DUALITY FOR DISTRIBUTIVE AND IMPLICATIVE SEMI-LATTICES

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Abstract. We develop a new duality for distributive and implicative meet semi-lattices. For distributive meet semi-lattices our duality generalizes Priestley’s duality for distributive lattices and provides an improvement of Celani’s duality. Our generalized Priestley spaces are similar to the ones constructed by Hansoul. Thus, one can view our duality for distributive meet semi-lattices as a completion of Hansoul’s work. For implicative meet semi-lattices our duality generalizes Esakia’s duality for Heyting algebras and provides an improvement of Vrancken-Mawet’s and Celani’s dualities. In the finite case it also yields Köhler’s duality. Thus, one can view our duality for implicative meet semi-lattices as a completion of Köhler’s work. As a consequence, we also obtain a new duality for Heyting algebras, which is an alternative to the Esakia duality.

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1. INTRODUCTION

Topological representation of distributive semi-lattices goes back to Stone’s pioneering work [15]. For distributive join semi-lattices with bottom it was worked out in detail in Grätzer [6, Section II.5, Theorem 8]. A full duality between meet semi-lattices with top (which are dual to join semi-lattices with bottom) and certain ordered topological spaces was developed by Celani [2]. The main novelty of [2] was the characterization of meet semi-lattice homomorphisms preserving top by means of certain binary relations. But the ordered topological spaces of [2] are rather difficult to work with, which is the main drawback of the paper. On the other hand, Hansoul [7] developed rather nice order topological duals of bounded join semi-lattices, but had no dual analogue of bounded join semi-lattice homomorphisms. It is our intention to develop duality for semi-lattices that improves both [2] and [7]. Like in [2], we work with meet semi-lattices which are dual to join semi-lattices. We generalize the notion of a Priestley space to that of a generalized Priestley space and develop duality between the category of bounded distributive meet semi-lattices and meet semi-lattice homomorphisms and the category of generalized Priestley spaces and generalized Priestley morphisms. In the particular case of bounded distributive lattices, our duality yields the well-known Priestley duality [13, 14].

The first duality for finite implicative meet semi-lattices was given by Köhler [9]. It was extended to the infinite case by Vrancken-Mawet [16] and Celani [1].\(^1\) The Vrancken-Mawet and Celani dualities are in terms of spectral-like ordered topological spaces. Both topologies are not Hausdorff, thus rather difficult to work with. We develop a new duality for bounded implicative meet

\(^1\)Celani’s paper contains a gap, which we correct at the end of Section 11.3.
semi-lattices that improves both Vrancken-Mawet’s and Celani’s dualities. It is obtained as a particular case of our duality for bounded distributive meet semi-lattices. We generalize the notion of an Esakia space to that of a generalized Esakia space and develop duality between the category of bounded implicative meet semi-lattices and implicative meet semi-lattice homomorphisms and the category of generalized Esakia spaces and generalized Esakia morphisms. In the particular case of Heyting algebras, our duality yields the well-known Esakia duality [3, 4]. Moreover, it provides a new duality for Heyting algebras, which is an alternative to the Esakia duality.

The paper is organized as follows. In Section 2 we provide all the needed information to make the paper self-contained. In particular, we recall basic facts about meet semi-lattices, distributive meet semi-lattices, and implicative meet semi-lattices, as well as the basics of Priestley’s duality for bounded distributive lattices and Esakia’s duality for Heyting algebras. In Section 3 we consider filters and ideals, as well as prime filters and prime ideals of a distributive meet semi-lattice and develop their theory. In Section 4 we introduce some of the main ingredients of our duality such as distributive envelops, Frink ideals, and optimal filters and give a detailed account of their main properties. The introduction of optimal filters is one of the main novelties of the paper. There are more optimal filters than prime filters of a bounded distributive meet semi-lattice $L$, and it is optimal filters and not prime filters that serve as points of the dual space of $L$, which allows us to prove that the Priestley-like topology of the dual of $L$ is compact, thus providing improvement of the previous work [6, 9, 16, 2, 1], in which the dual of $L$ was constructed by means of prime filters of $L$. In Section 5 we introduce a new class of homomorphisms between distributive meet semi-lattices, we call sup-homomorphisms, and provide an abstract characterization of distributive envelopes by means of sup-homomorphisms. In Section 6 we introduce generalized Priestley spaces, prove their main properties, and provide a representation theorem for bounded distributive meet semi-lattices by means of generalized Priestley spaces. In Section 7 we introduce generalized Esakia spaces and provide a representation theorem for bounded implicative meet semi-lattices by means of generalized Esakia spaces. In Section 8 we introduce generalized Priestley morphisms and show that the category of bounded distributive meet semi-lattices and meet semi-lattice homomorphisms is dually equivalent to the category of generalized Priestley spaces and generalized Priestley morphisms. We also introduce generalized Esakia morphisms and show that the category of bounded implicative meet semi-lattices and implicative meet semi-lattice homomorphisms is dually equivalent to the category of generalized Esakia spaces and generalized Esakia morphisms. In Section 9 we show that the subclasses of generalized Priestley morphisms which dually correspond to sup-homomorphisms can be characterized by means of special functions between generalized Priestley spaces, we
call strong Priestley morphisms, and show that the same also holds in the category of generalized Esakia spaces. In Section 10 we prove that in the particular case of bounded distributive lattices our duality yields the well-known Priestley duality, and that in the particular case of Heyting algebras it yields the Esakia duality. We also give an application to modal logic by showing that descriptive frames, which are duals of modal algebras, can be thought of as generalized Priestley morphisms of Stone spaces into themselves. Moreover, we introduce partial Esakia functions between Esakia spaces and show that generalized Esakia morphisms between Esakia spaces can be characterized in terms of partial Esakia functions. Furthermore, we show that Esakia morphisms can be characterized by means of special partial Esakia functions, we call partial Heyting functions, thus yielding a new duality for Heyting algebras, which is an alternative to the Esakia duality. We conclude the section by showing that partial functions are not sufficient for characterizing neither meet semi-lattice homomorphisms between bounded distributive lattices nor implicative meet semi-lattice homomorphisms between bounded implicative meet semi-lattices.

In Section 11 we show how our duality works by giving dual descriptions of Frink ideals, ideals, and filters, as well as 1-1 and onto homomorphisms. In Section 12 we show how to adjust our technique to handle the non-bounded case. Finally, in Section 13 we give a detailed comparison of our work with the relevant work by Grätzer [6], Köhler [9], Vrancken-Mawet [16], Celani [2, 1], and Hansoul [7], and correct a mistake in [1].

2. Preliminaries

In this section we recall basic facts about meet semi-lattices, distributive meet semi-lattices, and implicative meet semi-lattices. We also recall the basics of Priestley’s duality for bounded distributive lattices and Esakia’s duality for Heyting algebras.

We start by recalling that a meet semi-lattice is a commutative idempotent semigroup \( \langle S, \cdot \rangle \). For a given meet semi-lattice \( \langle S, \cdot \rangle \), we denote \( \cdot \) by \( \wedge \), and define a partial order \( \leq \) on \( S \) by \( a \leq b \) iff \( a = a \wedge b \). Then \( a \wedge b \) becomes the greatest lower bound of \( \{a, b\} \) and \( \langle S, \wedge \rangle \) can be characterized as a partially ordered set \( \langle S, \leq \rangle \) such that every nonempty finite subset of \( S \) has a greatest lower bound. Below we will be interested in meet semi-lattices with the greatest element \( \top \), i.e., in commutative idempotent monoids \( \langle M, \wedge, \top \rangle \). Let \( M \) denote the category of meet semi-lattices with \( \top \) and meet semi-lattice homomorphisms preserving \( \top \); that is \( M \) is the category of commutative idempotent monoids and monoid homomorphisms.

It is obvious that meet semi-lattices serve as a natural generalization of lattices: if \( \langle L, \wedge, \vee \rangle \) is a lattice, then \( \langle L, \wedge \rangle \) is a meet semi-lattice. Distributive meet semi-lattices serve as a natural generalization of distributive lattices. We recall that a meet semi-lattice \( L \) is distributive if for each \( a, b_1, b_2 \in L \) with
Set a \; \text{then} \; \text{Suppose that}

As follows from [6, Section II.5, Lemma 1], a lattice \langle L, \wedge, \vee \rangle is distributive iff the meet semi-lattice \langle L, \wedge \rangle is distributive. Let \text{DM} denote the category of distributive meet semi-lattices with \top and meet semi-lattice homomorphisms preserving \top. Obviously \text{DM} \subset \text{M}. However, unlike lattices, there is no meet semi-lattice identity expressing the difference, since the variety \text{M} is generated by the class \text{DM}. In fact, we have that the two element meet semi-lattice \text{2} generates \text{M} [8]. We list some of the interesting properties of the variety \text{M}. Since they are not directly related to our main purpose, we don’t provide any proofs. Some of the proofs can be found in [8, 6], the rest we leave as interesting exercises. The variety \text{M} is semi-simple (with \text{2} being a unique up to isomorphism nontrivial simple meet semi-lattice), finitely generated, has no proper non-trivial subvarieties, has the amalgamation property, and each epimorphism in \text{M} is surjective. On the other hand, \text{M} is neither congruence-distributive nor has the congruence extension property.

We recall that \text{M} \in \text{M} is an \textit{implicative meet semi-lattice} if for each \( a \in M \) the order-preserving map \( a \wedge (-) : M \to M \) has a right adjoint, denoted by \( a \to (-) \). If in addition \( M \) is a bounded lattice, then \( M \) is called a \textit{Heyting algebra}. The same way meet semi-lattices and distributive meet semi-lattices serve as a generalization of lattices and distributive lattices, respectively, implicative meet semi-lattices serve as a generalization of Heyting algebras: the \( (\wedge, \to, \top) \)-reduct of a Heyting algebra is an implicative meet semi-lattice.

For two implicative meet semi-lattices \( L \) and \( K \), we recall that a map \( h : L \to K \) is an \textit{implicative meet semi-lattice homomorphism} if \( h \) is a meet semi-lattice homomorphism and \( h(a \to b) = h(a) \to h(b) \) for each \( a, b \in L \). Since in an implicative meet semi-lattice we have that \( a \to a = \top \), it follows that each implicative meet semi-lattice homomorphism preserves \( \top \). Let \text{IM} denote the category of implicative meet semi-lattices and implicative meet semi-lattice homomorphisms.

We observe that the same way the lattice reduct of a Heyting algebra is distributive, the meet semi-lattice reduct of an implicative meet semi-lattice is distributive. This result belongs to folklore. We give a short proof of it for the lack of proper reference.

\textbf{Proposition 2.1.} The \( \wedge \)-reduct of an implicative meet semi-lattice is a distributive meet semi-lattice.

\textit{Proof.} Suppose that \( L \) is an implicative meet semi-lattice and \( a, b_1, b_2 \in L \) with \( b_1 \land b_2 \leq a \). Then \( b_1 \leq b_2 \to a \) and \( b_2 \leq b_1 \to a \). Set \( c = (b_1 \to a) \land (b_2 \to a) \).

Then \( a \leq c \), \( c \leq b_1 \to a \), and \( c \leq b_2 \to a \). Hence, \( b_1 \leq c \to a \) and \( b_2 \leq c \to a \).

Set \( c_1 = (c \to a) \land (b_2 \to a) \) and \( c_2 = (c \to a) \land (b_1 \to a) \). Since \( b_1 \leq c \to a \) and \( b_1 \leq b_2 \to a \), we have \( b_1 \leq c_1 \). Similarly \( b_2 \leq c_2 \).

Moreover, \( c_1 \land c_2 = (c \to a) \land (b_1 \to a) \land (b_2 \to a) = ((b_1 \to a) \land (b_2 \to a)) \land ((b_1 \to a) \land (b_2 \to a)) \to a = a \).
Therefore, there exist $c_1, c_2 \in L$ such that $b_1 \leq c_1$, $b_2 \leq c_2$, and $a = c_1 \wedge c_2$. Thus, $L$ is a distributive meet semi-lattice.

It follows from Proposition 2.1 that $\text{IM}$ is a subcategory of $\text{DM}$. Nevertheless, as a variety $\text{IM}$ behaves differently from $\text{DM}$. We list some of the properties of $\text{IM}$ without providing any proofs. Some of the proofs can be found in [12, 10, 11, 9], the rest we leave as interesting exercises. First of all, the lattice $F(L)$ of filters of an implicative meet-semi-lattice $L$ is isomorphic to the lattice $\Theta(L)$ of congruences of $L$. As a result, an implicative meet-semi-lattice $L$ is subdirectly irreducible iff the set $L - \{\top\}$ has a greatest element. Thus, $\text{IM}$ is neither semi-simple nor finitely generated. In fact, $\text{IM}$ is not even residually small, and the cardinality of the lattice of subvarieties of $\text{IM}$ is that of the continuum. On the other hand, $\text{IM}$ is locally finite and has equationally definable principal congruences. Hence, $\text{IM}$ is congruence-distributive and has the congruence extension property. It also has the amalgamation property and each epimorphism in $\text{IM}$ is surjective.

We conclude this section with a brief overview of Priestley’s duality for bounded distributive lattices and Esakia’s duality for Heyting algebras. For a partially ordered set $\langle X, \leq \rangle$ and $A \subseteq X$ let $\uparrow A = \{x \in X : \exists a \in A \text{ with } a \leq x\}$ and $\downarrow A = \{x \in X : \exists a \in A \text{ with } x \leq a\}$. If $A$ is the singleton $\{a\}$, then we write $\downarrow a$ and $\uparrow a$ instead of $\uparrow\{a\}$ and $\down\{a\}$, respectively. We call $A$ an upset (resp. downset) if $A = \uparrow A$ (resp. $A = \downarrow A$).

We recall that a Priestley space is an ordered topological space $X = \langle X, \tau, \leq \rangle$ which is compact and satisfies the Priestley separation axiom: if $x \not\leq y$, then there is a clopen (closed and open) upset $U$ of $X$ such that $x \in U$ and $y \not\in U$. It follows from the Priestley separation axiom that $X$ is in fact Hausdorff and that clopen sets form a basis for the topology. Thus, each Priestley space is a Stone space (compact Hausdorff 0-dimensional).

For two Priestley spaces $X$ and $Y$, a morphism $f : X \to Y$ is a Priestley morphism if $f$ is continuous and order-preserving. We denote the category of Priestley spaces and Priestley morphisms by $\text{PS}$. Let also $\text{BDL}$ denote the category of bounded distributive lattices and bounded lattice homomorphisms. Then we have that $\text{BDL}$ is dually equivalent to $\text{PS}$ [13]. We recall that the functors $(-)_* : \text{DL} \to \text{PS}$ and $(-)^* : \text{PS} \to \text{DL}$ establishing the dual equivalence are constructed as follows. If $L$ is a bounded distributive lattice, then $L_* = \langle X, \tau, \leq \rangle$, where $X$ is the set of prime filters of $L$, $\leq$ is set-theoretic inclusion, and $\tau$ is the topology generated by the subbasis $\{\varphi(a) : a \in L\} \cup \{\varphi(b)^c : b \in L\}$, where $\varphi(a) = \{x \in X : a \in x\}$ is the Stone map. If $h \in \text{hom}(L, K)$, then $h_* = h^{-1}$. If $X$ is a Priestley space, then $X^*$ is the lattice of clopen upsets of $X$, and if $f \in \text{hom}(X, Y)$, then $f^* = f^{-1}$. It follows from [13, 14] that the functors $(-)_*$ and $(-)^*$ are well-defined, and that they establish the dual equivalence of $\text{BDL}$ and $\text{PS}$.
Esakia’s duality is a restricted Priestley duality. We recall that an Esakia space is a Priestley space $X = (X, \tau, \leq)$ in which the downset of each clopen is again clopen. We also recall that an Esakia morphism from an Esakia space $X$ to an Esakia space $Y$ is a Priestley morphism $f$ such that for all $x \in X$ and $y \in Y$, from $f(x) \leq y$ it follows that there is $z \in X$ with $x \leq z$ and $f(z) = y$. We denote the category of Esakia spaces and Esakia morphisms by $\text{ES}$. Let also $\text{HA}$ denote the category of Heyting algebras and Heyting algebra homomorphisms. Then it was established in Esakia [3] that $\text{HA}$ is dually equivalent to $\text{ES}$. In fact, the same functors $(-)^*$ and $(-)^\ast$, restricted to $\text{HA}$ and $\text{ES}$, respectively, establish the desired equivalence.

3. Filters and ideals of meet semi-lattices

In this section we adapt the notions of a filter and an ideal of a lattice to meet semi-lattices. We also introduce the notion of a meet-prime filter of a distributive meet semi-lattice, which is an analogue of the notion of a prime filter of a distributive lattice, and present an analogue of the prime filter lemma for distributive meet semi-lattices. Most of these results are well-known. We present them here to keep the presentation as self-contained as possible.

Let $L$ be a meet semi-lattice. We recall that a nonempty subset $F$ of $L$ is a filter if (i) $a, b \in F$ implies $a \land b \in F$ and (ii) $a \in F$ and $a \leq b$ imply $b \in F$. Clearly $F$ is a filter of $L$ iff for each $a, b \in L$ we have $a, b \in F$ iff $a \land b \in F$. Similar to lattices, we have that $L$ is a filter of $L$, and if $\top \in L$, then an arbitrary intersection of filters of $L$ is again a filter of $L$. Therefore, for each $X \subseteq L$, there exists a least filter containing $X$, which we call the filter generated by $X$ and denote by $[X]$. It is obvious that

$$a \in [X] \text{ iff there exists a finite } Y \subseteq X \text{ such that } \bigwedge Y \leq a.$$ 

In particular, the filter generated by $x \in L$ is the upset $\uparrow x = \{ y \in L : x \leq y \}$. We also point out that if $X = \emptyset$, then $\bigwedge X = \top$, and so $[X] = \{ \top \}$.

Let $\mathcal{F}(L)$ denote the set of filters of $L$. Obviously the structure $(\mathcal{F}(L), \cap, \lor)$ forms a lattice, where $F_1 \lor F_2 = [F_1 \cup F_2]$, and $a \in F_1 \lor F_2$ iff there exist $a_1 \in F_1$ and $a_2 \in F_2$ such that $a_1 \land a_2 \leq a$. In particular, $\uparrow a \lor \uparrow b = \uparrow(a \land b)$. The following characterization of a distributive meet semi-lattice $L$ in terms of the filter lattice of $L$ follows from [6, Section II.5, Lemma 1]: A meet semi-lattice $L$ is distributive iff the lattice $(\mathcal{F}(L), \cap, \lor)$ is distributive. We call a filter $F$ of $L$ proper if $F \neq L$.

**Definition 3.1.** A proper filter $F$ of a meet semi-lattice $L$ is said to be meet-prime if for any two filters $F_1, F_2$ of $L$ with $F_1 \cap F_2 \subseteq F$, we have $F_1 \subseteq F$ or $F_2 \subseteq F$. 
In [2] meet-prime filters are called weakly irreducible filters. Meet-prime filters serve as an obvious generalization of prime filters of a lattice as the following proposition shows.

**Proposition 3.2.** A filter $F$ of a lattice $L$ is prime iff $F$ is meet-prime.

**Proof.** Suppose that $F$ is prime and $F_1 \cap F_2 \subseteq F$. If neither $F_1 \subseteq F$ nor $F_2 \subseteq F$, then there exist $a_1 \in F_1$ and $a_2 \in F_2$ with $a_1, a_2 \notin F$. From $a_1 \in F_1$ and $a_2 \in F_2$ it follows that $a_1 \vee a_2 \in F_1 \cap F_2 \subseteq F$. Since $F$ is prime, either $a_1 \in F$ or $a_2 \in F$, a contradiction. Thus, $F_1 \subseteq F$ or $F_2 \subseteq F$. Conversely, suppose that $F$ is meet-prime and $a \vee b \in F$. Since $\uparrow (a \vee b) = \uparrow a \cap \uparrow b$, we obtain $\uparrow a \cap \uparrow b \subseteq F$. As $F$ is meet-prime, the last inclusion implies $\uparrow a \subseteq F$ or $\uparrow b \subseteq F$. Thus, $a \in F$ or $b \in F$, and so $F$ is prime. \qed

From now on we will call meet-prime filters of a meet semi-lattice simply prime. Since a meet semi-lattice $L$ may not be a lattice and so the join of two elements of $L$ may not exist, the notion of an ideal of $L$ needs to be adjusted appropriately. For a subset $A$ of a meet semi-lattice $L$, let $A^u = \{ u \in L : \forall a \in A \text{ we have } a \leq u \}$ denote the set of upper bounds of $A$, and let $A^l = \{ l \in L : \forall a \in A \text{ we have } l \leq a \}$ denote the set of lower bounds of $A$. We call a nonempty subset $I$ of a meet semi-lattice $L$ an ideal if (i) $a \in I$ and $b \leq a$ imply $b \in I$ and (ii) $a, b \in I$ implies $(\uparrow a \cap \uparrow b) \cap I \neq \emptyset$. Our notion of ideal is dual to the notion of dual ideal of [6, p. 132]. In [2] ideals are called order ideals.

**Remark 3.3.** Condition (ii) of the definition of ideal is obviously equivalent to the following condition: (ii') $a, b \in I$ implies $\uparrow (a \cap b) \cap I \neq \emptyset$. Thus, we have that the following three conditions are equivalent:

1. $I$ is an ideal of $L$.
2. For each $a, b \in L$ we have $a, b \in I$ iff $\{ a, b \}^u \cap I \neq \emptyset$.
3. For each $a, b \in L$ we have $a, b \in I$ iff $(\uparrow a \cap \uparrow b) \cap I \neq \emptyset$.

We note that if $L$ has a top element, then $L$ itself is always an ideal. However, unlike the case with filters, a nonempty intersection of a family of ideals may not be an ideal as the following example shows.

**Example 3.4.** Let $L$ be the meet semi-lattice shown in Fig.1. Then each $\downarrow c_n$ is an ideal of $L$, but $\bigcap_{n \in \omega} \downarrow c_n = \{ \bot, a, b \}$ is not an ideal of $L$. 
Nevertheless, we have the following analogue of the prime filter lemma for
distributive meet semi-lattices. For a proof we refer to [6, Section II.5, Lemma
2] or [2, Theorem 8].

**Lemma 3.5** (Prime Filter Lemma). Suppose that $L$ is a distributive meet
semi-lattice. If $F$ is a filter and $I$ is an ideal of $L$ with $F \cap I = \emptyset$, then there
exists a prime filter $P$ of $L$ such that $F \subseteq P$ and $P \cap I = \emptyset$.

As a corollary, we obtain the following useful fact.

**Corollary 3.6.** Every proper filter $F$ of a distributive meet semi-lattice
$L$ is the intersection of the prime filters of $L$ containing $F$.

We call an ideal $I$ of a meet semi-lattice $L$ proper if $I \neq L$, and a proper
ideal $I$ of $L$ prime if for each $a, b \in L$ we have $a \land b \in I$ implies $a \in I$ or $b \in I$.
Then we have the following analogue of a well-known theorem for lattices.

**Proposition 3.7.** A subset $F$ of a meet semi-lattice $L$ is a prime filter iff
$I = L - F$ is a prime ideal.

**Proof.** Suppose that $F$ is a prime filter of $L$. Clearly $F \neq \emptyset, L$ implies $I = L - F \neq \emptyset, L$. Moreover, it is obvious that condition (i) of the definition of
filter implies that $I$ satisfies condition (i) of the definition of ideal. To show
that $I$ also satisfies condition (ii), suppose that $a, b \in I$. If $\uparrow a \cap \uparrow b \cap I = \emptyset$, then $\uparrow a \cap \uparrow b \subseteq F$. Since $F$ is prime, $\uparrow a \subseteq F$ or $\uparrow b \subseteq F$, so either $a \notin I$ or $b \notin I$, a contradiction. Thus, $(\uparrow a \cap \uparrow b) \cap I \neq \emptyset$, and so $I$ is an ideal. Finally,
to show that $I$ is prime, suppose that $a \land b \in I$. Then $a \land b \notin F$. By condition
(ii) of the definition of filter, either $a \notin F$ or $b \notin F$. Thus, $a \in I$ or $b \in I$.

Conversely, suppose that $I = L - F$ is a prime ideal. Clearly $I \neq \emptyset, L$ implies
$F \neq \emptyset, L$. Moreover, it is obvious that condition (i) of the definition of ideal
implies that $F$ satisfies condition (i) of the definition of filter. To show that
also satisfies condition (ii), suppose that \( a, b \in F \). Then \( a, b \notin I \). Since \( I \) is prime, we have \( a \land b \notin I \). Thus, \( a \land b \in F \), and so \( F \) is a filter. Finally, to show that \( F \) is prime, suppose that \( F_1 \cap F_2 \subseteq F \). If \( F_1 \not\subseteq F \) and \( F_2 \not\subseteq F \), then \( F_1 \cap I \neq \emptyset \) and \( F_2 \cap I \neq \emptyset \). Therefore, there exist \( a_1 \in F_1 \cap I \) and \( a_2 \in F_2 \cap I \). Obviously \( \uparrow a_1 \cap \uparrow a_2 \subseteq F_1 \cap F_2 \subseteq F \). On the other hand, since \( I \) is an ideal, \( (\uparrow a_1 \cap \uparrow a_2) \cap I \neq \emptyset \). This implies \( F \cap I \neq \emptyset \), a contradiction. Thus, either \( F_1 \subseteq F \) or \( F_2 \subseteq F \), and so \( F \) is prime. \( \square \)

4. Distributive envelopes, Frink ideals, and optimal filters

In this section we introduce three new concepts which play a fundamental role in developing our duality for distributive meet semi-lattices. The first concept we introduce is that of the distributive envelope \( D(L) \) of a distributive meet semi-lattice \( L \). We analyze how the filters and ideals of \( L \) are related to the filters and ideals of \( D(L) \). In general, it is not the case that to each ideal of \( D(L) \) there corresponds an ideal of \( L \). This paves a way for our second concept, that of Frink ideal, which is broader than our earlier concept of ideal. We establish the main properties of Frink ideals, and prove that the ordered set of Frink ideals of \( L \) is isomorphic to the ordered set of ideals of \( D(L) \).

