Econ Theory (2010) 42:523–538
DOI 10.1007/s00199-008-0403-3

RESEARCH ARTICLE

Self-enforced collusion through comparative cheap talk in simultaneous auctions with entry

Antonio Miralles

Received: 14 December 2006 / Accepted: 14 July 2008 / Published online: 21 August 2008

Abstract Campbell (J Econ Theory 82:425–450, 1998) develops a self-enforced collusion mechanism in simultaneous auctions based on complete comparative cheap talk and endogenous entry, with two bidders. His result is difficult to generalize to an arbitrary number of bidders, since the entry-decision stage of the game is characterized by strategic substitutes. This paper analyzes more-than-two-bidder, symmetric-prior cases. Two results are proved: (1) as the number of objects grows large, a full comparative cheap talk equilibrium exists and it yields asymptotically fully efficient collusion; and (2) there is always a partial comparative cheap talk equilibrium. All these results are supported by intuitive equilibria at the entry-decision stage (J Econ Theory 130:205–219, 2006; Math Soc Sci 2008, forthcoming). Numerical examples suggest that full comparative cheap talk equilibria are not uncommon even with few objects.

Keywords Auctions · Entry costs · Pre-play communication

JEL Classification D44 · D82 · C72

Antonio Miralles is grateful to Hsueh-Lyhn Huynh, Bart Lipman, Zvika Neeman, Gregory Pavlov and an anonymous referee for their comments and suggestions. During the research, he received financial support from the Spanish Ministry of Education (SEJ2006-04985), the Fulbright Commission and Boston University.

A. Miralles (✉)
Department of Economics,
Boston University, 270 Bay State Rd.,
Room 543, Boston, MA 02215, USA
e-mail: miralles@bu.edu

1 Introduction

Collusion in auctions is a recurrent research topic. Usually, researchers investigate so-called bidding rings (McAfee and McMillan 1992), in which some bidders coordinate their actions in order to improve their payoffs. Direct coordination of actions is not only illegal but also organizationally costly. For instance, commitment to a side-payments schedule is often necessary in order to prevent betrayals inside the bidding ring, in environments characterized by finitely many rounds of auctions. Sometimes the rules of the auction itself have to be used to coordinate actions.[1] This paper focuses instead on collusive mechanisms that are self-enforced and fairly simple. No side-payments nor trigger strategies are needed. We just need pre-play cheap talk, which is clearly difficult to prosecute by competition authorities.

Campbell (1998) studies coordination and collusion in simultaneous second-price auctions with entry costs. In the mechanism we extend here, the so-called ranking mechanism, (potential) bidders first engage into pre-play communication where they publicly signal their respective rankings of the objects on sale. In a second stage, each potential bidder decides in which auctions she participates. Finally, entry costs are paid, and the auctions are undertaken. Campbell shows that in the two-bidder case, a truthful ranking revelation (or full comparative cheap talk) equilibrium always exists. This mechanism provides asymptotically full efficiency to the cartel, as the number of objects on sale grows large.

Difficulties arise when one tries to generalize Campbell's results to more-than-two-bidder cases. Since the entry-decision stage of the game is characterized by strategic substitutes, nothing automatically guarantees that ranking an object higher is rewarded with all other bidders being less likely to participate in the auction for that object. Although ranking the object higher makes the bidder stronger in that auction, if a second bidder consequently reacts by being less likely to participate, a third bidder may be encouraged (due to strategic substitutes) by this second bidder's conservative reaction and may finally be more likely to participate. This fact could deter a truthful revelation of preferences over objects.

This paper shows results on those cases. First, a full comparative cheap talk equilibrium exists if the number of objects is large enough. Numerical examples suggest that such an equilibrium may exist even with a few objects. Second, we show that a partial cheap talk equilibrium always exists. In that equilibrium, each bidder splits the objects into two sets, the favorite one and the rest, and she lets the other bidders know about that split. All these equilibria are supported by a robust notion of equilibrium at the participation-decision stage which is named intuitive equilibrium (Tan and Yilankaya 2006; Miralles 2008).

Pesendorfer (2000) analyzes a setup that differs from Campbell's in that entry costs are not considered as a way to endogenize entry decisions. Instead, a third party

[1] The literature reports these complex collusive protocols in simultaneous auctions. One example is Grimm et al. (2003) concerning the 3G frequency auctions in Germany. Another example is Brusco and Lopomo (2002) for a series of simultaneous English auctions like the FCC spectrum auctions in the United States. The information arising during the auction process is useful to coordinate bidders. In this respect, see also Albano et al. (2001), and Engelbrecht-Wiggans and Kahn (2005), who analyze the SPaR (Stake, Protect and Revenge) strategies for a series of simultaneous English auctions.

receives bidders' ranking reports and decides who the sole final bidder will be for each auction. The author assumes that his model is a simplification of a dynamic environment where deviations from obedience are punished in subsequent rounds of auctions, and where the discount factor is arbitrarily close to 1.

Such a justification is plausible only with infinitely many repeated rounds, since the mechanism becomes unsustainable when the number of rounds into play is finite. Even in an infinite-round case, works like Aoyagi (2003), Blume and Heidhues (2004), and Skrzypacz and Hopenhayn (2004) have already designed collusion mechanisms that approach full collusive efficiency with infinitely many repeated rounds under certain conditions. There is no need for a ranking mechanism in that case.

The present paper analyzes collusion in weakly efficient auctions in the sense of Armstrong (2000), i.e. auctions that are efficient among the final participants. The use of this kind of auction mechanisms is justified. As Celik and Yilankaya (2005) and Lu (2008a) show for the case of symmetric bidders, a weakly efficient auction with proper subsidies and reservation prices is revenue-maximizing. Also, the ex ante efficient allocation can be obtained through an equilibrium of a weakly efficient auction with no reservation prices (Stegeman 1996; Lu 2008b).

The paper is organized as follows. Section 2 introduces and characterizes weakly efficient auctions with entry costs and risk-neutral bidders, under the IPV assumption. Section 3 presents Campbell (1998)'s ranking-revelation results and the issues appearing with just three potential bidders in the game. Numerical examples are presented however suggesting that full comparative cheap talk equilibria are not unusual. Section 4 presents an asymptotic generalization of Campbell's results. Section 5 proves the existence of a simpler communication equilibrium for an arbitrary, finite number of bidders and objects. Section 6 concludes. An Appendix contains long proofs.

