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A POLYCHORIC INSTRUMENTAL VARIABLE (PIV) ESTIMATOR FOR STRUCTURAL EQUATION MODELS WITH CATEGORICAL VARIABLES

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This paper presents a new polychoric instrumental variable (PIV) estimator to use in structural equation models (SEMs) with categorical observed variables. The PIV estimator is a generalization of Bollen's (Psychometrika 61:109–121, 1996) 2SLS/IV estimator for continuous variables to categorical endogenous variables. We derive the PIV estimator and its asymptotic standard errors for the regression coefficients in the latent variable and measurement models. We also provide an estimator of the variance and covariance parameters of the model, asymptotic standard errors for these, and test statistics of overall model fit. We examine this estimator via an empirical study and also via a small simulation study. Our results illustrate the greater robustness of the PIV estimator to structural misspecifications than the system-wide estimators that are commonly applied in SEMs.

Key words: latent variables, ordinal variables, dichotomous variables, instrumental variables, two-stage least squares (2SLS), factor analysis

1. Introduction

The Life Orientation Test (LOT) (Scheier & Carver, 1985) is an eight-item scale designed to measure two latent variables, optimism and pessimism. Each item is a five-category ordinal variable. Like these items, many measures in the social and behavioral sciences are ordinal or dichotomous variables rather than continuous. One of the challenges of Structural Equation Models (SEMs) is the incorporation of categorical observed variables such as these. If such variables are exogenous dummy variables, then they do not pose a problem (Jöreskog, 1973; Bollen, 1989, pp. 126–128). However, if noncontinuous observed variables are endogenous (i.e., influenced by other variables), then a SEM must take this into account. In recent decades researchers have made great strides in the incorporation of categorical endogenous variables in SEMs (e.g., Olsson, 1979; Jöreskog & Sörbom, 1984; Jöreskog, Sörbom, du Toit, & du Toit, 2001; Muthén, 1984; Muthén & Muthén, 2001; Poon & Lee, 1987; Maydeu-Olivares, 2001, 2006). These SEM approaches have at least four things in common:

(1) they assume that a continuous indicator underlies each categorical variable;
(2) they estimate category thresholds for each ordinal variable and estimate a polychoric correlation matrix among the underlying indicators;

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(3) they use an estimate of the asymptotic covariance matrix of the elements of the polychoric correlation matrix; and
(4) they employ a system-wide estimator (e.g., weighted least squares (WLS)) to estimate all parameters in all equations in the last stage.

In SEMs with *continuous* endogenous variables there has been some work on limited information equation-by-equation estimators that are more robust to structural misspecification than are the system-wide estimators such as full information maximum likelihood (FIML). See Madansky (1964), Hägglund (1982), Jöreskog (1983), and Bollen (1989, 1996a, 1996b, 2001). Though these estimators are not identical, they have in common that they transform the latent variables into observed variables and use Instrumental Variable (IV) and two-stage least squares (2SLS) estimators. Some of these estimators (e.g., Bollen, 1996a) can be implemented with widely available statistical software. An important advantage of the IV and 2SLS approaches is that they are less likely to spread the bias that occurs with misspecified structures throughout the system of equations than are the dominant full information, system-wide estimators like FIML (Bollen, Kirby, Curran, Paxton, & Chen, 2007). Given the approximate nature and the likely structural misspecification in virtually all SEMs, this greater robustness is a desirable feature of the IV/2SLS estimators. Yet, these limited information estimators have not been applied to latent variable, structural equation models with categorical observed variables.

The main purpose of this paper is to present a generalization of Bollen's 2SLS/IV estimator for continuous variables to SEMs that contain categorical endogenous variables. We refer to this extension as the Polychoric Instrumental Variable (PIV) estimator for SEMs. We will derive a consistent IV estimator and its asymptotic standard errors for the regression coefficients in the latent variable and measurement model. We also will provide an estimator of the variances and covariances, asymptotic standard errors for these, and test statistics of overall model fit. We will examine this estimator via an empirical study, and also, via a small simulation study.

The next section presents the model and assumptions. This is followed by a section that reviews the standard estimation and testing approach which culminates in a system-wide estimator. After this we have the section that develops the PIV estimator, its standard errors, and tests of overall model fit. The sections that follow will present an empirical example and a small simulation study. The conclusion summarizes the results and discusses remaining issues.

2. Model and Assumptions

Let $\mathbf{y}^* \sim N(\mathbf{0}, \mathbf{P})$ where $\mathbf{P}$ denotes a correlation matrix with elements $\rho_{ii'}$. Suppose that each y_j^*, $j = 1, \ldots, p$, has been categorized as $y_j = k_j$ if $\tau_{j_k} < y_j^* < \tau_{j_{k+1}}$, $k_j = 0, \ldots, K_j - 1$, where $\tau_{j0} = -\infty$, $\tau_{jK} = \infty$.

We assume the following linear structure for modeling the $\mathbf{y}^*$ as a function of m latent variables $\boldsymbol{\eta}$:

$$\mathbf{y}^* = \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \tag{1}$$

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\zeta}, \tag{2}$$

where

$$\begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\varsigma} \end{pmatrix} \sim N\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Theta} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi} \end{pmatrix}\right). \tag{3}$$

This is the model used in LISREL (Jöreskog & Sörbom, 2001) and Mplus (Muthén & Muthén, 2001) when there are no exogenous observed variables in a model. See the conclusion for comments on extending these models to exogenous observed variables.

Letting $\mathbf{\Theta} = \mathbf{I} - \mathrm{diag}(\mathbf{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Psi}(\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Lambda}')$, the covariance structure of the latent response variables $\mathbf{y}^*$ implied by the model in (1)–(3) is a correlation structure, $\mathbf{P}$, namely,

$$\mathbf{P} = \mathbf{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Psi}(\mathbf{I}-\mathbf{B})^{-1\prime}\mathbf{\Lambda}' + \mathbf{\Theta}. \tag{4}$$

3. Standard Estimation and Testing Approach

Given a random sample of size N from the above model, estimation generally proceeds using a multistage procedure. First, the tetrachoric/polychoric correlations $\boldsymbol{\rho}$ are consistently estimated along with their asymptotic covariance matrix $\mathbf{\Sigma}_{\rho\rho}$. Then the model parameters $\boldsymbol{\theta}$ (i.e., the mathematically independent parameters in $\mathbf{\Lambda}, \mathbf{B}, \mathbf{\Theta},$ and $\mathbf{\Psi}$) are estimated by minimizing

$$F = \big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}(\boldsymbol{\theta})\big)'\hat{\mathbf{W}}\big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}(\boldsymbol{\theta})\big), \tag{5}$$

where $\boldsymbol{\rho}(\boldsymbol{\theta})$ denotes the restrictions imposed on the $p^* = p(p-1)/2$ correlations by the q parameter vector $\boldsymbol{\theta}$, and $\hat{\mathbf{W}} \xrightarrow{p} \mathbf{W}$, a positive definite matrix. Some obvious choices of $\hat{\mathbf{W}}$ in (5) are $\hat{\mathbf{W}} = \widehat{\mathbf{\Sigma}}_{\rho\rho}^{-1}$ (WLS) (Muthén, 1978), $\hat{\mathbf{W}} = (\mathrm{diag}(\widehat{\mathbf{\Sigma}}_{\rho\rho}))^{-1}$ (diagonally weighted least squares (DWLS)) (Jöreskog & Sörbom, 1984; Gunsjö, 1994; Muthén, du Toit, & Spisic, 1997), and $\hat{\mathbf{W}} = \mathbf{I}$ (unweighted least squares (ULS)) (Muthén, 1993).