Frink ideals give rise to the third concept, that of optimal filters, which are set-theoretic complements of prime Frink ideals. We prove an analogue of the prime filter lemma for optimal filters and Frink ideals, which we call the optimal filter lemma, and show that the ordered set of optimal filters of \( L \) is isomorphic to the ordered set of prime filters of \( D(L) \). At the end of the section, we give a table of relations between different notions of filters and ideals of \( L \) and \( D(L) \), and conclude the section by giving an alternative construction of \( D(L) \), which will play a crucial role in developing our duality.

4.1. Distributive envelopes. Let \( L \) be a meet semi-lattice and let \( \Pr(L) \) denote the set of prime filters of \( L \). We define \( \sigma : L \to \mathcal{P}(\Pr(L)) \) by \( \sigma(a) = \{ x \in \Pr(L) : a \in x \} \) for each \( a \in L \). The next theorem goes back to Stone [15].

**Theorem 4.1.** If \( L \) is a meet semi-lattice, then \( \sigma : L \to \mathcal{P}(\Pr(L)) \) is a meet semi-lattice homomorphism. If \( L \) has top, then \( \sigma \) preserves top, and if \( L \) has bottom, then \( \sigma \) preserves bottom. In addition, if \( L \) is distributive, then \( \sigma \) is a meet semi-lattice embedding.

**Proof.** For a prime filter \( P \) of \( L \) we have:

\[
P \in \sigma(a \land b) \text{ iff } a \land b \in P
\]
\[
\text{iff } a, b \in P
\]
\[
\text{iff } P \in \sigma(a) \cap \sigma(b)
\]

Thus, \( \sigma \) is a meet semi-lattice homomorphism. Suppose that \( \top \in L \). Since each prime filter of \( L \) contains \( \top \), we have \( \sigma(\top) = \Pr(L) \). Therefore, \( \sigma \) preserves \( \top \).
Let \( \bot \in L \). As each prime filter of \( L \) does not contain \( \bot \), we have \( \sigma(\bot) = \emptyset \). Thus, \( \sigma \) preserves \( \bot \). Because \( \sigma \) is a meet semi-lattice homomorphism, we have that \( a \leq b \) implies \( \sigma(a) \subseteq \sigma(b) \). Suppose that \( L \) is a distributive meet semi-lattice and \( a \not\leq b \). Then \( \uparrow a \cap \uparrow b = \emptyset \), so by the prime filter lemma, there is a prime filter \( P \) of \( L \) such that \( a \in P \) and \( b \notin P \). Therefore, \( P \in \sigma(a) \) and \( P \notin \sigma(b) \). Thus, \( \sigma(a) \nsubseteq \sigma(b) \). Consequently, if \( L \) is distributive, \( a \leq b \) iff \( \sigma(a) \subseteq \sigma(b) \), and so \( \sigma \) is an embedding. \( \square \)

Let \( D(L) \) denote the sublattice of \( \mathcal{P}(\Pr(L)) \) generated by \( \sigma[L] \). Since \( \sigma[L] \) is closed under finite intersections, for each \( A \in \mathcal{P}(\Pr(L)) \), we have \( A \in D(L) \) iff \( A = \bigcup_{i=1}^{n} \sigma(a_i) \) for some \( a_i \in L \). It follows that \( \sigma[L] \) is join-dense in \( D(L) \).

Moreover, \( \sigma \) is a meet semi-lattice homomorphism from \( L \) to \( D(L) \), and \( \sigma \) is a meet semi-lattice embedding whenever \( L \) is distributive.

**Definition 4.2.** For a distributive meet semi-lattice \( L \), we call \( D(L) \) the **distributive envelope** of \( L \).

Whenever convenient we will identify a distributive meet semi-lattice \( L \) with \( \sigma[L] \) and consider \( L \) as a join-dense \( \wedge \)-subalgebra of \( D(L) \). Now we investigate the connection between filters and ideals of \( L \) and \( D(L) \).

**Lemma 4.3.** Let \( L \) be a distributive meet semi-lattice and let \( D(L) \) be the distributive envelope of \( L \). If \( F \) is a filter of \( L \), then \( \uparrow_{D(L)} \sigma[F] \) is a filter of \( D(L) \), and if \( I \) is an ideal of \( L \), then \( \downarrow_{D(L)} \sigma[I] \) is an ideal of \( D(L) \).

**Proof.** Let \( F \) be a filter of \( L \). Clearly \( \uparrow_{D(L)} \sigma[F] \) is an upset of \( D(L) \). Let \( A, B \in \uparrow_{D(L)} \sigma[F] \). Then there exist \( a, b \in F \) such that \( \sigma(a) \subseteq A \) and \( \sigma(b) \subseteq B \). Therefore, \( \sigma(a \wedge b) = \sigma(a) \cap \sigma(b) \subseteq A \cap B \). Since \( F \) is a filter of \( L \), we have \( a \wedge b \in F \). Thus, \( A \cap B \in \uparrow_{D(L)} \sigma[F] \). Consequently, \( \uparrow_{D(L)} \sigma[F] \) is a filter of \( D(L) \).

Let \( I \) be an ideal of \( L \). Clearly \( \downarrow_{D(L)} \sigma[I] \) is a downset of \( D(L) \). Let \( A, B \in \downarrow_{D(L)} \sigma[I] \). Then there exist \( a, b \in I \) such that \( A \subseteq \sigma(a) \) and \( B \subseteq \sigma(b) \). Since \( I \) is an ideal of \( L \), there exists \( e \in \{a, b\}^\wedge \cap I \). Thus, \( A, B \subseteq \sigma(e) \), implying that \( A \cup B \in \downarrow_{D(L)} \sigma[I] \). Consequently, \( \downarrow_{D(L)} \sigma[I] \) is an ideal of \( D(L) \). \( \square \)

**Lemma 4.4.** Let \( L \in \text{DM} \) and let \( D(L) \) be the distributive envelope of \( L \). If \( F \) is a filter of \( D(L) \), then \( \sigma^{-1}(F) \) is a filter of \( L \).

**Proof.** Let \( F \) be a filter of \( D(L) \). Since \( \top \in L \), we have \( \Pr(L) = \sigma(\top) \in D(L) \), so \( \sigma(\top) = \Pr(L) \in F \), and so \( \top \in \sigma^{-1}(F) \). Thus, \( \sigma^{-1}(F) \) is nonempty. Suppose that \( a \in \sigma^{-1}(F) \) and \( a \leq b \). Then \( \sigma(a) \in F \) and \( \sigma(a) \subseteq \sigma(b) \). Since \( F \) is an upset of \( D(L) \), it follows that \( \sigma(b) \in F \). Therefore, \( b \in \sigma^{-1}(F) \). For \( a, b \in \sigma^{-1}(F) \) we have \( \sigma(a), \sigma(b) \in F \). Since \( F \) is a filter of \( D(L) \), we have \( \sigma(a \wedge b) = \sigma(a) \cap \sigma(b) \in F \). Thus, \( a \wedge b \in \sigma^{-1}(F) \), and so \( \sigma^{-1}(F) \) is a filter of \( L \). \( \square \)
On the other hand, there exist ideals $I$ of $D(L)$ such that $\sigma^{-1}(I)$ is not an ideal of $L$ as the following example shows.

**Example 4.5.** Consider the distributive meet semi-lattice $L$ shown in Fig.1. The ordered set $(\text{Pr}(L), \subseteq)$ of prime filters of $L$ together with the distributive envelope $D(L)$ of $L$ is shown in Fig.2. We have that $I = \{\emptyset, \sigma(a), \sigma(b), \sigma(a) \cup \sigma(b)\}$ is an ideal of $D(L)$, but that $\sigma^{-1}(I) = \{\bot, a, b\}$ is not an ideal of $L$.

![Fig.2](image)

**Lemma 4.6.** Let $L$ be a distributive meet semi-lattice, $D(L)$ be the distributive envelope of $L$, and $I$ be an ideal of $D(L)$. Then $I$ is the ideal of $D(L)$ generated by $\sigma[\sigma^{-1}(I)] = I \cap \sigma[L]$.

*Proof.* Let $J$ be the ideal of $D(L)$ generated by $\sigma[\sigma^{-1}(I)] = I \cap \sigma[L]$. Obviously $J \subseteq I$. On the other hand, if $A \in I$, then as $A = \bigcup_{i=1}^{n} \sigma(b_i)$ for some $b_i \in L$, we have $\sigma(b_i) \in I \cap \sigma[L]$ for each $i \leq n$. Thus, $A \in J$. \qed

**Corollary 4.7.** Let $L$ be a distributive meet semi-lattice and let $D(L)$ be the distributive envelope of $L$. For ideals $I, J$ of $D(L)$ we have that the following conditions are equivalent:

1. $I = J$.
2. $\sigma^{-1}(I) = \sigma^{-1}(J)$.
3. $I \cap \sigma[L] = J \cap \sigma[L]$.

On the other hand, there exist filters $F$ of $D(L)$ such that $F$ is not generated by $\sigma[\sigma^{-1}(F)] = F \cap \sigma[L]$ as the following example shows.
Example 4.8. Consider the distributive meet semi-lattice $L$ and its distributive envelope $D(L)$ shown in Fig. 2. Then $F = \{ \sigma(a) \cup \sigma(b), \sigma(c_n), \Pr(L) : n \in \omega \}$ is a filter of $D(L)$ which is not generated by $\sigma[\sigma^{-1}(F)] = F \cap \sigma[L] = \{ \sigma(c_n), \Pr(L) : n \in \omega \}$.

For an ideal $I$ of $D(L)$, if $\sigma^{-1}(I)$ were an ideal of $L$, then Lemma 4.6 and Corollary 4.7 would imply that there is a 1-1 correspondence between ideals of $L$ and $D(L)$. However, $\sigma^{-1}(I)$ is not necessarily an ideal of $L$ as we have shown in Example 4.5. This forces us to introduce a weaker notion of an ideal of $L$, that of a Frink ideal.

4.2. Frink ideals.

Definition 4.9. (Frink [5, p. 227]) Let $L$ be a meet semi-lattice. A nonempty subset $I$ of $L$ is called a Frink ideal (F-ideal for short) if for each finite subset $A$ of $I$ we have $A^{ul} \subseteq I$. Equivalently, $I$ is a Frink ideal if for each $a_1, \ldots, a_n \in I$ and $c \in L$, whenever $\bigcap_{i=1}^n \uparrow a_i \subseteq \uparrow c$, we have $c \in I$. We call an F-ideal $I$ of $L$ proper if $I \neq L$, and we call $I$ prime if it is proper and $a \land b \in I$ implies $a \in I$ or $b \in I$ for each $a, b \in L$.

It is easy to verify that for each $a \in L$ we have $\downarrow a$ is an F-ideal. Moreover, unlike the case with ideals, a nonempty intersection of a family of F-ideals is again an F-ideal. Therefore, for each nonempty $X \subseteq L$, there exists a least F-ideal containing $X$. We call it the $F$-ideal generated by $X$, and denote it by $(X)$.

Lemma 4.10. Let $L$ be a meet semi-lattice and let $X$ be a nonempty subset of $L$. Then $(X) = \{ a \in L : \exists$ finite $A \subseteq X$ with $a \in A^{ul} \} = \{ a \in L : \exists a_1, \ldots, a_n \in X$ with $\bigcap_{i=1}^n \uparrow a_i \subseteq \uparrow a \}$.  

Proof. Suppose that $L$ is a meet semi-lattice and $X$ is a nonempty subset of $L$. Clearly $(X)$ is an F-ideal. Let $I$ be an F-ideal with $X \subseteq I$. We show that $(X) \subseteq I$. If $a \in (X)$, then there exist $a_1, \ldots, a_n \in X$ such that $\bigcap_{i=1}^n \uparrow a_i \subseteq \uparrow a$. Since $a_1, \ldots, a_n \in I$ and $I$ is an F-ideal, we have $a \in I$. Thus, $(X)$ is the F-ideal generated by $X$. \hfill \Box

Lemma 4.11. Let $L$ be a meet semi-lattice. Then each ideal of $L$ is an F-ideal and each F-ideal of $L$ is a downset. Moreover, if $L$ is a lattice, then the two notions coincide with the usual notion of an ideal of a lattice.
Proof. Suppose that $I$ is an ideal of $L$, $a_1, \ldots, a_n \in I$, $a \in L$, and $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow a$.

Since $I$ is an ideal, it is easy to prove by induction on $n$ that $\bigcap_{i=1}^{n} \uparrow a_i \cap I \neq \emptyset$.

Let $c \in \bigcap_{i=1}^{n} \uparrow a_i \cap I$. Then $c \in \uparrow a$, so $a \leq c$, and so $a \in I$. Therefore, $I$ is an F-ideal. That every F-ideal is a downset is obvious. Now suppose that $L$ is a lattice and $I$ is an F-ideal of $L$. Then $I$ is a downset. Moreover, for $a, b \in I$ we have $\uparrow a \cap \uparrow b = \uparrow (a \vee b)$. Thus, $a \vee b \in I$, and so the two notions coincide with the usual notion of an ideal of $L$. \qed

In particular, if a meet semi-lattice $L$ is finite and $\top \in L$, then $L$ is a lattice, and so each F-ideal of $L$ is an ideal. On the other hand, there exist meet semi-lattices for which not every F-ideal is an ideal. For example, if $L$ is the lattice shown in Fig.1, then $I = \{ \bot, a, b \}$ is an F-ideal which is not an ideal. The next lemma is useful in obtaining a 1-1 correspondence between F-ideals of a distributive meet semi-lattice $L$ and ideals of its distributive envelope $D(L)$.

Lemma 4.12. Let $L$ be a distributive meet semi-lattice. For each $a_1, \ldots, a_n$, $b \in L$ we have $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b$ iff $\sigma(b) \subseteq \bigcup_{i=1}^{n} \sigma(a_i)$.

Proof. First suppose that $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b$ and $x \in \sigma(b)$. Then $b \in x$, so $\uparrow b \subseteq x$.

Therefore, $\bigcap_{i=1}^{n} \uparrow a_i \subseteq x$, and since $x$ is a prime filter of $L$, there exists $i \leq n$ such that $\uparrow a_i \subseteq x$. Thus, $a_i \in x$, so $x \in \sigma(a_i)$, and so $x \in \bigcup_{i=1}^{n} \sigma(a_i)$. It follows that $\sigma(b) \subseteq \bigcup_{i=1}^{n} \sigma(a_i)$.

Conversely, suppose that $\sigma(b) \subseteq \bigcup_{i=1}^{n} \sigma(a_i)$ and $c \in \bigcap_{i=1}^{n} \uparrow a_i$. Then $a_i \leq c$ for each $i \leq n$. Therefore, $\sigma(a_i) \subseteq \sigma(c)$ for each $i \leq n$, and so $\bigcup_{i=1}^{n} \sigma(a_i) \subseteq \sigma(c)$.

Thus, $\sigma(b) \subseteq \sigma(c)$, so $b \leq c$, and so $c \in \uparrow b$. It follows that $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b$. \qed
Theorem 4.13. Let $L$ be a distributive meet semi-lattice and let $D(L)$ be its distributive envelope.

1. $I \subseteq L$ is an F-ideal of $L$ iff there is an ideal $J$ of $D(L)$ such that $I = \sigma^{-1}(J)$.
2. $I \subseteq L$ is a prime F-ideal of $L$ iff there is a prime ideal $J$ of $D(L)$ such that $I = \sigma^{-1}(J)$.

Proof. (1) Let $I$ be an F-ideal of $L$ and $J$ be the ideal of $D(L)$ generated by $\sigma[I] = \{\sigma(a) : a \in I\}$. We claim that $I = \sigma^{-1}(J)$. It is clear that $I \subseteq \sigma^{-1}(J)$. Let $b \in \sigma^{-1}(J)$. Then $\sigma(b) \in J$, so $\sigma(b) \subseteq \bigcup_{i=1}^{n} \sigma(a_i)$ for some $a_1, \ldots, a_n \in I$. By Lemma 4.12, $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b$. Since $I$ is an F-ideal, it follows that $b \in I$. Thus, $\sigma^{-1}(J) \subseteq I$. Consequently, $I = \sigma^{-1}(J)$. Conversely, let $I = \sigma^{-1}(J)$ for $J$ an ideal of $D(L)$. Clearly $I$ is nonempty. For $a_1, \ldots, a_n \in I$ and $c \in L$ with $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow c$, by Lemma 4.12, we have $\sigma(c) \subseteq \bigcup_{i=1}^{n} \sigma(a_i)$. Therefore, $\sigma(c) \in J$, and so $c \in I$. Thus, $I$ is an F-ideal of $L$.

(2) Let $I$ be a prime F-ideal of $L$ and let $J$ be the ideal of $D(L)$ generated by $\sigma[I]$. By (1), $I = \sigma^{-1}(J)$. We show that $J$ is a prime ideal of $D(L)$.

Suppose that $A \cap B \in J$. Then $A = \bigcup_{i=1}^{n} \sigma(a_i)$ and $B = \bigcup_{j=1}^{m} \sigma(b_j)$ for some $a_1, \ldots, a_n, b_1, \ldots, b_m \in L$. Therefore, $\bigcup_{i,j} (\sigma(a_i) \cap \sigma(b_j)) \in J$. It follows that $\sigma(a_i) \cap \sigma(b_j) \in \sigma[I]$ for all $i, j$. Since $I$ is prime, either $a_i \in I$ or $b_j \in I$. We look at $a_1 \wedge b_1, \ldots, a_1 \wedge b_m$. If $a_1 \notin I$, then $b_1, \ldots, b_m \in I$, so $B = \bigcup_{j=1}^{m} \sigma(b_j) \in J$. Therefore, without loss of generality we may assume that $a_1 \in I$. Now we look at $a_2 \wedge b_1, \ldots, a_2 \wedge b_m$. If $a_2 \notin I$, then $b_1, \ldots, b_m \in I$, so again $B = \bigcup_{j=1}^{m} \sigma(b_j) \in J$. Thus, without loss of generality we may assume that $a_1, a_2 \in I$. Going through all $a_1, \ldots, a_n$ we obtain that either $B = \bigcup_{j=1}^{m} \sigma(b_j) \in J$ or $A = \bigcup_{i=1}^{n} \sigma(a_i) \in J$. It follows that $J$ is a prime ideal of $D(L)$. The converse implication easily follows from (1) and the definition of prime F-ideals. \qed

As an immediate consequence of (the proof of) Theorem 4.13 and Corollary 4.7, we obtain the following:
Corollary 4.14. Let $L$ be a distributive meet semi-lattice and let $D(L)$ be its distributive envelope. The ordered set of Frink ideals of $L$ is isomorphic to the ordered set of ideals of $D(L)$, and the ordered set of prime Frink ideals of $L$ is isomorphic to the ordered set of prime ideals of $D(L)$.

4.3. Optimal filters.

Definition 4.15. Let $L$ be a distributive meet semi-lattice and let $D(L)$ be its distributive envelope. A filter $F$ of $L$ is said to be optimal if there exists a prime filter $P$ of $D(L)$ such that $F = \sigma^{-1}[P]$. We denote the set of optimal filters of $L$ by Opt($L$).

Clearly each optimal filter is proper.

Lemma 4.16 (Optimal Filter Lemma). Let $L$ be a distributive meet semi-lattice. If $F$ is a filter and $I$ is an $F$-ideal of $L$ with $F \cap I = \emptyset$, then there exists an optimal filter $G$ of $L$ such that $F \subseteq G$ and $G \cap I = \emptyset$.

Proof. Let $F$ be a filter and $I$ be an $F$-ideal of $L$ with $F \cap I = \emptyset$. Let also $\nabla$ be the filter and $\Delta$ be the ideal of $D(L)$ generated by $\sigma[F]$ and $\sigma[I]$, respectively. Suppose that there exists $A \in \nabla \cap \Delta$. Then there are $a_1, \ldots, a_n \in I$ and $b \in F$ such that $A = \bigcup_{i=1}^n \sigma(a_i)$ and $\sigma(b) \subseteq A$. Therefore, $\sigma(b) \subseteq \bigcup_{i=1}^n \sigma(a_i)$. By Lemma 4.12, $\bigcap_{i=1}^n \uparrow a_i \subseteq \uparrow b$. Since $I$ is an F-ideal, we obtain $b \in I$, a contradiction.

Thus, $\nabla \cap \Delta = \emptyset$, and so there is a prime filter $P$ of $D(L)$ such that $\nabla \subseteq P$ and $P \cap \Delta = \emptyset$. It follows that $F \subseteq \sigma^{-1}[P]$ and $\sigma^{-1}[P] \cap I = \emptyset$. If we set $G = \sigma^{-1}[P]$, then $G$ is the desired optimal filter.

Corollary 4.17. Every proper filter $F$ of a distributive meet semi-lattice is the intersection of the optimal filters containing $F$.

Proposition 4.18. Let $L$ be a distributive meet semi-lattice and let $F$ be a filter of $L$. Then the following conditions are equivalent:

1. $F$ is an optimal filter.
2. $F$ is a filter and $L - F$ is an $F$-ideal.
3. There is an $F$-ideal $I$ of $L$ such that $F \cap I = \emptyset$ and $F$ is maximal among the filters of $L$ with this property.

Proof. (1)$\Rightarrow$(2): Let $F$ be an optimal filter of $L$, $P$ be a prime filter of $D(L)$ such that $F = \sigma^{-1}[P]$, and $I = L - F$. Then $F$ is a proper filter of $L$, and so $I$ is nonempty. For $a_1, \ldots, a_n \in I$ and $c \in L$ with $\bigcap_{i=1}^n \uparrow a_i \subseteq \uparrow c$, Lemma 4.12
implies that $\sigma(c) \subseteq \bigcup_{i=1}^{n} \sigma(a_i)$. If $c \notin I$, then $c \in F$, and so $\sigma(c) \in P$. Therefore, $\bigcup_{i=1}^{n} \sigma(a_i) \in P$. Since $P$ is prime, $\sigma(a_i) \in P$ for some $a_i \in I$. Thus, $a_i \in F \cap I$, which is a contradiction. It follows that $c \in I$, and so $I$ is an $F$-ideal.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1): Suppose that $F$ is a filter and $I$ is an $F$-ideal of $L$ such that $F \cap I = \emptyset$ and $F$ is maximal among the filters of $L$ with this property. By the optimal filter lemma, there is an optimal filter $G$ of $L$ such that $F \subseteq G$ and $G \cap I = \emptyset$. Since $F$ is a maximal filter with this property, $F = G$. Thus, $F$ is optimal.

\[\square\]

Remark 4.19. Hansoul [7] defines a \textit{weakly prime} ideal $I$ of a distributive join semi-lattice $L$ as an ideal $I$ such that for each $a_1, \ldots, a_n \notin I$ and $b \in I$, there is $c \in \bigcap_{i=1}^{n} \downarrow a_i$ such that $c \nleq b$. Clearly an ideal $I$ is weakly prime iff for each $a_1, \ldots, a_n \notin I$ and $b \in L$, from $\bigcap_{i=1}^{n} \downarrow a_i \subseteq \downarrow b$ it follows that $b \notin I$. Since the dual $L^d$ of $L$ is a distributive meet semi-lattice and $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b$ if $\bigcap_{i=1}^{n} \downarrow a_i \subseteq \downarrow b$, it follows from Proposition 4.18 that an ideal $I$ of $L$ is weakly prime iff it is an optimal filter of $L^d$. A more detailed comparison with Hansoul’s work can be found in Section 11.1.

Lemma 4.20. Let $L$ be a distributive meet semi-lattice. Then each prime filter of $L$ is optimal. Moreover, if $L$ is a lattice, then the two notions coincide with the usual notion of a prime filter of a lattice.

\[\textit{Proof.}\] Suppose that $L$ is a distributive meet semi-lattice and $P$ is a prime filter of $L$. By Proposition 3.7, $I = L - P$ is a prime ideal, hence an $F$-ideal of $L$ by Lemma 4.11. Moreover, $P \cap I = \emptyset$ and by the construction of $I$, $P$ is maximal among the filters of $L$ with this property. Thus, $P$ is optimal by Proposition 4.18. Now let in addition $L$ be a lattice and $F$ be an optimal filter of $L$. By Proposition 4.18, $I = L - F$ is an $F$-ideal of $L$, and by Lemma 4.11, $I$ is actually an ideal of a lattice. Thus, the two notions coincide with the usual notion of a prime filter of a lattice. \[\square\]

In particular, if $L$ is a finite distributive meet semi-lattice and $\top \in L$, then $L$ is a finite distributive lattice, and so each optimal filter of $L$ is prime. On the other hand, there exist distributive meet semi-lattices in which not every optimal filter is prime. For example, in the distributive meet semi-lattice $L$
shown in Fig.1 one can easily check that $F = L - \{\bot, a, b\}$ is an optimal filter of $L$, but that it is not prime. Now we show that optimal filters of a distributive meet semi-lattice $L$ are in a 1-1 correspondence with prime filters of the distributive envelope $D(L)$ of $L$.

**Proposition 4.21.** Let $L$ be a distributive meet semi-lattice and let $D(L)$ be its distributive envelope.

1. If $P \in \Pr(D(L))$, then $\sigma^{-1}(P)$ is an optimal filter of $L$ and $\uparrow_{D(L)}(P \cap \sigma[L]) = P$.
2. If $F \in \Opt(L)$, then $\uparrow_{D(L)}[\sigma[F] \in \Pr(D(L))$.
3. For a filter $F$ of $D(L)$, $F \in \Pr(D(L))$ iff there is $G \in \Opt(L)$ such that $F = \uparrow_{D(L)}[\sigma[G]$.
4. If $P,Q \in \Pr(D(L))$, then the following conditions are equivalent:
   (a) $P \subseteq Q$.
   (b) $\sigma^{-1}(P) \subseteq \sigma^{-1}(Q)$.
   (c) $\sigma[\sigma^{-1}(P)] \subseteq \sigma[\sigma^{-1}(Q)]$.
   (d) $P \cap \sigma[L] \subseteq Q \cap \sigma[L]$. 
   (e) $\uparrow_{D(L)}(P \cap \sigma[L]) \subseteq \uparrow_{D(L)}(Q \cap \sigma[L])$.