2 Weakly efficient auctions with entry costs

Some introduction into efficient auctions with entry costs is convenient in order to gain familiarity with the notation and the concepts that will be use here.[2] For the moment consider the auction for a unique, indivisible object. This object is to be sold by a mechanism that is efficient among the final bidders (i.e. weakly efficient). There are N potential, risk-neutral bidders. There is no reservation price. Participating in the auction has a known cost c ($0 < c < v^*$) for each final participant.[3] Each bidder learns his valuation before deciding whether to participate or not in the auction. Each bidder i's valuation v_i is believed by other bidders to be drawn from a distribution F_i over the support $[0, v^*]$. We assume full support over this interval for ease of calculations, though noting that a relaxation of this assumption would not alter the main results. We invoke the IPV assumption, so valuations are independent among bidders.

[2] See Menezes and Monteiro (2000) and Levin and Smith (1994) for an introduction to the theory of auctions with entry costs. I rather follow the former paper in my work, since, unlike the latter model, it assumes that bidders learn their own valuations before taking the participation decision.

[3] As in Tan and Yilankaya (2006), we assume that entry costs are equal among bidders.

Since the mechanism is weakly efficient, the only relevant component of bidder i's strategy is the participation decision. She comes to know her valuation before taking this decision, so this strategy is related to a threshold (cut-off) value θ_i. Given i's participation in the auction, her expected payoff is increasing in her valuation, and therefore the participation decision is indeed a threshold strategy. Bidder i participates in the auction if and only if her valuation v_i lies above this threshold value (in the limit case $v_i = \theta_i$, we simplify by assuming that the bidder withdraws from the auction). Knowing others' cut-off strategies, if bidder i participates in the auction, her expected profits would be equal to

$$\int_0^{v_i} \Pr\left(x > \max_{j \neq i}[I\{v_j > \theta_j\} \cdot v_j]\right) \mathrm{d}x - c = \int_0^{v_i} \prod_{j \neq i} F_j\left(\max\{\theta_j, x\}\right) \mathrm{d}x - c$$

$I\{a\}$ is a function that takes value 1 if a is true, and 0 otherwise. The left-hand-side expression comes from (1) the fact that the marginal expected payoff is equal to the probability of winning, and (2) the fact that the auction is weakly efficient. The right-hand side follows from the IPV assumption.

Bidder i's best response $\theta_i^{\#}(\theta_{-i})$ to θ_{-i} (others' cut-off strategies) is characterized by

$$\int_0^{\theta_i^{\#}(\theta_{-i})} \prod_{j \neq i} F_j\left(\max\{\theta_j, x\}\right) \mathrm{d}x - c = (\leq) 0$$

The weak inequality only applies when the upper bound of the integral equals $v*$, as in further equations where the symbol appears in brackets. Observe that $\theta_i^{\#}(\cdot)$ is single-valued, continuous, strictly decreasing[4] (except in the corner case $\theta_i^{\#}(\theta_{-i}) = v*$) and belongs to $[c, v*]$. Observe as well that $\theta_i^{\#}(\theta_{-i}) = c \Leftrightarrow \theta_{-i} = (v*, \ldots, v*)$. A *cut-off equilibrium* is a vector $\Theta^* = (\theta_1^*, \ldots, \theta_N^*) \in [c, v*]^N$ such that

$$\int_0^{\theta_i^*} \prod_{j \neq i} F_j\left(\max\{\theta_j^*, x\}\right) \mathrm{d}x - c = (\leq) 0, \quad \forall i \in \{1, \ldots, N\}$$

The existence of a cut-off equilibrium is guaranteed by a simple use of Brower's fixed-point theorem. Let $\Theta^{\#}(\Theta)$ be the vector of best responses to a vector of cutoffs $\Theta \in [c, v*]^N$. It is clear that $\Theta^{\#}(\Theta) \in [c, v*]^N$ as well. Since $\Theta^{\#}(\cdot)$ is continuous, there must exist $\Theta^* \in [c, v*]^N$ such that $\Theta^{\#}(\Theta^*) = \Theta^*$, that is, a cut-off equilibrium.

We will particularly use an equilibrium concept found in Tan and Yilankaya (2006), the so-called *intuitive equilibrium*. Consider the auction for a single object. Let there be $s \in \{1, \ldots, N-1\}$ strong bidders (each one characterized by a distribution function G from which valuations are drawn) and $N - s$ weak ones (function F). G first order

[4] The entry stage is a game of strategic substitutes.

stochastically dominates F. There exists a cut-off equilibrium $\Theta^*(s)$ characterized by two values, $\theta_G^*(s)$ and $\theta_F^*(s)$. Bidders of identical strength play the same equilibrium strategy, due to symmetry among bidders of the same distribution type. An intuitive equilibrium is one in which $\theta_G^*(s) \leq \theta_F^*(s)$. The cut-off strategy is higher for weaker bidders, and so the probability of participating in the auction is lower for these weak bidders. Tan and Yilankaya show that such an equilibrium exists.

Miralles (2008) generalizes the concept of intuitive equilibrium to an arbitrary number of distribution types.

Lemma 1 (Miralles 2008). *If bidders can be ordered in a first order stochastic dominance ranking, such that*

$$F_1(v) \leq F_2(v) \leq \cdots \leq F_N(v), \quad \forall v \in [0, v^*],$$

then this game has at least one (generalized) intuitive cut-off equilibrium $\Theta^* = (\theta_1^*, \theta_2^*, \ldots, \theta_N^*)$ *in the sense that*

$$\theta_1^* \leq \theta_2^* \leq \cdots \leq \theta_N^*$$

If, additionally, there are sets of bidders $\{i, \ldots, i+k\}$ *such that* $F_i(v) = F_{i+1}(v) = \cdots = F_{i+k}(v)$, $\forall v \in [0, v^*]$, *then there is an intuitive cut-off equilibrium satisfying* $\theta_i^* = \theta_{i+1}^* = \cdots = \theta_{i+k}^*$ *for any of these sets.*

Proof See Miralles (2008). □

All the equilibria we depict hereafter are supported by intuitive cut-off equilibria. The predictive power of an intuitive equilibrium is arguably remarkable: it is unreasonable to expect a strong bidder to be less likely to participate than a weak bidder.

With these concepts in mind, the next sections are directly focused on the communication stage of the game.

