} \mathbb{1}\{|\frac{1}{m} \sum_{i \in B_j} \langle X_i, x - v_x \rangle| \geq r^2/2\} \leq \frac{1}{10}.$$

This may be proved by standard techniques of empirical processes.
Computing the proposed estimator is an interesting open problem. Coordinate descent does not quite do the job—it only guarantees $\|\hat{\mu}_n - \mu\|_{\infty} \leq r$. 
regression function estimation

Consider the standard statistical supervised learning problem under the squared loss.

Let \((X, Y)\) take values in \(\mathcal{X} \times \mathbb{R}\).

The goal is to predict \(Y\), upon observing \(X\), by \(f(X)\) for some \(f : \mathcal{X} \rightarrow \mathbb{R}\).

We measure the quality of \(f\) by the risk

\[
\mathbb{E}(f(X) - Y)^2.
\]

We have access to a sample \(\mathcal{D}_n = ((X_1, Y_1), \ldots, (X_n, Y_n))\).

We choose \(\hat{f}_n\) from a fixed class of functions \(\mathcal{F}\). The best function is

\[
f^* = \arg\min_{f \in \mathcal{F}} \mathbb{E}(f(X) - Y)^2.
\]
regression function estimation

We measure performance by either the mean squared error

$$\| \hat{f}_n - f^* \|_{L_2}^2 = \mathbb{E}((\hat{f}_n(X) - f^*(X))^2 | D_n)$$

or by the excess risk

$$R(\hat{f}_n) = \mathbb{E}((\hat{f}_n(X) - Y)^2 | D_n) - \mathbb{E}(f^*(X) - Y)^2.$$
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A procedure achieves accuracy $r$ with confidence $1 - \delta$ if

$$\mathbb{P}\left(\| \hat{f}_n - f^* \|_{L^2} \leq r\right) \geq 1 - \delta.$$

High accuracy and high confidence are conflicting requirements.

The accuracy edge is the smallest achievable accuracy with confidence $1 - \delta = 3/4$.

A quest with a long history has been to understand the tradeoff.
empirical risk minimization

The standard learning procedure is empirical risk minimization (ERM):

$$\hat{f}_n = \arg\min_{f \in \mathcal{F}} \sum_{i=1}^{n} (f(X_i) - Y_i)^2.$$  

ERM achieves near-optimal accuracy/confidence tradeoff for well-behaved distributions.

The performance of ERM is now well understood.

It works well if both $Y$ and $f(X)$ have sub-Gaussian tails (for all $f \in \mathcal{F}$).
The performance of ERM depends on the intricate interplay between the geometry of \( \mathcal{F} \) and the distribution of \((X, Y)\). We assume that \( \mathcal{F} \) is convex.

Let \( \mathcal{F}_{h,r} = \{f - h : f \in \mathcal{F}, \|f - h\|_{L^2} \leq r\} \) and let \( \mathcal{M}(\mathcal{F}_{h,r}, \epsilon) \) be the \( \epsilon \)-packing numbers.

For \( \kappa, \eta > 0 \), set

\[
\lambda_{Q}(\kappa, \eta) = \sup_{h \in \mathcal{F}} \inf r : \log \mathcal{M}(\mathcal{F}_{h,r}, \eta r) \leq \kappa^2 n \, .
\]

Similarly, let

\[
\lambda_{M}(\kappa, \eta) = \sup_{h \in \mathcal{F}} \inf r : \log \mathcal{M}(\mathcal{F}_{h,r}, \eta r) \leq \kappa^2 nr^2 \, .
\]
four complexity parameters

\[ r_E(\kappa) = \sup_{h \in \mathcal{F}} \inf \left\{ r : \mathbb{E} \sup_{u \in \mathcal{F}_{h,r}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i u(X_i) \right| \leq \kappa \sqrt{nr} \right\}, \]

Finally, let

\[ \tilde{r}_M(\kappa, h) = \inf \left\{ r : \mathbb{E} \sup_{u \in \mathcal{F}_{h,r}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i u(X_i) \cdot (h(X_i) - Y_i) \right| \leq \kappa \sqrt{nr^2} \right\}. \]

and

\[ \bar{r}_M(\kappa, \sigma) = \sup_{h \in \mathcal{F}_Y^{(\sigma)}} \tilde{r}_M(\kappa, h) \]

where \( \mathcal{F}_Y^{(\sigma)} = \{ f \in \mathcal{F} : \| f(X) - Y \|_{L_2} \leq \sigma \}. \)
Suppose $\| Y - f^*(X) \|_{L_2} \leq \sigma$ for a known constant $\sigma > 0$. Introduce the “complexity”

$$r^* = \max\{\lambda_Q(c_1, c_2), \lambda_M(c_1/\sigma, c_2), r_E(c_1), \tilde{r}_M(c_1, \sigma)\}.$$

Mendelson (2016) proved that $r^*$ is an upper bound for the accuracy edge (under a “small-ball” assumption).
linear regression–an example

Let $\mathcal{F} = \{\langle t, \cdot \rangle : t \in \mathbb{R}^d \}$ be the class of linear functionals.