**Proof.** (1) That $\sigma^{-1}(P)$ is an optimal filter of $L$ follows from the definition. We show that $\uparrow_{D(L)}(P \cap \sigma[L]) = P$. It is clear that $\uparrow_{D(L)}(P \cap \sigma[L]) \subseteq P$. To prove the other inclusion let $A \in P$. Then $A = \sigma(b_1) \cup \ldots \cup \sigma(b_n)$ for some $b_1,\ldots,b_n \in L$. Since $P$ is prime, there exists $i \leq n$ such that $\sigma(b_i) \in P$. Therefore, $\sigma(b_i) \in P \cap \sigma[L]$, and so $A \in \uparrow_{D(L)}(P \cap \sigma[L])$.

(2) Let $F \in \Opt(L)$ and let $Q \in \Pr(L)$ be such that $F = \sigma^{-1}(Q)$. Then $\sigma[F] = Q \cap \sigma[L]$, and by (1), $\uparrow_{D(L)}[\sigma[F] = \uparrow_{D(L)}(Q \cap \sigma[L]) = Q$. Thus, $\uparrow_{D(L)}[\sigma[F] \in \Pr(L)$.

(3) follows from (1) and (2).

(4) (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(e) is obvious, and (e)$\Rightarrow$(a) follows from (1). \hfill $\Box$

**Corollary 4.22.** Let $L$ be a distributive meet semi-lattice and let $D(L)$ be its distributive envelope. Then the map $P \mapsto \sigma^{-1}(P)$ is an order-isomorphism between $\langle \Pr(D(L)) \subseteq \rangle$ and $\langle \Opt(L), \subseteq \rangle$ whose inverse is the map $F \mapsto \uparrow_{D(L)}[\sigma[F]$. 

Thus, for a distributive meet semi-lattice $L$ and its distributive envelope $D(L)$, we have that $F$-ideals of $L$ correspond to ideals of $D(L)$, that prime $F$-ideals of $L$ correspond to prime ideals of $D(L)$, and that optimal filters of $L$ correspond to prime filters of $D(L)$.

**Lemma 4.23.** Let $L$ be a distributive meet semi-lattice and let $D(L)$ be its distributive envelope. If $P$ is a prime filter of $L$, then $\uparrow_{D(L)}[\sigma[P]$ is a prime filter of $D(L)$, and if $I$ is a prime ideal of $L$, then $\downarrow_{D(L)}[\sigma[I]$ is a prime ideal of $D(L)$.
Proof. Let $P$ be a prime filter of $L$. By Lemma 4.20, $P$ is optimal, and by Proposition 4.21.2, $\uparrow_{D(L)} \sigma [P]$ is a prime filter of $D(L)$. Now let $I$ be a prime ideal of $L$. By Lemma 4.3, $\downarrow_{D(L)} \sigma [I]$ is an ideal of $D(L)$. We show that $\downarrow_{D(L)} \sigma [I]$ is prime. Let $A \cap B \in \downarrow_{D(L)} \sigma [I]$, and let $A = \bigcup_{i=1}^{n} \sigma (a_i)$ and $B = \bigcup_{j=1}^{m} \sigma (b_j)$. From $\bigcup_{i=1}^{n} \sigma (a_i) \cap \bigcup_{j=1}^{m} \sigma (b_j) \in \downarrow_{D(L)} \sigma [I]$, and so $a_i \land b_j \in I$ for each $i, j$. Since $I$ is prime, either $a_i \in I$ or $b_j \in I$. We look at $a_1 \land b_1, \ldots, a_1 \land b_m$. If $a_1 \notin I$, then $b_1, \ldots, b_m \in I$, and so there exists $c \in \bigcap_{j=1}^{m} \uparrow b_j \cap I$. Therefore, $B = \bigcup_{j=1}^{m} \sigma (b_j) \subseteq \sigma (c) \in \sigma [I]$, so $B \in \downarrow_{D(L)} \sigma [I]$, and so without loss of generality we may assume that $a_1 \in I$. Now we look at $a_2 \land b_1, \ldots, a_2 \land b_m$. If $a_2 \notin I$, then $b_1, \ldots, b_m \in I$, so again $B \in \downarrow_{D(L)} \sigma [I]$. Thus, without loss of generality we may assume that $a_1, a_2 \in I$. Going through all $a_1, \ldots, a_n$ we obtain that either $B \in \downarrow_{D(L)} \sigma [I]$ or $A \in \downarrow_{D(L)} \sigma [I]$. It follows that $\downarrow_{D(L)} \sigma [I]$ is a prime ideal of $D(L)$. $\square$

For the reader’s convenience, we give a table of relations between different notions of filters and ideals of $L$ and $D(L)$.

**Correspondences between filters of $L$ and $D(L)$**

<table>
<thead>
<tr>
<th>$L$</th>
<th>filters</th>
<th>prime filters</th>
<th>optimal filters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(L)$</td>
<td>$\uparrow_{D(L)} \sigma [F]$, $F$ is a filter of $L$</td>
<td>$\uparrow_{D(L)} \sigma [F]$, $F$ is a prime filter of $L$</td>
<td>$\downarrow_{D(L)} \sigma [F]$, $F$ is an ideal of $L$</td>
</tr>
</tbody>
</table>

**Correspondences between ideals of $L$ and $D(L)$**

<table>
<thead>
<tr>
<th>$L$</th>
<th>F-ideals</th>
<th>prime ideals</th>
<th>ideals</th>
<th>prime ideals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(L)$</td>
<td>$\downarrow_{D(L)} \sigma [I]$, $I$ is an ideal of $L$</td>
<td>$\downarrow_{D(L)} \sigma [I]$, $I$ is a prime ideal of $L$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Let $L$ be a distributive meet semi-lattice. We define a map $\varphi : L \rightarrow \mathcal{P}(\text{Opt}(L))$ by
$$\varphi(a) = \{x \in \text{Opt}(L) : a \in x\}.$$ It is easy to verify that $\varphi$ is a meet semi-lattice homomorphism, that it preserves top whenever $L$ has a top, and that it preserves bottom whenever $L$ has a bottom. It also follows from the optimal filter lemma that $\varphi$ is 1-1. Thus, we obtain:

**Proposition 4.24.** Let $L$ be a distributive meet semi-lattice. Then $L$ is isomorphic to the meet semi-lattice $\langle \{\varphi(a) : a \in L\}, \cap \rangle$.

**Corollary 4.25.** For a distributive meet semi-lattice $L$, we have that the meet semi-lattices $\langle \{\sigma(a) : a \in L\}, \cap \rangle$ and $\langle \{\varphi(a) : a \in L\}, \cap \rangle$ are isomorphic.

**Lemma 4.26.** Let $L$ be a distributive meet semi-lattice and let $a, b_1, \ldots b_n \in L$. Then $\varphi(a) \subseteq \bigcup_{i=1}^{n} \varphi(b_i)$ iff $\bigcap_{i=1}^{n} \uparrow b_i \subseteq \uparrow a$.

**Proof.** First suppose that $\varphi(a) \subseteq \bigcup_{i=1}^{n} \varphi(b_i)$. If $c \in \bigcap_{i=1}^{n} \uparrow b_i$ and $c \notin \uparrow a$, then $b_i \leq c$ for each $i \leq n$ and $a \notin c$. By the optimal filter lemma, there exists an optimal filter $x$ of $L$ such that $a \in x$ and $c \notin x$. But then $b_i \notin x$ for each $i \leq n$. Therefore, $x \in \varphi(a)$ but $x \notin \bigcup_{i=1}^{n} \varphi(b_i)$, a contradiction. Thus, $a \leq c$, so $c \in \uparrow a$, and so $\bigcap_{i=1}^{n} \uparrow b_i \subseteq \uparrow a$. Now suppose that $\bigcap_{i=1}^{n} \uparrow b_i \subseteq \uparrow a$. If $x \in \varphi(a)$ and $x \notin \bigcup_{i=1}^{n} \varphi(b_i)$, then $a \in x$ and $b_i \notin x$ for each $i \leq n$. Since $x$ is an optimal filter, $L - x$ is an F-ideal by Proposition 4.18. So $b_i \in L - x$ for each $i \leq n$ and $\bigcap_{i=1}^{n} \uparrow b_i \subseteq \uparrow a$ imply $a \in L - x$, a contradiction. Thus, $x \in \bigcup_{i=1}^{n} \varphi(b_i)$, and so $\varphi(a) \subseteq \bigcup_{i=1}^{n} \varphi(b_i)$. □

**Proposition 4.27.** Let $L$ be a distributive meet semi-lattice and let $D(L)$ be its distributive envelope. Then the closure under finite unions of $\varphi[L]$ is isomorphic to $D(L)$. 
Proof. It follows from Lemmas 4.12 and 4.26 that \( \varphi(a_1) \cup \ldots \cup \varphi(a_n) = \varphi(b_1) \cup \ldots \cup \varphi(b_m) \)iff \( \sigma(a_1) \cup \ldots \cup \sigma(a_n) = \sigma(b_1) \cup \ldots \cup \sigma(b_m) \). Thus, we can define a map \( h \) from the closure under finite unions of \( \varphi[L] \) to \( D(L) \) by \( h(\varphi(a_1) \cup \ldots \cup \varphi(a_n)) = \sigma(a_1) \cup \ldots \cup \sigma(a_n) \). This map is clearly onto, and it follows from Lemmas 4.12 and 4.26 that it is an order-isomorphism. \( \square \)

5. Sup-homomorphisms and an abstract characterization of distributive envelopes

Let \( L \) and \( K \) be distributive meet semi-lattices and let \( h : L \to K \) be a meet semi-lattice homomorphism. If there exist \( a, b \in L \) such that \( a \lor b \) exists in \( L \) and \( h(a) \lor h(b) \) exists in \( K \), it is not necessary that \( h(a \lor b) = h(a) \lor h(b) \). Therefore, \( h \) may not be extended to a lattice homomorphism from \( D(L) \) to \( D(K) \). In this section we introduce a stronger notion of a homomorphism between distributive meet semi-lattices, we call a sup-homomorphism. We show that sup-homomorphisms have the property that they preserve all existing joins and that they can be extended to lattice homomorphisms between the distributive envelopes. We also give an abstract characterization of the distributive envelope by means of sup-homomorphisms, and prove that the category of distributive lattices and lattice homomorphisms is a reflective subcategory of the category of distributive meet semi-lattices and sup-homomorphisms.

Definition 5.1. Let \( L \) and \( K \) be distributive meet semi-lattices. We call a meet semi-lattice homomorphism \( h : L \to K \) a sup-homomorphism if

\[
\bigcap_{i=1}^n \uparrow a_i \subseteq \uparrow b \text{ implies } \bigcap_{i=1}^n \uparrow h(a_i) \subseteq \uparrow h(b)
\]

for each \( a_1, \ldots, a_n, b \in L \).

Proposition 5.2. Let \( L \) and \( K \) be distributive meet semi-lattices and \( h : L \to K \) be a meet semi-lattice homomorphism. Then the following conditions are equivalent.

1. \( h \) is a sup-homomorphism.
2. \( h^{-1}[I] \) is an F-ideal of \( L \) for each F-ideal \( I \) of \( K \).
3. \( h^{-1}[F] \) is an optimal filter of \( L \) for each optimal filter \( F \) of \( K \).

Proof. (1)\( \Rightarrow \)(2): Let \( h : L \to K \) be a sup-homomorphism and let \( I \) be an F-ideal of \( K \). We show that \( h^{-1}[I] \) is an F-ideal of \( L \). Suppose that \( a_1, \ldots, a_n \in h^{-1}[I] \) and \( b \in L \) are such that \( \bigcap_{i=1}^n \uparrow a_i \subseteq \uparrow b \). Then \( h(a_1), \ldots, h(a_n) \in I \). Since
Proposition 5.3. Let \( h : L \to K \) be a meet semi-lattice homomorphism. Then \( h \) is a sup-homomorphism, \( \bigcap_{i=1}^{n} \uparrow h(a_i) \subseteq \uparrow h(b) \). As \( I \) is an F-ideal, \( h(b) \in I \).

Therefore, \( b \in h^{-1}[I] \), and so \( h^{-1}[I] \) is an F-ideal of \( L \).

(2) \( \Rightarrow \) (3): Let \( h^{-1}[I] \) be an F-ideal of \( L \) for each F-ideal \( I \) of \( K \). We show that \( h^{-1}[F] \) is an optimal filter of \( L \) for each optimal filter \( F \) of \( K \). Suppose that \( F \) is an optimal filter of \( L \). Since \( h \) is a meet semi-lattice homomorphism, \( h^{-1}[F] \) is a filter of \( L \). Moreover, by Proposition 4.18, \( K - F \) is an F-ideal of \( K \). Therefore, \( h^{-1}[K - F] = L - h^{-1}[F] \) is an F-ideal of \( L \). This, by Proposition 4.18, means that \( h^{-1}[F] \) is an optimal filter of \( L \).

(3) \( \Rightarrow \) (1): Let \( h^{-1}[F] \) be an optimal filter of \( L \) for each optimal filter \( F \) of \( K \) and let \( \bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b \) for \( a_1, \ldots, a_n, b \in L \). If \( \bigcap_{i=1}^{n} \uparrow h(b_i) \not\subseteq \uparrow h(b) \), then, by Lemma 4.10, \( h(b) \) does not belong to the F-ideal generated by \( h(a_1), \ldots, h(a_n) \). By the optimal filter lemma, there is an optimal filter \( G \) of \( K \) such that \( h(b) \in G \) and \( h(a_1), \ldots, h(a_n) \not\in G \). Therefore, \( a_1, \ldots, a_n \not\in h^{-1}[G] \). But \( h^{-1}[G] \) is an optimal filter of \( L \). Thus, \( L - h^{-1}[G] \) is an F-ideal of \( L \), so \( b \in L - h^{-1}[G] \), and so \( h(b) \not\in G \), a contradiction. Consequently, \( \bigcap_{i=1}^{n} \uparrow h(b_i) \subseteq \uparrow h(b) \), implying that \( h \) is a sup-homomorphism.

We show that sup-homomorphisms are exactly those meet semi-lattice homomorphisms which preserve all existing joins. Let \( L \) and \( K \) be distributive meet semi-lattices and \( h : L \to K \) be a meet semi-lattice homomorphism. We say that \( h \) preserves all existing joins if for each \( a_1, \ldots, a_n \in L \), if \( a_1 \lor \ldots \lor a_n \) exists in \( L \), then \( h(a_1) \lor \ldots \lor h(a_n) \) exists in \( K \) and \( h(a_1 \lor \ldots \lor a_n) = h(a_1) \lor \ldots \lor h(a_n) \).

Proposition 5.3. Let \( L \) and \( K \) be distributive meet semi-lattices and \( h : L \to K \) be a meet semi-lattice homomorphism. Then \( h \) is a sup-homomorphism iff \( h \) preserves all existing joins.

Proof. Let \( h \) be a sup-homomorphism, \( a_1, \ldots, a_n \in L \), and \( a_1 \lor \ldots \lor a_n \) exist in \( L \). Then \( \bigcap_{i=1}^{n} \uparrow a_i = \uparrow (a_1 \lor \ldots \lor a_n) \). Since \( h \) is order-preserving, by the definition of sup-homomorphisms, the last equality implies that \( \bigcap_{i=1}^{n} \uparrow h(a_i) = \uparrow h(a_1 \lor \ldots \lor a_n) \). Therefore, \( h(a_1 \lor \ldots \lor a_n) \) is the join of \( h(a_1), \ldots, h(a_n) \) in \( K \). Thus, \( h \) preserves all existing joins. Conversely, suppose that \( h \) preserves all existing joins. Let \( a_1, \ldots, a_n, b \in L \) be such that \( \bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b \). Then, in the lattice of
filters of \( L \), we have \( \bigcap_{i=1}^{n} \uparrow a_i \lor \uparrow b = \uparrow b \). Since \( L \) is a distributive meet semi-lattice, the lattice of filters of \( L \) is distributive. Therefore, \( \bigcap_{i=1}^{n} (\uparrow a_i \lor \uparrow b) = \uparrow b \).

Since \( \uparrow a_i \lor \uparrow b = \uparrow (a_i \land b) \), we obtain \( \bigcap_{i=1}^{n} \uparrow (a_i \land b) = \uparrow b \) for each \( i = 1, \ldots, n \).

This implies that \( b \) is the join of \( a_1 \land b, \ldots, a_n \land b \) in \( L \). Therefore, since \( h \) preserves all existing joins, the join of \( h(a_1 \land b), \ldots, h(a_n \land b) \) exists in \( K \) and is equal \( h(b) \). Thus, \( \bigcap_{i=1}^{n} \uparrow h(a_i \land b) = \uparrow h(b) \), which means that

\[
\uparrow h(b) = \bigcap_{i=1}^{n} \uparrow (h(a_i) \land h(b)) = \bigcap_{i=1}^{n} (\uparrow h(a_i) \lor \uparrow h(b)).
\]

Using the distributivity of the lattice of filters of \( K \), we obtain

\[
\uparrow h(b) = \uparrow h(b) \lor \bigcap_{i=1}^{n} \uparrow h(a_i).
\]

Consequently, \( \bigcap_{i=1}^{n} \uparrow h(a_i) \subseteq \uparrow h(b) \), and so \( h \) is a sup-homomorphism. \( \square \)

**Remark 5.4.** In [7] Hansoul introduced the notion of a weak morphism for distributive join semi-lattices, which is a join semi-lattice homomorphism preserving all existing meets. It follows from Proposition 5.3 that \( h : L \to K \) is a sup-homomorphism iff \( h : L^d \to K^d \) is a weak morphism.

**Lemma 5.5.** Let \( L \) and \( K \) be distributive meet semi-lattices. If \( h : L \to K \) is a 1-1 sup-homomorphism, then

\[
\bigcap_{i=1}^{n} \uparrow h(a_i) \subseteq \uparrow h(b) \text{ implies } \bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b
\]

for each \( a_1, \ldots, a_n, b \in L \).

**Proof.** Suppose that \( \bigcap_{i=1}^{n} \uparrow h(a_i) \subseteq \uparrow h(b) \). If \( d \in \bigcap_{i=1}^{n} \uparrow a_i \), then \( h(d) \in \bigcap_{i=1}^{n} \uparrow h(a_i) \). Therefore, \( h(d) \in \uparrow h(b) \). Thus, \( h(b) \leq h(d) \). Since \( h \) is 1-1, this implies that \( b \leq d \). It follows that \( d \in \uparrow b \), and so \( \bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b \). \( \square \)
Proposition 5.6. Let \( L \) and \( K \) be distributive meet semi-lattices. If \( h : L \to K \) is a sup-homomorphism, then there is a unique lattice homomorphism \( D(h) : D(L) \to D(K) \) such that \( D(h) \circ \sigma = \sigma \circ h \). Moreover, if \( h \) is 1-1, then so is \( D(h) \).

Proof. Suppose that \( a_1, \ldots, a_n, b_1, \ldots, b_m \in L \) are such that \( \sigma(a_1) \cup \ldots \cup \sigma(a_n) = \sigma(b_1) \cup \ldots \cup \sigma(b_m) \). Then, for each \( 1 \leq i \leq n \), we have \( \sigma(a_i) \subseteq \sigma(b_1) \cup \ldots \cup \sigma(b_m) \).

Therefore, by Lemma 4.12, \( \bigcap_{j=1}^{m} \uparrow b_j \subseteq \uparrow a_i \). Since \( h \) is a sup-homomorphism, this implies that \( \bigcap_{j=1}^{m} \uparrow h(b_j) \subseteq \uparrow h(a_i) \). Thus, \( \sigma(h(a_i)) \subseteq \sigma(h(b_1)) \cup \ldots \cup \sigma(h(b_m)) \), and so \( \sigma(h(a_i)) \cup \ldots \cup \sigma(h(a_n)) \subseteq \sigma(h(b_1)) \cup \ldots \cup \sigma(h(b_m)) \). By a similar argument we obtain that \( \sigma(h(b_1)) \cup \ldots \cup \sigma(h(b_m)) \subseteq \sigma(h(a_1)) \cup \ldots \cup \sigma(h(a_n)) \). Consequently, \( \sigma(h(a_1)) \cup \ldots \cup \sigma(h(a_n)) = \sigma(h(b_1)) \cup \ldots \cup \sigma(h(b_m)) \), and so we can define \( D(h) : D(L) \to D(K) \) by

\[
D(h)(\sigma(a_1) \cup \ldots \cup \sigma(a_n)) = \sigma(h(a_1)) \cup \ldots \cup \sigma(h(a_n)).
\]

That \( D(h) \) is a lattice homomorphism from \( D(L) \) to \( D(K) \) and that \( D(h) \circ \sigma = \sigma \circ h \) is obvious.

If \( k : D(L) \to D(K) \) is a lattice homomorphism such that \( k \circ \sigma = \sigma \circ h \), then \( D(h)(\sigma(a_1) \cup \ldots \cup \sigma(a_n)) = \sigma(h(a_1)) \cup \ldots \cup \sigma(h(a_n)) = k(\sigma(a_1)) \cup \ldots \cup k(\sigma(a_n)) = k(\sigma(a_1) \cup \ldots \cup \sigma(a_n)) \). Therefore, \( k = D(h) \).

Finally, suppose that \( h \) is 1-1. If \( D(h)(\sigma(a_1) \cup \ldots \cup \sigma(a_n)) = D(h)(\sigma(b_1) \cup \ldots \cup \sigma(b_m)) \), then \( \sigma(h(a_1)) \cup \ldots \cup \sigma(h(a_n)) = \sigma(h(b_1)) \cup \ldots \cup \sigma(h(b_m)) \). This, by Lemma 4.12, means that \( \bigcap_{j=1}^{m} \uparrow h(b_j) \subseteq \uparrow h(a_i) \) for each \( 1 \leq i \leq n \) and \( \bigcap_{i=1}^{n} \uparrow h(a_i) \subseteq \uparrow h(b_j) \) for each \( 1 \leq j \leq m \). Since \( h \) is 1-1, by Lemma 5.5, we obtain that \( \bigcap_{j=1}^{m} \uparrow b_j \subseteq \uparrow a_i \) for each \( 1 \leq i \leq n \) and \( \bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow b_j \) for each \( 1 \leq j \leq m \). Therefore, \( \sigma(a_1) \cup \ldots \cup \sigma(a_n) = \sigma(b_1) \cup \ldots \cup \sigma(b_m) \). Thus, \( D(h) \) is 1-1. \( \square \)

As a consequence, we obtain that taking the distributive envelope of a distributive meet semi-lattice can be extended to a functor \( D \) from the category of distributive meet semi-lattices and sup-homomorphisms to the category of distributive lattices and lattice homomorphisms.

Noting that if \( K \) is a distributive lattice, then \( D(K) \) is (isomorphic to) \( K \), the following is an immediate consequence of Proposition 5.6.
Corollary 5.7. Let $L$ be a distributive meet semi-lattice and $D$ be a distributive lattice. If $h : L \to D$ is a sup-homomorphism, then there is a unique lattice homomorphism $D(h) : D(L) \to D$ such that $D(h) \circ \sigma = h$. Moreover, if $h$ is 1-1, then so is $D(h)$.

It follows that the functor $D$ is left adjoint to the inclusion functor. Consequently, the category of distributive lattices and lattice homomorphisms is a reflective subcategory of the category of distributive meet semi-lattices and sup-homomorphisms.

Theorem 5.8. Let $L$ be a distributive meet semi-lattice. The distributive envelope $D(L)$ of $L$ is up to isomorphism the unique distributive lattice $E$ for which there is a 1-1 sup-homomorphism $e : L \to E$ such that for each distributive lattice $D$ and a 1-1 sup-homomorphism $h : L \to D$, there is a unique 1-1 lattice homomorphism $k : E \to D$ with $k \circ e = h$.

Proof. As an immediate consequence of Corollary 5.7, we obtain that $D(L)$ has the property stated in the theorem because $\sigma : L \to D(L)$ is a 1-1 sup-homomorphism and for each distributive lattice $D$ and a 1-1 sup-homomorphism $h : L \to D$, there exists a unique 1-1 lattice homomorphism $D(h) : D(L) \to D$ with $D(h) \circ \sigma = h$. Now suppose that $E$ is a distributive lattice for which there is a 1-1 sup-homomorphism $e : L \to E$ such that for each distributive lattice $D$ and a 1-1 sup-homomorphism $h : L \to D$, there is a unique 1-1 lattice homomorphism $k : E \to D$ with $k \circ e = h$. Then there is a 1-1 lattice homomorphism $k : E \to D(L)$ such that $k \circ e = \sigma$. Also, by Corollary 5.7, there is a 1-1 lattice homomorphism $D(e) : D(L) \to E$ such that $D(e) \circ \sigma = e$. First we show that each element $a$ of $E$ has the form $e(a_1) \lor \ldots \lor e(a_n)$ for some $a_1, \ldots, a_n \in L$. Let $a \in E$. Then $k(a) \in D(L)$. Therefore, there exist $a_1, \ldots, a_n \in L$ such that $k(a) = \sigma(a_1) \lor \ldots \lor \sigma(a_n)$. Since $k \circ e = \sigma$, we have $D(e)(\sigma(a_1) \lor \ldots \lor \sigma(a_n)) = D(e)(e(a_1)) \lor \ldots \lor k(e(a_n)))$. As $D(e)$ is 1-1, the last equality implies $\sigma(a_1) \lor \ldots \lor \sigma(a_n) = k(e(a_1)) \lor \ldots \lor k(e(a_n)) = k(\sigma(a_1) \lor \ldots \lor \sigma(a_n))$. Because $k$ is 1-1, we obtain $a = e(a_1) \lor \ldots \lor e(a_n)$. Next we show that $E$ is isomorphic to $D(L)$. Since $k \circ e = \sigma$ and $D(e) \circ \sigma = e$, we have $D(e)(k(e(a_1)) \lor \ldots \lor e(a_n)) = D(e)(k(e(a_1))) \lor \ldots \lor k(e(a_n))) = D(e)(\sigma(a_1) \lor \ldots \lor \sigma(a_n)) = D(e)(\sigma(a_1)) \lor \ldots \lor D(e)(\sigma(a_n)) = e(a_1) \lor \ldots \lor e(a_n)$ and $k(D(e)(\sigma(a_1) \lor \ldots \lor \sigma(a_n))) = k(D(e)(\sigma(a_1))) \lor \ldots \lor D(e)(\sigma(a_n))) = k(e(a_1)) \lor \ldots \lor e(a_n) = k(e(a_1)) \lor \ldots \lor k(e(a_n)) = \sigma(a_1) \lor \ldots \lor \sigma(a_n)$. Therefore, the composition $D(e) \circ k$ is the identity function on $E$ and the composition $k \circ D(e)$ is the identity function on $D(L)$. Thus, $k : E \to D(L)$ and $D(e) : D(L) \to E$ are inverses of each other, and so $E$ is isomorphic to $D(L)$. \]

Theorem 5.8 provides an abstract characterization of the distributive envelope of a distributive meet semi-lattice. A similar characterization can also be found in Hansoul [7] for distributive join semi-lattices.
6. Generalized Priestley spaces

In this section we introduce one of the main concepts of this paper, that of generalized Priestley space. We show how to construct the generalized Priestley space $L_*$ from a bounded distributive meet semi-lattice $L$, and conversely, how a generalized Priestley space $X$ gives rise to the bounded distributive meet semi-lattice $X^*$. We further prove that a bounded distributive meet semi-lattice $L$ is isomorphic to $L_*$, thus providing a new representation theorem for bounded distributive meet semi-lattices. Furthermore, we show that a generalized Priestley space $X$ is order-isomorphic and homeomorphic to $X^*$.