This is the standard approach to estimate these models in such popular programs as LISREL (Jöreskog & Sörbom, 2001), Mplus (Muthén & Muthén, 2001), and EQS (Bentler, 1995). The estimation of the polychoric correlations is the same in LISREL and Mplus, although their estimated asymptotic covariance matrices are only equivalent asymptotically (Muthén & Satorra, 1995). In LISREL, the asymptotic covariance matrix is estimated as in Jöreskog (1994), whereas in Mplus it is estimated as in Muthén (1984). In EQS, the polychorics and their asymptotic covariance matrix are estimated as in Lee, Poon, and Bentler (1995). Throughout this paper we assume that the polychoric correlations and their asymptotic covariance matrix have been consistently estimated and that

$$\sqrt{N}(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}) \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma}_{\rho\rho}). \tag{6}$$

All procedures discussed above satisfy these conditions.

Let $\mathbf{H} = (\mathbf{\Delta}'\mathbf{W}\mathbf{\Delta})^{-1}\mathbf{\Delta}'\mathbf{W}$, where $\mathbf{\Delta} = \partial\boldsymbol{\rho}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}'$ a full column rank matrix so that the model is (locally) identified. From the consistency and asymptotic normality of the polychoric estimates and from standard results for weighted least squares estimators (e.g., Browne, 1984; Satorra, 1989; Satorra & Bentler, 1994) the estimator $\hat{\boldsymbol{\theta}}$ obtained by minimizing 5) is consistent and

$$\sqrt{N}\big(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\big) \xrightarrow{d} N(\mathbf{0}, \mathbf{H}\mathbf{\Sigma}_{\rho\rho}\mathbf{H}'). \tag{7}$$

In particular, when $\mathbf{W} = \mathbf{\Sigma}_{\rho\rho}^{-1}$, (7) simplifies to $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} N(\mathbf{0}, (\mathbf{\Delta}'\mathbf{\Sigma}_{\rho\rho}^{-1}\mathbf{\Delta})^{-1})$, and we obtain an estimator that asymptotically has minimum variance among the class of estimators (5). Letting $r = p^* - q$, in this special case, $T = N\hat{F} \xrightarrow{d} \chi^2_r$. In the general case, T is a mixture of one degree of freedom independent chi-square variates and a goodness of fit test can be obtained by matching its moments with those of a chi-square variable (Satorra & Bentler, 1994; Muthén, 1993).

Alternatively, following Browne (1984) another goodness of fit test of the model uses a quadratic form in the residual polychoric correlations. Let

$$T_B = N\big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}(\hat{\boldsymbol{\theta}})\big)'\hat{\mathbf{C}}\big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}(\hat{\boldsymbol{\theta}})\big), \tag{8}$$

where $\mathbf{C} = \mathbf{\Delta}_c(\mathbf{\Delta}_c' \mathbf{\Sigma}_{\rho\rho} \mathbf{\Delta}_c)^{-1} \mathbf{\Delta}_c'$, and $\mathbf{\Delta}_c$, a $p^* \times (p^* - q)$ matrix, is an orthogonal complement to $\mathbf{\Delta}$, such that $\mathbf{\Delta}_c' \mathbf{\Delta} = \mathbf{0}$. Then, $T_B \xrightarrow{d} \chi_r^2$. In practice, it is well known that the residual-based statistic converges very slowly to its asymptotic distribution. An asymptotically equivalent test statistic can be obtained following Yuan and Bentler (1997) as

$$T_{YB} = \frac{T_B}{1 + NT_B/(N-1)^2}. \tag{9}$$

The results of Yuan and Bentler suggest that T_{YB} may have a better small sample performance than T_B.

4. Polychoric Instrumental Variable (PIV) Estimator

In this section we develop the PIV estimator that builds on the approach taken in Bollen (1996a). However, one difference is that the estimator is based on the polychoric correlation matrix rather than the raw continuous data. Another difference is that we provide a method to estimate the variance and covariance model parameters and their standard errors. A final difference is that we also provide test statistics to assess the overall fit of the model.

To set the scale of the latent variables, we let

$$\mathbf{y}_1^* = \boldsymbol{\eta} + \boldsymbol{\varepsilon}_1, \tag{10}$$

that is, we scale the latent variables using the $m \times 1$ vector $\mathbf{y}_1^*$. Also, we partition $\boldsymbol{\varepsilon}$ and the matrices $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ according to the partitioning of $\mathbf{y}^* = \binom{\mathbf{y}_1^*}{\mathbf{y}_2^*}$. Thus, we write

$$\mathbf{y}_2^* = \mathbf{\Lambda}_2 \boldsymbol{\eta} + \boldsymbol{\varepsilon}_2. \tag{11}$$

Now, from (10) we can write

$$\boldsymbol{\eta} = \mathbf{y}_1^* - \boldsymbol{\varepsilon}_1. \tag{12}$$

Inserting (2) into (10) and using (12) we have

$$\mathbf{y}_1^* = \boldsymbol{\eta} + \boldsymbol{\varepsilon}_1 = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\zeta} + \boldsymbol{\varepsilon}_1 = \mathbf{B}(\mathbf{y}_1^* - \boldsymbol{\varepsilon}_1) + \boldsymbol{\zeta} + \boldsymbol{\varepsilon}_1 = \mathbf{B}\mathbf{y}_1^* + (\mathbf{I} - \mathbf{B})\boldsymbol{\varepsilon}_1 + \boldsymbol{\zeta}. \tag{13}$$

Also, we can insert (12) into (11) to obtain

$$\mathbf{y}_2^* = \mathbf{\Lambda}_2 \boldsymbol{\eta} + \boldsymbol{\varepsilon}_2 = \mathbf{\Lambda}_2(\mathbf{y}_1^* - \boldsymbol{\varepsilon}_1) + \boldsymbol{\varepsilon}_2 = \mathbf{\Lambda}_2 \mathbf{y}_1^* - \mathbf{\Lambda}_2 \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2. \tag{14}$$

Thus, we can write the latent variable and measurement models (1) and (2) as

$$\begin{pmatrix} \mathbf{y}_1^* \\ \mathbf{y}_2^* \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ \mathbf{\Lambda}_2 \end{pmatrix} \mathbf{y}_1^* + \begin{pmatrix} (\mathbf{I} - \mathbf{B}) & \mathbf{0} & \mathbf{I} \\ -\mathbf{\Lambda}_2 & \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \\ \boldsymbol{\zeta} \end{pmatrix}, \tag{15}$$

$$\mathbf{y}^* = \boldsymbol{\theta}_1 \mathbf{y}_1^* + \mathbf{u}, \tag{16}$$

where $\boldsymbol{\theta}_1$ contains all coefficients and $\mathbf{u}$ is a composite disturbance. The resulting equation corresponds to that in Bollen (1996a) with the exception that in the previous paper the $\mathbf{y}^*$ variables are observed. However, here we can consistently estimate the polychoric correlation matrix and its asymptotic covariance matrix. We can use these to estimate the model parameters. Estimation of the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2')'$ proceeds in two stages. In the first stage, $\boldsymbol{\theta}_1$ is estimated. These are the mathematically independent parameters in $\boldsymbol{\Lambda}_2$ and $\mathbf{B}$; that is, the coefficient parameters. In a second stage, $\boldsymbol{\theta}_2$ is estimated given the first-stage estimates $\hat{\boldsymbol{\theta}}_1$. $\boldsymbol{\theta}_2$ consists of the mathematically independent parameters in $\boldsymbol{\Theta}$ and $\boldsymbol{\Psi}$; that is, variance and covariance model parameters.

