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Limited information estimation and testing of Thurstonian models for preference data

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Abstract

Thurstonian models provide a rich representation of choice behavior that does not assume that stimuli are judged independently of each other, and they have an appealing substantive interpretation. These models can be seen as multivariate standard normal models that have been discretized using a set of thresholds and that impose certain restrictions on these thresholds and on the inter-correlations among the underlying normal variates. In this paper we provide a unified framework for modeling preference data under Thurstonian assumptions and we propose a limited information estimation and testing framework for it. Although these methods have a long tradition in psychometrics, until recently only their application to rating data has been considered. Here we shall give an overview of how these methods can be readily applied to fit not only rating data, but also paired comparison and ranking data. The limited information methods discussed here are appealing because they are extremely fast, they are able to estimate models essentially of any size, they can easily accommodate external information about the stimuli and/or respondents, and in simulations they have been found to be very robust to data sparseness.

Keywords: Categorical data analysis; LISREL

1. Introduction

In 1927 Thurstone suggested a class of models for fitting paired comparison data that has been highly influential in the literature (see Bock and Jones, 1968). His model is characterized by three assumptions (Thurstone, 1927): (a) whenever a pair of stimuli is presented to a subject it elicits a continuous preference (utility function, or in Thurstone's terminology, *discriminal process*) for each stimulus; (b) the stimulus whose value is larger at the moment of the comparison will be preferred by the subject; (c)

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these unobserved preferences are normally distributed in the population. Later on, he suggested (Thurstone, 1931) that this could be a suitable model for fitting ranking data. Furthermore, by introducing a threshold relationship linking the unobserved preferences to the observed responses Thurstone suggested a similar model for fitting rating data. This model can be traced back to Thurstone and Chave (1929).

In this paper we provide a unified framework for modeling preference data under Thurstonian assumptions and we propose an estimation method for it. We shall show how this general model includes as special cases the three experimental procedures for preference data collection just mentioned (paired comparison, ranking and ratings) and discuss its limited information estimation and testing.

Thurstonian models provide a rich representation of choice behavior that does not assume that stimuli are judged independently of each other. Also, they have an appealing substantive interpretation. Yet, they have not been extensively employed in their full generality in applications. This is because with ranking data, and with paired comparison and rating data when each subject evaluates every stimuli (what Bock and Jones refer to as *multiple judgment sampling*), it is necessary to evaluate high dimensional multivariate normal integrals to obtain the preference patterns' probabilities for Thurstonian models. To overcome this estimation problem, we propose employing limited information estimation and testing procedures that make use of only the first and second order marginals of the observed contingency tables, thus avoiding the numerical evaluation of high dimensional multivariate normal integrals. These limited information methods have a long tradition in psychometrics (Christoffersson, 1975; Christoffersson and Gunsjö, 1983, 1996; Jöreskog, 1994; Lee et al., 1995; Muthén, 1978, 1982, 1984, 1993; Muthén et al., 1997; Olsson, 1979) although until recently only their application to rating data had been considered. As these methods have become increasingly popular, however, interest has developed in their application to other types of preference data. Brady (1989), Chan and Bentler (1998) and Maydeu-Olivares (1999a) have all proposed limited information procedures for estimating Thurstonian ranking models, whereas Maydeu-Olivares (2001a,b, 1999c) discussed limited information estimation of Thurstonian models for dichotomous (binary) and polytomous paired comparison data.

This paper is organized as follows. First, we briefly describe the restrictions required to fit dichotomous paired comparison, ranking and dichotomous rating data. Next, we provide a unified Thurstonian model that includes these models as special cases. Then, we review a three-stage procedure for estimating and testing these models from the first and second order marginals of the contingency table. The estimation procedure is due to Muthén (1978, 1993) and the proposed goodness of fit tests are due to Maydeu-Olivares (2001a). After this, we briefly discuss how to accommodate polytomous paired comparison, ranking and rating data in a Thurstonian framework. Similarly, we present an extension of the three-stage estimation and testing procedure suitable for polytomous data. Christoffersson and Gunsjö (1983, 1996) and Jöreskog (1994) considered the use of this estimation procedure to fit models for polytomous data involving restrictions only on the correlations among the underlying normal variates. Maydeu-Olivares (1999b) considered its use to fit models, such as those of interest here, that involve restrictions in both the thresholds and the underlying correlations of normal variates and proposed the testing procedures described here. All these procedures are based on grouped data. When ancillary information on the stimuli and/or subjects is to be modeled simultaneously with the preference data, then it is preferable to estimate the model using ungrouped data, as in Muthén (1984) (see also Küsters, 1987; Bermann, 1993; Muthén and Satorra, 1995). Estimation using ungrouped data will be discussed in the last section of the paper.

2. Thurstonian modeling of dichotomous preference data

2.1. Sampling theory

Suppose we wish to model the preferences of a homogeneous population for a set of n stimuli using a random sample of N individuals from this population. We may investigate these preferences using a variety of experimental procedures. For brevity, in this paper we only consider three of them. Consider first a complete paired comparison experiment. In this experiment,

$$\tilde{n} = \binom{n}{2} = \frac{n(n-1)}{2}$$

pairs of stimuli are constructed and each pair is presented to each individual in the sample who is asked to choose one stimulus in each pair. The outcomes of each paired comparison will be represented by a dichotomous random variable y_l indicating whether for each ordered pairwise combination of stimuli l a subject chooses stimulus i or i'

$$y_{lj} = \begin{cases} 1 \text{ if subject } j \text{ chooses object } i \\ 0 \text{ if subject } j \text{ chooses object } i' \end{cases} \quad l = 1, \ldots, \tilde{n};\ j = 1, \ldots, N \tag{1}$$

where $l \equiv (i,\ i')$, $(i = 1, \ldots,\ n-1;\ i' = i+1, \ldots,\ n)$. Clearly, each dichotomous variable y_l is a Bernoulli random variable, and the joint distribution of the $\tilde{n}$ random variables $\mathbf{y}$ is multivariate Bernoulli (Teugels, 1990).