Let $L$ be a bounded distributive meet semi-lattice and let $D(L)$ be its distributive envelope. Then $D(L)$ is a bounded distributive lattice. We let

$$L_* = \text{Opt}(L), \quad L_+ = \text{Pr}(L), \quad \text{and} \quad D(L)_* = \text{Pr}(D(L)).$$

By Lemma 4.20, $L_+ \subseteq L_*$, and by Corollary 4.22, $\langle L_*, \subseteq \rangle \simeq \langle D(L)_*, \subseteq \rangle$. We recall that $\varphi : L \to \mathcal{P}(L_*)$ is defined by $\varphi(a) = \{ x \in L_* : a \in x \}$. We define $\varphi_D : D(L) \to \mathcal{P}(D(L)_*)$ by $\varphi_D(A) = \{ x \in D(L)_* : A \in x \}$. For $a \in L$ and $x \in D(L)_*$, we have

$$x \in \varphi_D(\sigma(a)) \iff \sigma(a) \in x \iff a \in \sigma^{-1}(x).$$

Therefore,

$$\varphi(a) = \{ \sigma^{-1}(x) : x \in \varphi_D(\sigma(a)) \}.$$

Let $\mathfrak{B}_L = \varphi[L]$. Then $h : \mathfrak{B}_L \to \{ \varphi_D(\sigma(a)) : a \in L \}$ given by $h(\varphi(a)) = \varphi_D(\sigma(a))$ is a bounded meet semi-lattice isomorphism, so $\mathfrak{B}_L$ and $\{ \{ \varphi_D(\sigma(a)) : a \in L \}, \cap, D(L)_*, \emptyset \}$ are isomorphic to each other and to $L$.

We recall that $\{ \varphi_D(A) - \varphi_D(B) : A, B \in D(L) \}$ is a basis for the Priestley topology $\tau_p$ on $D(L)_*$.

**Proposition 6.1.** Let $L$ be a bounded distributive meet semi-lattice and let $D(L)$ be its distributive envelope. Then the Priestley topology on $D(L)_*$ has $\{ \varphi_D(\sigma(a)) : a \in L \} \cup \{ \varphi_D(\sigma(b))^c : b \in L \}$ as a subbasis.

**Proof.** Let $A, B \in D(L)$. Then there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in L$ such that $A = \bigcup_{i=1}^n \sigma(a_i)$ and $B = \bigcup_{j=1}^m \sigma(b_j)$. Therefore, $\varphi_D(A) = \varphi_D(\bigcup_{i=1}^n \sigma(a_i)) = \bigcup_{i=1}^n \varphi_D(\sigma(a_i))$ and $\varphi_D(B) = \varphi_D(\bigcup_{j=1}^m \sigma(b_j)) = \bigcup_{j=1}^m \varphi_D(\sigma(b_j))$. Thus, $\varphi_D(A) - \varphi_D(B) = \bigcup_{i=1}^n \varphi_D(\sigma(a_i)) - \bigcup_{j=1}^m \varphi_D(\sigma(b_j)) = \bigcup_{i=1}^n (\varphi_D(\sigma(a_i)) \cap \bigcup_{j=1}^m \varphi_D(\sigma(b_j))^c)$. It follows that the elements of the basis $\{ \varphi_D(A) - \varphi_D(B) : A, B \in D(L) \}$ of $D(L)_*$ are finite intersections of the elements of $\{ \varphi_D(\sigma(a)) : a \in L \} \cup \{ \varphi_D(\sigma(b))^c : b \in L \}$.
Theorem 6.4. Let $L$ be a bounded distributive meet semi-lattice and let $D(L)$ be its distributive envelope.

1. $\langle L_*, \tau, \subseteq \rangle$ is order-isomorphic and homeomorphic to $\langle D(L)_*, \tau_P, \subseteq \rangle$.
2. $\langle L_*, \tau, \subseteq \rangle$ is a Priestley space.
3. $L_+$ is dense in $\langle L_*, \tau \rangle$.

Proof. (1) By Corollary 4.22, $\langle L_*, \subseteq \rangle$ is order-isomorphic to $\langle D(L)_*, \subseteq \rangle$, and by Proposition 6.1, $\langle L_*, \tau \rangle$ is homeomorphic to $\langle D(L)_*, \tau_P \rangle$. The result follows.

(2) follows from (1) and the well-known fact that $\langle D(L)_*, \tau_P, \subseteq \rangle$ is a Priestley space.

(3) Since $\mathcal{S} = \{ \varphi(a) : a \in L \} \cup \{ \varphi(b)^c : b \in L \}$ is a subbasis for $\tau$ and $\{ \varphi(a) : a \in L \}$ is closed under finite intersections, an element of the basis for $\tau$ that $\mathcal{S}$ generates has the form $\varphi(a) \cap \varphi(b_1)^c \cap \ldots \cap \varphi(b_n)^c$ for some $a, b_1, \ldots, b_n \in L$. If $\varphi(a) \cap \varphi(b_1)^c \cap \ldots \cap \varphi(b_n)^c \neq \emptyset$, then $\varphi(a) \not\subseteq \bigcup_{i=1}^{n} \varphi(b_i)$.

Therefore, by Lemma 4.26, $\bigcap_{i=1}^{n} \uparrow b_i \not\subseteq \uparrow a$. Thus, by Lemma 4.12, $\sigma(a) \not\subseteq \bigcup_{i=1}^{n} \sigma(b_i)$. Hence, there is $y \in L_+$ such that $a \in y$ and $b_1, \ldots, b_n \not\in y$. It follows that $\varphi(a) \cap \varphi(b_1)^c \cap \ldots \cap \varphi(b_n)^c \cap L_+ \neq \emptyset$, and so $L_+$ is dense in $L_*$.

Lemma 6.3. For a bounded distributive meet semi-lattice $L$ we have that every open upset of $L_*$ is a union of elements of $\mathfrak{B}_L$.

Proof. Let $U$ be an open upset of $L_*$ and let $x \in U$. It is sufficient to find $a \in L$ such that $x \in \varphi(a) \subseteq U$. For each $y \not\in U$ we have $x \not\subseteq y$. Therefore, there is $y \in L$ such that $a_y \in x$ and $a_y \not\in y$. Thus, $\bigcap \{ \varphi(a_y) : y \not\in U \} \cap U^c = \emptyset$. This by compactness of $L_*$ implies that there exist $a_1, \ldots, a_n \in L$ such that $\varphi(a_1) \cap \cdots \cap \varphi(a_n) \cap U^c = \emptyset$. Moreover, $x \in \varphi(a_i)$ for each $i \leq n$. Therefore, $x \in \varphi(a_1 \wedge \cdots \wedge a_n) \subseteq U$, and so there exists $a = a_1 \wedge \cdots \wedge a_n$ in $L$ with $x \in \varphi(a) \subseteq U$.

Let $D(\mathfrak{B}_L)$ denote the distributive lattice generated (in $\mathcal{P}(L_*)$) by $\mathfrak{B}_L$. Then $A \in D(\mathfrak{B}_L)$ iff $A$ is a finite union of elements of $\mathfrak{B}_L$. Let also $\mathfrak{CU}(L_*)$ denote the lattice of clopen upsets of $L_*$.

Theorem 6.4. For a bounded distributive meet semi-lattice $L$ we have $D(L) \simeq D(\mathfrak{B}_L) = \mathfrak{CU}(L_*)$. 

Therefore, by Lemma 4.26, $\bigcap_{i=1}^{n} \uparrow b_i \not\subseteq \uparrow a$. Thus, by Lemma 4.12, $\sigma(a) \not\subseteq \bigcup_{i=1}^{n} \sigma(b_i)$. Hence, there is $y \in L_+$ such that $a \in y$ and $b_1, \ldots, b_n \not\in y$. It follows that $\varphi(a) \cap \varphi(b_1)^c \cap \ldots \cap \varphi(b_n)^c \cap L_+ \neq \emptyset$, and so $L_+$ is dense in $L_*$.

Lemma 6.3. For a bounded distributive meet semi-lattice $L$ we have that every open upset of $L_*$ is a union of elements of $\mathfrak{B}_L$.

Proof. Let $U$ be an open upset of $L_*$ and let $x \in U$. It is sufficient to find $a \in L$ such that $x \in \varphi(a) \subseteq U$. For each $y \not\in U$ we have $x \not\subseteq y$. Therefore, there is $a_y \in L$ such that $a_y \in x$ and $a_y \not\in y$. Thus, $\bigcap \{ \varphi(a_y) : y \not\in U \} \cap U^c = \emptyset$. This by compactness of $L_*$ implies that there exist $a_1, \ldots, a_n \in L$ such that $\varphi(a_1) \cap \cdots \cap \varphi(a_n) \cap U^c = \emptyset$. Moreover, $x \in \varphi(a_i)$ for each $i \leq n$. Therefore, $x \in \varphi(a_1 \wedge \cdots \wedge a_n) \subseteq U$, and so there exists $a = a_1 \wedge \cdots \wedge a_n$ in $L$ with $x \in \varphi(a) \subseteq U$. 

Let $D(\mathfrak{B}_L)$ denote the distributive lattice generated (in $\mathcal{P}(L_*)$) by $\mathfrak{B}_L$. Then $A \in D(\mathfrak{B}_L)$ iff $A$ is a finite union of elements of $\mathfrak{B}_L$. Let also $\mathfrak{CU}(L_*)$ denote the lattice of clopen upsets of $L_*$.

Theorem 6.4. For a bounded distributive meet semi-lattice $L$ we have $D(L) \simeq D(\mathfrak{B}_L) = \mathfrak{CU}(L_*)$. 

Therefore, by Lemma 4.26, $\bigcap_{i=1}^{n} \uparrow b_i \not\subseteq \uparrow a$. Thus, by Lemma 4.12, $\sigma(a) \not\subseteq \bigcup_{i=1}^{n} \sigma(b_i)$. Hence, there is $y \in L_+$ such that $a \in y$ and $b_1, \ldots, b_n \not\in y$. It follows that $\varphi(a) \cap \varphi(b_1)^c \cap \ldots \cap \varphi(b_n)^c \cap L_+ \neq \emptyset$, and so $L_+$ is dense in $L_*$.
Proof. It follows from Proposition 4.27 that \( D(L) \) is isomorphic to \( D(\mathfrak{B}_L) \). By Lemma 6.3, each element of \( \mathfrak{C}L(L_a) \) is a union of elements of \( \mathfrak{B}_L \). Now since each element of \( \mathfrak{C}L(L_a) \) is compact, it is a finite union of elements of \( \mathfrak{B}_L \). Thus, \( D(\mathfrak{B}_L) = \mathfrak{C}L(L_a) \).

We now study some properties of the tuple \( \langle L_\ast, \tau, \subseteq, L_+ \rangle \) which will motivate the notion of a generalized Priestley space we introduce below.

**Proposition 6.5.** Let \( L \) be a bounded distributive meet semi-lattice. Then for each \( x \in L_\ast \), there is \( y \in L_+ \) such that \( x \subseteq y \).

**Proof.** Since \( x \) is an optimal filter of \( L \), we have \( L - x \) is nonempty. Let \( a \in L - x \). Then \( x \) is disjoint from the ideal \( \downarrow a \), and by the prime filter lemma, there is a prime filter \( y \) of \( L \) such that \( x \subseteq y \) and \( a \notin y \). \( \square \)

**Proposition 6.6.** Let \( L \) be a bounded distributive meet semi-lattice and let \( U \) be a clopen upset of \( L_\ast \). Then the following conditions are equivalent:

1. \( U = \varphi(a) \) for some \( a \in L \).
2. \( L_a - U = \downarrow(L_+ - U) \).
3. \( \max(L_a - U) \subseteq L_+ \).

**Proof.** \((1) \Rightarrow (2)\): Let \( U = \varphi(a) \). Since \( L_a \subseteq L_\ast \), we have \( L_+ - \varphi(a) \subseteq L_a - \varphi(a) \), and as \( \varphi(a) \) is an upset, \( L_a - \varphi(a) \) is a downset, so \( \downarrow(L_+ - \varphi(a)) \subseteq L_a - \varphi(a) \). Conversely, if \( x \in L_a - \varphi(a) \), then \( x \notin \varphi(a) \). Therefore, \( a \notin x \). So \( x \cap \downarrow a = \emptyset \), and by the prime filter lemma, there is a prime filter \( y \) of \( L \) such that \( x \subseteq y \) and \( a \notin y \). Thus, \( x \subseteq y \) and \( y \in L_+ - \varphi(a) \), implying that \( x \in \downarrow(L_+ - \varphi(a)) \). It follows that \( L_a - \varphi(a) = \downarrow(L_+ - \varphi(a)) \).

\((2) \Rightarrow (3)\): Let \( L_a - U = \downarrow(L_+ - U) \) and let \( x \in \max(L_a - U) \). Then \( x \in \downarrow(L_+ - U) \), and as \( x \) is a maximal point of \( L_a - U \), we have \( x \in L_+ - U \subseteq L_+ \).

\((3) \Rightarrow (1)\): Let \( U \) be a clopen upset of \( L_\ast \) and let \( \max(L_a - U) \subseteq L_+ \). Then there exist \( a_1, \ldots, a_n \in L \) such that \( U = \varphi(a_1) \cup \ldots \cup \varphi(a_n) \). Let \( F \) be the filter \( \bigcap_{i=1}^{n} \uparrow a_i \) and let \( I \) be the Frink ideal generated by \( a_1, \ldots, a_n \). If \( F \cap I = \emptyset \), then, by the optimal filter lemma, there exists an optimal filter \( x \) of \( L \) such that \( F \subseteq x \) and \( x \cap I = \emptyset \). Since \( a_i \in I \), we have that \( x \notin a_i \) for each \( i \leq n \), so \( x \notin \varphi(a_1) \cup \ldots \cup \varphi(a_n) = U \). Therefore, \( x \in L_a - U \). Since \( \langle L_\ast, \tau, \subseteq \rangle \) is a Priestley space and \( L_\ast - U \) is a closed subset of \( L_\ast \), there exists \( y \in \max(L_a - U) \) such that \( x \subseteq y \). But then \( y \in L_+ \) and \( y \notin U \). Moreover, \( \bigcap_{i=1}^{n} \uparrow a_i \subseteq x \subseteq y \) and as \( y \) is prime, there is \( i \leq n \) such that \( \uparrow a_i \subseteq y \). Hence, \( y \in \varphi(a_i) \subseteq U \), which is a
contradiction. Therefore, there is \( a \in F \cap I \). Thus, \( a = \bigcap_{i=1}^{n} \uparrow a_i \) and \( \bigcap_{i=1}^{n} \downarrow a_i \subseteq \uparrow a \).

So \( \bigcap_{i=1}^{n} \uparrow a_i = \uparrow a \), \( a = a_1 \lor \ldots \lor a_n \), and so \( \varphi(a) = \varphi(a_1) \cup \ldots \cup \varphi(a_n) = U \). □

Proposition 6.6 can be restated as follows:

**Proposition 6.7.** Let \( L \) be a bounded distributive meet semi-lattice and let \( V \) be a clopen downset of \( L_* \). Then the following conditions are equivalent:

1. \( V = \varphi(a)^c \) for some \( a \in L \).
2. \( V = \uparrow (L_+ \cap V) \).
3. \( \max(V) \subseteq L_+ \).

The following is an immediate consequence of Proposition 6.6.

**Corollary 6.8.** Let \( L \) be a bounded distributive meet semi-lattice. Then:

\[
\mathcal{B}_L = \{ U \in \mathcal{Cl}(L_*) : L_* - U = \uparrow (L_+ - U) \} = \{ U \in \mathcal{Cl}(L_*) : \max(L_+ - U) \subseteq L_+ \}.
\]

For a family \( \mathcal{F} = \{ \varphi(a_i) : a_i \in L \} \) we recall that \( \mathcal{F} \) is directed if for each \( \varphi(a_i), \varphi(a_j) \in \mathcal{F} \) there exists \( \varphi(a_k) \in \mathcal{F} \) such that \( \varphi(a_i) \cup \varphi(a_j) \subseteq \varphi(a_k) \). Clearly \( \mathcal{F} \) is directed iff for each \( \varphi(a_{i_1}), \ldots, \varphi(a_{i_n}) \in \mathcal{F} \) there exists \( \varphi(a_k) \in \mathcal{F} \) such that \( \varphi(a_{i_1}) \cup \ldots \cup \varphi(a_{i_n}) \subseteq \varphi(a_k) \). For each \( x \in L_* \) let \( \mathcal{I}_x = \{ \varphi(a) : x \notin \varphi(a) \} \).

**Proposition 6.9.** Let \( L \) be a bounded distributive meet semi-lattice. Then \( x \in L_+ \) iff \( \mathcal{I}_x \) is directed.

**Proof.** Let \( x \in L_+ \) and let \( \varphi(a), \varphi(b) \in \mathcal{I}_x \). Then \( x \notin \varphi(a), \varphi(b) \). Therefore, \( a, b \notin x \). Since \( x \) is a prime filter of \( L \), it follows that \( \uparrow a \cap \uparrow b \not\subseteq x \). Thus, there exists \( c \in \uparrow a \cap \uparrow b \) such that \( c \notin x \). Consequently, \( \varphi(a) \cup \varphi(b) \subseteq \varphi(c) \) and \( x \notin \varphi(c) \). So \( \varphi(c) \in \mathcal{I}_x \), and so \( \mathcal{I}_x \) is directed. Conversely, suppose that \( \mathcal{I}_x \) is directed. We show that \( x \in L_+ \). If not, then there exist filters \( F_1 \) and \( F_2 \) of \( L \) such that \( F_1 \cap F_2 \subseteq x \) but \( F_1 \not\subseteq x \) and \( F_2 \not\subseteq x \). Let \( a_1 \in F_1 - x \) and \( a_2 \in F_2 - x \). Then \( x \notin \varphi(a_1), \varphi(a_2) \), and so \( \varphi(a_1), \varphi(a_2) \in \mathcal{I}_x \). Since \( \mathcal{I}_x \) is directed, there exists \( \varphi(a) \in \mathcal{I}_x \) such that \( \varphi(a_1) \cup \varphi(a_2) \subseteq \varphi(a) \). From \( \varphi(a) \in \mathcal{I}_x \) it follows that \( a \notin x \), and from \( \varphi(a_1) \cup \varphi(a_2) \subseteq \varphi(a) \) it follows that \( a \in \uparrow a_1 \cap \uparrow a_2 \subseteq F_1 \cap F_2 \subseteq x \). The obtained contradiction proves that \( x \in L_+ \). □

The results we have established about the dual space of a bounded distributive semi-lattice \( L \) justify the following definition of a generalized Priestley space. Let \( \langle X, \tau, \leq \rangle \) be a Priestley space and let \( X_0 \) be a dense subset of \( X \). For a clopen subset \( U \) of \( X \), we say that \( X_0 \) is cofinal in \( U \) if \( \max(U) \subseteq X_0 \). We call a clopen upset \( U \) of \( X \) admissible if \( X_0 \) is cofinal in \( U^c \).
Let $X^*$ denote the set of admissible clopen upsets of $X$. We note that $U \in X^*$ iff $U$ is a clopen upset such that $\max(U^c) \subseteq X_0$, which happens iff $U$ is a clopen upset such that $U^c = \downarrow(X_0 - U)$. For $x \in X$ let $I_x = \{U \in X^* : x \notin U\}$.

**Definition 6.10.** We call a quadruple $X = \langle X, \tau, \leq, X_0 \rangle$ a generalized Priestley space if:

1. $\langle X, \tau, \leq \rangle$ is a Priestley space.
2. $X_0$ is a dense subset of $X$.
3. For each $x \in X$ there is $y \in X_0$ such that $x \leq y$.
4. $x \in X_0$ iff $I_x$ is directed.
5. For all $x, y \in X$, we have $x \leq y$ iff ($\forall U \in X^*) (x \in U \Rightarrow y \in U)$.

**Remark 6.11.** Clearly condition (3) of Definition 6.10 is equivalent to $\max X \subseteq X_0$, which means that $\emptyset$ is admissible. Also, when in a generalized Priestley space $X$ we have $X_0 = X$, then $X^* = \mathcal{U}(X)$, so conditions (2)–(5) of Definition 6.10 become redundant, and so $X$ becomes a Priestley space. Thus, the notion of a generalized Priestley space generalizes that of a Priestley space.

**Proposition 6.12.** For a bounded distributive meet semi-lattice $L$, the quadruple $L^* = \langle L^*, \tau, \subseteq, L^+ \rangle$ is a generalized Priestley space.

**Proof.** It follows from Lemma 6.2 that $\langle L^*, \tau, \subseteq \rangle$ is a Priestley space, and that $L^+$ is dense in $\langle L^*, \tau \rangle$. Therefore, $L_*$ satisfies conditions (1) and (2) of Definition 6.10. By Propositions 6.5 and 6.9, $L_*$ satisfies conditions (3) and (4) of Definition 6.10. Lastly it follows from Proposition 6.6 that $L_*^* = \varphi[L]$. Clearly $x \subseteq y$ iff ($\forall a \in L) (a \in x \Rightarrow b \in x)$, thus $L_*$ satisfies condition (5) of Definition 6.10. Consequently, $L_*$ is a generalized Priestley space.

Since for each bounded distributive meet semi-lattice $L$ we have $L_*^* = \varphi[L]$, we immediately obtain:

**Theorem 6.13.** (Representation Theorem) For each bounded distributive meet semi-lattice $L$ we have $L \simeq L_*^*$; that is, for each bounded distributive meet semi-lattice $L$ there exists a generalized Priestley space $X$ such that $L$ is isomorphic to $X^*$.

**Proposition 6.14.** Let $X$ be a generalized Priestley space. Then $X^* = \langle X^*, \cap, X, \emptyset \rangle$ is a bounded distributive meet semi-lattice.

**Proof.** First we show that $X^*$ is closed under $\cap$. If $U, V \in X^*$, then $\max((U \cap V)^c) = \max(U^c \cup V^c) \subseteq \max(U^c) \cup \max(V^c) \subseteq X_0$. Thus, $U \cap V \in X^*$. Next $\max(X^c) = \max(\emptyset) = \emptyset \subseteq X_0$, so $X \in X^*$. Also, $\max(\emptyset^c) = \max(X) \subseteq X_0$ by condition (3) of Definition 6.10, so $\emptyset \in X^*$. Lastly we show that $(X^*, \cap)$ is distributive. Let $U, V, W \in X^*$ with $U \cap V \subseteq W$. Then $W^c \subseteq U^c \cup V^c$. For each
$x \in \max(W^c)$ we have that $x \in U^c$ or $x \in V^c$. Therefore, $W \in \mathcal{I}_x$ and either $U \in \mathcal{I}_x$ or $V \in \mathcal{I}_x$. Since $x \in X$, by condition (4) of Definition 6.10, from $W, U \in \mathcal{I}_x$ it follows that there exists $U_x \in \mathcal{I}_x$ such that $W \cup U \subseteq U_x$; and from $W, V \in \mathcal{I}_x$ it follows that there exists $V_x \in \mathcal{I}_x$ such that $W \cup V \subseteq V_x$. Thus, $W^c = \bigcup \{K_x : x \in \max(W^c)\}$, where $K_x = U_x^c$ or $K_x = V_x^c$. Since $W^c$ is compact and each $K_x$ is open, there exist finite subsets $A, B$ of Definition 6.10, there is $CU$ such that $ CU \cap V = \emptyset$. Clearly we have $ CU \cap V = \emptyset$. Let $CU = \bigcap \{U_x : x \in A\}$ and $V' = \bigcap \{V_x : x \in B\}$. Clearly $U \subseteq U'$, $V \subseteq V'$, and $U, V \in X^*$. Moreover, $W^c = \bigcup \{U_x^c : x \in A\} \cup \bigcup \{V_x^c : x \in B\}$. Let $U' = \bigcap \{U_x : x \in A\}$ and $V = \bigcap \{V_x : x \in B\}$. Thus, $\langle X^*, \cap \rangle$ is distributive. Consequently, $\langle X^*, \cap, X, \emptyset \rangle$ is a bounded distributive meet lattice. \qed

Proposition 6.15. Let $X$ be a generalized Priestley space. Then the closure of $X^*$ under finite unions is $\mathcal{U}(X)$.

Proof. Clearly we have $X^* \subseteq \mathcal{U}(X)$, so the closure of $X^*$ under finite unions is contained in $\mathcal{U}(X)$. Conversely, suppose that $U \in \mathcal{U}(X)$. Since $X, \emptyset \in X^*$, we may assume without loss of generality that $U \neq X, \emptyset$. For each $x \in U$ and $y \notin U$, if $x \in V$ is an upset, we have $x \notin y$. Therefore, by condition (5) of Definition 6.10, there is $V_y \in X^*$ such that $x \in V_y$ and $y \notin V_y$. Then $U^c \cap \bigcap \{V_y : y \notin U\} = \emptyset$. Since $X$ is compact, there are $y_1, \ldots, y_n \in U^c$ such that $U^c \cap V_{y_1} \cap \ldots \cap V_{y_n} = \emptyset$. Therefore, $x \in V_{y_1} \cap \ldots \cap V_{y_n} \subseteq U$. Let $W_x = V_{y_1} \cap \ldots \cap V_{y_n}$. Since $X^*$ is closed under finite intersections, $W_x \in X^*$. Then $U = \bigcup \{W_x : x \in U\}$, and as $X$ is compact, there are $x_1, \ldots, x_k \in U$ such that $U = W_{x_1} \cup \ldots \cup W_{x_k}$. Thus, $U$ is a finite union of elements of $X^*$. \qed

Corollary 6.16. For a generalized Priestley space $X$, the family $X^* \cup \{U^c : U \in X^*\}$ is a subbasis for the topology on $X$.