4.1. *Polychoric Instrumental Variable (PIV) Estimator of Regression Model Parameters*

Consider a single equation j from (15):

$$y_j^* = \mathbf{z}_j' \boldsymbol{\theta}_1^{(j)} + u_j, \tag{17}$$

where $\boldsymbol{\theta}_1^{(j)}$ is the $q_1^{(j)} \times 1$ column vector formed from the nonzero coefficients in the jth row of $\boldsymbol{\theta}_1$. The $\mathbf{z}_j'$ is a $1 \times q_1^{(j)}$ vector of latent response variables from $\mathbf{y}_1^*$ that have nonzero coefficients in the jth equation and u_j is the scalar composite disturbance for the jth equation of (16).

Let $\mathrm{COV}(\mathbf{z}_j, \mathbf{z}_j') = \mathbf{P}_{zz}$ and $\mathrm{COV}(\mathbf{z}_j, y_j^*) = \mathbf{P}_{zy^*}$. Estimation of $\boldsymbol{\theta}_1^{(j)}$ with the ordinary least squares (OLS) estimator leads to

$$\hat{\boldsymbol{\theta}}_1^{(j)} = \hat{\mathbf{P}}_{zz}^{-1} \hat{\mathbf{P}}_{zy^*} \tag{18}$$

(see Bollen, 1989, pp. 161–162). However, this is an inconsistent estimator as the composite disturbance, u_j, correlates with variables in $\mathbf{z}_j$. We make use of IVs to "purge" the explanatory variables of that part which is correlated with the disturbances. The intuitive idea behind IVs is to construct predicted versions of the original explanatory variables that unlike the original explanatory variables are uncorrelated with the disturbances. To do this, the predicted versions of the explanatory variables are linear combinations of the IVs for that equation, where the IVs are uncorrelated with the disturbances.

In an identified model there will be a subset of the $\mathbf{y}^*$ variables that qualify as IVs.[1] Define $\mathbf{v}_j$ to be the IVs that correspond to $\mathbf{z}_j$. Instrumental Variables are $\mathbf{y}^*$ variables that meet three conditions:

(1) $\mathbf{v}_j$ must correlate with $\mathbf{z}_j$ ($\mathrm{COV}(\mathbf{z}_j, \mathbf{v}_j') \neq \mathbf{0}$);
(2) $\mathbf{v}_j$ must not correlate with u_j ($\mathrm{COV}(u_j, \mathbf{v}_j') = \mathbf{0}$); and
(3) there must be at least as many variables in $\mathbf{v}_j$ as there are in $\mathbf{z}_j$.

The potential pool of IVs is a subset of all $\mathbf{y}^*$ variables in a model. The structure of the model implies that some of these $\mathbf{y}^*$ variables satisfy conditions (1) and (2) and others do not. Those that do are the *model implied instrumental variables* (Bollen, 1996a, 2001).

Condition (1) is easy to check by using the sample correlations of the IVs or by regressing the variables in $\mathbf{z}_j$ on those in $\mathbf{v}_j$ and checking for nonzero R^2s. In addition, these R^2s should not be too low (not less than, say, 0.10). Condition (3) is just a counting rule that is tied to the identification of the parameters in an equation. If there are fewer IVs than variables in $\mathbf{z}_j$ that correlate with u_j, then the equation is underidentified in the absence of other restrictions on the coefficients. If there are exactly the same number, the equation meets a necessary condition

[1] We have not come across any situations where an identified model does not have IVs. In the context of simultaneous equation models without latent variables where all equation disturbances are allowed to correlate, the rank condition of identification used for such models ensures sufficient number of IVs (Bowden & Turkington, 1984, p. 129). However, it might be possible to construct a more general example where there are constraints on variance or covariance parameters that enable model identification, but that do not permit a sufficient number of IVs to estimate coefficients. If so, we expect this to rarely occur in practice.

for exact identification of its unconstrained coefficients. More IVs than variables in $\mathbf{z}_j$ leads to overidentification.

Condition (2), that the IVs are uncorrelated with u_j, is the most problematic condition. The disturbance term is a composite disturbance that is some function of the measurement errors or the latent variable disturbances. We must choose as IVs those $\mathbf{y}^*$ variables that do not correlate with the elements of the composite disturbance term for the jth equation.[2] A variable that is neither directly nor indirectly influenced by any elements in the composite disturbance of the equation (i.e., u_j) is a potential IV. Researchers can determine this for each equation via the equations of the model or the path diagram. It is possible to automate the selection of model-implied IVs and Bollen and Bauer (2004) provide an algorithm and an SAS macro to do so. We will illustrate the selection of IVs in the empirical example.

The IVs will often differ from one equation to the next rather than being the same set for all equations. Hence the use of j to distinguish the IVs for the jth equation from those of the others. Define $\mathrm{COV}(\mathbf{v}_j, \mathbf{z}_j') = \mathbf{P}_{vz}$, $\mathrm{COV}(\mathbf{v}_j, \mathbf{v}_j') = \mathbf{P}_{vv}$, and $\mathrm{COV}(y_j^*, \mathbf{v}_j) = \mathbf{P}_{y^*v}$. We can represent the 2SLS as an IV estimator (see, e.g., Bowden & Turkington, 1984) that in our notation is

$$\boldsymbol{\theta}_1^{(j)} = \left(\mathbf{P}_{vz}'\mathbf{P}_{vv}^{-1}\mathbf{P}_{vz}\right)^{-1}\mathbf{P}_{vz}'\mathbf{P}_{vv}^{-1}\mathbf{P}_{vy^*}. \tag{19}$$

With $\hat{\mathbf{P}}_{zv}$, $\hat{\mathbf{P}}_{vv}^{-1}$, and $\hat{\mathbf{P}}_{vy^*}$ consistent estimators of $\mathbf{P}_{zv}$, $\mathbf{P}_{vv}^{-1}$, and $\mathbf{P}_{vy^*}$, respectively, the PIV estimator, $\hat{\boldsymbol{\theta}}_1^{(j)}$, is

$$\hat{\boldsymbol{\theta}}_1^{(j)} = \left(\hat{\mathbf{P}}_{vz}'\hat{\mathbf{P}}_{vv}^{-1}\hat{\mathbf{P}}_{zv}\right)^{-1}\hat{\mathbf{P}}_{vz}'\hat{\mathbf{P}}_{vv}^{-1}\hat{\mathbf{P}}_{vy^*}. \tag{20}$$