Consider now a complete rating experiment. In this experiment, the n stimuli are presented separately to each individual in the sample who is asked to express his or her preferences using a dichotomous rating scale. When the outcomes of each rating are represented by a dichotomous random variable $y_i = \{0,1\}$, the joint distribution of the n random variables $\mathbf{y}$ is again multivariate Bernoulli.

Finally, consider a ranking experiment. In this case, the respondents are asked to order the stimuli as a function of a given preference criterion. In this case, since there are $n!$ possible permutations of the original ordering of the stimuli, the sampling distribution of the observed rankings is multinomial with $n!$ possible outcomes. We can transform this multinomial distribution to an $\tilde{n}$ dimensional multivariate Bernoulli distribution as follows: Construct a dichotomous variable y_l for each ordered pairwise combination of stimuli to indicate which stimulus was ranked above the other

$$y_{lj} = \begin{cases} 1 \text{ if object } i \text{ is ranked above object } i' \\ 0 \text{ if object } i \text{ is ranked below object } i'. \end{cases} \quad l = 1, \ldots, \tilde{n};\ j = 1, \ldots, N \tag{2}$$

In this case, however, suitable constraints must be imposed on the parameters of the multivariate Bernoulli distribution, as since there are only $n!$ possible ranking patterns, the observed binary contingency table must have $2^{\tilde{n}} - n!$ structural zeroes.

2.2. Paired comparison data

A Thurstonian model for the observed patterns of paired comparisons is obtained by considering the joint distribution of an n-dimensional vector of unobserved continuous preferences $\mathbf{t}$ and an $\tilde{n}$-dimensional vector of random errors $\mathbf{e}$ associated with each specific paired comparison (Takane, 1987; Takane and de Leeuw, 1987; Maydeu-Olivares, 2001b). We assume that

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{e} \end{pmatrix} \sim N\left[\begin{pmatrix} \boldsymbol{\mu}_t \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_t & \\ \mathbf{0} & \omega \mathbf{I} \end{pmatrix} \right]. \tag{3}$$

These random variables are linearly transformed using

$$\mathbf{y}^* = \mathbf{A}\mathbf{t} + \mathbf{e} \tag{4}$$

where $\mathbf{A}$ is an $\tilde{n} \times n$ matrix of contrasts such that each column corresponds to one of the stimuli being compared and each row to one of the paired comparisons. For example, when $n = 4$, $\mathbf{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{5}$$

Finally, the binary variables $\mathbf{y}$ are obtained from the unobserved variables $\mathbf{y}^*$ using

$$y_{lj} = \begin{cases} 1 & \text{if} \quad y^*_{lj} \geq 0 \\ 0 & \text{if} \quad y^*_{lj} < 0 \end{cases}. \tag{6}$$

Hence, according to Thurstone's model, for any paired comparison pattern,

$$\begin{aligned} \Pr\left(\bigcap_{l=1}^{\tilde{n}} y_l \right) &= \int \underset{\mathbf{R}}{\cdots} \int \phi_{\tilde{n}}(\mathbf{y}^*\colon \boldsymbol{\mu}_{y^*}, \boldsymbol{\Sigma}_{y^*})\, \mathrm{d}\mathbf{y}^* \\ &= \int \underset{\mathbf{R}}{\cdots} \int \phi_{\tilde{n}}(\mathbf{y}^*\colon \mathbf{A}\boldsymbol{\mu}_t, \mathbf{A}\boldsymbol{\Sigma}_t\mathbf{A}' + \omega\mathbf{I})\, \mathrm{d}\mathbf{y}^* \end{aligned} \tag{7}$$

where $\phi_{\tilde{n}}(\bullet)$ denotes a normal density and the intervals of the area of integration $\mathbf{R}$ are

$$R_l = \begin{cases} (0, \infty) & \text{if} \quad y_l = 1 \\ (-\infty, 0) & \text{if} \quad y_l = 0 \end{cases}. \tag{8}$$

The pattern probabilities (7) are unchanged when we standardize $\mathbf{y}^*$ using

$$\mathbf{z}^* = \mathbf{D}(\mathbf{y}^* - \boldsymbol{\mu}_{y^*}) \qquad \mathbf{D} = \mathrm{diag}(\boldsymbol{\Sigma}_{y^*})^{-1/2} \tag{9}$$

where at $y_l^* = 0$, $\tau_l := (-\mu_l^* / \sqrt{\sigma_{ll}^*})$, and σ_{ll}^* denotes a diagonal element of $\boldsymbol{\Sigma}_{y*}$. As a result, $\boldsymbol{\mu}_{z*} = \mathbf{0}$ and

$$\boldsymbol{\tau} = -\mathbf{D}\boldsymbol{\mu}_{y*} = -\mathbf{D}\mathbf{A}\boldsymbol{\mu}_t \quad \mathbf{P}_{z*} = \mathbf{D}\boldsymbol{\Sigma}_{y*}\mathbf{D} = \mathbf{D}(\mathbf{A}\boldsymbol{\Sigma}_t\mathbf{A}' + \omega\mathbf{I})\,\mathbf{D}. \tag{10}$$

We use $\mathbf{P}$ in (10) to indicate that the covariance matrix of $\mathbf{z}^*$ has ones along its diagonal. Thus, (7) can be re-written as

$$\Pr\left(\bigcap_{l=1}^{\tilde{n}} y_l\right) = \int \underset{\breve{\mathbf{R}}}{\cdots} \int \phi_{\tilde{n}}\,(\mathbf{z}^*\colon \mathbf{0}, \mathbf{P}_{z*})\,\mathrm{d}\mathbf{z}^* \tag{11}$$

and the limits of integration in (11) are

$$\breve{R}_l = \begin{cases} (\tau_l, \infty) & \text{if} \quad y_l = 1 \\ (-\infty, \tau_l) & \text{if} \quad y_l = 0 \end{cases}. \tag{12}$$