3 The game: problems with three bidders

The game that is to be played (generically denoted as $\Gamma_{\text{“}M\text{”}}\{D, c, N, L\}$) works as follows. Let there be $L \geq 2$ objects to be sold and $N \geq 2$ risk-neutral (potential) bidders. The objects are to be sold by a weakly efficient auction mechanism. No entry fees nor reservation prices are allowed.[5] Each auction has the same entry cost $c \in (0, v^*)$ across objects and bidders. Valuations are i.i.d. over bidders and objects, each drawn from a distribution function D with full support over the range $[0, v^*]$ and associated density d. Hence, bidders are ex-ante symmetric. Pre-play (public) communication occurs once all bidders learn their own (but not others') valuations for all objects and before they take the participation decisions. Messages are sent simultaneously.

[5] This is for ease of calculations. It should be noted that the results found here are robust to entry fees and reservation prices, as long as they are equal among objects and bidders.

After communication, bidders update information, and each one decides in which auctions she participates. Participation decisions are also taken simultaneously. For each auction, each final bidder pays the entry cost, and the auction is undertaken.

A bidder's strategy is characterized, skipping subindices for the moment, as a triple function $(\mu, \beta, \tilde{\Theta})$. $\mu : [0, v^*]^L \rightarrow M$ is the message function, mapping from bidder's valuations to the message space. The nature of M defines the specific communication game that is played. $\beta : M^{N-1} \rightarrow \mathcal{F}^{(N-1)L}$ is the belief function, that maps from the received messages to the family of distribution functions in the support $[0, v^*]$, for each other bidder and for each object. $\tilde{\Theta} : \mathcal{F}^{(N-1)L} \rightarrow [c, v^*]^L$ is the cut-off vector function, that maps updated beliefs into own cut-off values.

For any (potential) bidder i, let us denote her interim expected global payoff as a function $\tilde{\Pi}_i(V_i, (\mu_j, \beta_j, \tilde{\Theta}_j)_{j\in\{1,\dots,N\}})$,[6] where V_i stands for bidder i's vector of valuations. A Bayesian equilibrium $(\mu_j^*, \beta_j^*, \tilde{\Theta}_j^*)_{j\in\{1,\dots,N\}}$ is then characterized by: (1) given $(V_j, \mu_j^*, \beta_j^*)_{j\in\{1,\dots,N\}}$, for each $l \in \{1, \dots, L\}$ the profile $\{\tilde{\Theta}_{jl}^*\}_{j\in\{1,\dots,N\}}$ constitutes a cut-off equilibrium given $[\beta_j^*[(\mu_k^*(V_k))_{k\neq j}]]_{j\in\{1,\dots,N\}}$; (2) for each $i \in \{1, \dots, N\}$, β_i^* correctly updates the distribution functions given the message strategies $(\mu_j^*)_{j\neq i}$; (3) for any $i \in \{1, \dots, N\}$, given $(\beta_j^*, \tilde{\Theta}_j^*)_{j\in\{1,\dots,N\}}$ and for each $V_i \in [0, v^*]^L$, $\mu_i^*(V_i)$ maximizes the interim expected global payoff $\tilde{\Pi}_i(V_i, (\mu_i, \beta_i^*, \tilde{\Theta}_i^*), (\mu_j^*, \beta_j^*, \tilde{\Theta}_j^*)_{j\neq i})$.[7]

In this section and the next, we deal with full ranking revelation games.

Definition 1 A *full ranking revelation game* $\Gamma_{FR}\{D, c, N, L\}$ with N potential bidders, L objects, initial distribution D and entry cost c is characterized by $M = \wp\{1, \dots, L\}$, where $\wp\{1, \dots, L\}$ is the family of all permutations on $\{1, \dots, L\}$. A bidder i's message in this game is denoted as a vector $(r_{i1}^L, \dots, r_{iL}^L) = r_i^L \in \wp\{1, \dots, L\}$.

A *truthful ranking revelation equilibrium* (or *full comparative cheap talk equilibrium*) in this game is a Bayesian Equilibrium in which message strategies satisfy the following property: $v_{ih} > v_{ik} \Rightarrow r_{ih}^L > r_{ik}^L$, $\forall i \in \{1, \dots, N\}$, $\forall h, k \in \{1, \dots, L\}$.

Proposition 4.2 in Campbell (1998) states that two risk-neutral bidders are able to obtain asymptotically full collusive efficiency in simultaneous, weakly efficient auctions with entry costs, via full ranking-revelation equilibrium with voluntary participation decisions (i.e. self-enforcement). In this equilibrium, each bidder reveals his complete ranking to the other, before the auctions take place. After updating beliefs, each bidder decides what auctions she enters. The proof of this proposition relies on a property of cut-off equilibria with two bidders, which he calls the *monotonicity property*.

This monotonicity property is described as follows. Following the model of the previous section, consider any cut-off equilibrium (call it the benchmark equilibrium). Switch one bidder's distribution function to another one that first order stochastically dominates her original distribution function. That is, this bidder becomes "stronger".

[6] The global payoff is the sum of all object-specific payoffs (denoted as $\tilde{\pi}(\cdot)$), since it is assumed that there are no synergies among objects.

[7] In further parts of the paper (see Proof of Proposition 3), the notation of this expression is simplified to $\tilde{\Pi}_i(V_i; \mu_i)$.

Then, there will exist a new cut-off equilibrium in which the other bidder's cut-off strategy is increased with respect to the benchmark.

As a consequence of this property, and with respect to the benchmark equilibrium, the strengthened bidder's cut-off strategy is reduced, her interim expected payoff increases whatever his valuation, and this difference of payoffs is in turn increasing in his valuation. The latter property is indeed defined as *supermodularity*. Let us select an object and a bidder (we skip subindices) and let us define $\tilde{\pi}(v; m)$ as the interim expected payoff for this object and this bidder, if her valuation is v and her message is $m \in \{1, \ldots, L\}$. A higher m indicates a higher position in the ranking of objects. Supermodularity is defined as: for $v' \geq v$ and $m' \geq m$, $\tilde{\pi}(v'; m') + \tilde{\pi}(v; m) \geq \tilde{\pi}(v'; m) + \tilde{\pi}(v; m')$.

This property and its consequences ensure that the incentive compatibility constraint holds. In a comparative cheap talk equilibrium, each bidder publicly announces a (probably partial) ordering of the objects. This ordering truthfully reflects ordinal preferences among objects. Given the messages, bidders correctly update their prior beliefs. Finally, cut-off equilibria are played in the auction stage of the game.