Proof. Since $X$ is a Priestley space, $\{U - V : U, V \in \mathcal{U}(X)\}$ is a basis for the topology on $X$. By Proposition 6.15, $U = \bigcup_{i=1}^n U_i$ and $V = \bigcup_{j=1}^m V_j$ for some $U_1, \ldots, U_n, V_1, \ldots, V_m \in X^*$. Therefore, $U - V = \bigcup_{i=1}^n U_i - \bigcup_{j=1}^m V_j = \bigcup_{i=1}^n (U_i \cap \bigcap_{j=1}^m V_j^c)$. Thus, the elements of the basis $\{U - V : U, V \in \mathcal{U}(X)\}$ of $X$ are finite intersections of elements of $X^* \cup \{U^c : U \in X^*\}$. Consequently, $X^* \cup \{U^c : U \in X^*\}$ is a subbasis for the topology on $X$. \qed
Lemma 6.17. Let $X$ be a generalized Priestley space, and let $U, U_1, \ldots, U_n \in X^*$. Then $\bigcap_{i=1}^n \uparrow U_i \subseteq \uparrow U$ iff $U \subseteq \bigcup_{i=1}^n U_i$.

Proof. Suppose that $\bigcap_{i=1}^n \uparrow U_i \subseteq \uparrow U$. We show that $U \cap X_0 \subseteq \bigcup_{i=1}^n U_i$. Let $x \in U \cap X_0$. If $x \notin \bigcup_{i=1}^n U_i$, then $U_1, \ldots, U_n \in I_x$, so by condition (4) of Definition 6.10, there exists $V \in I_x$ such that $\bigcap_{i=1}^n \uparrow U_i \subseteq V$. Therefore, $V \in \bigcap_{i=1}^n \uparrow U_i$, and so $V \subseteq U$. Thus, $U \subseteq V$, implying that $x \in V$, a contradiction. Consequently, $x \in \bigcup_{i=1}^n U_i$, and so $U \cap X_0 \subseteq \bigcup_{i=1}^n U_i$. By condition (2) of Definition 6.10, $X_0$ is a dense subset of $X$. Since $U$ is open in $X$, we have $U \cap X_0$ is dense in $U$. Therefore, $U \cap X_0 \subseteq \bigcup_{i=1}^n U_i$ implies $U \subseteq \bigcup_{i=1}^n U_i$. Conversely, suppose that $U \subseteq \bigcup_{i=1}^n U_i$. For $V \in X^*$ with $V \in \bigcap_{i=1}^n \uparrow U_i$, we have $\bigcup_{i=1}^n U_i \subseteq V$. Therefore, $U \subseteq V$, implying that $V \in \uparrow U$. Thus, $\bigcap_{i=1}^n \uparrow U_i \subseteq \uparrow U$. □

Corollary 6.18. Let $X$ be a generalized Priestley space. Then $\mathcal{U}(X)$ is isomorphic to $D(X^*)$.

Proof. By Proposition 6.15, $\mathcal{U}(X)$ is the closure of $X^*$ under finite unions. Let $U_1, \ldots, U_n, V_1, \ldots, V_m \in X^*$. It follows from Lemmas 4.12 and 6.17 that $U_1 \cup \ldots \cup U_n = V_1 \cup \ldots \cup V_m$ iff $\sigma(U_1) \cup \ldots \cup \sigma(U_n) = \sigma(V_1) \cup \ldots \cup \sigma(V_m)$. Thus, we can define a map $h : \mathcal{U}(X) \to D(X^*)$ by $h(U_1 \cup \ldots \cup U_n) = \sigma(U_1) \cup \ldots \cup \sigma(U_n)$. This map is clearly onto, and it follows from Lemmas 4.12 and 6.17 that it is an order-isomorphism. □

Let $X$ be a generalized Priestley space. We define $\varepsilon : X \to X^*$ by

$$\varepsilon(x) = \{U \in X^* : x \in U\}.$$ 

First we show that $\varepsilon$ is well-defined.

Proposition 6.19. Let $X$ be a generalized Priestley space. For each $x \in X$, we have $\varepsilon(x)$ is an optimal filter of $X^*$. Moreover, if $x \in X_0$, then $\varepsilon(x)$ is a prime filter of $X^*$.
Proof. Let \( X \) be a generalized Priestley space. It follows from Proposition 6.14 that \( (X^*, \cap, X, \emptyset) \) is a bounded distributive meet-semilattice. Clearly \( \varepsilon(x) \) is a filter of \( X^* \) and \( X^* - \varepsilon(x) \) is nonempty. We show that \( X^* - \varepsilon(x) \) is an F-ideal of \( X^* \). Suppose that \( U_1, \ldots, U_n \in X^* - \varepsilon(x) \), \( U \in X^* \), and \( \bigcap_{i=1}^n \uparrow U_i \subseteq \uparrow U \). By Lemma 6.17, \( U \subseteq \bigcup_{i=1}^n U_i \). Since \( x \notin \bigcup_{i=1}^n U_i \), it follows that \( x \notin U \). Therefore, \( U \in X^* - \varepsilon(x) \), so \( X^* - \varepsilon(x) \) is an F-ideal, and so, by Proposition 4.18, \( \varepsilon(x) \) is an optimal filter. Now suppose that \( x \in X_0 \) and that \( \varepsilon(x) \) is not a prime filter of \( X^* \). Then there exist filters \( F_1 \) and \( F_2 \) of \( X^* \) such that \( F_1 \cap F_2 \subseteq \varepsilon(x) \), but \( F_1 \nsubseteq \varepsilon(x) \) and \( F_2 \nsubseteq \varepsilon(x) \). Let \( U_1 \in F_1 - \varepsilon(x) \) and \( U_2 \in F_2 - \varepsilon(x) \). Then \( x \notin U_1, U_2 \), and so \( U_1, U_2 \in \mathcal{I}_x \). By condition (4) of Definition 6.10, there exists \( V \in \mathcal{I}_x \) such that \( U_1 \cup U_2 \subseteq V \). Hence, \( V \in \uparrow U_1 \cap \uparrow U_2 \subseteq F_1 \cap F_2 \subseteq \varepsilon(x) \). Thus, \( x \in V \), a contradiction. We conclude that \( \varepsilon(x) \) is a prime filter of \( X^* \). \( \square \)

**Proposition 6.20.** The map \( \varepsilon : X \rightarrow X^*_\ast \) is 1-1 and onto. Moreover, if \( P \) is a prime filter of \( X^* \), then \( \varepsilon^{-1}(P) \in X_0 \).

**Proof.** It follows from condition (5) of Definition 6.10 that \( \varepsilon \) is 1-1. We show that \( \varepsilon \) is onto. Suppose that \( P \) is an optimal filter of \( X^* \). Let \( I = X^* - P \). By Proposition 4.18, \( I \) is an F-ideal of \( X^* \). Let \( G \) be the filter of \( \mathcal{U}(X) \) generated by \( P \) and \( J \) be the ideal of \( \mathcal{U}(X) \) generated by \( I \). We claim that \( G \cap J = \emptyset \). If not, then there exist \( V \in \mathcal{U}(X) \), \( U \in P \), and \( U_1, \ldots, U_n \in I \) such that \( U \subseteq V \) and \( V \subseteq U_1 \cup \ldots \cup U_n \). By Lemma 6.17, \( \bigcap_{i=1}^n \uparrow U_i \subseteq \uparrow V \subseteq \uparrow U \). Since \( I \) is an F-ideal, we have \( U \in I \), so \( U \notin P \), a contradiction. Thus, by the prime filter lemma, there is a prime filter \( F \) of \( \mathcal{U}(X) \) such that \( G \subseteq F \) and \( F \cap J = \emptyset \). By the Priestley duality, there exists \( x \in X \) such that \( F = \{ U \in \mathcal{U}(X) : x \in U \} \). We show that \( P = F \cap X^* \). It is clear that \( P \subseteq F \cap X^* \). Conversely, if \( U \in F \cap X^* \) and \( U \notin P \), then \( U \in I \), which is a contradiction since \( F \) is disjoint from \( I \). Thus, \( P = F \cap X^* \) \( = \{ U \in X^* : x \in U \} \). Consequently \( \varepsilon(x) = P \), and so \( \varepsilon \) is onto.

Now suppose that \( P \) is a prime filter of \( X^* \). Since \( \varepsilon \) is onto, there exists \( x \in X \) such that \( \varepsilon(x) = P \). If \( x \notin X_0 \), then by condition (4) of Definition 6.10, \( \mathcal{I}_x \) is not directed, so there exist \( U, V \in \mathcal{I}_x \) such that for no \( W \in \mathcal{I}_x \) we have \( U \cup V \subseteq W \). Therefore, for each \( W \in X^* \), from \( W \in \uparrow U \cap \uparrow V \) it follows that \( x \in W \), and so \( W \in \varepsilon(x) \). Thus, \( \uparrow U \cap \uparrow V \subseteq \varepsilon(x) \), and as \( \varepsilon(x) \) is a prime filter of \( X^* \), \( \uparrow U \subseteq \varepsilon(x) \) or \( \uparrow V \subseteq \varepsilon(x) \). Consequently, \( U \in \varepsilon(x) \) or \( V \in \varepsilon(x) \), and so \( x \in U \) or \( x \in V \), a contradiction. Thus, \( x \in X_0 \). \( \square \)
Theorem 6.21. For a generalized Priestley space $X$, the map $\varepsilon : X \to X^*$ is an order-isomorphism and a homeomorphism. Moreover, $\varepsilon[X_0] = X^*_+$. 

Proof. It follows from Propositions 6.19 and 6.20 that $\varepsilon$ is a bijection and that $\varepsilon[X_0] = X^*_+$. Condition (5) of Definition 6.10 implies that for each $x, y \in X$, we have $x \leq y$ iff $\varepsilon(x) \subseteq \varepsilon(y)$. Thus, $\varepsilon$ is an order-isomorphism. We show that $\varepsilon$ is a homeomorphism. By Corollary 6.16, $X^* \cup \{U^c : U \in X^*\}$ is a subbasis for the topology on $X$, and $\{\varphi(U) : U \in X^*\} \cup \{\varphi(U)^c : U \in X^*\}$ is a subbasis for the topology on $X^*_+$. For $U \in X^*$ we have:

$$x \in \varepsilon^{-1}(\varphi(U)) \iff \varepsilon(x) \in \varphi(U) \iff U \in \varepsilon(x) \iff x \in U.$$ 

Thus, $\varepsilon^{-1}(\varphi(U)) = U$ and $\varepsilon^{-1}(\varphi(U)^c) = U^c$. It follows that $\varepsilon$ is continuous. Now since $\varepsilon$ is a continuous map between compact Hausdorff spaces, $\varepsilon$ is a homeomorphism. \hfill \□

7. Generalized Esakia spaces

In this section we introduce our second main concept of the paper, that of generalized Esakia space, which we obtain by augmenting the concept of generalized Priestley space. As with bounded distributive meet semi-lattices, we show how to construct the generalized Esakia space $L_*$ from a bounded implicative meet semi-lattice $L$, and conversely, how a generalized Esakia space $X$ gives rise to the bounded implicative meet semi-lattice $X^*$. We also show that a bounded implicative meet semi-lattice $L$ is isomorphic to $L^*_*$, thus providing a new representation theorem for bounded implicative meet semi-lattices, and prove that a generalized Esakia space $X$ is order-isomorphic and homeomorphic to $X^*_+$. We conclude the section by showing that the distributive envelope of a bounded implicative meet semi-lattice may not be a Heyting algebra.

Let $L$ be a bounded implicative meet semi-lattice. For $a, b \in L$, let

$$\varphi(a) \rightarrow \varphi(b) = [\downarrow (\varphi(a) - \varphi(b))]^c = \{x \in L_* : \uparrow x \cap \varphi(a) \subseteq \varphi(b)\}.$$ 

Lemma 7.1. Let $L$ be a bounded implicative meet semi-lattice and let $a, b \in L$. Then $\varphi(a \rightarrow b) = \varphi(a) \rightarrow \varphi(b)$.

Proof. First suppose that $x \in \varphi(a \rightarrow b)$ and $y \in \uparrow x \cap \varphi(a)$. Then $a \rightarrow b \in x$, $x \subseteq y$, and $a \in y$. Therefore, $a, a \rightarrow b \in y$, so $b \in y$, and so $y \in \varphi(b)$. Thus, $\uparrow x \cap \varphi(a) \subseteq \varphi(b)$, so $x \in \varphi(a) \rightarrow \varphi(b)$, and so $\varphi(a \rightarrow b) \subseteq \varphi(a) \rightarrow \varphi(b)$. Now suppose that $x \in \varphi(a) \rightarrow \varphi(b)$. If $x \notin \varphi(a \rightarrow b)$, then $a \rightarrow b \notin x$. Let $F$ be the filter of $L$ generated by $\{a\} \cup \{x\}$. If there is $c \in F \cap \downarrow b$, then there is $d \in x$ such that $a \land d \leq c \leq b$. Therefore, $d \leq a \rightarrow b$, and so $a \rightarrow b \in x$, a contradiction. Thus, $F \cap \downarrow b = \emptyset$, and by the prime filter lemma, there is $y \in L_+ \subseteq L_*$ such that $F \subseteq y$ and $b \notin y$. It follows that $x \subseteq y$, $a \in y$, and $b \notin y$. Therefore, $y \in \uparrow x \cap \varphi(a)$ and $y \notin \varphi(b)$. Thus, $\uparrow x \cap \varphi(a) \subseteq \varphi(b)$, and so $x \notin \varphi(a) \rightarrow \varphi(b)$,
a contradiction. We conclude that $x \in \varphi(a \rightarrow b)$, so $\varphi(a) \rightarrow \varphi(b) \subseteq \varphi(a \rightarrow b)$, and so $\varphi(a \rightarrow b) = \varphi(a) \rightarrow \varphi(b)$. □

Let $X$ be a Priestley space. We recall that each clopen $U$ in $X$ has the form $\bigcup_{i=1}^{n}(U_i - V_i)$, where $U_i, V_i \in \mathcal{U}(X)$, and that $X$ is an Esakia space whenever $\downarrow U$ is clopen for each clopen $U$ in $X$.

Let $X$ be a generalized Priestley space. Then each clopen $U$ in $X$ has the form $\bigcap_{i=1}^{m}(U_i - V_i)$, where $U_i, V_j \in X^*$.

**Definition 7.2.** Let $X$ be a generalized Priestley space and $U$ be clopen in $X$. We call $U$ **Esakia clopen** if $U = \bigcup_{i=1}^{n}(U_i - V_i)$ for some $U_1, \ldots, U_n, V_1, \ldots, V_n \in X^*$.

**Lemma 7.3.** If $X$ is a generalized Priestley space and $U$ is Esakia clopen in $X$, then $\max(U) \subseteq X_0$.

**Proof.** Let $U$ be Esakia clopen. Then there exist $U_1, \ldots, U_n, V_1, \ldots, V_n \in X^*$ such that $U = \bigcup_{i=1}^{n}(U_i - V_i)$. Therefore, $\max(U) = \max[\bigcup_{i=1}^{n}(U_i - V_i)] \subseteq \bigcup_{i=1}^{n}\max(U_i - V_i) = \bigcup_{i=1}^{n}\max(U_i \cap V_i^c)$. Since $U_i$ is an upset, $V_i^c$ is a downset, and $V_i \in X^*$, we have $\max(U_i \cap V_i^c) \subseteq \max(V_i^c) \subseteq X_0$. Therefore, $\max(U) \subseteq \bigcup_{i=1}^{n}\max(U_i \cap V_i^c) \subseteq \bigcup_{i=1}^{n}\max(V_i^c) \subseteq X_0$. □

On the other hand, it is worth pointing out that the converse of Lemma 7.3 is not true in general.

**Definition 7.4.** We call a generalized Priestley space $X$ a **generalized Esakia space** if for each Esakia clopen $U$ in $X$, we have $\downarrow U$ is clopen.

For a generalized Esakia space $X$ and $U, V \in X^*$, let $U \rightarrow V = [\downarrow(U - V)]^c = \{x \in X : \uparrow x \cap U \subseteq V\}$.

**Proposition 7.5.**

1. If $L$ is a bounded implicative meet semi-lattice, then $L_* = \langle L_*, \tau, \subseteq, L_+ \rangle$ is a generalized Esakia space.

2. If $X$ is a generalized Esakia space, then $X^* = \langle X^*, \cap, \rightarrow, X, \emptyset \rangle$ is a bounded implicative meet semi-lattice.
Proof. (1) Suppose that $L$ is a bounded implicative meet semi-lattice. Then $L$ is a bounded distributive meet semi-lattice, and so $L_*$ is a generalized Priestley space. Let $U$ be Esakia clopen in $L_*$. Then $U = \bigcup_{i=1}^n (\varphi(a_i) - \varphi(b_i))$ for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in L$. By Lemma 7.1, $\varphi(a_i) \to \varphi(b_i) = [\downarrow (\varphi(a_i) - \varphi(b_i))]^c \in L_*^\circ$. Therefore, $\downarrow (\varphi(a_i) - \varphi(b_i))$ is clopen in $L_*$ for each $i \leq n$. Thus, $\downarrow U = \bigcup_{i=1}^n (\varphi(a_i) - \varphi(b_i))$ is clopen in $L_*$, and so $L_*$ is a generalized Esakia space.

(2) Suppose that $X$ is a generalized Esakia space. Then $X$ is a generalized Priestley space, and so $\langle X^*, \cap, X, \emptyset \rangle$ is a bounded distributive meet semi-lattice. Let $U, V \in X^*$. Then $U - V$ is Esakia clopen. Therefore, $\downarrow (U - V)$ is clopen in $X$ and $\max \downarrow (U - V) = \max (U - V) \subseteq X_0$. Therefore, $U \to V = [\downarrow (U - V)]^c \in X^*$. Moreover, it is routine to verify that for $U, V, W \in X^*$ we have $U \cap W \subseteq V$ if $W \subseteq U \to V$. Thus, $\langle X^*, \cap, \to, X, \emptyset \rangle$ is a bounded implicative meet semi-lattice.

It follows that for each bounded implicative meet semi-lattice $L$ we have $L_*^\circ = \varphi[L]$. Thus, we immediately obtain:

**Theorem 7.6.** (Representation Theorem) For each bounded implicative meet semi-lattice $L$ we have $L \simeq L_*^\circ$; that is, for each bounded implicative meet semi-lattice $L$ there exists a generalized Esakia space $X$ such that $L$ is isomorphic to $X^*$.

We show that there exist generalized Esakia spaces which are not Esakia spaces. This implies that the distributive envelope $D(L)$ of a bounded implicative meet semi-lattice $L$ may not be a Heyting algebra.

**Example 7.7.** Consider the implicative meet semi-lattice $L$, its distributive envelope $D(L)$, and its dual space $L_*$ shown in Fig.3, where the black circles indicate the elements of $L$ and the white circles indicate the elements of $D(L) - L$. Then $L_*$ is a generalized Esakia space order-homeomorphic to the dual space $D(L)_*$ of $D(L)$. We denote by $P$ the filter $\{\top, g_1, g_2, \ldots\}$ and by $Q$ the filter $\uparrow a \cup \bigcup \{\uparrow f_n : n \in \omega\}$. It is easy to see that $Q$ is the only optimal filter of $L$ which is not prime. Thus, $L_+ = L_* - \{Q\}$. It is also easy to calculate that $a \to b = b$ and $a \to c = c$ in $L$, but that $\sigma(a) \to (\sigma(b) \cup \sigma(c))$ does not exist in $D(L)$. Consequently, $D(L)$ is not a Heyting algebra. Put dually, $U = \{\uparrow a, \uparrow f_1, \uparrow f_2, \ldots, Q\}$ is clopen in $L_*$, but $\downarrow U = U \cup \{P\}$ is not clopen in $L_*$. Thus, $L_*$ is not an Esakia space. Of course, $U$ is not Esakia clopen because $L_*$ is a generalized Esakia space.
This section consists of two subsections. In the first one we introduce generalized Priestley morphisms, total generalized Priestley morphisms, and functional generalized Priestley morphisms, the category $\text{GPS}$ of generalized Priestley spaces and generalized Priestley morphisms, the category $\text{GPS}^T$ of generalized Priestley spaces and total generalized Priestley morphisms, and the category $\text{GPS}^F$ of generalized Priestley spaces and functional generalized Priestley morphisms. We also introduce the category $\text{BDM}$ of bounded distributive meet semi-lattices and semi-lattice homomorphisms preserving top, the category $\text{BDM}^\perp$ of bounded distributive meet semi-lattices and bounded semi-lattice homomorphisms, and the category $\text{BDM}^S$ of bounded distributive meet semi-lattices and sup-homomorphisms. We prove that $\text{BDM}$ is dually equivalent to $\text{GPS}$, that $\text{BDM}^\perp$ is dually equivalent to $\text{GPS}^T$, and that $\text{BDM}^S$ is dually equivalent to $\text{GPS}^F$. In the second subsection we introduce generalized Esakia morphisms, total generalized Esakia morphisms, and functional generalized Esakia morphisms, the category $\text{GES}$ of generalized Esakia spaces and generalized Esakia morphisms, the category $\text{GES}^T$ of generalized Esakia spaces and total generalized Esakia morphisms, and the category $\text{GES}^F$ of generalized Esakia spaces and functional generalized Esakia morphisms. We also
introduce the category $\text{BIM}$ of bounded implicative meet semi-lattices and implicative meet semi-lattice homomorphisms, the category $\text{BIM}^+ \downarrow$ of bounded implicative meet semi-lattices and bounded implicative meet semi-lattice homomorphisms, and the category $\text{BIM}^5$ of bounded implicative meet semi-lattices and implicative meet semi-lattice sup-homomorphisms. We prove that $\text{BIM}$ is dually equivalent to $\text{GES}$, that $\text{BIM}^+ \downarrow$ is dually equivalent to $\text{GES}^T$, and that $\text{BIM}^5$ is dually equivalent to $\text{GES}^F$.

8.1. The categories $\text{GPS}$, $\text{GPS}^T$, and $\text{GPS}^F$. Let $X$ and $Y$ be nonempty sets. Given a relation $R \subseteq X \times Y$, for each $A \subseteq Y$ we define

$$\Box_R A = \{x \in X : (\forall y \in Y)(xRy \Rightarrow y \in A)\} = \{x \in X : R[x] \subseteq A\}.$$ 

It is easy to verify that for each $A, B \subseteq Y$ we have $\Box_R (A \cap B) = \Box_R A \cap \Box_R B$ and $\Box_R (Y) = X$.

Let $L$ and $K$ be bounded distributive meet semi-lattices and let $h : L \to K$ be a meet semi-lattice homomorphism preserving top. We define $R_h \subseteq K \times L$ by

$$xR_h y \iff h^{-1}(x) \subseteq y$$

for each $x \in K_*$ and $y \in L_*$. We call $R_h$ the dual of $h$.

**Proposition 8.1.** Let $L$ and $K$ be bounded distributive meet semi-lattices and let $h : L \to K$ be a meet semi-lattice homomorphism preserving top. Then:

1. $(\subseteq_K, \circ_R h) \subseteq R_h$.
2. $(R_h \circ \subseteq_L ) \subseteq R_h$.
3. If $x R_h y$, then there is $a \in L$ such that $y \notin \varphi(a)$ and $R_h[x] \subseteq \varphi(a)$.
4. $\varphi(h(a)) = \Box_R \varphi(a)$.

**Proof.** (1) Suppose that $x, y \in K_*$, $z \in L_*$, and $y R_h z$. Then $h^{-1}(x) \subseteq h^{-1}(y) \subseteq z$. Thus, $x R_h z$.

(2) is proved similarly to (1).

(3) Suppose that $x R_h y$. Then $h^{-1}(x) \not\subseteq y$, so there is $a \in L$ such that $a \in h^{-1}(x)$ and $a \notin y$. Therefore, $y \notin \varphi(a)$, and if $x R_h z$, then $a \in z$. Thus, $R_h[x] \subseteq \varphi(a)$.

(4) If $x \in \varphi(h(a))$, then $a \in h^{-1}(x)$. Therefore, for each $z \in L_*$ with $x R_h z$, we have $a \in z$. Thus, $R_h[x] \subseteq \varphi(a)$, and so $\varphi(h(a)) \subseteq \Box_R \varphi(a)$. Conversely, if $x \in \Box_R \varphi(a)$, then $R_h[x] \subseteq \varphi(a)$. If $x \notin \varphi(h(a))$, then $a \notin h^{-1}(x)$. So $h^{-1}(x) \cap \downarrow a = \emptyset$, and by the prime filter lemma, there exists $y \in L_+ \subseteq L_*$ such that $h^{-1}(x) \subseteq y$ and $a \notin y$. But $h^{-1}(x) \subseteq y$ implies $y \in R_h[x]$, so $a \in y$, which is a contradiction. We conclude that $x \in \varphi(h(a))$. Thus, $\Box_R \varphi(a) \subseteq \varphi(h(a))$, and so $\varphi(h(a)) = \Box_R \varphi(a)$. \qed

**Definition 8.2.** Let $X$ and $Y$ be generalized Priestley spaces. A relation $R \subseteq X \times Y$ is called a **generalized Priestley morphism** if the following conditions are satisfied:
Lemma 8.3. Let $X$ and $Y$ be generalized Priestley spaces and $R \subseteq X \times Y$ be a generalized Priestley morphism. Then:

1. $(\leq_X \circ R) \subseteq R$.
2. $(R \circ \leq_Y) \subseteq R$.