To establish the consistency of the PIV estimator, $\hat{\boldsymbol{\theta}}_1^{(j)}$, note that

$$\begin{aligned}
\mathbf{P}_{vy^*} &= \mathrm{COV}\left(\mathbf{v}_j, \mathbf{z}_j'\boldsymbol{\theta}_1^{(j)} + u_j\right) \\
&= \mathrm{COV}\left(\mathbf{v}_j, \mathbf{z}_j'\boldsymbol{\theta}_1^{(j)}\right) + \mathrm{COV}(\mathbf{v}_j, u_j) \\
&= \mathbf{P}_{vz}\boldsymbol{\theta}_1^{(j)},
\end{aligned} \tag{21}$$

where the last line follows from the IVs being uncorrelated with u_j. Using this result and equation (19) we have:

$$\begin{aligned}
\left(\mathbf{P}_{vz}'\mathbf{P}_{vv}^{-1}\mathbf{P}_{vz}\right)^{-1}\mathbf{P}_{vz}'\mathbf{P}_{vv}^{-1}\mathbf{P}_{vy^*} &= \left(\mathbf{P}_{vz}'\mathbf{P}_{vv}^{-1}\mathbf{P}_{vz}\right)^{-1}\mathbf{P}_{vz}'\mathbf{P}_{vv}^{-1}\mathbf{P}_{vz}\boldsymbol{\theta}_1^{(j)} \\
&= \boldsymbol{\theta}_1^{(j)}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
plim\left(\hat{\boldsymbol{\theta}}_1^{(j)}\right) &= plim\left[\left(\hat{\mathbf{P}}_{vz}'\hat{\mathbf{P}}_{vv}^{-1}\hat{\mathbf{P}}_{zv}\right)^{-1}\hat{\mathbf{P}}_{vz}'\hat{\mathbf{P}}_{vv}^{-1}\hat{\mathbf{P}}_{vy^*}\right] \\
&= \left(\mathbf{P}_{vz}'\mathbf{P}_{vv}^{-1}\mathbf{P}_{vz}\right)^{-1}\mathbf{P}_{vz}'\mathbf{P}_{vv}^{-1}\mathbf{P}_{vy^*} \\
&= \boldsymbol{\theta}_1^{(j)}.
\end{aligned} \tag{23}$$

Thus, $\hat{\boldsymbol{\theta}}_1^{(j)}$ is a consistent estimator of $\boldsymbol{\theta}_1^{(j)}$.

[2] A simple illustration of variables that can always serve as IVs are exogenous observed variables. These are observed variables that have no measurement error and that are predetermined in the model (i.e., no other variables influence them). For instance, gender or race are often exogenous variables in social and behavioral science research. Exogenous variables are uncorrelated with all disturbances in a model so they will qualify as IVs in all equations. We discuss exogenous observed variables in the conclusion.

4.1.1. Asymptotic Standard Errors of PIV Regression Parameter Estimates
Collecting all $p - m$ equations (23) we can write

$$\hat{\boldsymbol{\theta}}_1 = \boldsymbol{\gamma}(\widehat{\boldsymbol{\rho}}). \tag{24}$$

The asymptotic distribution of $\hat{\boldsymbol{\theta}}_1 = \boldsymbol{\gamma}(\widehat{\boldsymbol{\rho}})$ then is obtained via a first-order expansion, $\hat{\boldsymbol{\theta}}_1 \stackrel{a}{=} \boldsymbol{\theta}_1 + (\partial\boldsymbol{\gamma}(\boldsymbol{\rho})/\partial\boldsymbol{\rho}')(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho})$. Now, letting $\mathbf{K} = \partial\boldsymbol{\gamma}(\boldsymbol{\rho})/\partial\boldsymbol{\rho}'$, we can write

$$\sqrt{N}\big(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1\big) \stackrel{a}{=} \mathbf{K}\sqrt{N}(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}). \tag{25}$$

From (6) and (25) we obtain $\sqrt{N}(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1) \stackrel{d}{\to} N(\mathbf{0}, \mathbf{K}\boldsymbol{\Sigma}_{\rho\rho}\mathbf{K}')$. Thus, standard errors for $\hat{\boldsymbol{\theta}}_1$ are obtained using

$$\widehat{\mathrm{Acov}}\big(\hat{\boldsymbol{\theta}}_1\big) = \frac{1}{N}\hat{\mathbf{K}}\widehat{\boldsymbol{\Sigma}}_{\rho\rho}\hat{\mathbf{K}}', \tag{26}$$

and we consistently estimate $\mathbf{K}$ by evaluating it at the sample polychoric correlations.

4.2. Estimation of PIV Variance and Covariance Model Parameters

To estimate $\boldsymbol{\theta}_2$ given $\hat{\boldsymbol{\theta}}_1$ we shall simply minimize

$$F_2 = \big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}\big(\boldsymbol{\theta}_2, \widehat{\boldsymbol{\theta}}_1\big)\big)'\big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}\big(\boldsymbol{\theta}_2, \widehat{\boldsymbol{\theta}}_1\big)\big). \tag{27}$$

Let $\boldsymbol{\Delta}_2 = \partial\boldsymbol{\rho}(\boldsymbol{\theta}_2, \hat{\boldsymbol{\theta}}_1)/\partial\boldsymbol{\theta}_2'$ and $\mathbf{H}_2 = (\boldsymbol{\Delta}_2'\boldsymbol{\Delta}_2)^{-1}\boldsymbol{\Delta}_2'$. From standard results

$$\sqrt{N}\big(\hat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}_2\big) \stackrel{a}{=} \mathbf{H}_2\sqrt{N}\big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}\big(\boldsymbol{\theta}_2, \widehat{\boldsymbol{\theta}}_1\big)\big). \tag{28}$$

Now, the asymptotic distribution of $\sqrt{N}(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}(\boldsymbol{\theta}_2, \widehat{\boldsymbol{\theta}}_1))$ is obtainable as follows: Let $\boldsymbol{\Delta}_1 = \partial\boldsymbol{\rho}(\boldsymbol{\theta}_1)/\partial\boldsymbol{\theta}_1'$. We have $\boldsymbol{\rho}(\boldsymbol{\theta}_2, \hat{\boldsymbol{\theta}}_1) \stackrel{a}{=} \boldsymbol{\rho} + (\partial\boldsymbol{\rho}(\boldsymbol{\theta}_1)/\partial\boldsymbol{\theta}_1')\ (\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1)$. Thus, from (25), $\boldsymbol{\rho}(\boldsymbol{\theta}_2, \hat{\boldsymbol{\theta}}_1) - \boldsymbol{\rho} \stackrel{a}{=} \boldsymbol{\Delta}_1\mathbf{K}\sqrt{N}(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho})$. Also, $\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}(\boldsymbol{\theta}_2, \hat{\boldsymbol{\theta}}_1) = [\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}] - [\boldsymbol{\rho}(\boldsymbol{\theta}_2, \hat{\boldsymbol{\theta}}_1) - \boldsymbol{\rho}]$. Therefore,

$$\sqrt{N}\big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}\big(\boldsymbol{\theta}_2, \widehat{\boldsymbol{\theta}}_1\big)\big) \stackrel{a}{=} (\mathbf{I} - \boldsymbol{\Delta}_1\mathbf{K})\sqrt{N}(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}), \tag{29}$$

and from (6) we obtain the desired result:

$$\sqrt{N}\big(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}\big(\boldsymbol{\theta}_2, \widehat{\boldsymbol{\theta}}_1\big)\big) \stackrel{d}{\to} N\big(\mathbf{0}, (\mathbf{I} - \boldsymbol{\Delta}_1\mathbf{K})\widehat{\boldsymbol{\Sigma}}_{\rho\rho}(\mathbf{I} - \boldsymbol{\Delta}_1\mathbf{K})'\big). \tag{30}$$

4.2.1. Asymptotic Standard Errors of PIV Variance and Covariance Parameter Estimates
From (6), (28), and (29),

$$\sqrt{N}\big(\hat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}_2\big) \stackrel{d}{\to} N\big(\mathbf{0}, \mathbf{H}_2(\mathbf{I} - \boldsymbol{\Delta}_1\mathbf{K})\boldsymbol{\Sigma}_{\rho\rho}(\mathbf{I} - \boldsymbol{\Delta}_1\mathbf{K})'\mathbf{H}_2'\big).$$

Thus, standard errors for $\hat{\boldsymbol{\theta}}_2$ can be obtained using

$$\widehat{\mathrm{Acov}}\big(\hat{\boldsymbol{\theta}}_2\big) = \frac{1}{N}\hat{\mathbf{H}}_2(\mathbf{I} - \hat{\boldsymbol{\Delta}}_1\hat{\mathbf{K}})\widehat{\boldsymbol{\Sigma}}_{\rho\rho}(\mathbf{I} - \hat{\boldsymbol{\Delta}}_1\hat{\mathbf{K}})'\hat{\mathbf{H}}_2', \tag{31}$$

where $\mathbf{H}_2$, $\boldsymbol{\Delta}_1$, and $\mathbf{K}$ are evaluated at $\hat{\boldsymbol{\theta}}$.