Eqs. (11) and (12) show that with observed binary data only the thresholds $\boldsymbol{\tau}$ and the tetrachoric correlations $\mathbf{P}_{z*}$ are identified. Furthermore, to investigate the identification of the parameters in $\boldsymbol{\mu}_t$, $\boldsymbol{\Sigma}_t$ and ω, one needs only to investigate the identification of the structures in (10). Now, since the variances of $\mathbf{z}^*$ are not identified, the parameter ω is not identified, and the diagonal elements of $\boldsymbol{\Sigma}_t$ are also not identified. Thus, to identify the model we let $\omega = 1$ and $\sigma_{ii} = 1, \forall i$, so that we estimate $\mathbf{P}_t$, a correlation matrix, rather than $\boldsymbol{\Sigma}_t$. Furthermore, the elements of $\boldsymbol{\mu}_t$ suffer from a location indeterminacy. To solve this, we arbitrarily let $\mu_n = 0$.

2.3. Ranking data

The parameters of Thurstone's model for paired comparison data are: (1) μ_i and σ_{ii} the means and variances of the population's unobserved continuous preferences for each stimulus, (2) $\rho_{ii'}$ the correlations in that population between the preferences for any two stimuli i and i', and (c) ω the common variance in the population of the errors $e_{ii'}$ associated with each paired comparison. The random errors $\mathbf{e}$ in (4) are crucial in modeling paired comparison data. Their inclusion allows the modeling of intransitive patterns of paired comparisons. A pattern of binary preferences is said to be *transitive* when, given the pattern, it is possible to rank order the stimuli, and intransitive otherwise. Substantively, a random error $e_{ii'}$ reflects that a subject's preference for a stimulus can change during the paired comparison experiment as the stimulus is presented next to different stimuli, thus giving rise to intransitivities. We assume that these errors are uncorrelated with the continuous preferences $\mathbf{t}$, uncorrelated with each other, and with a common variance ω.

Mathematically, the addition of $\omega\mathbf{I}$ to $\mathbf{A}\boldsymbol{\Sigma}_t\mathbf{A}'$ yields a positive definite matrix that when dichotomized assigns non-zero probabilities to all binary paired comparison patterns. Maydeu-Olivares (1999a) showed that when $\mathbf{e} = \mathbf{0}$, the model defined by Eqs. (3)–(6) above assigns non-zero probabilities only to transitive paired comparison patterns, and thus yields a model suitable for fitting ranking data, as it imposes the required $2^{\tilde{n}} - n!$ structural zeroes in the $\tilde{n}$-dimensional multivariate Bernoulli distribution.

Thus, after transforming the ranking patterns into binary patterns using (2), the probability of any such pattern is also given by (7) with (8), although in this case the covariance structure is $\mathbf{\Sigma}_{y*} = \mathbf{A\Sigma}_t\mathbf{A}'$. To this expression, we may apply the same change of variable of integration (9) as with paired comparison data without changing the ranking probabilities. Finally, to identify the model we also let $\sigma_{ii} = 1$, $\forall i$, and $\mu_n = 0$. With ranking data, however, the non-diagonal elements of $\mathbf{P}_t$ also suffer from a location indeterminacy. To solve this indeterminacy, following Dansie (1986), we let $\rho_{n,n-1} = 0$.

2.4. Rating data

Consider an n-dimensional vector of unobserved continuous preferences $\mathbf{t} \sim N(\boldsymbol{\mu}_t, \mathbf{\Sigma}_t)$. We assume that the binary variables $\mathbf{y}$ are obtained from the unobserved preferences $\mathbf{t}$ by the following threshold relationship

$$y_{ij} = \begin{cases} 1 & \text{if} \quad t_{ij} \geq \alpha \\ 0 & \text{if} \quad t_{ij} < \alpha \end{cases}. \tag{13}$$

The probability of any such pattern of binary ratings is now given by

$$\Pr\left(\bigcap_{i=1}^{n} y_l\right) = \int \underset{\mathbf{R}}{\cdots} \int \phi_n(\mathbf{y}^*{:}\boldsymbol{\mu}_t, \mathbf{\Sigma}_t)\, \mathrm{d}\mathbf{y}^* \tag{14}$$

with intervals of integration

$$R_i = \begin{cases} (\alpha, \infty) & \text{if} \quad y_i = 1 \\ (-\infty, \alpha) & \text{if} \quad y_i = 0 \end{cases}. \tag{15}$$

As in the previous two experimental procedures, we may perform the change of variable of integration (9) to (14) without changing the pattern probabilities where now at $y_l^* = \alpha$,

$$\tau_i := \frac{\alpha - \mu_i^*}{\sqrt{\sigma_{ii}^*}} = \frac{\alpha - \mu_i}{\sqrt{\sigma_{ii}}}.$$

Finally, to identify this model, as before, we let $\sigma_{ii} = 1$, $\forall i$ so that again a correlation matrix $\mathbf{P}_t$, is estimated rather than $\mathbf{\Sigma}_t$, and since the $\boldsymbol{\mu}_t$ suffer from a location indeterminacy we arbitrarily let $\mu_n = 0$.