Proof. (1) Suppose that $x \leq_X y$ and $yRz$. If $xRz$, then by condition (1) of Definition 8.2, there is $U \in Y^*$ such that $R[x] \subseteq U$ and $z \notin U$. By condition (2) of Definition 8.2, $\square_R U \in X^*$. From $R[x] \subseteq U$ it follows that $x \in \square_R U$. Because $\square_R U \in X^*$, then $\square_R U$ is an upset. Therefore, $y \in \square_R U$, so $R[y] \subseteq U$, and so $z \in U$, a contradiction. Thus, $xRz$.

(2) Suppose that $xRy$ and $y \leq_Y z$. If $xRz$, then, by condition (1) of Definition 8.2, there is $U \in Y^*$ such that $R[x] \subseteq U$ and $z \notin U$. Therefore, $y \notin U$. On the other hand, $R[x] \subseteq U$ implies $y \in U$. The obtained contradiction proves that $xRz$.

Remark 8.4. It is easy to verify that condition (1) of Lemma 8.3 is equivalent to saying that for each $B \subseteq Y$ we have $R^{-1}[B]$ is a downset of $X$, that condition (2) of Lemma 8.3 is equivalent to saying that for each $A \subseteq X$ we have $R[A]$ is an upset of $Y$, and that conditions (1) and (2) together are equivalent to $(\leq_X \circ R \circ \leq_Y) \subseteq R$.

Given a generalized Priestley morphism $R \subseteq X \times Y$, we define a map $h_R : Y^* \to X^*$ by

$$h_R(U) = \square_R U$$

for each $U \in Y^*$.

Lemma 8.5. If $R \subseteq X \times Y$ is a generalized Priestley morphism, then $h_R : Y^* \to X^*$ is a meet semi-lattice homomorphism preserving top.

Proof. Let $U, V \in Y^*$. Then $h_R(U \cap V) = \square_R (U \cap V) = \square_R U \cap \square_R V = h_R(U) \cap h_R(V)$. Moreover, $h_R(Y) = \square_R Y = X$. \hfill \Box

Proposition 8.6. Let $L$ and $K$ be bounded distributive meet semi-lattices and let $h : L \to K$ be a meet semi-lattice homomorphism preserving top. Then for each $a \in L$ we have $\varphi(h(a)) = h_{R_b}(\varphi(a))$.

Proof. We have $x \in \varphi(h(a))$ iff $h(a) \in x$ iff $a \in h^{-1}(x)$, and $x \in h_{R_b}(\varphi(a))$ iff $x \in \square_{R_b} \varphi(a)$ iff $(\forall y \in L_*)(xR_b y \Rightarrow a \in y)$ iff $(\forall y \in L_*)(h^{-1}(x) \subseteq y \Rightarrow a \in y)$. Now either $h^{-1}(x) = L$ or $h^{-1}(x)$ is a proper filter of $L$. If $h^{-1}(x) = L$, then for all $y \in L_*$ we have $h^{-1}(x) \not\subseteq y$. Therefore, both $a \in h^{-1}(x)$ and
filters of then by the optimal filter lemma, \(x\) for each \(\varphi\) and \(X\), let morphisms may not be a generalized Priestley morphism. Therefore, we have \(h\) that \(S\) be generalized Priestley morphisms. Define \(\epsilon\) such that \(R\) is \(\epsilon\). Now let \(xRy\). Then, by condition (1) of Definition 8.2, there is \(U \in Y^*\) such that \(y \notin U\) and \(R[x] \subseteq U\). Therefore, \(y \notin U\) and \(x \in h_R(U)\). Thus, we have \(U \notin \epsilon(y)\) and \(h_R(U) \in \epsilon(x)\). It follows that \(h_R^{-1}(\epsilon(x)) \subseteq \epsilon(y)\). Consequently, \(xRy\) if \(\epsilon(x)R_{h_R}\epsilon(y)\).

Unfortunately, the usual set-theoretic composition of two generalized Priestley morphisms may not be a generalized Priestley morphism. Therefore, we introduce the composition of two generalized Priestley morphisms as follows. Let \(X, Y,\) and \(Z\) be generalized Priestley spaces, and \(R \subseteq X \times Y\) and \(S \subseteq Y \times Z\) be generalized Priestley morphisms. Define \(S*R \subseteq X \times Z\) by

\[
x(S*R)z \iff \epsilon(x)R_{(h_R \circ h_S)}\epsilon(z).
\]

Note that

\[
x(S*R)z \iff (\forall U \in Z^*)(x \in \Box_R \Box_S U \Rightarrow z \in U).
\]

**Lemma 8.8.** If \(X, Y,\) and \(Z\) are generalized Priestley spaces, and \(R \subseteq X \times Y\) and \(S \subseteq Y \times Z\) are generalized Priestley morphisms, then for each \(U \in Z^*\), we have

\[
\Box_R \Box_S U = (h_R \circ h_S)(U) = \Box_{(S*R)} U.
\]

**Proof.** Let \(U \in Z^*\). Clearly \(\Box_R \Box_S U = (h_R \circ h_S)(U)\). On the other hand, for \(x \in U\), we have:

\[
x \in (h_R \circ h_S)(U) \iff (h_R \circ h_S)(U) \in \epsilon(x)
\iff U \in (h_R \circ h_S)^{-1}[\epsilon(x)]
\iff (\forall z \in Z)(\epsilon(x)R_{(h_R \circ h_S)}\epsilon(z) \Rightarrow z \in U)
\iff (\forall z \in Z)(x(S*R)z \Rightarrow z \in U)
\iff (S*R)[x] \subseteq U
\iff x \in \Box_{(S*R)} U.
\]

Therefore, \((h_R \circ h_S)(U) = \Box_{(S*R)} U\).
Lemma 8.9. If $X, Y,$ and $Z$ are generalized Priestley spaces, and $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are generalized Priestley morphisms, then $S*R \subseteq X \times Z$ is a generalized Priestley morphism.

Proof. To see that condition (1) of Definition 8.2 is satisfied, let $(x, y) \not\in S*R$. Then there is $U \in Z^*$ such that $x \in \Box_R \Box_S U$ and $z \not\in U$. By Lemma 8.8, this means that $(S*R)[x] \subseteq U$ and $z \not\in U$. To see that condition (2) of Definition 8.2 is also satisfied, let $U \in Z^*$. Then $(h_R \circ h_S)(U) \in X^*$, which, by Lemma 8.8, means that $\Box_{(S*R)} U \in X^*$. Thus, $S*R$ is a generalized Priestley morphism. \qed

Lemma 8.10. Let $X, Y, Z,$ and $W$ be generalized Priestley spaces, and $R \subseteq X \times Y, S \subseteq Y \times Z,$ and $T \subseteq Z \times W$ be generalized Priestley morphisms. Then $T*(S*R) = (T*S)*R$.

Proof. Let $x \in X$ and $w \in W$. Then, by Lemma 8.8, we have:

\begin{align*}
x[T*(S*R)]w &\iff (\forall U \in W^*)(x \in \Box_{(S*R)} \Box_T U \Rightarrow w \in U) \\
&\iff (\forall U \in W^*)(x \in \Box_R \Box_S \Box_T U \Rightarrow w \in U) \\
&\iff (\forall U \in W^*)(x \in \Box_R \Box_{(T*S)} U \Rightarrow w \in U) \\
&\iff x[(T*S)*R]w.
\end{align*}

\qed

Lemma 8.11. Let $X$ be a generalized Priestley space. Then:

1. $\leq_X \subseteq X \times X$ is a generalized Priestley morphism.
2. If $R \subseteq X \times Y$ is a generalized Priestley morphism, then $\leq_X \circ R = R$.
3. If $S \subseteq Y \times X$ is a generalized Priestley morphism, then $S \circ \leq_X = S$.

Proof. (1) follows from the definition of a generalized Priestley space, while (2) and (3) follow from Lemma 8.3 and the reflexivity of $\leq_X$. \qed

As an immediate consequence of Lemmas 8.8, 8.9, 8.10, and 8.11, we obtain that generalized Priestley spaces and generalized Priestley morphisms form a category, in which * is the composition of two morphisms and $\leq_X$ is the identity morphism for each object $X$. We denote this category by GPS. Let also BDM denote the category of bounded distributive meet semi-lattices and semi-lattice homomorphisms preserving top.

We show that BDM is dually equivalent to GPS. Define two functors $(\_)_* : BDM \to GPS$ and $(\_)^* : GPS \to BDM$ as follows. For a bounded distributive meet semi-lattice $L$, set $L_* = \langle L_*, \tau, \subseteq, L_+ \rangle$, and for a meet semi-lattice homomorphism $h$ preserving top, set $h_* = R_h$; for a generalized Priestley space $X$, let $X^*$ be the bounded distributive meet semi-lattice of admissible clopen upsets of $X$, and for a generalized Priestley morphism $R$, let $R^* = h_R$. 
In order to prove that the functors \((-\_\) and \((-)^*\) establish the dual equivalence of BDM and GPS, we define the natural transformations from the identity functor \(\text{id}_{\text{BDM}} : \text{BDM} \to \text{BDM}\) to the functor \((-\_)^* : \text{BDM} \to \text{BDM}\) and from the identity functor \(\text{id}_{\text{GPS}} : \text{GPS} \to \text{GPS}\) to the functor \((-)^*_\_ : \text{GPS} \to \text{GPS}\).

The first natural transformation associates with each object \(L\) of BDM the isomorphism \(\varphi_L : L \to L^*\); and the second natural transformation associates with each object \(X\) of GPS the generalized Priestley morphism \(R_{\epsilon_X} \subseteq X \times X^{**}\) given by

\[xR_{\epsilon_X}\epsilon(y) \iff \epsilon_X(x) \subseteq \epsilon_X(y)\]

for each \(x, y \in X\).

**Theorem 8.12.** The functors \((-\_\) and \((-)^*\) establish the dual equivalence of BDM and GPS.

**Proof.** It follows from Propositions 6.12 and 8.1 that \((-\_\) is well-defined. Proposition 6.14 and Lemma 8.5 imply that \((-)^*\) is well-defined. Now apply Theorems 6.13 and 6.21 and Propositions 8.6 and 8.7. \(\square\)

Now we turn our attention to meet semi-lattice homomorphisms preserving bottom and to sup-homomorphisms.

**Lemma 8.13.** Let \(L\) and \(K\) be bounded distributive meet semi-lattices and let \(h : L \to K\) be a meet semi-lattice homomorphism preserving top. Then:

1. \(h\) preserves bottom iff \(R_h^{-1}[L_\_] = K_\_\).
2. \(h\) is a sup-homomorphism iff \(R_h[x]\) has a least element for each \(x \in K_\_\).

**Proof.** (1) By Proposition 8.6, we have:

\[h\text{ preserves bottom } \iff \begin{align*}
    h(\bot) &= \bot \\
    \varphi(h(\bot)) &= \varphi(\bot) \\
    h_{R_h}(\varphi(\bot)) &= \varphi(\bot) \\
    h_{R_h}(\emptyset) &= \emptyset \\
    R_h^{-1}[L_\_] &= K_\_.
\end{align*}\]

(2) For each \(x \in K_\_,\) we show that \(R_h[x]\) has a least element iff \(h^{-1}[x] \in L_\_.\) If \(h^{-1}[x] \in L_\_,\) then it is clear that \(h^{-1}[x]\) is the least element of \(R_h[x]\). Conversely, let \(y\) be the least element of \(R_h[x]\). By the optimal filter lemma, \(h^{-1}[x] = \bigcap \{z \in L_\_ : h^{-1}[x] \subseteq z\} = \bigcap R_h[x] = y.\) Therefore, \(h^{-1}[x] \in L_\_.\) By Proposition 5.2, \(h\) is a sup-homomorphism iff \(h^{-1}[x] \in L_\_\) for each \(x \in K_\_.\) Thus, \(h\) is a sup-homomorphism iff \(R_h[x]\) has a least element for each \(x \in K_\_.\) \(\square\)

**Definition 8.14.** Let \(X\) and \(Y\) be generalized Priestley spaces and let \(R \subseteq X \times Y\) be a generalized Priestley morphism.

1. We call \(R\) total if \(R^{-1}[Y] = X.\)
2. We call \( R \) functional if for each \( x \in X \) there is \( y \in Y \) such that \( R[x] = \uparrow y \).

Obviously \( R \) is functional iff \( R[x] \) has a least element. It is also clear that each functional generalized Priestley morphism is total. As an immediate consequence of Theorem 8.12 and Lemma 8.13, we obtain:

**Corollary 8.15.** Let \( X \) and \( Y \) be generalized Priestley spaces and \( R \subseteq X \times Y \) be a generalized Priestley morphism. Then:

1. \( h_R \) preserves bottom iff \( R \) is total.
2. \( h_R \) is a sup-homomorphism iff \( R \) is functional.

In particular, it follows that each sup-homomorphism preserves bottom.

**Lemma 8.16.** Let \( X, Y, \) and \( Z \) be generalized Priestley spaces, and \( R \subseteq X \times Y \) and \( S \subseteq Y \times Z \) be generalized Priestley morphisms.

1. \( S \ast R \) is total whenever \( R \) and \( S \) are total.
2. \( S \ast R \) is functional whenever \( R \) and \( S \) are functional.

*Proof.* (1) Let \( R \) and \( S \) be total generalized Priestley morphisms and let \( z \in Z \). Since \( S \) and \( R \) are total, there exist \( y \in Y \) and \( x \in X \) such that \( ySz \) and \( xWy \). We show that \( x(S \ast R)z \). Let \( U \in \mathcal{Z}^* \) be such that \( x \in \mathbf{a}R \mathbf{a}S \mathbf{a}U \). Then \( y \in \mathbf{a}S \mathbf{a}U \), and so \( z \in U \). Therefore, \( x(S \ast R)z \), which implies that \( S \ast R \) is total.

(2) Let \( R \) and \( S \) be functional generalized Priestley morphisms and let \( x \in X \). Then there exist \( y \in Y \) and \( z \in Z \) such that \( R[x] = \uparrow y \) and \( S[y] = \uparrow z \). We show that \((S \ast R)[x] = \uparrow z \). Let \( U \in \mathcal{Z}^* \) and let \( x \in \mathbf{a}R \mathbf{a}S \mathbf{a}U \). From \( xWy \) it follows that \( y \in \mathbf{a}S \mathbf{a}U \); and \( ySz \) implies \( z \in U \). Therefore, \( x(S \ast R)z \). Thus, \( \uparrow z \subseteq (S \ast R)[x] \). Conversely, let \( x(S \ast R)u \). If \( z \not\subseteq u \), then there exists \( V \in \mathcal{Z}^* \) such that \( z \in V \) and \( u \not\in V \). Therefore, \( x \not\in \mathbf{a}R \mathbf{a}S \mathbf{a}V \). Thus, there exist \( y' \in Y \) and \( z' \in Z \) such that \( xRy' \), \( y'Sz' \), and \( z' \not\in V \). Since \( R[x] = \uparrow y \), we obtain \( y \leq y' \), and so \( ySz' \). From \( S[y] = \uparrow z \) it follows that \( z \leq z' \). Therefore, \( z' \in V \), a contradiction. We conclude that \( u \in \uparrow z \), so \((S \ast R)[x] \subseteq \uparrow z \), and so \((S \ast R)[x] = \uparrow z \). Thus, \( S \ast R \) is functional. \( \square \)

**Remark 8.17.** Let \( R \) and \( S \) be functional generalized Priestley morphisms. Then it is easy to verify that \( S \circ R \) is again a functional generalized Priestley morphism. Therefore, \( S \ast R = S \circ R \).

Let \( \text{GPS}^T \) denote the category of generalized Priestley spaces and total generalized Priestley morphisms. It follows from Lemma 8.16.1 that \( \text{GPS}^T \) forms a category, which is obviously a proper subcategory of \( \text{GPS} \). Let also \( \text{GPS}^F \) denote the category of generalized Priestley spaces and functional generalized Priestley morphisms. By Lemma 8.16.2, \( \text{GPS}^F \) forms a category, which is clearly a proper subcategory of \( \text{GPS}^T \).
We let $\text{BDM}^\perp$ denote the category of bounded distributive meet semi-lattices and bounded semi-lattice homomorphisms, and $\text{BDM}^\cap$ denote the category of bounded distributive meet semi-lattices and sup-homomorphisms. Similarly, we have that $\text{BDM}^\cap$ is a proper subcategory of $\text{BDM}^\perp$, and that $\text{BDM}^\perp$ is a proper subcategory of $\text{BDM}$.

**Theorem 8.18.**

1. $\text{BDM}^\perp$ is dually equivalent to $\text{GPS}^\top$.
2. $\text{BDM}^\cap$ is dually equivalent to $\text{GPS}^\cap$.

**Proof.** (1) Let $(-)_*: \text{BDM}^\perp \to \text{GPS}^\top$ and $(-)^*: \text{GPS}^\top \to \text{BDM}^\perp$ denote the restrictions of $(-)_*$ and $(-)^*$ to $\text{BDM}^\perp$ and $\text{GPS}^\top$, respectively. By Lemma 8.13.1 and Corollary 8.15.1, whenever $h: L \to K$ preserves bottom, then $R_h$ is total, and conversely, whenever $R \subseteq X \times Y$ is total, then $h_R$ preserves bottom. Now apply Theorem 8.12.

(2) Let $(-)_*: \text{BDM}^\cap \to \text{GPS}^\cap$ and $(-)^*: \text{GPS}^\cap \to \text{BDM}^\cap$ denote the restrictions of $(-)_*$ and $(-)^*$ to $\text{BDM}^\cap$ and $\text{GPS}^\cap$, respectively. By Lemma 8.13.2 and Corollary 8.15.2, whenever $h: L \to K$ is a sup-homomorphism, then $R_h$ is functional, and conversely, whenever $R \subseteq X \times Y$ is functional, then $h_R$ is a sup-homomorphism. Now apply Theorem 8.12. □

### 8.2. The categories $\text{GES}$, $\text{GES}^\top$, and $\text{GES}^\cap$.

**Proposition 8.19.** Let $L$ and $K$ be bounded implicative meet semi-lattices and let $h: L \to K$ be an implicative meet semi-lattice homomorphism. Then for each $x \in K_+$ and $y \in L_+$ we have $xR_hy$ implies there exists $z \in K_+$ such that $x \subseteq z$ and $R_h[z] = \uparrow y$.

**Proof.** Suppose that $x \in K_+$, $y \in L_+$, and $xR_hy$. Then $h^{-1}(x) \subseteq y$. We show that $\downarrow_K h(L - y)$ is an ideal of $K$. Let $a, b \in \downarrow_K h(L - y)$. Then there exist $c, d \in L - y$ such that $a \leq h(c)$ and $b \leq h(d)$. Since $y$ is a prime filter of $L$, we have $L - y$ is a (prime) ideal of $L$, so $\uparrow_L c \cap \uparrow_L d \cap (L - y) \neq \emptyset$. Therefore, there exists $e \in \uparrow_L c \cap \uparrow_L d$ such that $e \notin y$. Thus, $a \leq h(e)$, $b \leq h(e)$, and $h(e) \in \downarrow g h(L - y)$. It follows that $\uparrow_K a \cap \uparrow_K b \cap \downarrow_K h(L - y) \neq \emptyset$, and so $\downarrow_K h(L - y)$ is an ideal of $K$. Let $F$ be the filter of $K$ generated by $x \cup h(y)$. If $a \in F \cap \downarrow_K h(L - y)$, then there exist $b \in x$, $c \in y$, and $d \in L - y$ such that $b \land h(c) \leq a \leq h(d)$. Then $b \leq h(c \to d)$. Therefore, $h(c \to d) \in x$, and so $c \to d \in y$. Now $c \in y$ and $c \to d \in y$ imply $d \in y$, a contradiction. Thus, $F \cap \downarrow_K h(L - y) = \emptyset$, and so, by the prime filter lemma, there exists a prime filter $z$ of $K$ such that $F \subseteq z$ and $z \cap \downarrow_K h(L - y) = \emptyset$. Then $x \subseteq z$ and $h^{-1}(z) = y$. Thus, there exists $z \in K_+$ such that $x \subseteq z$ and $R_h[z] = \uparrow y$. □

Let $L$ and $K$ be bounded implicative meet semi-lattices and let $h: L \to K$ be an implicative meet semi-lattice homomorphism. Consider the following
condition, which is a strengthening of the condition of Proposition 8.19: For each \( x \in K \) and \( y \in L \), we have \( xRhy \) implies there exists \( z \in K \) such that \( x \subseteq z \) and \( R_h[z] = \uparrow y \). The next example shows that this stronger condition does not necessarily hold.

**Example 8.20.** Let \( X \) and \( Y \) be generalized Esakia spaces shown in Fig.4, where \( Y_0 = Y - \{ z \} \), and as a topological space, \( Y \) is the one-point compactification of \( Y_0 \) (as a discrete space). Then each point of \( Y_0 \) is an isolated point of \( Y \), and \( z \) is the only limit point of \( Y \). Let \( R \subseteq X \times Y \) be defined as follows: \( R[x_1] = \uparrow y_1 \), \( R[x_2] = \{ y_2 \} \), and \( R[x_3] = \{ y_3 \} \). It is easy to verify that \( R \) is a generalized Esakia morphism, that \( x_1Rz \), but that there is no \( x \in X \) such that \( R[z] = \uparrow z \). Thus, the condition above is not satisfied. The dual implicative meet semi-lattice \( X^* \simeq D(X^*) \) of \( X \), which is a Heyting algebra, together with the dual implicative meet semi-lattice \( Y^* \) and its distributive envelope \( D(Y^*) \) are shown in Fig.4. The elements of \( Y^* \) are depicted in the black circles, while the only element of \( D(Y^*) - Y^* \) is depicted in the white circle. The implicative meet semi-lattice homomorphism \( h_R : Y^* \to X^* \) is shown in Fig.4 by means of brown arrows. Note that \( h_R \) is undefined on \( \{ y_2, y_3 \} \), thus \( h_R \) is not a homomorphism from \( D(Y^*) \) to \( D(X^*) \simeq X^* \).

**Definition 8.21.** Let \( X \) and \( Y \) be generalized Esakia spaces. We call a generalized Priestley morphism \( R \subseteq X \times Y \) a generalized Esakia morphism if for each \( x \in X \) and \( y \in Y_0 \), from \( xRy \) it follows that there exists \( z \in X_0 \) such that \( x \leq z \) and \( R[z] = \uparrow y \).

**Proposition 8.22.** Let \( X \) and \( Y \) be generalized Esakia spaces and let \( R \subseteq X \times Y \) be a generalized Esakia morphism. Then \( h_R : Y^* \to X^* \) is an implicative meet semi-lattice homomorphism.
Proof. Since \( R \subseteq X \times Y \) is a generalized Esakia morphism, it is a generalized Priestley morphism, so \( h_R : Y^* \to X^* \) is a meet semi-lattice homomorphism preserving top. We show that for each \( U, V \in Y^* \), we have \( h_R(U \to V) = h_R(U) \to h_R(V) \). Suppose that \( x \in h_R(U \to V) \). Then \( R[x] \subseteq U \to V \). If \( x \notin h_R(U) \to h_R(V) \), then there exists \( y \in \uparrow x \cap h_R(U) \) such that \( y \notin h_R(V) \). Therefore, \( R[y] \subseteq U \) and \( R[y] \not\subseteq V \). Thus, there exists \( z \in Y \) such that \( yRz \) and \( z \notin V \). Since \( x \leq y \), we have \( R[y] \subseteq R[x] \). Therefore, \( R[y] \subseteq U \to V \), and so \( z \in U \to V \). This together with \( z \in U \) imply \( z \in V \), a contradiction. Thus, \( x \in h_R(U) \to h_R(V) \), and so \( h_R(U \to V) \subseteq h_R(U) \to h_R(V) \). Conversely, suppose that \( x \in h_R(U) \to h_R(V) \) and \( x \notin h_R(U \to V) \). Then \( \uparrow x \cap h_R(U) \subseteq h_R(V) \) and \( R[x] \not\subseteq U \to V \). Therefore, there exists \( y \in Y \) such that \( xRy \) and \( y \notin U \to V = [\{(U - V)^c\}]^c \). Thus, \( y \in \{U - V\} \). Since \( Y \) is a generalized Esakia space, \( \{U - V\} \) is clopen in \( Y \) and \( \max[\{(U - V)\}] = \max(U - V) \subseteq Y_0 \). Therefore, there exists \( z \in \max(U - V) \subseteq Y_0 \) such that \( y \leq z \). Then \( xRy \leq z \in Y_0 \), so \( xRxz \in Y_0 \), and since \( R \) is a generalized Esakia morphism, there exists \( u \in X_0 \) such that \( x \leq u \) and \( R[u] = \uparrow z \). As \( z \in U \), we have \( R[u] \subseteq U \), so \( u \in h_R(U) \). Therefore, \( u \in \uparrow x \cap h_R(U) \subseteq h_R(V) \), so \( R[u] \subseteq V \). Thus, \( z \in \uparrow z = R[u] \subseteq V \), a contradiction. Consequently, our assumption that \( x \notin h_R(U \to V) \) is false, so \( h_R(U) \to h_R(V) \subseteq h_R(U \to V) \), and so \( h_R(U \to V) = h_R(U) \to h_R(V) \). \( \square \)

Lemma 8.23. Let \( X, Y, \) and \( Z \) be generalized Esakia spaces, and \( R \subseteq X \times Y \) and \( S \subseteq Y \times Z \) be generalized Esakia morphisms. Then \( S \circ R \) is a generalized Esakia morphism.

Proof. By Proposition 8.22, \( h_R : Y^* \to X^* \) and \( h_S : Z^* \to Y^* \) are implicative meet semi-lattice homomorphisms. Therefore, \( h_R \circ h_S : Z^* \to X^* \) is an implicative meet semi-lattice homomorphism. By Lemma 8.8, \( h_{(S \circ R)} = h_R \circ h_S \). Therefore, \( h_{(S \circ R)} \) is an implicative meet semi-lattice homomorphism. By Proposition 8.19, \( R_{h_{(S \circ R)}} \) is a generalized Esakia morphism. This, by Theorem 8.12, implies that \( S \circ R \) is a generalized Esakia morphism. \( \square \)

Let \( \text{GES} \) denote the category of generalized Esakia spaces and generalized Esakia morphisms, in which \( \circ \) is the composition of two morphisms and \( \leq_X \) is the identity morphism for each object \( X \). Let also \( \text{GES}^T \) denote the subcategory of \( \text{GES} \) whose objects are generalized Esakia spaces and whose morphisms are total generalized Esakia morphisms, and \( \text{GES}^F \) denote the subcategory of \( \text{GES}^T \) whose objects are generalized Esakia spaces and whose morphisms are functional generalized Esakia morphisms. Clearly \( \text{GES}^F \) is a proper subcategory of \( \text{GES}^T \) and \( \text{GES}^T \) is a proper subcategory of \( \text{GES} \).