Lemma 2 *A comparative cheap talk equilibrium exists if and only if $\tilde{\pi}(v; m)$ is supermodular.*

Proof It directly follows from Chakraborty Harbaugh's (2007) proof of their Theorem 1. It is easy to see that the definition of supermodularity is equivalent to (interim) incentive compatibility for pairwise comparison across objects. And ranking revelation is equivalent to an array of pairwise comparisons. □

With more than two bidders, the monotonicity property is not guaranteed. Just three bidders are enough to complicate the extension of Campbell's findings.

To see this most easily, consider the following situation. Let bidder 1 become "stronger" in the sense that the distribution function for her valuation increases in the first order dominance sense. Consequently, bidders 2 and 3 switch their best response functions upwards. Let us assume that there is an equilibrium in which bidder 2's cut-off is increased with respect to the original equilibrium and bidder 1 lowers her own cut-off. Let us focus on what bidder 3 does. We can easily see that there is a trade-off. On the one hand, bidder 1's "strength" and her new cut-off induces bidder 3 to increase his cut-off with respect to the original situation. On the other hand, the fact that bidder 2 increases her cut-off induces bidder 3 to lower her cut-off, due to the fact that cut-off strategies are strategic substitutes. The result is uncertain, and the monotonicity property is not guaranteed, since it may be the case that bidder 3 lowers her cut-off with respect to the original equilibrium. Supermodularity of $\tilde{\pi}(v; m)$ is not ensured and therefore the incentive compatibility constraint might be violated for some types.

In this respect, there are two pieces of good news (for the bidders). First, numerical examples show that the monotonicity property holds in many cases. Second, the monotonicity property is just a sufficient, not necessary, condition for supermodularity.

The following example lists all possible intuitive cut-off equilibria when the prior D corresponds to the uniform distribution ($D(v) = v$, $v \in [0, 1]$), $N = 3$, and $L = 3$.

Example 1 Intuitive equilibria with uniform prior, with bidders i, j, k, for an arbitrary object $h \in \{1, 2, 3\}$ and with entry cost $c = 0.1$.

Case	r^L_{ih}	r^L_{jh}	r^L_{kh}	θ^*_{ih}	θ^*_{jh}	θ^*_{kh}
1	3	3	3	0.7197	0.7197	0.7197
2	3	3	2	0.5645	0.5645	0.9275
3	3	3	1	0.5624	0.5624	0.9281
4	3	2	2	0.1209	0.8141	0.8141
5	3	2	1	0.1127	0.7962	0.8178
6	3	1	1	0.1017	0.7968	0.7968
7	2	2	2	0.4731	0.4731	0.4731
8	2	2	1	0.3587	0.3587	0.6296
9	2	1	1	0.1249	0.5278	0.5278
10	1	1	1	0.2691	0.2691	0.2691

For each case, the intuitive equilibrium is unique in the class of equilibria depicted in Lemma 1. Campbell's monotonicity property is accomplished here. Each time a bidder increases the declared ranking, the other bidders become less likely to participate. Therefore, a full ranking revelation equilibrium exists.

Additional numerical examples have been analyzed. For the family of prior distributions $D(v) = v^a$ and a grid constituted by $c \in [0.05, 0.5]$ and $a \in [0.5, 2.5]$, intuitive equilibria have been calculated. The results are analogous to the example shown above: the monotonicity property holds in all cases. Hence, the existence of a complete comparative cheap-talk equilibrium does not appear to be an exceptional fact.

While it is not impossible to construct cases where that property is not present, doing so does not constitute an easy exercise. It generally requires the density at the equilibrium cut-off for some distribution type to be very high related to the cumulative distribution. Such a case is difficult to find if one assumes certain regularity conditions on the priors, usually affecting the hazard rate.

4 Asymptotic efficient collusion

In this section, we present the following result: with sufficiently many objects, complete rankings are truthfully revealed.[8] Consequently, bidders asymptotically obtain an ex post bidder-efficient allocation, in the sense that the bidder that has the highest valuation for some object always keeps it at zero price, provided this valuation is higher than the entry cost.

There is an intuitive explanation for the result we find. Asymptotically, revealing ranking positions is equivalent to revealing the valuations themselves. Therefore, this induces cut-off equilibria for each object in which only the bidder revealing the highest valuation for the object takes part in the auction, as long as this valuation is higher than c. Imagine that bidder i has to reveal valuations for two objects 1 and 2, where

8 A richer set of results in this respect can be found in Miralles (2005).

$v_1 > v_2$. For each object, she has to allocate different values that are either $\bar{v}$ or $\underline{v}$, where $\bar{v} > \underline{v}$, knowing that the other bidders will believe her. The higher the value she attaches to an object, the more likely she will win the object for free. Given that, she prefers to give the highest probability of winning to the object she really appreciates more, by a simple convex combination argument. Translating this argument to every possible pair of objects, it is seen that the bidder tends to reveal the truth.

Proposition 1 *Consider a game $\Gamma_{FR}\{F, c, N, L\}$, fixing N constant. In this context, $\exists \tilde{L}$ such that $\forall L \geq \tilde{L}$ there exists a truthful ranking revelation equilibrium supported by intuitive cut-off equilibria. Denote E_{NLO} as the event "valuations are such that with N potential bidders and $L \geq \tilde{L}$ objects, this truthful ranking revelation equilibrium is ex post efficient among bidders, for a fixed set O of objects", defined on the event-space $[0, v*]^{NL}$. Then the following is true:*

$$\forall \varepsilon > 0, \exists \bar{L} \geq \tilde{L} : \forall L \geq \bar{L}, \Pr(E_{LNO}) > 1 - \varepsilon$$

Proof See the Appendix. □

5 A partial comparative cheap talk equilibrium

In this section, we focus on weaker properties of weakly efficient auctions with entry costs. We will use them to construct partial comparative cheap talk equilibria.

We concentrate on the simplest comparative cheap talk messages, in which objects are split into two groups of respectively most- and least-preferred objects. We will show that such a partial comparative cheap talk can be sustained in equilibrium for any sizes of groups, as long as these sizes are equal among bidders.

In the game, we center our attention to the following partial ranking messages. Let $h \in \{1, \ldots, L-1\}$ be the number of elements in the set of most-preferred objects, so the set of least-preferred objects contains $L-h$ elements. Let this number be the same for each bidder. The message that bidder i sends consists of a set $H_i \subset \{1, \ldots, L\}$ containing h elements. We call such messages two-group messages, since such a message partitions the set of objects into two groups.