2.5. A unified Thurstonian model for dichotomous preference data

It is clear that the Thurstonian models for the three experimental designs discussed in the previous sections can be unified as follows: For any pattern of binary preferences,

$$\Pr\left(\bigcap_{i=1}^{m} y_l\right) = \int \underset{\mathbf{R}}{\cdots} \int \phi_m(\mathbf{z}^*)\, \mathrm{d}\mathbf{z}^* \tag{16}$$

$$R_i = \begin{cases} (\tau_i, \infty) & \text{if} \quad y_i = 1 \\ (-\infty, \tau_i) & \text{if} \quad y_i = 0 \end{cases} \quad i = 1, \ldots, m \tag{17}$$

with $m = n$ (rating data) or $\tilde{n}$ (paired comparison or ranking data). Thus, Thurstonian models simply consist of structured multivariate normal densities that have been discretized. What changes from one experimental procedure to another (paired comparisons, rankings or ratings) are (a) the structure imposed on the thresholds, (b) the structure imposed on the correlations among the underlying variables $\mathbf{z}^*$, and (c) the restrictions needed for identification. These are

Experimental design	$\boldsymbol{\tau}$	$\mathbf{P}_{z^*}$	Additional restrictions
Paired comparison	$\mathbf{DA}\boldsymbol{\mu}_t$	$\mathbf{DAP}_t\mathbf{A}'\mathbf{D} + \mathbf{D}^2$	$\mu_n = 0$
Ranking	$\mathbf{DA}\boldsymbol{\mu}_t$	$\mathbf{DAP}_t\mathbf{A}'\mathbf{D}$	$\mu_n = 0$, $\rho_{n,n-1} = 0$
Rating	$\alpha\mathbf{1} - \boldsymbol{\mu}_t$	$\mathbf{P}_t$	$\mu_n = 0$

(18)

2.6. Restricted Thurstonian models

So far we have discussed a Thurstonian model for preference data with no restrictions in the mean and covariance structure of the underlying unobserved preferences except for minimal restrictions needed to ensure its identification. We refer to this model as an *unrestricted Thurstonian model*. When this model is found to yield a reasonable fit to the observed preference patterns, then one may consider introducing restrictions in the parameters of this model. A number of restricted Thurstonian models have been proposed, some of them by Thurstone (1927) himself. Takane (1987) gives a good overview of these models.

Some of these restricted models only impose structure on the covariance matrix $\boldsymbol{\Sigma}_t$. For instance, Thurstone (1927) proposed the so-called Case V and Case III models in which $\boldsymbol{\Sigma}_t = \sigma^2\mathbf{I}$ and $\boldsymbol{\Sigma}_t = \boldsymbol{\Psi}$, respectively, where $\boldsymbol{\Psi}$ is a diagonal matrix. But low rank approximations $\boldsymbol{\Sigma}_t = \boldsymbol{\Lambda\Lambda}'$ (Takane, 1980; Heiser and de Leeuw, 1981), and a common factor model structure $\boldsymbol{\Sigma}_t = \boldsymbol{\Lambda\Lambda}' + \boldsymbol{\Psi}$ (Takane, 1987) have also been proposed. Alternatively, one may consider models that include restrictions in the mean vector as well as in the covariance matrix of the underlying unobserved preferences. For instance, in the ideal point model $\boldsymbol{\mu}_t = \boldsymbol{\mu}_\iota - \text{diag}(\boldsymbol{\Lambda\Lambda}')\mathbf{1}_n$ (Brady, 1989; Böckenholt, 1993), in the wandering vector model $\boldsymbol{\mu}_t = \boldsymbol{\Lambda}\boldsymbol{\mu}_v$ (Carroll, 1980; De Soete and Carroll, 1983), and in the wandering ideal point vector model $\boldsymbol{\mu}_t = \boldsymbol{\Lambda}\boldsymbol{\mu}_v - 1/2\ \text{diag}(\boldsymbol{\Lambda\Lambda}')\mathbf{1}_n$ (De Soete et al., 1986). These three models share the same covariance structure, namely, $\boldsymbol{\Sigma}_t = \boldsymbol{\Lambda\Lambda}' + \boldsymbol{\Psi}$. For further details, see Takane (1987).

It is beyond the scope of this paper to provide identification restrictions for all these models for the three experimental procedures considered here. Identification restrictions for some of these models are discussed in Maydeu-Olivares (1999a, 2001a,b), Tsai (2000) and Tsai and Böckenholt (1999).

3. Limited information estimation and testing for dichotomous data

Consider the model (16) with (17). Also, consider the following parametric structure for this model, $\boldsymbol{\kappa}(\boldsymbol{\theta}) = (\boldsymbol{\tau}(\boldsymbol{\theta}), \boldsymbol{\rho}(\boldsymbol{\theta}))'$, where $\boldsymbol{\rho}$ denotes a vector obtained by stacking the lower diagonal elements of $\mathbf{P}_{z*}$ in (18). From (16) and (17),

$$\pi_i := \Pr(y_i = 1) = \Phi_1(-\tau_i) \quad i = 1, \ldots, m \tag{19}$$

$$\pi_{ii'} := \Pr[(y_i = 1) \cap (y_{i'} = 1)] = \Phi_2(-\tau_i, -\tau_{i'}, \rho_{ii'}) \quad i = 2, \ldots, m; \; i' = 1, \ldots, i-1 \tag{20}$$

where $\Phi_m(\bullet)$ denotes an m-dimensional standard normal distribution function. Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2)'$ where $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are the m-dimensional and $m(m-1)/2$-dimensional vectors of first and second order marginal probabilities defined by (19) and (20), respectively, with sample counterparts $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)'$.