4.3. Goodness of Fit Tests

The above procedure yields consistent estimates $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_1', \hat{\boldsymbol{\theta}}_2')'$. The consistency of $\hat{\boldsymbol{\theta}}_1$ was described above, whereas the consistency of $\hat{\boldsymbol{\theta}}_2$ follows from the consistency of the residual vector $\sqrt{N}(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}(\boldsymbol{\theta}_2, \widehat{\boldsymbol{\theta}}_1))$ and standard results for minimum distance estimators.

Since the estimator is consistent, to assess the goodness of fit of the restrictions imposed by the model on the matrix of polychoric correlations we can simply use the statistics T_B and T_{YB} given in (8) and (9). In this paper we consistently estimate $\boldsymbol{\Sigma}_{\rho\rho}$ as in Jöreskog (1994).

Alternatively, we can assess the overall goodness of fit of the model by matching the asymptotic moments of a test statistic with those of a chi-square distribution (see Satorra & Bentler, 1994; Cai, Maydeu-Olivares, Coffman, & Thissen, 2006). Let $\hat{\mathbf{e}} = \hat{\boldsymbol{\rho}} - \boldsymbol{\rho}(\hat{\boldsymbol{\theta}})$, and consider the statistic

$$T = \hat{\mathbf{e}}'\hat{\mathbf{e}}. \tag{32}$$

This statistic is the ULS fit function, evaluated at the PIV parameter estimates. Now, let $\boldsymbol{\Delta} = \partial\boldsymbol{\rho}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}'$ and $\mathbf{G} = \begin{pmatrix} \mathbf{K} \\ \mathbf{I} - \boldsymbol{\Delta}_1\mathbf{K} \end{pmatrix}$. We have $\boldsymbol{\rho}(\hat{\boldsymbol{\theta}}) \stackrel{a}{=} \boldsymbol{\rho} + (\partial\boldsymbol{\rho}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}')(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$, and since $\hat{\mathbf{e}} = [\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}] - [\boldsymbol{\rho}(\hat{\boldsymbol{\theta}}) - \boldsymbol{\rho}]$, $\hat{\mathbf{e}} \stackrel{a}{=} (\mathbf{I} - \boldsymbol{\Delta}\mathbf{G})(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho})$. Finally, let

$$\mathbf{M} = (\mathbf{I} - \boldsymbol{\Delta}\mathbf{G})\Sigma_{\rho\rho'}(\mathbf{I} - \boldsymbol{\Delta}\mathbf{G})'. \tag{33}$$

The first two asymptotic moments of T are $\mathrm{Tr}(\mathbf{M})$ and $2\,\mathrm{Tr}(\mathbf{M}^2)$.

The mean corrected scaled statistic is therefore $T_s = T/(\mathrm{Tr}(\mathbf{M})/r)$ which is referred to a chi-square distribution with r degrees of freedom. The mean and variance adjusted statistic is $T_a = T/(\mathrm{Tr}(\mathbf{M}^2)/r)$, which is referred to a chi-square distribution with $d = \mathrm{Tr}(\mathbf{M})^2/(\mathrm{Tr}(\mathbf{M}^2)/r)$ degrees of freedom.[3]

5. Empirical Example: PIV vs. ULS Estimation of the Lot Data

We mentioned the Life Orientation Test (LOT) in the Introduction to motivate this paper. The LOT consists of eight items that measure optimism and pessimism where each item consists of a five-category ordinal variable. Chang, D'Zurilla, and Maydeu-Olivares (1994) fitted a confirmatory two-factor model to this questionnaire: $\mathbf{P}(\boldsymbol{\theta}) = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}$, where $\boldsymbol{\Theta} = \mathbf{I} - \mathrm{diag}(\boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}')$ is a diagonal matrix,

$$\boldsymbol{\Lambda}' = \begin{pmatrix} 1 & \lambda_{21} & \lambda_{31} & \lambda_{41} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda_{62} & \lambda_{72} & \lambda_{82} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi} = \begin{pmatrix} \psi_{11} & \psi_{21} \\ \psi_{21} & \psi_{22} \end{pmatrix}.$$

The clusters correspond to the positively and to the negatively worded items of the questionnaire, respectively. That is, the factors measure optimism and pessimism, respectively.

Chang et al. (1994) used WLS and found that this model reproduced well the polychoric matrix. Their data (389 observations) are reanalyzed here using the ULS and the PIV estimators.

This confirmatory factor analysis is overidentified and for all of the equations there are more than enough model-implied IVs to estimate the factor loadings. To illustrate the selection of

[3] Although we do not consider it further, another model test statistic that could be used is the vanishing tetrad test for polychoric correlations (Hipp & Bollen, 2003).

model-implied IVs, consider the equations for the model,

$$y_1^* = \eta_1 + \epsilon_1, \qquad y_5^* = \eta_2 + \epsilon_5, \tag{34}$$

$$y_2^* = \lambda_{21}\eta_1 + \epsilon_2, \qquad y_6^* = \lambda_{62}\eta_2 + \epsilon_6, \tag{35}$$

$$y_3^* = \lambda_{31}\eta_1 + \epsilon_3, \qquad y_7^* = \lambda_{72}\eta_2 + \epsilon_7, \tag{36}$$

$$y_4^* = \lambda_{41}\eta_1 + \epsilon_4, \qquad y_8^* = \lambda_{82}\eta_2 + \epsilon_8. \tag{37}$$

From (34) we have $\eta_1 = y_1^* - \epsilon_1$. Substituting this into the y_2^* equation leads to

$$y_2^* = \lambda_{21} y_1^* - \lambda_{21}\epsilon_1 + \epsilon_2. \tag{38}$$

To estimate λ_{21} we need IVs for y_1^*. The IVs must correlate with y_1^* and be uncorrelated with $-\lambda_{21}\epsilon_1 + \epsilon_2$. Given that $\mathbf{\Theta}$ is diagonal (all ϵs are uncorrelated), y_3^* to y_8^* are uncorrelated with ϵ_1 and ϵ_2, and hence all these variables meet one condition of being an IV. We also want indicators that correlate with y_1^*. A correlation between η_1 and η_2 implies that y_3^* to y_8^* correlate with y_1^*. Thus, all of these indicators (y_3^* to y_8^*) are model-implied IVs for y_1^*. Bollen et al.'s (2007) results suggest that in small- to moderate-sized samples the IV estimator is relatively unbiased when using $p_i + 1$ or $p_i + 2$ IVs for each equation, where p_i denotes the number of $\mathbf{z}$ variables for each equation. Those model-implied IVs that are most strongly related to the scaling indicator that they will predict are usually the best IVs to use when choosing a subset of eligible ones. In a valid model any subset of the model-implied IVs will result in a consistent estimator of the factor loading even though in a given sample the estimates will differ depending on the subset of IVs chosen due to sampling fluctuations. In the following table we list the IVs we used in each equation:

y	**z**	**IV**
2	1	3, 4
3	1	2, 4
4	1	2, 3
6	5	7, 8
7	5	6, 8
8	5	6, 7

Table 1 shows that compared to the ULS estimator, we obtained similar goodness of fit results when employing the PIV estimator. This is particularly true in the case of the T_B and T_{YB} statistics. Table 2 contains the parameter estimates and standard errors using the IV estimator and for comparison using ULS. As can be seen in this table, very similar parameter estimates and standard errors are obtained with both estimators.