Let $X$ be an isotropic random vector in $\mathbb{R}^d$ such that
$$\| \langle X, t \rangle \|_{L_4} \leq L \| \langle X, t \rangle \|_{L_2}.$$ 

Suppose $Y = \langle t_0, X \rangle + W$ for some $t_0 \in \mathbb{R}^d$ and symmetric independent noise $W$ with variance $\sigma^2$. 
linear regression

Given $n$ independent samples $(X_i, Y_i)$, least-squares regression (ERM) finds $\hat{t}_n$ such that

$$\|\hat{t}_n - t\| \leq c \frac{\sigma}{\delta} \sqrt{\frac{d}{n}}$$

with probability $1 - \delta - e^{-cd}$.

Note the weak accuracy/confidence tradeoff.

Lecué and Mendelson (2016) show that this is essentially optimal.

However, if everything is sub-Gaussian, one has

$$\|\hat{t}_n - t\| \leq c\sigma \sqrt{\frac{d}{n}}$$

with probability $1 - e^{-cd}$.

We introduce a procedure that achieves the same performance as sub-Gaussian ERM but under the general fourth-moment condition.
A natural idea is to replace ERM by minimization of the median-of-means estimate of the risk $\mathbb{E}(f(X) - Y)^2$. Difficult to analyze—may be suboptimal.
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Difficult to analyze—may be suboptimal.

Instead, we run a median-of-means tournament.

The idea is that, based on a median-of-means estimate of the difference $\mathbb{E}(f(X) - Y)^2 - \mathbb{E}(h(X) - Y)^2$,

we can have a good guess if $f$ or $h$ has a smaller risk.
median-of-means tournament

To make the idea work, we design a (two- or) three-step procedure. Each step uses an independent sample so before starting we split the data into (two or) three equal parts.

The procedure has a parameter $r > 0$, the desired accuracy level.

The main steps of the procedure are:

- Distance referee
- Elimination phase
- Champions league
For each pair $f, h \in \mathcal{F}$, one may use define a median-of-means estimate $\Phi_n(f, h)$ using $\left( |f(X_i) - h(X_i)| \right)_{i=1}^n$ such that, with “high probability”, for all $\Phi_n(f, h)$,

$$\text{if } \Phi_n(f, h) \geq \beta r \text{ then } \|f - h\|_{L_2} \geq r$$

and

$$\text{if } \Phi_n(f, h) < \beta r \text{ then } \|f - h\|_{L_2} < \alpha r$$

for some constants $\alpha, \beta$.

Matches are only allowed between $f, h \in \mathcal{F}$ if $\Phi_n(f, h) \geq \beta r$. 

**step 1: the distance referee**
step 2: elimination phase

For any pair $f, h \in \mathcal{F}$, if the distance referee allows a match, calculate the median-of-means estimate based on the samples

$$(f(X_i) - Y_i)^2 - (h(X_i) - Y_i)^2.$$ 

if the estimate is negative, $f$ wins the match otherwise $h$ wins.

$f \in \mathcal{F}$ is a champion if it wins all its matches. Let $\mathcal{H}$ be the set of all champions.

If one only cares about the mean squared error $\|\hat{f}_n - f^*\|_{L^2}$, then one may select any champion $\hat{f}_n \in \mathcal{H}$.

One may show that, with “high probability”, $\mathcal{H}$ contains $f^*$ and possibly other functions within distance $O(r)$ of $f^*$.

If the excess risk also matters, all champions in $\mathcal{H}$ advance to the Champions League for the playoffs.
To select a champion with a small excess risk, we use the simple fact that, for any $f \in \mathcal{F}$,

$$\mathbb{E}(f(X) - Y)^2 - \mathbb{E}(f^*(X) - Y)^2 \\ \leq -2\mathbb{E}(f^*(X) - f(X))(f(X) - Y).$$

The Champions League winner is selected based on median-of-means estimates of $\mathbb{E}(h(X) - f(X))(f(X) - Y)$ for all pairs $f, h \in \mathcal{F}$. 
Suppose that $\mathcal{F}$ is a convex class of functions and

- for every $f, h \in \mathcal{F}$, $\|f - h\|_{L^4} \leq L \|f - h\|_{L^2}$;
- for every $f \in \mathcal{F}$, $\|f - Y\|_{L^4} \leq L \|f - Y\|_{L^2}$;

Then the median-of-means tournament achieves an essentially optimal accuracy/confidence tradeoff.

For any $r > r^*$, with probability at least

$$1 - \exp(-c_0 n \min\{1, \sigma^{-2} r^2\})$$

$$\|\hat{f} - f^*\|_{L^2} \leq cr$$

and

$$\mathbb{E}((\hat{f}(X) - Y)^2|\mathcal{D}_n) \leq \mathbb{E}(f^*(X) - Y)^2 + (cr)^2.$$
Recall the example $\mathcal{F} = \{\langle t, \cdot \rangle : t \in \mathbb{R}^d \}$ with $X$ isotropic such that $\| \langle X, t \rangle \|_{L^4} \leq L \| \langle X, t \rangle \|_{L^2}$ and $Y = \langle t_0, X \rangle + W$. We obtain

$$\| \hat{t}_n - t \| \leq c\sigma \sqrt{\frac{d}{n}}$$

with probability $1 - e^{-cd}$ and also

$$\mathbb{E}(\langle f(X) - Y \rangle^2 | \mathcal{D}_n) - \mathbb{E}(f^*(X) - Y)^2 \leq c\sigma^2 \frac{d}{n}.$$
algorithmic challenge

Find an algorithmically efficient version of the median-of-means tournament.
G. Lugosi and S. Mendelson.  
Sub-Gaussian estimators of the mean of a random vector.  

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