Muthén (1978, 1993) and Muthén et al. (1997) considered the following three-stage estimator for the parameter vector $\boldsymbol{\theta}$. Using (19), in a first stage, each element of $\boldsymbol{\tau}$ is estimated separately from its corresponding univariate sample proportion as

$$\hat{\tau}_i = -\Phi_1^{-1}(p_i). \tag{21}$$

In a second stage, each $\rho_{ii'}$ (tetrachoric correlation) is estimated separately from its corresponding bivariate proportion $p_{ii'}$ given the first stage estimates as

$$\hat{\rho}_{ii'} = \Phi_2^{-1}(p_{ii'} | -\hat{\tau}_i, -\hat{\tau}_{i'}). \tag{22}$$

Finally, $\boldsymbol{\theta}$ is estimated by minimizing

$$F = (\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}(\boldsymbol{\theta}))'\hat{\mathbf{W}}(\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}(\boldsymbol{\theta})) \tag{23}$$

where $\hat{\mathbf{W}}$ is a matrix converging in probability to $\mathbf{W}$, a non-negative definite matrix. Muthén (1978) showed that the asymptotic covariance matrix of $\hat{\boldsymbol{\kappa}}$ is

$$\boldsymbol{\Xi} = \boldsymbol{\Delta}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Delta}^{-1'} \quad \boldsymbol{\Delta} = \frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\kappa}'} \tag{24}$$

where $\boldsymbol{\Gamma}$ denotes the asymptotic covariance matrix of $\mathbf{p}$. Then, obvious choices of $\hat{\mathbf{W}}$ in (23) are $\hat{\mathbf{W}} = \hat{\boldsymbol{\Xi}}^{-1}$ (WLS: Muthén, 1978), $\hat{\mathbf{W}} = \operatorname{diag}(\hat{\boldsymbol{\Xi}})^{-1}$ (DWLS: Muthén et al., 1997), and $\hat{\mathbf{W}} = \mathbf{I}$ (ULS: Muthén, 1993).

Muthén (1993) showed that

$$\sqrt{N}(\hat{\theta} - \boldsymbol{\theta}) \xrightarrow{d} N(\mathbf{0}, \mathbf{H}\boldsymbol{\Xi}\mathbf{H}') \quad \mathbf{H} = (\tilde{\boldsymbol{\Delta}}'\mathbf{W}\tilde{\boldsymbol{\Delta}})^{-1}\tilde{\boldsymbol{\Delta}}'\mathbf{W} \tag{25}$$

where $\xrightarrow{d}$ denotes convergence in distribution, and $\tilde{\boldsymbol{\Delta}} = (\partial \boldsymbol{\kappa})/(\partial \boldsymbol{\theta}')$. He also considered goodness of fit tests for the structural restrictions of the model $\boldsymbol{\kappa}(\boldsymbol{\theta})$. Limited information tests of the overall restrictions of the model have been proposed by Maydeu-Olivares (2001a). He showed that

$$\sqrt{N}\hat{\boldsymbol{\varepsilon}} := \sqrt{N}(\mathbf{p} - \boldsymbol{\pi}(\hat{\boldsymbol{\theta}})) \xrightarrow{d} N(\mathbf{0}, \tilde{\mathbf{M}}) \tag{26}$$

$$\tilde{\mathbf{M}} = \left(\mathbf{I} - \boldsymbol{\Delta}\tilde{\boldsymbol{\Delta}}\mathbf{H}\boldsymbol{\Delta}^{-1}\right)\boldsymbol{\Gamma}\left(\mathbf{I} - \boldsymbol{\Delta}\tilde{\boldsymbol{\Delta}}\mathbf{H}\boldsymbol{\Delta}^{-1}\right). \tag{27}$$

Consider the statistic $T = N\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}$. This is asymptotically distributed as a mixture of r independent chi-square distributions with one degree of freedom, where

$$r = \frac{m(m+1)}{2} - q$$

and q is the number of mathematically independent elements in $\boldsymbol{\theta}$. To test H_0: $\boldsymbol{\pi} = \boldsymbol{\pi}(\boldsymbol{\theta})$, Maydeu-Olivares (2001a) proposed scaling T by its mean or adjusting it by its mean and variance so that it approximates a chi-square distribution using

$$T_s = \frac{r}{\mathrm{Tr}\left[\tilde{\mathbf{M}}\right]} T \quad T_a = \frac{\mathrm{Tr}[\tilde{\mathbf{M}}]}{\mathrm{Tr}[\tilde{\mathbf{M}}^2]} T. \tag{28}$$

T_s is to be referred to a chi-square distribution with r degrees of freedom, whereas T_a is to be referred to a chi-square distribution with

$$d = \frac{\left(\mathrm{Tr}\left[\tilde{\mathbf{M}}\right]\right)^2}{\mathrm{Tr}\left[\tilde{\mathbf{M}}^2\right]}$$

degrees of freedom.

The goodness of fit tests (28) are based on those proposed by Satorra and Bentler (1994) for covariance structure analysis for continuous variables. In this literature, simulation studies suggest that these tests statistics clearly outperform tests based on T (Bentler, 1994) and tests based on the generalized Wald statistic proposed by Browne (1984) (see Satorra and Bentler, 1994). Also, although the proposed reference distributions for T_s and T_a are not their actual (and presently unknown) asymptotic distributions, it has been repeatedly shown in simulation studies (e.g., Satorra and Bentler, 1994; Muthén, 1993) that these reference distributions work well in practice.