TABLE 1.
Goodness of fit tests for the LOT data.

	PIV			ULS		
Stat.	Value	*df*	*p*	Value	*df*	*p*
T_B	25.30	19	0.15	25.01	19	0.16
T_{YB}	23.75	19	0.21	23.49	19	0.21
T_s	23.56	19	0.21	24.6	19	0.17
T_a	12.20	9.83	0.26	14.66	11.33	0.22

TABLE 2.
Parameter estimates and standard errors for the LOT data.

	PIV estimation		ULS estimation	
Par.	Est.	*SE*	Est.	*SE*
λ_{21}	1.23	0.11	1.25	0.13
λ_{31}	1.05	0.10	1.15	0.13
λ_{41}	0.63	0.10	0.69	0.11
λ_{62}	1.32	0.12	1.38	0.14
λ_{72}	1.32	0.11	1.32	0.13
λ_{82}	1.07	0.11	1.18	0.13
ψ_{11}	0.46	0.06	0.42	0.07
ψ_{21}	−0.24	0.04	−0.22	0.04
ψ_{22}	0.43	0.07	0.40	0.07

Notes: $N = 389$.

6. Simulation

In this section we investigate the finite sample behavior of the PIV parameter estimates, standard errors, and goodness of fit tests by means of a small simulation study with correctly and incorrectly specified models. We use a standard multivariate normal (MVN) density with $p = 8$ variables where each variable is categorized into $K = 2$ categories using the thresholds $\tau_i = 0$. The following correlation structure was used: $\mathbf{P}(\boldsymbol{\theta}) = \mathbf{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Psi}(\mathbf{I} - \mathbf{B})^{-1\prime}\mathbf{\Lambda}' + \mathbf{\Theta}$. The parameters for the correct specification were

$$\mathbf{\Lambda}' = \begin{pmatrix} 1 & \lambda_{21} & \lambda_{31} & \lambda_{41} & 0 & 0 & 0 & \lambda_{81} \\ 0 & 0 & 0 & \lambda_{42} & 1 & \lambda_{62} & \lambda_{72} & \lambda_{82} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ \beta_{21} & 0 \end{pmatrix}, \quad \mathbf{\Psi} = \begin{pmatrix} \psi_{11} & 0 \\ 0 & \psi_{22} \end{pmatrix}, \quad \text{and} \quad \mathbf{\Theta} = \mathbf{I} - \operatorname{diag}\big(\mathbf{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Psi}(\mathbf{I} - \mathbf{B})^{-1\prime}\mathbf{\Lambda}'\big).$$

The second column of Table 3 contains the true parameter values.

We fitted two models to the simulated data: a correctly specified model, and a misspecified model where $\lambda_{42} = \lambda_{81} = 0$. We pitted the performance of the PIV estimator against the ULS estimator as it is a natural system-wide counterpart of the PIV estimator, since both estimators are based on unweighted/ordinary least squares.

6.1. *Correctly Specified Model*

The estimated parameter estimates and asymptotic standard errors for the PIV estimator and the system-wide ULS estimator are reported in Table 3 for the *correctly specified* models for an N of 200, 1000, and 5000.

The results shown in Table 3 reveal that there is little to choose from between these two estimators in this correctly specified model. However, the PIV estimates have slightly less bias than the ULS for $N = 200$. Both PIV and ULS estimators are essentially unbiased in the larger sample sizes. The standard errors for ULS are slightly more accurate than for PIV.

To shed some light onto whether PIV is more accurate than ULS in very small samples, we considered the case $N = 50$. Also, we pitted the PIV estimator against the DWLS estimator. Table 6.1 summarizes the results. In this table we provide the median absolute parameter bias as well as the median absolute standard error bias for each of the conditions. Here, we define parameter and standard error bias as $(\bar{x}_{\hat{\theta}} - \theta_0)/\theta_0$ and $(\bar{x}_{SE(\hat{\theta})} - sd_{\hat{\theta}})/sd_{\hat{\theta}}$, respectively. Consistent

TABLE 3.
Simulation results for the correctly specified model: Parameter estimates and standard errors.

		PIV estimator									ULS estimator								
		$N=200$			$N=1000$			$N=5000$			$N=200$			$N=1000$			$N=5000$		
Par.	True	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*
λ_{21}	0.8	0.80	0.13	0.12	0.80	0.06	0.06	0.80	0.02	0.03	0.81	0.14	0.14	0.80	0.06	0.06	0.80	0.03	0.03
λ_{31}	0.8	0.80	0.13	0.13	0.80	0.06	0.07	0.80	0.02	0.02	0.82	0.14	0.14	0.80	0.06	0.06	0.80	0.03	0.03
λ_{41}	0.8	0.80	0.16	0.16	0.80	0.07	0.08	0.80	0.03	0.03	0.81	0.12	0.15	0.80	0.06	0.06	0.80	0.03	0.03
λ_{42}	0.4	0.40	0.20	0.19	0.40	0.08	0.05	0.40	0.04	0.04	0.40	0.12	0.12	0.40	0.05	0.05	0.40	0.02	0.02
λ_{62}	0.8	0.79	0.11	0.11	0.80	0.05	0.05	0.80	0.02	0.02	0.80	0.12	0.12	0.80	0.05	0.05	0.80	0.02	0.02
λ_{72}	0.8	0.79	0.11	0.11	0.80	0.05	0.09	0.80	0.02	0.02	0.81	0.12	0.12	0.80	0.05	0.05	0.80	0.02	0.02
λ_{81}	0.4	0.40	0.21	0.20	0.40	0.09	0.06	0.40	0.04	0.04	0.40	0.12	0.13	0.40	0.05	0.05	0.40	0.02	0.02
λ_{82}	0.8	0.79	0.14	0.13	0.80	0.06	0.06	0.80	0.03	0.03	0.81	0.12	0.12	0.80	0.05	0.05	0.80	0.02	0.02
β_{21}	0.3	0.30	0.16	0.16	0.30	0.07	0.07	0.30	0.03	0.03	0.31	0.12	0.12	0.30	0.05	0.05	0.30	0.02	0.02
ψ_{11}	0.7	0.71	0.12	0.12	0.70	0.06	0.06	0.70	0.03	0.02	0.70	0.13	0.13	0.70	0.06	0.06	0.70	0.03	0.03
ψ_{22}	0.7	0.70	0.14	0.13	0.70	0.06	0.06	0.70	0.03	0.03	0.69	0.12	0.12	0.70	0.05	0.05	0.70	0.02	0.02

Notes: 1000 replications per condition. All replications converged except for $N=200$, where 993 replications converged for PIV and 991 for ULS.

TABLE 4.
Simulation results for the correctly specified model: Median absolute bias for parameter estimates and standard errors.