Definition 2 A *two-group revelation game with size h* $\Gamma_{TGh}\{D, c, N, L\}$ with N potential bidders, L objects, initial distribution D and entry cost c is characterized by $M = \{H \subset \{1, \ldots, L\} : \#H = h\}$.

A Bayesian equilibrium of this game is a *Two-group h-sized Comparative Cheap-talk Equilibrium* ($TCCE_h$) if message strategies in this equilibrium have, for any bidder i, H_i equal to the set of her h most-preferred objects.

We are to show that a truthful revelation of the partial ranking contained in such messages is supported as a part of a Bayesian equilibrium. Not only that, but we will also show that such an equilibrium is supported by intuitive cut-off equilibria.

But first, in order to get the main results, we need a variation of Campbell's monotonicity property that can be applied to an arbitrary number of bidders and just two (first order dominance ordered) distribution functions G and F (where G dominates F). This property suffices to ensure that the $\tilde{\pi}(v; m)$ function (recall the preceding sections) is supermodular.

We proceed to analyze what happens to any intuitive equilibrium when one bidder switches from one type to the other. We first define some notation. Let $\varphi_t(\theta; s)$ denote the (unique) symmetric best cut-off response that any bidder of distribution type $t \in \{G, F\}$ would play if the bidders of the other type played the cut-off strategy θ, in a game with s strong bidders and $N - s$ weak ones. That is, if we rearrange bidders properly,

$$\varphi_t(\theta; s) = \theta_t^{\#}(\underbrace{\varphi_t(\theta; s), \ldots, \varphi_t(\theta; s)}_{t\text{-type bidders}}, \underbrace{\theta, \ldots, \theta}_{\text{rest}})$$

Note that $\varphi_t(\theta; s)$ is continuous and strictly decreasing in θ (except for the corner case $\varphi_t(\theta; s) = v^*$). Finally, denote by $\pi_t(v; s)$ the expected payoff of any t-type bidder if her valuation is v, given an intuitive equilibrium in a game with s strong bidders.[9] The following proposition is our variation from Campbell's monotonicity property.

Proposition 2 Supermodularity for two distribution types. *Consider any intuitive equilibrium* $\Theta^*(s)$ *for a game with* $s \in \{0, \ldots, N-1\}$ *strong bidders.*[10] *Then any intuitive equilibrium* $\Theta^*(s+1)$ *for a game with* $s+1$ *strong bidders meets the following conditions:* (a) $\theta_G^*(s+1) \le \theta_F^*(s)$, (b) $\pi_G(v; s+1) \ge \pi_F(v; s)$ *for any* $v \in [0, v^*]$, *and* (c) $\pi_G(v; s+1) - \pi_F(v; s)$ *is increasing in* v *(i.e.* $\pi_\cdot(\cdot)$ *is supermodular in bidder's valuation and distribution type).*

Proof See the Appendix. □

Before passing to the main results, it is worth commenting on what this proposition states. Let there be s strong bidders and $N - s$ weak bidders (and any corresponding benchmark intuitive equilibrium), and suppose that one weak bidder changes to strong. In any new intuitive equilibrium and as compared to the benchmark equilibrium, this bidder will be more likely to participate. She will in turn obtain a higher interim expected payoff. Admittedly, it is not shown that every other bidder's participation becomes more unlikely (that is, the monotonicity property). However, the interim expected payoff is supermodular in the sense that its increase when a weak bidder becomes strong is nondecreasing in the bidder's valuation. And this is enough to guarantee the supermodularity of $\tilde{\pi}(\cdot) \equiv E_{(\text{others' types})}\pi_\cdot(\cdot)$.

We now present the main result of this section, i.e. the existence of a partial comparative cheap talk equilibrium for an arbitrary and finite number of bidders and objects. The reader may already have it clear in mind that the proof directly follows from Proposition 2 and Lemma 2.

Proposition 3 *For any* $h \in \{1, \ldots, L-1\}$, *there exists a Two-group h-sized Comparative Cheap-talk Equilibrium* ($TCCE_h$) *supported by intuitive cut-off equilibria.*

Proof See the Appendix. □

[9] We have abused notation again. Please recall that this is different from the $\tilde{\pi}(v; m)$ function we observed in previous sections.

[10] When $s = 0$ or $s = N$, an intuitive equilibrium is interpreted as the unique symmetric equilibrium.

6 Conclusion

Pesendorfer (2000)'s analysis of ranking revelation prior to simultaneous auctions is not complete because it assumes that bidders will obey the coordinator's allocation directions. In the absence of commitment, his assumption only holds for an infinitely repeated auction game. Campbell (1998) solves the enforcement problem by introducing entry costs in a set of weakly efficient auctions. Yet he only analyzes and presents his ranking revelation result for the two-bidder case.

Extending ranking revelation (complete comparative cheap chalk) to environments with arbitrary number of bidders is a cumbersome task. The entry-decision stage in an auction with entry costs is a game with strategic substitutes. With more than two bidders, this means that a bidder's object-specific interim expected payoff need not be supermodular in both valuation and ranking position. This supermodularity is indeed a necessary and sufficient condition for a truthful revelation of comparative information.

Despite this difficulty, we have been able to prove the following results. First, full comparative cheap talk equilibrium exists if the number of objects is large enough. Numerical examples suggest that the equilibrium may hold even for few objects. Second, a partial cheap talk equilibrium always exists, where each bidder publicly splits the objects into two sets: the set of favorite objects and the rest. All these equilibria are supported by intuitive cut-off equilibria (Tan and Yilankaya 2006; Miralles 2008) at the participation-decision stage.

In many contexts the equilibria here shown are very easy to understand (by the potential bidders) and very difficult to detect (by competition authorities). Learning information about others' ordinal preferences is good for the bidders (therefore bad for the sellers). It is remarkable that such simple, self-enforced collusion mechanisms enable each bidder to face less competition on average for the objects she prefers.

Appendix

Proof of Proposition 1. We can assume that objects are ordered in a sequence. Thus, we analyze a sequence of games, starting from a two-object game, then adding one object and playing it again, and so on. We then proceed through five steps. First, we state that ranking revelation induces an increasingly precise estimation of valuations. With this, a second step shows that for a chosen object this leads to a cut-off equilibrium where only one (if any) bidder participates, asymptotically. Third, we extend this result to all objects. Fourth, we show that this imply asymptotic supermodularity, proving the existence of the full ranking revelation equilibrium. Fifth and last, we prove asymptotic efficiency: the only bidder (if any) that participates in an auction is the one with the highest valuation for the object, and she pays the participation cost only.