Although these methods were proposed for fitting rating data, they can be applied to estimate any of the Thurstonian models described in the previous section as well. Maydeu-Olivares (1999a, 2001a,b) has used these methods to fit some models, such as the unrestricted model (18), to ranking and paired comparison data, respectively. However, because of the structural zeroes in the contingency table when fitting ranking data, the degrees of freedom available for testing ranking models must be adjusted using (Maydeu-Olivares, 1999a)

$$r = \frac{\tilde{n}(\tilde{n}+1)}{2} - \sum_{x=2}^{n-1} \binom{x}{2} - q. \tag{29}$$

The proposed estimation and testing procedure are based on the sample means, $\mathbf{p}_1$, and on the sample bivariate joint moments, $\mathbf{p}_2$, of the multivariate Bernoulli distribution. Teugels (1990) showed that there is a one-to-one correspondence between the cell proportions of the contingency table and the sample moments of this distribution. $\mathbf{p}_1$ and $\mathbf{p}_2$ can be reasonably well estimated in small samples, whereas the cell proportions which depend on joint sample moments up to order n require large samples to be well estimated. This may give an edge to limited information estimation and testing procedures over procedures that make full use of all available data when large models are estimated using small samples. To support this conjecture consider the three estimators WLS, DWLS and ULS described above in (23). The WLS estimator has asymptotically smallest variance among the estimators based on $\mathbf{p}$. However, WLS estimates depend on third and fourth order joint moments of the multivariate Bernoulli distribution through $\mathbf{\Gamma}$ (Christoffersson, 1975). When these moments are consistently estimated using sample moments, it has been repeatedly shown in simulation studies (e.g., Muthén, 1993; Maydeu-Olivares, 2001a) that the resulting parameter estimates, standard errors and goodness of fit tests are seriously biased unless the sample size is very large and the size of the model is small. On the other hand, much smaller sample sizes are required to obtain unbiased parameter estimates, standard errors and goodness of fit tests when DWLS or ULS is employed. For instance, Maydeu-Olivares (2001a,b) has shown that appropriate parameter estimates, standard errors and goodness of fit tests for a paired comparison model for seven stimuli estimated by DWLS or ULS can be obtained with as few as 100 observations.

In closing this section, we would like to point out the similarities between the estimation procedure described here and classical least squares procedures proposed for estimating Thurstonian models (see Bock and Jones, 1968; Arbuckle and Nugent, 1973, and references therein). Classical procedures make use only of the means of the multivariate Bernoulli distribution and assume they are independent. In contrast, the procedure described here employs both the first and second joint sample moments of this distribution, and also takes into account their interdependencies to obtain asymptotically correct standard errors and goodness of fit tests.

4. Thurstonian modeling of polytomous preference data

We now consider experimental procedures that yield patterns of polytomous preference data. That is, each of the observed categorical variables $\mathbf{y}$ takes $k>2$ values. One of these procedures is the polytomous paired comparison task. In this task respondents are presented pairs of stimuli and they are asked to express the degree of their preference for one stimuli or the other using a rating scale. Another procedure is a polytomous ranking task (Böckenholt and Dillon, 1997: Appendix B) in which respondents are asked to order the stimuli and then to assess on a rating scale the distance between the stimuli with adjacent rank positions. Yet another example is a polytomous rating task. Thurstonian models for polytomous ranking data do not seem to be amenable to the estimation framework presented here and thus they will not be considered in the following discussion.

4.1. Paired comparison data

Now, each of the y_l variables is multinomial with k categories. As in the binary case, we assume (3) and (4), but now to accommodate the fact that the observed variables consist of $k>2$ categories instead of (6) we assume

$$y_{lj} = h \quad \text{if} \quad \alpha_h < y^*_{lj} < \alpha_{h+1} \qquad h = 0, \ldots, k-1 \tag{30}$$

where $\alpha_0 = -\infty$, $\alpha_k = \infty$. Furthermore, if there are no presentation order effects $\alpha_h = -\alpha_{k-h}$ and $\alpha_{k/2} = 0$ (Agresti, 1992; Böckenholt and Dillon, 1997).

Also, the probability of observing a pattern of polytomous paired comparisons is given by (7) but now the integration intervals are

$$R_l = (\alpha_h, \alpha_{h+1}) \quad \text{if} \quad y_l = h \tag{31}$$

instead of (8). As in the case of dichotomous data, we may apply the change of variable transformation (9) without altering the pattern probabilities where at $y^*_l = \alpha_h$, $\tau_{l_h} := (\alpha_h - \mu^*_l)/(\sqrt{\sigma^*_{ll}})$. As a result, $\boldsymbol{\mu}_{z*}$ and $\mathbf{P}_{z*}$ equal the expressions obtained in the dichotomous case, but now

$$\boldsymbol{\tau}_h := \left(\tau_{1_h}, \ldots, \tau_{l_h}\right)' = \mathbf{D}\left(\alpha_h \mathbf{1} - \boldsymbol{\mu}_{y*}\right) \quad h = 0, \ldots, k-1. \tag{32}$$

Also, the same identification restrictions discussed for the dichotomous case can be employed to identify this model.

4.2. Rating data

We assume that an n-dimensional vector of unobserved continuous preferences $\mathbf{t} \sim N(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$ has been categorized using

$$y_{ij} = h \quad \text{if} \quad \alpha_h < t_{ij} < \alpha_{h+1} \qquad h = 0, \ldots, k-1 \tag{33}$$

where $\alpha_0 = -\infty$, $\alpha_k = \infty$. As a result, the probability of observing any pattern of polytomous ratings is given by (14) where instead of (15), the integration intervals are now

$$R_i = (\alpha_h, \alpha_{h+1}) \quad \text{if} \quad y_i = h. \tag{34}$$

Again we may perform the change of variable of integration (9) in the model where at $y^*_i = \alpha_h$,

$$t_{i_h} := \frac{\alpha_h - \mu^*_i}{\sqrt{\sigma^*_{ii}}} = \frac{\alpha_h - \mu_i}{\sqrt{\sigma_{ii}}},$$

or in matrix form, (32), $\boldsymbol{\mu}_{z*}$ and $\mathbf{P}_{z*}$ remain unchanged from the binary case, and we may apply the identification restrictions discussed for the binary case to identify this model.