Estimator	N	Converged replications	Est.	*SEs*
PIV	50	945	.036	.066
	200	993	.0060	.025
	1000	1000	.0038	.025
	5000	1000	.0008	.014
ULS	50	940	.056	.092
	200	991	.010	.021
	1000	1000	.0012	.018
	5000	1000	.0015	.011
DWLS	50	938	.045	.10
	200	993	.010	.024
	1000	1000	.0016	.013
	5000	1000	.0010	.0065

with previous comparisons between DWLS and ULS (Maydeu-Olivares, 2001), Table 4 shows that the differences between these estimators are small. The differences between these estimators and the PIV estimator are also small. It is worth mentioning, however, that the PIV estimator minimally outperforms the other estimators in number of convergent replications and parameter estimation accuracy in small samples ($N \leq 200$).

The small sample behavior of the goodness of fit test statistics is reported in Table 5. The mean and variance of the tests statistics under consideration, as well as empirical rejection rates in the critical region 1% to 20%, are provided in this table. For $N = 200$, rejection rates using the T_s and T_a tend to be closest to the expected rejection rates for PIV and ULS. Indeed, for the ULS estimator, the mean and variance adjusted statistic T_a provides a fairly accurate test of the model for $N = 200$. In the largest samples, T_B and T_{YB} tend to be the most accurate for PIV and ULS.

For comparison, we also investigated the performance of moment corrections to the weighted statistic

$$T = \hat{\mathbf{e}}'\big(\operatorname{diag}(\hat{\boldsymbol{\Sigma}}_{\rho\rho})\big)^{-1}\hat{\mathbf{e}}.$$

This statistic is the DWLS fit function, evaluated at the PIV parameter estimates. With this statistic instead of (32), M in equation (33) becomes

$$\mathbf{M} = \big(\operatorname{diag}(\hat{\boldsymbol{\Sigma}}_{\rho\rho})\big)^{-1}(\mathbf{I} - \boldsymbol{\Delta}\mathbf{G})\widehat{\boldsymbol{\Sigma}}_{\rho\rho'}(\mathbf{I} - \boldsymbol{\Delta}\mathbf{G})'.$$

The behavior of the moment corrections to this weighted statistic is very similar to the behavior of the unweighted statistic reported in Table 5.

6.2. *Incorrectly Specified Model*

Previous literature suggests that 2SLS/IV is more robust to specification error than are system-wide estimators (e.g., Cragg, 1968; Bollen et al., 2007). Bollen (2001) gives conditions for his 2SLS/IV estimator to be robust to structural misspecifications. Adapting these to our notation, they are as follows:

Suppose that for the jth equation in the correctly specified model, the model-implied IVs are in $\mathbf{v}_j$. The 2SLS/IV estimator of the coefficients in $\hat{\boldsymbol{\theta}}_1^{(j)}$ is robust for any structural misspecifications in other equations under two conditions:

TABLE 5.
Simulation results for the correctly specified model: Goodness of fit tests of the structural restrictions.

Estimator	N	Stat.	Mean	Variance	Rejection rates 1%	5%	10%	20%
PIV	200	T_B	18.5	40.3	1.5	9.1	16.4	28.9
		T_{YB}	16.8	27.5	0.3	2.5	7.6	17.7
		T_s	15.7	61.9	3.1	8.3	12.5	18.2
		T_a	6.7	14.2	0.7	2.9	6.2	13.9
	1000	T_B	17.2	33.7	1.2	4.7	10.3	21.1
		T_{YB}	16.9	31.2	1.0	4.1	8.6	20.0
		T_s	16.6	71.2	4.2	9.5	13.9	21.4
		T_a	7.6	17.7	1.5	4.5	8.3	16.4
	5000	T_B	17.0	35.5	1.2	5.0	10.9	20.5
		T_{YB}	16.93	34.97	1.1	4.9	10.4	20.0
		T_s	16.95	77.3	5.2	10.6	14.2	22.3
		T_a	7.7	16.9	2	5.6	9.8	16.9
ULS	200	T_B	18.8	41.4	1.6	9.6	18.7	30.5
		T_{YB}	17.0	28.1	0.4	2.7	7.9	20.0
		T_s	17.2	46.2	3.0	7.2	12.7	22.7
		T_a	11.7	21.6	0.9	4.6	9.2	19.5
	1000	T_B	17.2	34.1	1.2	5.0	10.2	21.1
		T_{YB}	16.9	31.6	0.9	4.3	8.8	20.0
		T_s	16.9	42.9	2.4	6.6	12.0	20.7
		T_a	12.4	23.19	0.9	4.5	8.9	18.7
	5000	T_B	17.0	35.6	1.1	5.2	10.7	20.6
		T_{YB}	17.0	35.1	1	4.9	10.2	20.1
		T_s	17.1	47.1	2.7	7.2	13.0	23.0
		T_a	12.7	26.1	1	5.5	10.4	20.7

Notes: 17 *df*. 1000 replications per condition. All replications converged except for N = 200, where 993 replications converged for PIV and 991 for ULS.

(1) *the equation being estimated is correctly specified; and*
(2) *the misspecifications in the other equations do not alter the variables in* $\mathbf{v}_j$.

These same robustness conditions will hold for the PIV estimator of the coefficients. They apply to coefficients ($\boldsymbol{\theta}_1$) and not necessarily to variance and covariance parameters ($\boldsymbol{\theta}_2$). Indeed our estimator of the variances and covariances in $\boldsymbol{\theta}_2$ utilizes a system-wide estimator that is conditional on the values in $\hat{\boldsymbol{\theta}}_1$. Since at least some parameters in $\hat{\boldsymbol{\theta}}_1$ will be biased in an incorrect model, $\hat{\boldsymbol{\theta}}_2$ need not be robust to structural misspecifications. In system-wide estimators like ULS, robustness conditions are not known but simulation evidence for system-wide estimators suggests that it is more prone to spread structural misspecifications to both coefficients and variance/covariance parameters.

We return to our simulation data to illustrate these properties. We used the same simulated data as before but estimate a misspecified model where $\lambda_{42} = \lambda_{81} = 0$. The parameter estimates and asymptotic standard errors for the PIV estimator and the system-wide ULS estimator are reported in Table 6. This model meets the conditions for robustness to misspecification set forth in Bollen (2001).

The above robustness conditions predict that the PIV estimator of coefficients will be robust for the β_{21} coefficient that links the two latent variables, and for all factor loadings except in the two structurally misspecified equations for y_4 and y_8. In fact, it is, and the PIV estimates for

TABLE 6.
Simulation results for the incorrectly specified model: Parameter estimates and standard errors.