We let \( \text{BIM} \) denote the category of bounded implicative meet semi-lattices and implicative meet semi-lattice homomorphisms, \( \text{BIM}^+ \) denote the category
of bounded implicative meet semi-lattices and bounded implicative meet semi-lattice homomorphisms, and \( \text{BIM}^S \) denote the category of bounded implicative meet semi-lattices and implicative meet semi-lattice sup-homomorphisms. We have that \( \text{BIM}^S \) is a proper subcategory of \( \text{BIM}^\perp \) and that \( \text{BIM}^\perp \) is a proper subcategory of \( \text{BIM} \).

**Theorem 8.24.**

1. The category \( \text{BIM} \) is dually equivalent to the category \( \text{GES} \).
2. The category \( \text{BIM}^\perp \) is dually equivalent to the category \( \text{GES}^T \).
3. The category \( \text{BIM}^S \) is dually equivalent to the category \( \text{GES}^F \).

**Proof.** Apply Theorems 8.12 and 8.18 and Propositions 7.5, 8.19, and 8.22. \(\square\)

### 9. Functional morphisms

In this section we show that functional generalized Priestley (resp. Esakia) morphisms can be characterized by means of special functions between generalized Priestley (resp. Esakia) spaces we call strong Priestley (resp. Esakia) morphisms.

#### 9.1. Functional generalized Priestley morphisms

Let \( X \) and \( Y \) be Priestley spaces. We recall that a map \( f : X \to Y \) is a Priestley morphism if \( f \) is continuous and order-preserving.

**Definition 9.1.** Let \( X \) and \( Y \) be generalized Priestley spaces. We call a map \( f : X \to Y \) a strong Priestley morphism if \( f \) is order-preserving and \( U \in Y^* \) implies \( f^{-1}(U) \in X^* \).

Since \( X^* \cup \{U^c : U \in X^*\} \) and \( Y^* \cup \{V^c : V \in Y^*\} \) form subbases for the Priestley topologies on \( X \) and \( Y \), respectively, and \( f^{-1}(V^c) = f^{-1}(V)^c \) for each \( V \subseteq Y \), it follows that each strong Priestley morphism is a continuous function, hence a Priestley morphism. We note that the composition of strong Priestley morphisms is again a strong Priestley morphism, and that the identity map \( \text{id}_X : X \to X \) is a strong Priestley morphism. Therefore, generalized Priestley spaces and strong Priestley morphisms form a category in which composition is the usual set-theoretic composition of functions and the identity morphism is the usual identity function. We denote this category by \( \text{PS}^S \).

Let \( X \) and \( Y \) be generalized Priestley spaces and \( R \subseteq X \times Y \) be a functional generalized Priestley morphism. We define \( f^R : X \to Y \) by

\[
f^R(x) = \text{the least element of } R[x].
\]

**Lemma 9.2.** If \( X \) and \( Y \) are generalized Priestley spaces and \( R \subseteq X \times Y \) is a functional generalized Priestley morphism, then \( f^R : X \to Y \) is a strong Priestley morphism. Moreover, if \( X, Y, \) and \( Z \) are generalized Priestley spaces
and $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are functional generalized Priestley morphisms, then $f^{S \circ R} = f^S \circ f^R$.

**Proof.** Let $x, y \in X$. If $x \leq y$, then $R[y] \subseteq R[x]$, and so $f^R(x) \leq f^R(y)$. Therefore, $f^R$ is order-preserving. Now let $U \in Y^*$. Then $x \in (f^R)^{-1}(U)$ iff $f^R(x) \in U$ iff $\uparrow f^R(x) \subseteq U$ iff $R[x] \subseteq U$ iff $x \in \Box_R(U)$. Thus, $(f^R)^{-1}(U) = \Box_R U$, so $(f^R)^{-1}(U) \in X^*$, and so $f^R$ is a strong Priestley morphism. Moreover, since for functional generalized Priestley morphisms we have that $\ast$ coincides with $\circ$, it is easy to see that if $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are functional generalized Priestley morphisms, then $f^{S \circ R} = f^{R \circ S} = f^S \circ f^R$. \qed

Now let $X$ and $Y$ be generalized Priestley spaces and $f : X \to Y$ be a strong Priestley morphism. Define $R^f \subseteq X \times Y$ by

$$xR^fy \iff f(x) \leq y.$$  

**Lemma 9.3.** If $X$ and $Y$ are generalized Priestley spaces and $f : X \to Y$ is a strong Priestley morphism, then $R^f$ is a functional generalized Priestley morphism. Moreover, if $X$, $Y$, and $Z$ are generalized Priestley spaces and $f : X \to Y$ and $g : Y \to Z$ are strong Priestley morphisms, then $R^{g \circ f} = R^g \ast R^f$.

**Proof.** Let $x \in X$ and $y \in Y$. If $xR^fy$, then $f(x) \leq y$. Therefore, there exists $U \in Y^*$ such that $f(x) \in U$ and $y \notin U$. Thus, $R^f[x] = \uparrow f(x) \subseteq U$ and $y \notin U$, and so condition (1) of Definition 8.2 is satisfied. Let $U \in Y^*$. For $x \in X$ we have $x \in \Box_{R^f} U$ iff $R^f[x] \subseteq U$ iff $\uparrow f(x) \subseteq U$ iff $f(x) \in U$ iff $x \in f^{-1}(U)$. Therefore, $\Box_{R^f}(U) = f^{-1}(U)$, so $\Box_{R^f}(U) \in X^*$ and so condition (2) of Definition 8.2 is satisfied. Thus, $R^f$ is a generalized Priestley morphism. Now, since $R^f[x] = \uparrow f(x)$ for each $x \in X$, we have that $R^f$ is a functional generalized Priestley morphism. Moreover, since for functional generalized Priestley morphisms we have that $\ast$ coincides with $\circ$, it is easy to see that if $f : X \to Y$ and $g : Y \to Z$ are strong Priestley morphisms, then $R^{g \circ f} = R^f \circ R^g = R^g \ast R^f$. \qed

**Lemma 9.4.** Let $X$ and $Y$ be generalized Priestley spaces, $R \subseteq X \times Y$ be a functional generalized Priestley morphism, and $f : X \to Y$ be a strong Priestley morphism. Then $R^{f^R} = R$ and $f^{R^f} = f$.

**Proof.** For each $x \in X$ let $l_x$ be the least element of $R[x]$. Then $xR^{f^R}y$ iff $f^{R^f}(x) \leq y$ iff $l_x \leq y$ iff $xRl_x \leq y$ iff $xRy$. Thus, $R^{f^R} = R$. Also, $f^{R^f}(x) = \{ f(x) \}$, so $f^{R^f} = f$. \qed

As an immediate consequence of Lemmas 9.2, 9.3, and 9.4, we obtain:

**Proposition 9.5.** The categories $\text{GPS}^F$ and $\text{GPS}^S$ are isomorphic.
This together with Theorem 8.18.2, immediately give us:

**Theorem 9.6.** The categories BDM$^5$ and GPS$^5$ are dually equivalent.

An explicit construction of functors from BDM$^5$ to GPS$^5$ and vice versa can be obtained based on the following observation.

**Lemma 9.7.**

1. Let $X$ and $Y$ be generalized Priestley spaces and $f : X \to Y$ be a strong Priestley morphism. Then for each $U \in Y^*$ we have $h_{R^f}(U) = f^{-1}(U)$.

2. Let $L$ and $K$ be bounded distributive meet semi-lattices and $h : L \to K$ be a sup-homomorphism. Then $f^{K_h}(y) = h^{-1}(y)$ for each $y \in K_*$.

**Proof.** (1) Let $x \in h_{R^f}(U) = \Box_{R^f} U$. Then $xR^f y$ implies $y \in U$. Since $xR^f f(x)$, it follows that $f(x) \in U$, and so $x \in f^{-1}(U)$. Now let $x \in f^{-1}(U)$. Then $f(x) \in U$. Therefore, if $xR^f y$, then $f(x) \leq y$, and so $y \in U$. Thus, $x \in \Box_{R^f} U = h_{R^f}(U)$, and so $h_{R^f}(U) = f^{-1}(U)$.

(2) Since $h$ is a sup-homomorphism, $R_h$ is a functional generalized Priestley morphism, and $h^{-1}(y)$ is the least element of $R_h[y]$. Thus, $a \in f^{K_h}(y)$ iff $a$ belongs to the least element of $R_h[y]$ iff $a \in h^{-1}(y)$.

Now we can define the functors $(-)_* : \text{BDM}^5 \to \text{GPS}^5$ and $(-)^* : \text{GPS}^5 \to \text{BDM}^5$ explicitly as follows: If $L$ is a bounded distributive meet semi-lattice, then $L_* = L_*$ and if $h : L \to K$ is a sup-homomorphism, then $h_* = h^{-1}$; also, if $X$ is a generalized Priestley space, then $X^* = X^*$, and if $f : X \to Y$ is a strong Priestley morphism, then $f^* = f^{-1}$. Therefore, the functors $(-)_* : \text{BDM}^5 \to \text{GPS}^5$ and $(-)^* : \text{GPS}^5 \to \text{BDM}^5$ behave exactly like the Priestley functors $(-)_* : \text{BDL} \to \text{PS}$ and $(\cdot)^* : \text{PS} \to \text{BDL}$.

**9.2. Functional generalized Esakia morphisms.** Our next task is to discuss the strong Priestley morphisms between generalized Esakia spaces which correspond to the functional generalized Esakia morphisms.

**Lemma 9.8.** Let $X$ and $Y$ be generalized Esakia spaces and $f : X \to Y$ be a strong Priestley morphism. Then $R^f$ is a generalized Esakia morphism iff for each $x \in X$ and $y \in Y_0$, from $f(x) \leq y$ it follows that there exists $z \in X_0$ such that $x \leq z$ and $f(z) = y$.

**Proof.** Suppose that $R^f$ is a generalized Esakia morphism. Let $x \in X$ and $y \in Y_0$. If $f(x) \leq y$, then $xR^f y$. Therefore, there exists $z \in X_0$ such that $x \leq z$ and $R^f[z] = \uparrow y$. It follows from the definition of $R^f$ that $f(z) = y$. Conversely, suppose that for each $x \in X$ and $y \in Y_0$, from $f(x) \leq y$ it follows that there exists $z \in X_0$ such that $x \leq z$ and $f(z) = y$. Let $x \in X$ and $y \in Y_0$. If $xR^f y$, then $f(x) \leq y$. Therefore, there exists $z \in X_0$ such that $x \leq z$ and $f(z) = y$. Thus, $R^f[z] = \uparrow y$, and so $R^f$ is a generalized Esakia morphism. □
Definition 9.9. Let $X$ and $Y$ be generalized Esakia spaces. We call a map $f : X \to Y$ a strong Esakia morphism if it is a strong Priestley morphism such that for each $x \in X$ and $y \in Y_0$, from $f(x) \leq y$ it follows that there exists $z \in X_0$ such that $x \leq z$ and $f(z) = y$.

As an immediate consequence of Lemmas 9.4 and 9.8, we obtain:

Lemma 9.10. Let $X$ and $Y$ be generalized Esakia spaces and $R \subseteq X \times Y$ be a functional generalized Priestley morphism. Then $f^R$ is a strong Esakia morphism iff $R$ is a generalized Esakia morphism.

Let $GES^S$ denote the category of generalized Esakia spaces and strong Esakia morphisms. By Proposition 9.5 and Lemmas 9.8 and 9.10, we obtain:

Proposition 9.11. The categories $GES^S$ and $GES^F$ are isomorphic.

Let also $BIM^S$ denote the category of bounded implicative meet semi-lattices and implicative meet semi-lattice sup-homomorphisms. As an immediate consequence of Theorem 8.18 and Proposition 9.11, we obtain:

Theorem 9.12. The categories $BIM^S$ and $GES^S$ are dually equivalent.

10. Priestley and Esakia dualities as particular cases

In this section we show how the well-known Priestley and Esakia dualities are particular cases of our dualities for bounded distributive meet semi-lattices and bounded implicative meet semi-lattices, respectively. We also obtain an application to modal logic by showing that descriptive frames, which are duals of modal algebras, are exactly generalized Priestley morphisms of Stone spaces into themselves. Finally, we give a new dual representation of Heyting algebra homomorphisms by means of special partial functions. This yields a new duality for Heyting algebras, which is an alternative to the Esakia duality.

10.1. Priestley duality as a particular case. We start by showing how the Priestley duality between the category $BDL$ of bounded distributive lattices and bounded lattice homomorphisms on the one hand and the category $PS$ of Priestley spaces and Priestley morphisms on the other follows from Theorem 9.6.

Let $L$ be a bounded distributive lattice. Then $L_* = L_+$, and so $\langle L_*, \tau, \leq \rangle = \langle L_+, \tau, \subseteq \rangle$ is a Priestley space. Conversely, if $X = \langle X, \tau, \leq \rangle$ is a Priestley space, then $X^* = \mathfrak{U}(X)$. Therefore, given two Priestley spaces $X$ and $Y$, a map $f : X \to Y$ is a strong Priestley morphism iff $f$ is order-preserving and $V \in \mathfrak{U}(Y)$ implies $f^{-1}(V) \in \mathfrak{U}(X)$. Because $\mathfrak{U}(X) \cup \{U^c : U \in \mathfrak{U}(X)\}$ and $\mathfrak{U}(Y) \cup \{V^c : V \in \mathfrak{U}(Y)\}$ are subbases for the Priestley topologies on $X$ and $Y$, respectively, the last condition is equivalent to $f$ being continuous. Thus, the notions of a strong Priestley morphism and of a Priestley morphism coincide.
Lemma 10.1. Let $L$ and $K$ be bounded distributive lattices and $h : L \to K$ be a bounded meet semi-lattice homomorphism. Then the following conditions are equivalent:

1. $h$ preserves $\lor$.
2. $h$ is a sup-homomorphism.
3. $h^{-1}(x) \in L_+$ for each $x \in K_+$.

Proof. This is an immediate consequence of Propositions 5.2 and 5.3 and the fact that $L_* = L_+$ and $K_* = K_+$. □

The well-known Priestley duality is now an immediate consequence of Theorem 9.6 and Lemmas 9.7 and 10.1.

Corollary 10.2. The category $BDL$ is dually equivalent to the category $PS$.

Let $BDL^{\wedge,T}$ denote the category of bounded distributive lattices and meet semi-lattice homomorphisms preserving top and let $BDL^{\wedge,T,\perp}$ denote the category of bounded distributive lattices and bounded meet semi-lattice homomorphisms. Clearly $BDL$ is a proper subcategory of $BDL^{\wedge,T,\perp}$ and $BDL^{\wedge,T}$ is a proper subcategory of $BDL^{\wedge,T}$. Let $PS^R$ denote the category of Priestley spaces and generalized Priestley morphisms, $PS^T$ denote the category of Priestley spaces and total generalized Priestley morphisms, and $PS^F$ denote the category of Priestley spaces and functional generalized Priestley morphisms. Clearly $PS^F$ is a proper subcategory of $PS^T$ and $PS^T$ is a proper subcategory of $PS$. As an immediate consequence of Theorems 8.12 and 8.18 and Proposition 9.5, we obtain:

Corollary 10.3.

1. The category $BDL^{\wedge,T}$ is dually equivalent to the category $PS^R$.
2. The category $BDL^{\wedge,T,\perp}$ is dually equivalent to the category $PS^T$.
3. The category $BDL$ is dually equivalent to the category $PS^F$, which is isomorphic to $PS$.

10.1.1. Application to modal logic. Our results have an application to modal logic. Let $(B, \Box)$ be a modal algebra; that is, $B$ is a Boolean algebra and $\Box : B \to B$ satisfies $\Box(a \land b) = \Box(a) \land \Box(b)$ and $\Box \top = \top$. It follows from the well-known duality for modal algebras that the dual objects corresponding to modal algebras are descriptive frames $(X, R)$, where $X$ is a Stone space and $R$ is a binary relation on $X$ such that (i) $R[x]$ is closed for each $x \in X$, and (ii) $R^{-1}[U]$ is clopen whenever $U$ is clopen. Obviously $\Box : B \to B$ is a particular case of a meet semi-lattice homomorphism (between Boolean algebras) preserving top. We show that for Stone spaces $X$, generalized Priestley morphisms $R \subseteq X \times X$ are exactly those binary relations $R$ on $X$ for which $(X, R)$ is a descriptive frame. Let $X$ be a Stone space and let $R \subseteq X \times X$ be a generalized Priestley
morphism. Since $\leq_X$ is simply equality, $X^*$ coincides with the Boolean algebra of clopen subsets of $X$. Therefore, condition (2) of Definition 8.2 is equivalent to condition (ii). We show that condition (1) of Definition 8.2 is equivalent to condition (i). Suppose that condition (1) is satisfied and $y \not\in R[x]$. Then $x R y$, and by condition (1), there exists a clopen subset $U$ of $X$ such that $R[x] \subseteq U$ and $y \notin U$. Thus, $R[x]$ is closed and condition (i) is satisfied. Conversely suppose that condition (i) is satisfied and $x R y$. Then $y \notin R[x]$. Since $R[x]$ is closed and $X$ is a Stone space, there exists a clopen subset $U$ of $X$ such that $R[x] \subseteq U$ and $y \notin U$. Thus, condition (1) is satisfied.

10.2. Esakia duality as a particular case. Now we show that Esakia’s duality between the category $\mathcal{HA}$ of Heyting algebras and Heyting algebra homomorphisms and the category $\mathcal{ES}$ of Esakia spaces and Esakisa morphisms follows from Theorem 9.12. As with Priestley spaces, Esakia spaces are simply generalized Esakis spaces $X = \langle X, \leq, \tau, X_0 \rangle$ in which $X_0 = X$. Consequently, the concepts of an Esakia morphism and of a strong Esakia morphism coincide. Thus, by Lemmas 9.2, 9.3, 9.8, and 9.10, for Esakia spaces $X$ and $Y$, if $R \subseteq X \times Y$ is a functional generalized Esakia morphism, then $f^R : X \to Y$ is an Esakia morphism, and if $f : X \to Y$ is an Esakia morphism, then $R^f \subseteq X \times Y$ is a functional generalized Esakia morphism. This together with Theorem 9.12 gives us the well-known Esakia duality:

**Corollary 10.4.** The category $\mathcal{HA}$ of Heyting algebras and Heyting algebra homomorphisms is dually equivalent to the category $\mathcal{ES}$.

Let $\mathcal{HA}^{\land, \to}$ denote the category of Heyting algebras and implicative meet semi-lattice homomorphisms, and let $\mathcal{HA}^{\land, \to, \bot}$ denote the category of Heyting algebras and bounded implicative meet semi-lattice homomorphisms. Clearly $\mathcal{HA}$ is a proper subcategory of $\mathcal{HA}^{\land, \to, \bot}$ and $\mathcal{HA}^{\land, \to, \bot}$ is a proper subcategory of $\mathcal{HA}^{\land, \to}$. Let $\mathcal{ES}^R$ denote the category of Esakia spaces and generalized Esakia morphisms, $\mathcal{ES}^T$ denote the category of Esakia spaces and total generalized Esakia morphisms, and $\mathcal{ES}^F$ denote the category of Esakia spaces and functional generalized Esakia morphisms. Clearly $\mathcal{ES}^F$ is a proper subcategory of $\mathcal{ES}^T$ and $\mathcal{ES}^T$ is a proper subcategory of $\mathcal{ES}^R$. As an immediate consequence of Theorem 8.24 and Corollary 10.3, we obtain:

**Corollary 10.5.**

1. The category $\mathcal{HA}^{\land, \to}$ is dually equivalent to the category $\mathcal{ES}^R$.
2. The category $\mathcal{HA}^{\land, \to, \bot}$ is dually equivalent to the category $\mathcal{ES}^T$.
3. The category $\mathcal{HA}$ is dually equivalent to the category $\mathcal{ES}^F$, which is isomorphic to $\mathcal{ES}$.

10.3. Partial Esakia functions. Now we give an alternative description of morphisms of $\mathcal{ES}^R$, $\mathcal{ES}^T$, and $\mathcal{ES} \simeq \mathcal{ES}^F$ by means of special partial functions
between Esakia spaces, thus obtaining new dualities for HA\(^{\land,-}\), HA\(^{\land,-,\perp}\), and HA. The new duality for HA is an alternative to the Esakia duality.

Let X and Y be Esakia spaces and let \( R \subseteq X \times Y \) be a generalized Esakia morphism. We define an equivalence relation \( E_R \) on X by

\[
x E_R y \text{ iff } R[x] = R[y].
\]

**Lemma 10.6.** Let X and Y be Esakia spaces and let \( R \subseteq X \times Y \) be a generalized Esakia morphism. For each \( x \in X \) we have that the equivalence class \( E_R[x] \) is closed.

**Proof.** Let \( y \notin E_R[x] \). Then \( R[x] \nsubseteq R[y] \) or \( R[y] \nsubseteq R[x] \). Therefore, there exists \( u \in Y \) such that \( xRu \) and \( yRu \) or there exists \( v \in Y \) such that \( xRv \) and \( yRv \). Since \( R \) is a generalized Esakia morphism, hence a generalized Priestley morphism, there exists a clopen upset \( U \) of \( Y \) such that \( R[y] \subseteq U \) and \( u \notin U \) or there exists a clopen upset \( V \) of \( Y \) such that \( R[x] \subseteq V \) and \( v \notin V \). Thus, \( y \in \Box_R U \) and \( E_R[x] \cap \Box_R U = \emptyset \) or \( E_R[x] \subseteq \Box_R V \) and \( y \notin \Box_R V \), and so \( E_R[x] \cap (\Box_R V)^c = \emptyset \) and \( y \in (\Box_R V)^c \). In either case, there exists a clopen subset \( W \) of \( X \) (\( W = \Box_R U \) or \( W = (\Box_R V)^c \)) such that \( y \in W \) and \( E_R[x] \cap W = \emptyset \). Consequently, \( E_R[x] \) is a closed subset of \( X \).

Since \( X \) is an Esakia space, hence a Priestley space, and \( E_R[x] \) is closed, for each \( y \in E_R[x] \) there exists \( z \in \max E_R[x] \) such that \( y \leq z \). We define a partial function \( f_R : X \to Y \) as follows. Let

\[
dom(f_R) = \{ x \in X : R[x] \text{ has a least element and } x \in \max E_R[x] \},
\]

and for \( x \in \dom(f_R) \) let

\[
f_R(x) = \text{the least element of } R[x].
\]

For \( x, y \in X \) we use the standard notation \( x < y \) whenever \( x \leq y \) and \( x \neq y \).

**Lemma 10.7.** Let X and Y be Esakia spaces and let \( R \subseteq X \times Y \) be a generalized Esakia morphism. Then:

1. For \( x, y \in \dom(f_R) \), if \( x < y \), then \( f_R(x) < f_R(y) \).
2. For \( x \in \dom(f_R) \) and \( y \in Y \), from \( f_R(x) < y \) it follows that there exists \( z \in \dom(f_R) \) such that \( x < z \) and \( f_R(z) = y \).
3. For \( x \in X \) and \( y \in Y \), we have \( xRy \) iff there exists \( z \in \dom(f_R) \) such that \( x \leq z \) and \( f_R(z) = y \).
4. If \( U \in \Cu(Y) \), then \( \Box_R U = (\downarrow f_R^{-1}(U^c))^c \).
5. For \( x \in X \) and \( y \in Y \), if \( y \notin f_R[x] \), then there exists \( U \in \Cu(Y) \) such that \( x \in (\downarrow f_R^{-1}(U^c))^c \) and \( y \notin U \).