4.3. A unified Thurstonian model for polytomous preference data

To summarize our current presentation, the probability of observing a pattern of polytomous paired comparison or ratings is given by (16) with

$$R_i = \left(\tau_{i_h}, \tau_{i_{h+1}}\right) \quad \text{if} \quad y_i = h \qquad i = 1, \ldots, m; h = 0, \ldots, k-1 \tag{35}$$

where $\tau_{i_0} = -\infty$, $\tau_{i_k} = \infty$, $m = n$ (rating data) or $\tilde{n}$ (paired comparison data). However, the structure imposed on the thresholds and on the correlations among the underlying variables $\mathbf{z}^*$, as well as the identification restrictions, change across experimental procedures. These are

Experimental design	$\boldsymbol{\tau}_h$	$\mathbf{P}_{z^*}$	Additional restrictions
Paired comparison	$\mathbf{D}(\alpha_h \mathbf{1} - \mathbf{A}\boldsymbol{\mu}_t)$	$\mathbf{D}\mathbf{A}\mathbf{P}_t\mathbf{A}'\mathbf{D} + \mathbf{D}^2$	$\mu_n = 0$
Rating	$\alpha_h \mathbf{1} - \boldsymbol{\mu}_t$	$\mathbf{P}_t$	$\mu_n = 0$

(36)

The model we have just described is an unrestricted Thurstonian model. If this model yields a reasonable approximation to the data at hand, one may consider fitting some of the restricted models discussed in Section 2.6. For instance, Böckenholt and Dillon (1997) discussed fitting a Case V model to ordinal paired comparison data, whereas Tsai and Böckenholt (1999) discussed fitting factor and ideal point models to these data.

5. Limited information estimation and testing for polytomous data

Consider the model (16) with (35). Also, consider the parametric structure $\boldsymbol{\kappa}(\boldsymbol{\theta}) = (\boldsymbol{\tau}(\boldsymbol{\theta}), \boldsymbol{\rho}(\boldsymbol{\theta}))'$ where $\boldsymbol{\tau}(\boldsymbol{\theta}) = (\boldsymbol{\tau}_1(\boldsymbol{\theta}), \ldots, \boldsymbol{\tau}_{k-1}(\boldsymbol{\theta}))'$. According to this model,

$$\pi_{i_h} := \Pr(y_i = h) = \int_{\tau_{i_h}}^{\tau_{i_{h+1}}} \phi_1(z_i^*)\, \mathrm{d}z_i^* \tag{37}$$

$$\pi_{i_h i'_{h'}} := \Pr[(y_i = h) \cap (y_{i'} = h')] = \int_{\tau_{i_h}}^{\tau_{i_{h+1}}} \int_{\tau_{i'_{h'}}}^{\tau_{i_{h'+1}}} \phi_2(z_i^*, z_{i'}^*: \rho_{ii'})\, \mathrm{d}z_i^*\, \mathrm{d}z_{i'}^*. \tag{38}$$

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2)'$ where now $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are used to denote the km-dimensional and $k^2 m(m-1)/2$-dimensional vectors of first and second order marginal probabilities defined by (37) and (38), respectively, with sample counterparts $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)'$.

Consider the following three-stage estimator for the parameter vector $\boldsymbol{\theta}$ (Christoffersson and Gunsjö, 1983, 1996; Jöreskog, 1994; Maydeu-Olivares, 1999b). Using (37), in a first stage, the elements of $\boldsymbol{\tau}$ are estimated for each univariate contingency table at a time by minimizing

$$L(\boldsymbol{\tau}_i) = N \sum_{h=1}^{k} p_{i_h} \ln \pi_{i_h}(\boldsymbol{\tau}_i). \tag{39}$$

In a second stage, each $\rho_{ii'}$ (polychoric correlation) is estimated separately from its corresponding bivariate contingency table given the first stage estimates by minimizing

$$L(\rho_{ii'}|\hat{\boldsymbol{\tau}}_i, \hat{\boldsymbol{\tau}}_{i'}) = N \sum_{h=1}^{k} \sum_{h'=1}^{k} p_{i_h i'_{h'}} \ln \pi_{i_h i'_{h'}} (\rho_{ii'}|\hat{\boldsymbol{\tau}}_i, \hat{\boldsymbol{\tau}}_{i'}). \tag{40}$$

Note that for each variable, $\hat{\boldsymbol{\tau}}_i$ is the maximum likelihood estimate of $\boldsymbol{\tau}_i$, whereas for each pair of variables $\hat{\rho}_{ii'}$ is the pseudo-maximum likelihood (in the sense of Gong and Samaniego, 1981) estimate of $\rho_{ii'}$.

Finally, in a third stage $\boldsymbol{\theta}$ is estimated using (23). Again, one may employ WLS, DWLS or ULS in this third stage. Maydeu-Olivares (1999b) showed that the asymptotic covariance matrix of $\hat{\boldsymbol{\kappa}}$ obtained using (39) and (40) is

$$\boldsymbol{\Xi} = \mathbf{G}\boldsymbol{\Gamma}\mathbf{G}' \quad \mathbf{G} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ -\mathbf{B}_2\boldsymbol{\Delta}_{21}\mathbf{B}_1 & \mathbf{B}_2 \end{pmatrix} \tag{41}$$

where $\boldsymbol{\Gamma}$ denotes the asymptotic covariance matrix of $\mathbf{p}$,

$$\mathbf{B}_1 = (\boldsymbol{\Delta}'_{11}\mathbf{D}_1\boldsymbol{\Delta}_{11})^{-1}\boldsymbol{\Delta}'_{11}\mathbf{D}_1 \quad \mathbf{B}_2 = (\boldsymbol{\Delta}'_{22}\mathbf{D}_2\boldsymbol{\Delta}_{22})^{-1}\boldsymbol{\Delta}'_{22}\mathbf{D}_2 \tag{42}$$

and $\mathbf{D}_1 = \mathrm{Diag}(\boldsymbol{\pi}_1)^{-1}$, $\mathbf{D}_2 = \mathrm{Diag}(\boldsymbol{\pi}_2)^{-1}$, $\boldsymbol{\Delta}_{11} = (\partial\boldsymbol{\pi}_1)/(\partial\boldsymbol{\tau}')$, $\boldsymbol{\Delta}_{21} = (\partial\boldsymbol{\pi}_2)/(\partial\boldsymbol{\tau}')$ and $\boldsymbol{\Delta}_{22} = (\partial\boldsymbol{\pi}_2)/(\partial\boldsymbol{\rho}')$. In the dichotomous case, Eqs. (41) with (42) reduce to Eq. (24) due to Muthén (1978). Also, Christoffersson and Gunsjö (1983) and Jöreskog (1994) considered the special case of (41) with (42) in which only the asymptotic covariance matrix of $\hat{\boldsymbol{\rho}}(\boldsymbol{\theta})$ is considered (see Maydeu-Olivares, 1999b).