		PIV estimator									ULS estimator								
		$N=200$			$N=1000$			$N=5000$			$N=200$			$N=1000$			$N=5000$		
Par	*True*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*	$\bar{x}est$	$\bar{x}SE$	*sd est*
λ_{21}	0.8	0.80	0.13	0.13	0.80	0.06	0.05	0.80	0.02	0.02	0.83	0.14	0.14	0.82	0.06	0.06	0.82	0.02	0.02
λ_{31}	0.8	0.79	0.13	0.13	0.80	0.06	0.05	0.80	0.02	0.03	0.83	0.14	0.14	0.82	0.06	0.06	0.82	0.02	0.02
λ_{41}	0.8	0.91	0.14	0.14	0.92	0.06	0.06	0.92	0.03	0.03	1.31	0.12	0.21	1.30	0.08	0.09	1.30	0.04	0.04
λ_{62}	0.8	0.79	0.11	0.11	0.80	0.05	0.05	0.80	0.02	0.02	0.82	0.12	0.13	0.82	0.05	0.05	0.82	0.02	0.02
λ_{72}	0.8	0.80	0.11	0.11	0.80	0.05	0.05	0.80	0.02	0.02	0.83	0.12	0.12	0.82	0.05	0.05	0.82	0.02	0.02
λ_{82}	0.8	0.90	0.11	0.11	0.91	0.05	0.05	0.91	0.02	0.02	1.22	0.14	0.16	1.21	0.06	0.07	1.21	0.03	0.03
β_{21}	0.3	0.29	0.16	0.16	0.30	0.07	0.07	0.30	0.03	0.03	0.58	0.12	0.12	0.57	0.05	0.05	0.57	0.02	0.02
ψ_{11}	0.7	0.79	0.13	0.14	0.80	0.07	0.06	0.80	0.03	0.03	0.55	0.11	0.12	0.55	0.05	0.05	0.55	0.02	0.02
ψ_{22}	0.7	0.69	0.12	0.13	0.69	0.05	0.05	0.69	0.02	0.02	0.44	0.09	0.10	0.45	0.04	0.04	0.45	0.02	0.02

Notes: 1000 replications per condition. The number of converged replications for $N=200$, 1000, and 5000 is 969, 976, and 953 for PIV; and 955, 968, and 951 for ULS.

TABLE 7.
Simulation results for the incorrectly specified model: Median absolute bias for parameter estimates and standard errors.

Estimator	N	Converged replications	Est.	*SEs*
PIV	200	969	.013	.017
	1000	976	.0057	.018
	5000	953	.0022	.020
ULS	200	955	.21	.035
	1000	968	.22	.064
	5000	951	.22	.052

these correct equations are identical in the correct and incorrect models. In contrast in the misspecified model, the system-wide ULS estimator spreads bias from the omitted cross-loadings. In this model the bias in the factor loadings for the *correctly specified* equations are not large, leading to a slightly higher value than the population parameter (estimate of 0.82 or 0.83 for 0.80 parameter). But the β_{21} estimate shifts from 0.30 in the correct model to 0.57 or 0.58 in the misspecified one, a nearly 100% bias and enough to change our assessment of the relation between these latent variables. In addition, the factor loadings for the indicator equations with omitted cross-loadings exhibit more bias in ULS than they do for the PIV estimates. For example, for $N = 1000$ the mean of the ULS estimator of λ_{41} is 1.30 compared to the mean of 0.92 for PIV where the true parameter value is 0.80. For this structurally misspecified equation, both are biased as expected, but the bias in ULS is far greater than for PIV.

Table 7 provides a summary of the bias for the incorrectly specified model by giving the median absolute bias for the parameter estimates and standard errors. Across all sample sizes and for estimates and standard errors, the bias is consistently greater for the ULS estimator than for the PIV. Concealed by this table but revealed in Table 6 is that the bias in the PIV estimator is confined to the structurally misspecified equations and the variance estimates. The remaining factor loadings are essentially unbiased and robust to the structural misspecification. With the ULS estimator the bias spreads across all parameters, even those from correctly specified equations.

7. Discussion and Conclusion

In this paper we developed the PIV estimator for SEMs when one or more endogenous observed variables are dichotomous or ordinal. Unlike the system-wide estimators such as polychoric ULS, WLS, or DWLS, the PIV estimator is an equation-by-equation estimator of the coefficient parameters. The variance and covariance parameters in the model are estimated given the PIV coefficient parameter estimates. Furthermore, we provide methods to estimate the asymptotic standard errors of all estimated parameters and developed chi-square tests of the models overidentification restrictions. The PIV estimator of the model parameters and their asymptotic standard errors performed well in our empirical example and simulation. The T_{YB} and T_B chi-square test performs reasonably well in moderate-sized samples, but were less accurate in smaller samples.

Our presentation concentrated on SEMs without exogenous observed covariates. For ease of exposition, the present account has focused on correlation structure models where all the observed variables are categorical. However, the PIV procedure described here ($\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\zeta}$) can be generalized to $\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\mathbf{x} + \boldsymbol{\zeta}$ where $\mathbf{x}$ consists of observed exogenous covariates along the lines of Bollen (2002). In this latter model, the assumption of underlying multivariate normality is replaced by an assumption of underlying continuous variables conditional on the set of exogenous variables (see, e.g., Muthén, 1984). Also, we can treat structural models with restrictions on the mean vector, covariance matrix, and (probit) regression slopes. For further details of this general model, see Muthén (1984), Browne and Arminger (1995), and Küsters (1987). For details on how mean and covariance structures can be estimated with IV estimators, see Bollen (1996a).

It is interesting to contrast the properties of the PIV estimator for categorical endogenous variables compared to those of the 2SLS/IV estimator of Bollen (1996a) from which it builds. It is worth noting that the approach to estimating variances and covariances and the chi-square tests of overall model fit that we developed for the PIV estimator could be modified to apply to Bollen's (1996a) 2SLS/IV estimator for continuous variables. The 2SLS/IV estimator for continuous variables is robust to excess kurtosis in the observed variables in that the consistency of the parameter estimator and the accuracy of the asymptotic standard errors still hold. In contrast, the usual polychoric correlation estimator is consistent when each pair of variables comes from a bivariate normal distribution. Nonnormality of the underlying variables can undermine the consistency of the PIV estimator or of the standard WLS or DWLS estimators. Work by Quiroga (1992), Flora (2002), Flora and Curran (2004), and Maydeu-Olivares (2006) suggests that the polychoric matrix is robust to some forms of nonnormality, but this is an area that requires further research. If alternative distributional assumptions are made for $\mathbf{y}^*$, a new estimator of the correlation matrix for these variables is possible and these could be subject to PIV estimation. But if the estimator of the polychoric correlation matrix is not a consistent estimator, then the PIV estimator, like the ULS, DWLS, and WLS estimators, need not be consistent.

Another area of robustness, besides distributional robustness, is the robustness to *structural misspecifications* in a model. Structural misspecifications refer to the use of the wrong structure in an SEM such as omitted paths, omitted correlated disturbances, or the wrong dimensionality of measures. In a previous section we noted that the structural misspecification robustness conditions given in Bollen (2001) carry over to the PIV estimator of the regression parameters. However, the same robustness does not carry over for $\hat{\boldsymbol{\theta}}_2$ since its consistency depends on the consistency of *all* regression parameters. Though many of the PIV estimators of $\hat{\boldsymbol{\theta}}_1^{(i)}$ could be consistent under a structural misspecification, at least some, particularly in the misspecified equation, will not be. Similar conditions for robustness to structural misspecification are not known for the system-wide estimators such as polychoric ULS, WLS, or DWLS, but based on experience with continuous variables (Cragg, 1968; Bollen et al., 2007) these system-wide estimators are likely to be less robust than the PIV estimator for regression and factor loading parameters. When structural misspecifications of a model are suspected, we recommend the PIV estimator as the parameter estimator for the factor loadings and regression coefficients since they are likely to be closer to their parameter values than are those for system-wide estimators.

We end with a cautionary note. The properties that we have established for the PIV estimator are asymptotic. Other analytical and simulation work will need to examine when and under what conditions these asymptotic properties take hold. But the simplicity of the estimator and its greater robustness of regression parameter estimates to structural specification errors suggest that the PIV should be further examined. In addition, its finite sample behavior should be compared to the more common system-wide estimators that are in use.

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