(1) $\frac{r_{ih}^L}{L} \xrightarrow{\text{prob}} F(v_{ih})$, or equivalently, for any $\gamma, \eta > 0, \exists L_0^h(\gamma, \eta) : \forall L \geq L_0^h(\gamma, \eta)$,

$$\Pr\left(\left|F^{-1}\left(\frac{r_{ih}^L}{L}\right) - v_{ih}\right| > \gamma \text{ for some } i \in \{1, \ldots, L\}\right) < \eta$$

(2) Define the set $W_h^L \equiv \arg\max_{i\in\{1,\dots,N\}} \frac{r_{ih}^L}{L}$, and randomly choose some $i_h^L \in W_h^L$. We claim that for low enough values γ, η, if ranking values are revealed truthfully, for L big enough we have a cut-off equilibrium Θ_h^{*L} for object h (out of L) in which, with probability higher than $1-\eta$, only bidder i_h^L (if any, depending on whether or not $v_{i_h^L h} > c$) takes part in hth-object auction.

We prove this. Recall that $\frac{r_{i_h^L h}^L}{L} \geq \frac{r_{jh}^L}{L}, \forall j \neq i_h^L$. Let θ_{jh}^L denote bidder j's threshold strategy in the auction for object h (out of L objects). Consider cut-off strategies $\theta_{i_h^L h}^L \in \left[c, F^{-1}\left(\frac{r_{i_h^L h}^L}{L}\right)\right)$, a bidder $m \neq i_h^L$, and $\theta_{jh}^L = v^* \,\forall j \notin \{i_h^L, m\}$. Then, the expression

$$\int_0^{F^{-1}\left(\frac{r_{mh}^L}{L}\right)} \Pr\left(x > I\{v_{i_h^L h} > \theta_{i_h^L h}^L\} \cdot v_{i_h^L h} \;\Big|\; \left\{\frac{r_{ih}^L}{L}\right\}_{i\in\{1,\dots,N\}}\right) \mathrm{d}x - c$$

is negative for L big enough, given the result in part 1 of the proof, even if $\frac{r_{mh}^L}{L} = \frac{r_{i_h^L h}^L}{L}$ (due to the participation cost). Therefore, $\forall m \neq i_h^L$, and for L big enough, some $\theta_{mh}^L \geq F^{-1}\left(\frac{r_{mh}^L}{L}\right) + \alpha_{mh}^L$ is m's best response to some $\theta_{i_h^L h}^L \leq F^{-1}\left(\frac{r_{i_h^L h}^L}{L}\right) - \alpha_{i_h^L h}^L \geq c$, being $(\alpha_{1h}^L, \dots, \alpha_{Lh}^L) \gg 0$. This holds regardless what cutoffs all other j's$\notin \{m, i_h^L\}$ choose. Consequently, for some $\beta_{i_h^L h}^L \geq 0$ small enough, and for L big enough, bidder i_h^L's best response to $\left\{F^{-1}\left(\frac{r_{mh}^L}{L}\right) + \alpha_{mh}^L\right\}_{m\neq i_h^L}$ is equal to $c + \beta_{i_h^L h}^L \leq F^{-1}\left(\frac{r_{i_h^L h}^L}{L}\right) - \alpha_{i_h^L h}^L$ whenever $\theta_{mh}^L > F^{-1}\left(\frac{r_{lh}^L}{L}\right) \forall m \neq i_h^L$. Thus, the set

$$M_h^L \equiv \left[c,\ c + \beta_{i_h^L h}^L\right] \times \prod_{m\neq i_h} \left[\min\left\{F^{-1}\left(\frac{r_{mh}^L}{L}\right) + \alpha_{mh}, v^*\right\},\ v^*\right]$$

is closed under best responses. For L big enough, there exists at least one (intuitive) cut-off equilibrium $\Theta_h^{*L}(r_h^L)$ in this set M_h^L. Importantly, notice that $\beta_{i_h^L h}^L$ becomes arbitrarily close to zero as L increases, since the probability that the other bidders do not ever participate in that auction approaches 1 in M_h^L. Therefore, $\theta_{i_h^L h}^{*L}(r_h^L)$ asymptotically approaches c. Also, any other bidder m's equilibrium strategy in turn converges to $\min\left\{F^{-1}\left(\frac{r_{i_h^L h}^L}{L}\right) + c, v^*\right\}$. Such equilibrium results with probability higher than $1-\eta$ in an outcome where nobody but i_h (if any) takes part in the auction.

The proof of the claim above is completed by considering cases where $F^{-1}\left(\frac{r^L_{i^L_h h}}{L}\right) \leq c$. We use Lemma 1 to select an equilibrium $\Theta^{*L}_h(r^L_h)$ such that $c \leq \theta^{*L}_{i^L_h h}(r^L_h) \leq \theta^{*L}_{mh}(r^L_h)$, $\forall m \neq i^L_h$. In any case, it must happen $\theta^{*L}_{mh}(r^L_h) > c$, $\forall m \neq i^L_h$, since no two bidders can both have the same cut-off equilibrium c. It is clear that i^L_hth-bidder's equilibrium strategy is arbitrarily close to c , as L grows large, since $F^{-1}\left(\frac{r^L_{mh}}{L}\right) \leq c$ for $m \neq i^L_h$ and thus i^L_h expects other bidders to withdraw from the auction with probability close to 1. Consequently, as L grows large, the probability of the desired result becomes higher than $1 - \eta$.

Denote by $\Pr(A^{NL}_h \mid \{r^L_{ih}\}_{i\in\{1,\ldots,N\}})$ the probability that, in the cut-off equilibrium, no bidder $j \neq i^L_h$ takes part in the auction for object h out of L, with N potential bidders and the available information about ranking values. We have proved that there exists L^h_1 such that, $\forall L \geq L^h_1$, $\Pr(A^{NL}_h \mid \{r^L_{ih}\}_{i\in\{1,\ldots,N\}}) > 1 - \eta$.

(3) Set $L_1(L) \geq \max_{h\in\{1,\ldots,L\}} L^h_1$ (we eliminate the arguments of L^h_1 in order to avoid excessive notation). For L big enough (say $L \geq L_1$), we get $L_1(L) \leq L$. Thus, for any $L \geq L_1$, we have an equilibrium for each object h which satisfies the result of part (3) of the proof.