Standard errors for $\hat{\boldsymbol{\theta}}$ and test statistics for assessing the goodness of fit of the model to the first and second order marginals of the contingency table are given, as the binary case, by (25) and (28) with (26), with $\tilde{\mathbf{H}} = \boldsymbol{\Delta}\tilde{\boldsymbol{\Delta}}\mathbf{HG}$ n (27), where

$$\boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{\Delta}_{11} & \mathbf{0} \\ \boldsymbol{\Delta}_{21} & \boldsymbol{\Delta}_{22} \end{pmatrix}.$$

The degrees of freedom available for testing in the case of polytomous data are

$$r = (k-1)m + \frac{(k-1)^2 m(m-1)}{2} - q. \tag{43}$$

This is because for each variable, there are $(k-1)$ independent probabilities, and for each pair of variables, there are only $(k-1)^2$ independent probabilities given the univariate probabilities.

6. Limited information estimation and testing for preference data with ancillary information

One of the greatest advantages of employing a limited information framework to model preference data is that one can easily model simultaneously the observed preference patterns along with ancillary information concerning the respondents and/or the stimuli. Takane (1987) discusses how to incorporate this information into the model. When incorporating ancillary information into the model we obtain a multivariate ordinal probit model to which we introduce the restrictions discussed in Sections 2 and 4 to accommodate the characteristics of the experimental task used to obtain the preference patterns. Also, in this case it is convenient to distinguish between dependent variables $\mathbf{z}^*$ and purely exogenous variables $\mathbf{x}$. In so doing we may use a conditional estimation approach based on a multivariate normality assumption for $\mathbf{z}^*|\mathbf{x}$. This is less restrictive than an assumption of joint multivariate normality for $\mathbf{z}^*$ and $\mathbf{x}$, which may be untenable if for instance some of the $\mathbf{x}$ variables are dummy-coded (Arminger and Browne, 1995). Since in a conditional approach $\boldsymbol{\mu}_x$ and $\boldsymbol{\Sigma}_x$ are not explicitly estimated, if these parameters are of interest for some ancillary variables, then these ancillary variables must be treated as endogenous variables and modeled along with the variables $\mathbf{z}^*$ arising from the experimental procedure used to gather the preference data.

The three stage estimation setup described in the previous section can be easily extended to accommodate the modeling ancillary variables within a conditional approach as follows: In a first stage, we estimate all thresholds, regression intercepts and regression slopes separately for each observed dependent variable using an expression akin to (39). Given these first stage estimates, in a second stage we estimate separately for each pair of variables a conditional polyserial or polychoric correlation using an expression akin to (40). Finally, in a third stage $\boldsymbol{\theta}$ is estimated from the parameters estimated in the first two stages using (23). However, when modeling ancillary information it is computationally more convenient to perform the first and second stages of the estimation using ungrouped data (individual observations) than grouped data (as in (39) and (40)) due to data sparseness. For a discussion of this topic see Muthén (1982).

Three different expressions for the asymptotic covariance matrix of $\hat{\boldsymbol{\kappa}}$, $\boldsymbol{\Xi}$, estimated using ungrouped data are given by Muthén (1984; see also Muthén and Satorra, 1995), Küsters (1987; see also Küsters, 1990) and Bermann (1993). Standard errors for $\hat{\boldsymbol{\theta}}$ can be obtained using (25), and test statistics for assessing the goodness of fit of the structural restrictions of the model $\boldsymbol{\kappa}(\hat{\boldsymbol{\theta}})$ can be obtained following Satorra and Bentler (1994).

7. Conclusions

Thurstonian modeling of preference data obtained under what Bock and Jones (1968) referred to as *multiple judgment sampling* has been hampered by the lack of adequate estimation procedures. In this paper we have described one estimation approach suitable for dealing with these models that has a long tradition in psychometrics, yet until very recently has been applied to preference data other than ratings.

The limited information methods discussed here are appealing because they are extremely fast, they are able to estimate models essentially of any size, they can easily accommodate external information about the stimuli and/or respondents, and in simulations they have been found to be very robust to data sparseness. Furthermore, these methods have been implemented in commercially available software such as MPLUS (Muthén and Muthén, 1998) and MECOSA (Arminger et al., 1996), the latter being a set of GAUSS modules.

However, although the small sample behavior of these estimators has been relatively well studied using simulation studies in the case of rating data (e.g. Muthén, 1993;

Muthén et al., 1997), little is known as to the small sample behavior of these estimators when modeling other types of preference data. Also, it is clear that additional work is needed when estimating a multivariate ordinal probit model using a limited information approach. For instance, it is not clear which of the three available expressions for the covariance matrix of the statistics employed in the third stage performs better in small sample applications. Also, it is not clear how one should assess the distributional assumptions of the model.

It is hoped that with the recent availability of suitable estimation procedures, such as those described here or resampling methods such as those employed by Yao and Böckenholt (1999) and Tsai and Böckenholt (1999), we shall be able to better understand the advantages and limitations of the full family of Thurstonian models in fitting preference data.

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