(4) Let, $r^L_{ih} > \hat{r}^L_{ih}$, and consider other bidders' (unknown) ranking positions $r^L_{-ih} \equiv \{r^L_{jh} : j \neq i\}$. Then, in the cut-off equilibria described in part (3), bidder i's interim expected profits regarding any object h are:

$$\begin{aligned}
\tilde{\pi}^L_{ih}(v_{ih}; r^L_{ih}) = {} & \Pr\left(i = i^L_h \mid r^L_{ih}\right) \cdot E_{r^L_{-ih}}\Big[\Pr(A^{LL}_h \mid r^L_{ih}, r^L_{-ih}) \\
& \cdot I\left\{v_{ih} > \theta^{*L}_{ih}(r^L_{ih}, r^L_{-ih})\right\} \cdot (v_{ih} - c) + \left(1 - \Pr(A^{LL}_h \mid r^L_{ih}, r^L_{-ih})\right) \cdot g^L_{ih}(\cdot)\Big] \\
& + \Pr\left(i \neq i^L_h \mid r^L_{ih}\right) \cdot E_{r^L_{-ih}}\left[\left(1 - \Pr(A^{LL}_h \mid r^L_{ih}, r^L_{-ih})\right) \cdot d^L_{ih}(\cdot)\right] \\
\tilde{\pi}^L_{ih}(v_{ih}; \hat{r}^L_{ih}) = {} & \Pr\left(i = i^L_h \mid \hat{r}^L_{ih}\right) \cdot E_{r^L_{-ih}}\Big[\Pr(A^{LL}_h \mid \hat{r}^L_{ih}, r^L_{-ih}) \\
& \cdot I\left\{v_{ih} > \theta^{*L}_{ih}(\hat{r}^L_{ih}, r^L_{-ih})\right\} \cdot (v_{ih} - c) + \left(1 - \Pr(A^{LL}_h \mid \hat{r}^L_{ih}, r^L_{-ih})\right) \cdot \hat{g}^L_{ih}(\cdot)\Big] \\
& + \Pr\left(i \neq i^L_h \mid \hat{r}^L_{ih}\right) \cdot E_{r^L_{-ih}}\left[\left(1 - \Pr(A^{LL}_h \mid \hat{r}^L_{ih}, r^L_{-ih})\right) \cdot \hat{d}^L_{ih}(\cdot)\right]
\end{aligned}$$

where $g, d, \hat{g}, \hat{d}$ are non-negative bounded functions of several arguments. As $L \to \infty$, $\tilde{\pi}^L_{ih}(v_{ih}; r^L_{ih}) - \tilde{\pi}^L_{ih}(v_{ih}; \hat{r}^L_{ih})$ converges to $[\Pr\left(i = i^L_h \mid r^L_{ih}\right) - \Pr\left(i = i^L_h \mid \hat{r}^L_{ih}\right)] \cdot I\{v_{ih} > c\} \cdot (v_{ih} - c)$, which is nondecreasing in v_{ih}. Therefore, there exists $\tilde{L} \geq L_1$ such that for any bidder i and any object h and for any $L \geq \tilde{L}$, $\tilde{\pi}^L_{ih}(\cdot; \cdot)$ is supermodular. This shows the existence of a full comparative cheap talk equilibrium for L large enough.

(5) We now prove efficiency. Concentrate on $L \geq \tilde{L}$. Fix $\hat{L}$ and a set of objects $O \subset \{1, \ldots, \hat{L}\}$. It is true that

$$\forall \varphi > 0, \exists L^O_2 \geq \#O : \forall L \geq L^O_2,\ \Pr\left(\#W^L_h = 1, \forall h \in O\right) > 1 - \varphi,$$

where # stands for the number of elements of a set. So we concentrate on the states where for each object in O only one bidder has the maximum ranking position among bidders. In these states, for $L \geq L_2^O$, we get $\frac{r^L_{i^L_h h}}{L} > \frac{r^L_{jh}}{L}, \forall j \neq i^L_h$, for any $h \in O$. Given that, it can be said that $\forall \delta > 0, \exists L_3^O \geq L_2^O : \forall L \geq L_3^O$,

$$\Pr\left(v_{i^L_h h} > v_{jh}, \forall j \neq i^L_h, \forall h \in O \;\middle|\; r^L, \#W^L_h = 1 \; \forall h \in O\right) > 1 - \delta$$

Therefore, for all object in O, the probability that each object taker is the person that gives to it the highest valuation exceeds $1 - \delta$. Then, with a probability higher than

$$1 - \varepsilon \equiv (1 - \eta)(1 - \varphi)(1 - \delta),$$

we obtain the efficiency result, for $L \geq \bar{L} \equiv \max\left\{L_3^O, \tilde{L}\right\}$, completing the proof. Observe that in the cut-off equilibria we are dealing with throughout the proof, the sellers of objects belonging to O finally face a unique buyer (if any) in each auction, and therefore they obtain zero profits, with a probability also higher than $1 - \varepsilon$. □

Proof of Proposition 2. If $N = 2$, the proof is in Campbell (1998), so we restrict attention to $N > 2$. Consider $s \in \{1, \ldots, N - 2\}$. Define μ_i^s as the unique value that solves $\varphi_i(\mu_i^s; s) = \mu_i^s$. Note that $\mu_i^s < v^*$ always and it exists. More importantly, notice that $\mu_G^{s+1} = \mu_F^s$, since they both solve the same equation:

$$G(\mu)^s F(\mu)^{N-s-1} \mu = c$$

Since $\varphi_i(\cdot; s)$ is decreasing, and since by the intuitive equilibrium concept $\theta_G^*(s + 1) \leq \theta_F^*(s + 1)$, we must have $\theta_G^*(s + 1) \leq \mu_G^{s+1}$. Following an analogous reasoning, we get $\theta_F^*(s) \geq \mu_F^s$.

Consider $s = 0$, so all bidders are weak and reach a symmetric equilibrium. Compared to $s = 1$, we also obtain the result of point (a). To see that, observe that since $\theta_G^*(1) \leq \theta_F^*(1)$, we must have $\theta_G^*(1) \leq \mu_G^1$. But once again $\mu_G^1 = \mu_F^0$, and $\mu_F^0 = \theta_F^*(0)$. Therefore we obtain the result. Finally, consider $s = N - 1$. An analogous reasoning will lead to the same conclusion when comparing it to the case $s = N$. So we have shown (a).

We now go to points (b) and (c). Interim expected profits follow the formulae[11]

$$\pi_G(v; s + 1) = I\{v > \theta_G^*(s + 1)\} \int_{\theta_G^*(s+1)}^{v} G(x)^s F(\max\{\theta_F^*(s + 1), x\})^{N-s-1} \mathrm{d}x$$

$$\pi_F(v; s) = I\{v > \theta_F^*(s)\} \int_{\theta_F^*(s)}^{v} G(x)^s F(x)^{N-s-1} \mathrm{d}x$$

[11] WLOG take $\theta_F^*(N) = v^*$.

The formulae follow from (1) the expression for expected payoff given participation, and (2) the cut-off equilibrium condition. Both are in Sect. 2. Since $\theta_G^*(s+1) \leq \theta_F^*(s)$, points (b) and (c) clearly hold. □

Proof of Proposition 3. For any believed message $\tilde{H}$, updated beliefs imply a distribution function (say G) for each object in $\tilde{H}$ and a function (say F) for each object outside $\tilde{H}$. It is clear that G first order stochastically dominates F.

Consider valuations $v_a \geq v_b$. Point (c) in Proposition 3 implies that

$$\pi_G(v_a; s+1) + \pi_F(v_b; s) \geq \pi_G(v_b; s+1) + \pi_F(v_a; s), \quad \forall s < N$$

Take some bidder k. Let H be the set of bidder k's h most preferred objects. Consider a set $H' \neq H$, where H' also has h elements. Take an object $b \in H' \backslash H$ and another object $a \in H \backslash H'$, to construct the set $H'' = [H' \backslash \{b\}] \cup \{a\}$. It is clear that $v_a \geq v_b$, where v_j denotes bidder k's valuation for object j. Denote the global interim expected profits of bidder k given his valuation vector V and his message $\tilde{H}$ by $\tilde{\Pi}(V; \tilde{H})$. These are equal to the sum of the interim expected payoffs from each object j, denoted as $\tilde{\pi}^j(v_j; \tilde{H})$. Then,

$$\begin{aligned}
\Delta\tilde{\Pi} &\equiv \tilde{\Pi}(V; H'') - \tilde{\Pi}(V; H') \\
&= \tilde{\pi}^a(v_a; H'') + \tilde{\pi}^b(v_b; H'') - [\tilde{\pi}^b(v_b; H') + \tilde{\pi}^a(v_a; H')] \\
&= E_{\hat{s}}\pi_G(v_a; \hat{s}+1) + E_{\hat{s}}\pi_F(v_b; \hat{s}) \\
&\quad -[E_{\hat{s}}\pi_G(v_b; \hat{s}+1) + E_{\hat{s}}\pi_F(v_a; \hat{s})] \\
&= E_{\hat{s}}[\pi_G(v_a; \hat{s}+1) + \pi_F(v_b; \hat{s}) \\
&\quad -[\pi_G(v_b; \hat{s}+1) + \pi_F(v_a; \hat{s})]] \\
&\geq 0
\end{aligned}$$

Here the operator $E_{\hat{s}}$ is the expectation over the random variable $\hat{s}$, the number of strong bidders in the auction apart from bidder k, given the fact that everyone else reveals his true list. We conclude that $H \in \arg\max_{\tilde{H} \subset \{1,\ldots,L\}, \#\tilde{H}=h} \tilde{\Pi}(V; \tilde{H})$. This proves the proposition. □

References

Albano, G.L., Romano, F., Lovo, S.: On some collusive and signaling equilibria in ascending auctions for multiple objects, The Fondazione Eni Enrico Mattei Note di Lavoro Series 62.2001 (2001)
Aoyagi, M.: Bid rotation and collusion in repeated auctions. J Econ Theory **112**, 79–105 (2003)
Armstrong, M.: Optimal multi-object auctions. Rev Econ Stud **67**, 455–481 (2000)
Blume, A., Heidhues, P.: Private monitoring in auctions. In: 13th WZB Conference on Markets and Political Economy: Collusion and Cartels (2004)
Brusco, S., Lopomo, G.: Collusion via signaling in simultaneous ascending bid auctions with heterogeneous objects, with and without complementarities. Rev Econ Stud **69**, 407–436 (2002)
Campbell, C.M.: Coordination in auctions with entry. J Econ Theory **82**, 425–450 (1998)
Celik, G., Yilankaya, O.: Optimal Auctions with Participation Costs, mimeo. Vancouver, BC: University of British Columbia (2005)
Chakraborty, A., Harbaugh, R.: Comparative cheap talk. J Econ Theory **132**, 70–94 (2007)

Engelbrecht-Wiggans, R., Kahn, C.: Low revenue equilibria in simultaneous ascending price auctions. Manage Sci **51**, 508–518 (2005)
Grimm, V., Riedel, F., Wolfstetter, E.: Low price equilibrium in multi-unit auctions: the GSM spectrum auction in Germany. Int J Ind Organ **21**, 1557–1569 (2003)
Levin, D., Smith, J.L.: Equilibrium in auctions with entry. Am Econ Rev **84**, 585–599 (1994)
Lu, J.: Auction design with opportunity cost. Econ Theory, forthcoming (2008a)
Lu, J.: Why a simple second-price auction induces efficient endogenous entry. Theory Decis, forthcoming (2008b)
McAfee, R.P., McMillan, J.: Bidding rings. Am Econ Rev **82**, 579–599 (1992)
Menezes, F.M., Monteiro, P.K.: Auctions with endogenous participation. Rev Econ Des **5**, 71–89 (2000)
Miralles, A.: Easy Collusion Mechanisms in Auctions with Entry, Mimeo. Boston, MA: Boston University (2005)
Miralles, A.: Intuitive and noncompetitive equilibria in weakly efficient auctions with entry costs, Math Soc Sci, forthcoming (2008)
Pesendorfer, M.: A study of collusion in first-price auctions. Rev Econ Stud **67**, 381–411 (2000)
Skrzypacz, A., Hopenhayn, H.: Tacit collusion in repeated auctions. J Econ Theory **114**, 153–169 (2004)
Stegeman, M.: Participation costs and efficient auctions. J Econ Theory **71**, 228–259 (1996)
Tan, G., Yilankaya, O.: Equilibria in second price auctions with participation costs. J Econ Theory **130**, 205–219 (2006)