In addition, if \( R \) is total, then \( \max X \subseteq \dom(f_R) \).
Proof. Let $X$ and $Y$ be Esakia spaces and let $R \subseteq X \times Y$ be a generalized Esakia morphism. To see (1), let $x, y \in \text{dom}(f_R)$ with $x < y$. Then $x \leq y$, and so $R[y] \subseteq R[x]$. Thus, $f_R(y) \in R[x]$, and so $f_R(x) \leq f_R(y)$. If $f_R(x) = f_R(y)$, then $R[x] = R[y]$. Therefore, $xE_Ry$ and $x < y$, so $x \notin \max E_R[x]$, a contradiction. Thus, $f_R(x) < f_R(y)$. To see (2), let $x \in \text{dom}(f_R)$, $y \in Y$, and $f_R(x) < y$. Then $f_R(x) \leq y$, so $xRy$. Since $R$ is a generalized Esakia morphism, there exists $z \in X$ such that $x \leq z$ and $R[z] = \uparrow y$. If $x = z$, then $R[x] = R[z]$, and so $R[x] = \uparrow y$, which implies that $f_R(x) = y$, a contradiction. Thus, $x < z$. Let $u \in \max E_R[z]$ be such that $z \leq u$. Then $x < u$, $u \in \text{dom}(f_R)$, and $R[u] = R[z] = \uparrow y$. Thus, $f_R(u) = y$. For (3), it is clear that if there exists $z \in \text{dom}(f_R)$ such that $x \leq z$ and $f_R(z) = y$, then $x \leq zRy$, and so $xRy$. Conversely, if $xRy$, then as $R$ is a generalized Esakia morphism, there exists $z \in X$ such that $x \leq z$ and $R[z] = \uparrow y$. Let $u \in \max E_R[z]$ be such that $z \leq u$. Then $x \leq u$, $u \in \text{dom}(f_R)$, and $f_R(u) = y$. To see (4), let $U \in \mathcal{U}(Y)$. We have that $x \in \Box_R U$ iff $R[x] \subseteq U$, and that $x \in (\downarrow f_R^{-1}(U^c))^c$ iff $(\forall z \in \text{dom}(f_R))(x \leq z$ implies $f_R(z) \in U)$. First suppose that $x \in \Box_R U$, $z \in \text{dom}(f_R)$, and $x \leq z$. Then $f_R(z) \in R[z] \subseteq R[x] \subseteq U$, so $f_R(z) \in U$, and so $x \in (\downarrow f_R^{-1}(U^c))^c$. Now suppose that $x \in (\downarrow f_R^{-1}(U^c))^c$ and $xRy$. Since $R$ is a generalized Esakia morphism, there exists $z \in X$ such that $x \leq z$ and $R[z] = \uparrow y$. Let $m \in \max E_R[z]$ be such that $z \leq m$. Then $m \in \text{dom}(f_R)$, $x \leq m$, and $f_R(m) = y$. Thus, $f_R(m) \in U$, so $y \in U$, and $x \in \Box_R U$. To see (5), let $x \in X$, $y \in Y$, and $y \notin f_R[x]$. By (3), $f_R[x] = R[x]$. Therefore, $y \notin f_R[x]$ implies $xRy$. Since $R$ is a generalized Esakia morphism, hence a generalized Priestley morphism, there exists $U \in \mathcal{U}(Y)$ such that $x \in \Box_R U$ and $y \notin U$. By (4), $\Box_R U = (\downarrow f_R^{-1}(U^c))^c$. Thus, $x \in (\downarrow f_R^{-1}(U^c))^c$ and $y \notin U$. Finally, let $R$ be total and let $x \in \text{max}X$. Then there exists $y \in Y$ such that $xRy$. From $y \in Y$ it follows that there exists $u \in \text{max}Y$ such that $y \leq u$. So $xRy \leq u$, implying that $xRu$. Since $R$ is a generalized Esakia morphism, there exists $z \in X$ such that $x \leq z$ and $R[z] = \uparrow u = \{u\}$. But $x \in \text{max}X$ and $x \leq z$ imply $x = z$. Therefore, $x \in \max E_R[x]$ and $R[x] = \{u\}$. Thus, $x \in \text{dom}(f_R)$ and $f_R(x) = u$. Consequently, $\text{max}X \subseteq \text{dom}(f_R)$. \qed

Lemma 10.7 motivates the following definition of a partial Esakia function between Esakia spaces.

**Definition 10.8.** Let $X$ and $Y$ be Esakia spaces and let $f : X \to Y$ be a partial function. We call $f$ a partial Esakia function if $f$ satisfies the following four conditions:

1. For $x, y \in \text{dom}(f)$, if $x < y$, then $f(x) < f(y)$.
2. For $x \in \text{dom}(f)$ and $y \in Y$, from $f(x) < y$ it follows that there exists $z \in \text{dom}(f)$ such that $x < z$ and $f(z) = y$.
3. If $U \in \mathcal{U}(Y)$, then $(\downarrow f_R^{-1}(U^c))^c \in \mathcal{U}(X)$.
4. For \( x \in X \) and \( y \in Y \), if \( y \notin f([x]) \), then there exists \( U \in \mathfrak{U}(Y) \) such that \( f([x]) \subseteq U \) and \( y \notin U \).

If in addition \( \max X \subseteq \text{dom}(f) \), then we call \( f \) \textit{well}.

Let \( X \) and \( Y \) be Esakia spaces and let \( f : X \to Y \) be a partial Esakia function. We define \( R_f \subseteq X \times Y \) by

\[
x R_f y \text{ if and only if } \exists z \in \text{dom}(f) \text{ such that } x \leq z \text{ and } f(z) = y.
\]

**Lemma 10.9.** Let \( X \) and \( Y \) be Esakia spaces and let \( f : X \to Y \) be a partial Esakia function. Then \( R_f \) is a generalized Esakia morphism. If in addition \( f \) is well, then \( R_f \) is total.

**Proof.** Let \( x, y \in X \), \( u \in Y \), \( x \leq y \), and \( y R_f u \). Then there exists \( z \in \text{dom}(f) \) such that \( y \leq z \) and \( f(z) = u \). Therefore, \( x \leq z \) and \( f(z) = u \), implying that \( x R_f u \). Thus, condition (1) of Definition 8.2 is satisfied. Let \( x \in X \), \( u, v \in Y \), \( x R_f u \), and \( u \leq v \). Then there exists \( y \in \text{dom}(f) \) such that \( x \leq y \) and \( f(y) = u \). Therefore, \( f(y) \leq v \), and as \( R \) is a generalized Esakia morphism, there exists \( z \in \text{dom}(f) \) such that \( y \leq z \) and \( f(z) = v \). Thus, \( x \leq z \) and \( f(z) = v \), so \( x R_f v \), and so condition (2) of Definition 8.2 is satisfied. Let \( x \in X \), \( y \in Y \), and \( x R_f y \). Therefore, \( y \notin f([x]) \). Since \( f \) is a partial Esakia function, there exists \( U \in \mathfrak{U}(Y) \) such that \( f([x]) \subseteq U \) and \( y \notin U \). Thus, \( x \in (\downarrow f^{-1}(U^c))^c \) and \( y \notin U \). We show that \( \Box R_f U = (\downarrow f^{-1}(U^c))^c \). For \( x \in X \) we have \( x \in \Box R_f U \) if \( R_f[x] \subseteq U \), and \( x \in (\downarrow f^{-1}(U^c))^c \) iff \( \forall z \in \text{dom}(f)(x \leq z \) implies \( f(z) \in U \)). First suppose that \( x \in \Box R_f U \), \( z \in \text{dom}(f) \), and \( x \leq z \). Then \( x R_f f(z) \), so \( f(z) \in U \), and so \( x \in (\downarrow f^{-1}(U^c))^c \). Now suppose that \( x \in (\downarrow f^{-1}(U^c))^c \) and \( x R_f y \). Then there exists \( z \in \text{dom}(f) \) such that \( x \leq z \) and \( f(z) = y \). Therefore, \( f(z) \in U \), so \( y \in U \), and so \( x \in \Box R_f U \). Consequently, \( \Box R_f U = (\downarrow f^{-1}(U^c))^c \). But then \( x \in \Box R_f U \) and \( y \notin U \), and so condition (3) of Definition 8.2 is satisfied. Let \( U \in \mathfrak{U}(Y) \). Since \( \Box R_f U = (\downarrow f^{-1}(U^c))^c \) and \((\downarrow f^{-1}(U^c))^c \subseteq \mathfrak{U}(X) \), we have \( \Box R_f U \in \mathfrak{U}(X) \), and so condition (4) of Definition 8.2 is satisfied. It follows that \( R_f \) is a generalized Priestley morphism. To see that \( R_f \) is a generalized Esakia morphism, let \( x \in X \), \( y \in Y \), and \( x R_f y \). Then there exists \( z \in \text{dom}(f) \) such that \( x \leq z \) and \( f(z) = y \). We show that \( R_f[x] = \uparrow y \). Clearly \( \uparrow y \subseteq R_f[x] \). If \( x R_f u \), then there exists \( v \in \text{dom}(f) \) such that \( x \leq v \) and \( f(v) = u \). Therefore, \( f(x) \leq u \), so \( y \leq u \), and so \( u \in \uparrow y \). Thus, \( R_f[x] \subseteq \uparrow y \), and so \( R_f[x] = \uparrow y \). Consequently, there exists \( z \in X \) such that \( x \leq z \) and \( R_f[z] = \uparrow y \), and so \( R_f \) is a generalized Esakia morphism. Lastly suppose that \( f \) is a well partial Esakia function and \( x \in X \). Then there exists \( z \in \max X \) such that \( x \leq z \). Since \( f \) is well, \( z \in \text{dom}(f) \). Therefore, \( f(z) \in Y \) and \( x R_f f(z) \). Thus, there exists \( y = f(z) \) in \( Y \) such that \( x R_f y \), so \( R_f^{-1}[Y] = X \), and so \( R_f \) is a total generalized Esakia morphism. \qed
Lemma 10.10. Let $X$ and $Y$ be Esakia spaces, $R \subseteq X \times Y$ be a generalized Esakia morphism, and $f : X \rightarrow Y$ be a partial Esakia function. Then $R_{f_{R}} = R$ and $f_{R_{R}} = f$.

Proof. Let $x \in X$ and $y \in Y$. By Lemma 10.7, $xR_{y}$ iff there exists $z \in \text{dom}(f_{R})$ such that $x \leq z$ and $f_{R}(z) = y$, which by the definition of $R_{f_{R}}$ means that $xR_{f_{R}y}$. Thus, $R_{f_{R}} = R$. Now let $x \in \text{dom}(f_{R})_{y}$ and let $f_{R_{y}}(x) = y$. Then $x \in \text{max}E_{R_{y}}[x]$ and $R_{f}[x] = \uparrow_{y}$. Therefore, $xR_{f_{R}}y$, and so there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $f(z) = y$. Thus, $R_{f}[z] = \uparrow_{y}$. It follows that $z \in E_{R_{y}}[x]$ and since $x \in \text{max}E_{R_{y}}[x]$ and $x \leq z \in E_{R_{y}}[x]$, we obtain $x = z$. Thus, $x \in \text{dom}(f)$ and $f_{R_{y}}(x) = f(x)$, so $f_{R_{f}} = f$. Conversely, let $x \in \text{dom}(f)$ and let $f(x) = y$. Then $R_{f}[x] = \uparrow_{y}$. We show that $x \in \text{max}E_{R_{f}}[x]$. Let $z \in E_{R_{f}}[x]$ and $x \leq z$. Then $R_{f}[z] = R_{f}[x] = \uparrow_{y}$. Therefore, $zR_{f}y$, and so there exists $z' \in \text{dom}(f)$ such that $z \leq z'$ and $f(z') = y$. Thus, $x \leq z'$ and $f(x) = f(z')$. This implies $x = z = z'$, and so $x \in \text{max}E_{R_{f}}[x]$. Therefore, $x \in \text{dom}(f_{R_{f}})$ and $f_{R_{f}}(x) = y = f(x)$. It follows that $\text{dom}(f_{R_{f}}) = \text{dom}(f)$ and $f_{R_{f}}(x) = f(x)$ for each $x \in \text{dom}(f_{R_{f}}) = \text{dom}(f)$. Thus, $f_{R_{f}} = f$. \qed

Definition 10.11. Let $X$ and $Y$ be Esakia spaces and let $f : X \rightarrow Y$ be a partial Heyting function. We call $f$ a partial Heyting function if for each $x \in X$ there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $f(\uparrow z) = \uparrow f(z)$.

It is easy to verify that each partial Heyting function is well.

Lemma 10.12. Let $X$ and $Y$ be Esakia spaces. If $R \subseteq X \times Y$ is a functional generalized Esakia morphism, then the partial Esakia function $f_{R}$ is a partial Heyting function. Conversely, if $f : X \rightarrow Y$ is a partial Heyting function, then the generalized Esakia morphism $R_{f}$ is a functional generalized Esakia morphism.

Proof. Let $R \subseteq X \times Y$ be a functional generalized Esakia morphism and let $x \in X$. Then $R[x] = \uparrow_{y}$ for some $y \in Y$. Therefore, $xR_{y}$, and so there exists $z \in \text{dom}(f_{R})$ such that $x \leq z$ and $f_{R}(z) = y$. Thus, $f_{R}[\uparrow x] = R[x] = \uparrow_{y} = \uparrow f_{R}(z)$. Consequently, $f_{R}$ is a partial Heyting function. Now let $f : X \rightarrow Y$ be a partial Heyting function and let $x \in X$. Then there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $f(\uparrow z) = \uparrow f(z)$. Thus, $R[x] = f(\uparrow z) = \uparrow f(z)$, and so $R$ is a functional generalized Esakia morphism. \qed

Let $X, Y$, and $Z$ be Esakia spaces and $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be partial Esakia functions. We define $g * f : X \rightarrow Y$ as the partial Esakia function $g * f = f_{(R_{g} * R_{f})}$. By Lemma 10.10, it is not difficult to see that Esakia spaces and partial Esakia functions form a category, we denote by $\mathbf{ES}^{P}$. It is also clear that if both $f$ and $g$ are well (resp. Heyting), then so is $g * f$. Therefore, Esakia spaces and well partial Esakia functions as well as Esakia spaces and
partial Heyting functions also form categories, we denote by $\text{ES}^W$ and $\text{ES}^H$, respectively. Obviously $\text{ES}^H$ is a proper subcategory of $\text{ES}^W$ and $\text{ES}^W$ is a proper subcategory of $\text{ES}^P$. As an immediate consequence of Corollary 10.5 and Lemmas 10.7, 10.9, 10.10, and 10.12, we obtain:

**Corollary 10.13.**
1. The category $\text{ES}^P$ is isomorphic to the category $\text{ES}^P$.
2. The category $\text{ES}^T$ is isomorphic to the category $\text{ES}^W$.
3. The categories $\text{ES}$, $\text{ES}^F$, and $\text{ES}^H$ are isomorphic.

Now putting Corollaries 10.5 and 10.13 together, we obtain:

**Corollary 10.14.**
1. The category $\text{HA}^{\land,-}$ is dually equivalent to the category $\text{ES}^P$.
2. The category $\text{HA}^{\land,-,\bot}$ is dually equivalent to the category $\text{ES}^W$.
3. The category $\text{HA}$ is dually equivalent to the category $\text{ES}^H$.

In particular, this provides an alternative to the Esakia duality. We give a direct proof of this by establishing that $\text{ES}$ is isomorphic to $\text{ES}^H$, thus providing an explicit construction of an Esakia morphism from a partial Heyting function and vice versa. Let $X$ and $Y$ be Esakia spaces and let $f : X \to Y$ be a partial Heyting function. We define a function $g_f : X \to Y$ as follows. Let $x \in X$. Since $f$ is a partial Heyting function, there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $f([x]) = \uparrow f(z)$. Such a $z$ may not be unique, but all such $z$’s have the same $f$-image. Thus, we set $g_f(x) = f(z)$.

**Lemma 10.15.** If $X$ and $Y$ are Esakia spaces and $f : X \to Y$ is a partial Heyting function, then $g_f : X \to Y$ is an Esakia morphism.

**Proof.** We need to show that $g_f$ is continuous, order-preserving, and satisfies the following condition: For $x \in X$ and $y \in Y$, from $g_f(x) \leq y$ it follows that there exists $z \in X$ with $x \leq z$ and $g_f(z) = y$. Let $x, y \in X$ and $x \leq y$. Since $f$ is a partial Heyting function, there exist $u, v \in \text{dom}(f)$ such that $x \leq u$, $y \leq v$, $f([x]) = \uparrow f(u)$, and $f([y]) = \uparrow f(v)$. Therefore, $\uparrow f(v) \subseteq \uparrow f(u)$, so $f(u) \leq f(v)$, and so $g_f(x) \leq g_f(y)$. Now let $x \in X$, $y \in Y$, and $g_f(x) \leq y$. Then there exists $u \in \text{dom}(f)$ such that $x \leq u$ and $g_f(x) = f(u)$. Therefore, $f(u) \leq y$, and since $f$ is a partial Heyting function, hence a partial Esakia function, there is $z \in \text{dom}(f)$ such that $u \leq z$ and $f(z) = y$. Thus, $x \leq z$ and $g_f(z) = f(z) = y$. Next suppose that $U$ is a clopen upset of $Y$. Then $x \in g_f^{-1}(U)$ iff $g_f(x) \in U$ iff $f([x]) \subseteq U$ iff $x \in (\downarrow f^{-1}(U^c))^c$. Thus, $g_f^{-1}(U) = (\downarrow f^{-1}(U^c))^c$, and so $g_f^{-1}(U)$ is a clopen upset of $X$. Now suppose that $U$ is clopen in $Y$. Then $U = \bigcup_{i=1}^{n}(U_i - V_i)$, where $U_i, V_i$ are clopen upsets of $X$. Therefore, $g_f^{-1}(U) = g_f^{-1}((\bigcup_{i=1}^{n}(U_i - V_i))) =$
\[ \bigcup_{i=1}^{n}(g_i^{-1}(U_i) - g_i^{-1}(V_i)) = \bigcup_{i=1}^{n}((\downarrow f_i^{-1}(U_i^{-c}))^c - (\downarrow f_i^{-1}(V_i^{-c}))^c) \]

and so \( g_f^{-1}(U) \) is clopen in \( X \). Thus, \( g_f \) is continuous, and so \( g_f \) is an Esakia morphism. \( \square \)

Now let \( X \) and \( Y \) be Esakia spaces and let \( g : X \to Y \) be an Esakia morphism. We define a partial function \( f_g : X \to Y \) as follows. We let

\[ \text{dom}(f_g) = \{ x \in X : x \in \text{max} g^{-1}(g(x)) \}, \]

and for \( x \in \text{dom}(f_g) \) we set \( f_g(x) = g(x) \).

**Lemma 10.16.** If \( X \) and \( Y \) are Esakia spaces and \( g : X \to Y \) is an Esakia morphism, then \( f_g : X \to Y \) is a partial Heyting function.

**Proof.** Let \( x, y \in \text{dom}(f_g) \) and let \( x < y \). Then \( x \in \text{max} g^{-1}g(x) \) and \( y \in \text{max} g^{-1}g(y) \). From \( x < y \) it follows that \( x \leq y \), so \( g(x) \leq g(y) \). If \( g(x) = g(y) \), then \( x, y \in \text{max} g^{-1}g(x) \), which together with \( x < y \) leads to a contradiction. Thus, \( g(x) < g(y) \), so \( f_g(x) < f_g(y) \), and so condition (1) of Definition 10.8 is satisfied. Now let \( x \in \text{dom}(f_g) \), \( y \in Y \), and \( f_g(x) < y \). Then \( g(x) < y \), so \( g(x) \leq y \), and since \( g \) is an Esakia morphism, there exists \( z \in X \) such that \( x \leq z \) and \( g(z) = y \). Since \( g \) is continuous, \( g^{-1}g(z) \) is closed, so there exists \( u \in \text{max} g^{-1}g(z) \) such that \( z \leq u \). Therefore, \( x \leq u \), \( u \in \text{dom}(f_g) \), and \( f_g(u) = g(u) = y \). If \( x = u \), then \( g(x) = g(u) = y \), a contradiction. Thus, \( x < u \) and \( f_g(u) = y \), and so condition (2) of Definition 10.8 is satisfied. Next let \( U \) be a clopen upset of \( Y \). We show that \( (\downarrow f_g^{-1}(U^{-c}))^c = g^{-1}(U) \). We have \( x \in (\downarrow f_g^{-1}(U^{-c}))^c \) iff \( (\forall z \in \text{dom}(f_g))(x \leq z \Rightarrow f_g(z) \in U) \) and \( x \in g^{-1}(U) \) iff \( g(x) \in U \). First let \( x \in g^{-1}(U) \). Then for each \( z \in \text{dom}(f_g) \) with \( x \leq z \) we have \( g(x) \leq g(z) = f_g(z) \). Thus, \( f_g(z) \in U \), and so \( x \in (\downarrow f_g^{-1}(U^{-c}))^c \). Conversely, let \( x \notin g^{-1}(U) \). Then \( g(x) \notin U \). Let \( z \in \text{max} g^{-1}g(x) \) be such that \( x \leq z \). Then \( z \in \text{dom}(f_g) \), \( x \leq z \), and \( f_g(z) = g(z) = g(x) \notin U \). Thus, \( x \notin (\downarrow f_g^{-1}(U^{-c}))^c \). Consequently, \( (\downarrow f_g^{-1}(U^{-c}))^c = g^{-1}(U) \), so \( (\downarrow f_g^{-1}(U^{-c}))^c \) is clopen in \( X \), and so condition (3) of Definition 10.8 is satisfied. Next we show that \( f_g \) satisfies the condition of Definition 10.11. Let \( x \in X \) and \( z \in \text{max} g^{-1}g(x) \) be such that \( x \leq z \). Then \( z \in \text{dom}(f_g) \), \( x \leq z \), and \( g(x) = g(z) \). Thus, \( f_g[z] = \uparrow g(x) = \uparrow g(z) = \uparrow f_g(z) \), and so \( f_g \) satisfies the condition of Definition 10.11. Lastly, let \( x \in X \), \( y \in Y \), and \( y \notin f_g[x] \). By the above, there exists \( z \in \text{dom}(f_g) \) such that \( x \leq z \) and \( f_g[z] = \uparrow f_g(z) \). Therefore, \( y \notin \uparrow f_g(z) \). So \( f_g(z) \notin y \), and by the Priestley separation axiom, there exists a clopen upset \( U \) of \( Y \) such that \( f_g(z) \in U \) and \( y \notin U \). Thus, \( f_g[z] = \uparrow f_g(z) \subseteq U \) and \( y \notin U \). Consequently, \( f_g \) satisfies condition (4) of Definition 10.8, so \( f_g \) is a partial Esakia function satisfying the condition of Definition 10.11, and so \( f_g \) is a partial Heyting function. \( \square \)
Lemma 10.17. Let $X$ and $Y$ be Esakia spaces, let $g : X \to Y$ be an Esakia morphism, and let $f : X \to Y$ be a partial Heyting function. Then $g_{f^g} = g$ and $f_{g^f} = f$.

Proof. Let $g : X \to Y$ be an Esakia morphism and let $x \in X$. Since $f_g$ is a partial Heyting function, there exists $z \in \text{dom}(f_g)$ such that $x \leq z$ and $f_g[\uparrow x] = \uparrow f_g(z)$. Therefore, $z \in \text{max}g^{-1}f_g(x)$ and $f_g(z) = g(z)$. But $f_g[\uparrow x] = \uparrow g(x)$. Thus, $\uparrow g(x) = \uparrow g(z)$, and so $g(x) = g(z)$. It follows that $g_{f_g}(x) = f_g(z) = g(z) = g(x)$, and so $g_{f_g} = g$. Now let $f : X \to Y$ be a partial Heyting function. If $x \in \text{dom}(f_{g_{f^g}})$, then $x \in \text{max}g_f^{-1}g_f(x)$ and $f_{g_{f^g}}(x) = g_f(x)$. Since $f$ is a partial Heyting function, there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $f[\uparrow x] = \uparrow f(z)$. But then $g_f(x) = f(z) = g_f(z)$, so $x = z$ as $x \in \text{max}g_f^{-1}g_f(x)$. Thus, $x \in \text{dom}(f)$ and $f_{g_{f^g}}(x) = f(z) = f(x)$. If $x \in \text{dom}(f)$, then $f(x) = g_f(x)$. So $x \in \text{max}g_f^{-1}g_f(x)$ and $f_{g_{f^g}}(x) = g_f(x) = f(x)$. Thus, $\text{dom}(f_{g_{f^g}}) = \text{dom}(f)$ and for each $x \in \text{dom}(f_{g_{f^g}}) = \text{dom}(f)$ we have $f_{g_{f^g}}(x) = f(x)$. Consequently, $f_{g_{f^g}} = f$. □

As an immediate consequence of Lemmas 10.15, 10.16, and 10.17, we obtain a direct proof of the fact that $\text{ES}$ is isomorphic to $\text{ES}^H$. For the reader’s convenience we give a table that gathers together the dual equivalences of different categories that we obtained in the last two sections. For two categories $C$ and $D$, we use $C \simeq D$ to denote that $C$ is dually equivalent to $D$, and $C \cong D$ to denote that $C$ is isomorphic to $D$. 
## Categories of algebras:

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<tr>
<th>Category</th>
<th>Objects</th>
<th>Morphisms</th>
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</thead>
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<td>Bounded distributive meet semi-lattices</td>
<td>Top-preserving meet semi-lattice homomorphisms</td>
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<tr>
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<tr>
<td>BDM^S</td>
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<td>Sup-homomorphisms</td>
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<tr>
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<td>HA</td>
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## Categories of spaces:

<table>
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<th>Morphisms</th>
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<td>Generalized Priestley morphisms</td>
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<td>Partial Heyting functions</td>
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<tr>
<td>(\text{ES}^)</td>
<td>“______”</td>
<td>Esakia morphisms</td>
</tr>
</tbody>
</table>
Dualities:

\[
\begin{align*}
\text{BDM} & \sim d \text{ GPS} \\
\text{BDM} & \sim d \text{ GPS}^T \\
\text{BDM}^d & \sim \text{ GPS}^F \cong \text{ GPS}^S \\
\text{BDM}^\wedge,\wedge,_,_,_ & \sim \text{ PS}^R \\
\text{BDM}^\wedge,\wedge,\_,\_,_ & \sim \text{ PS}^T \\
\text{BDL} & \sim d \text{ PS}^F \cong \text{ PS} \\
\text{BIM} & \sim d \text{ GES} \\
\text{BIM}^\wedge & \sim d \text{ GES}^T \\
\text{BIM}^d & \sim \text{ GES}^F \cong \text{ GES}^S \\
\text{HA}^\wedge,\rightarrow & \sim \text{ ES}^R \cong \text{ ES}^P \\
\text{HA}^\wedge,\rightarrow,\_,\_,_ & \sim \text{ ES}^T \cong \text{ ES}^W \\
\text{HA} & \sim d \text{ ES}^F \cong \text{ ES}^H \cong \text{ ES}
\end{align*}
\]

We conclude this section by giving two counterexamples. The first one shows that it is impossible to characterize generalized Priestley morphisms between Priestley spaces in terms of partial functions, and the second one shows that it is impossible to characterize generalized Esakia morphisms between generalized Esakia spaces in terms of partial functions. Thus, there is no generalization of Corollary 10.13 to neither Priestley spaces nor generalized Esakia spaces.

**Example 10.18.** Consider the Priestley spaces \( X \) and \( Y \) and the generalized Priestley morphism \( R \subseteq X \times Y \) shown in Fig. 5. The corresponding top preserving meet semi-lattice homomorphism \( h_R : \mathcal{U}(Y) \to \mathcal{U}(X) \) is also shown in Fig. 5. There are only three partial functions from \( X \) to \( Y \): the empty function, the total function sending \( x \) to \( y \), and the total function sending \( x \) to \( z \). It is easy to see that their corresponding meet semi-lattice homomorphisms from \( \mathcal{U}(Y) \) to \( \mathcal{U}(X) \) are different from \( h_R \). Thus, it is impossible to characterize \( R \subseteq X \times Y \) in terms of partial functions from \( X \) to \( Y \).
Example 10.19. Consider the generalized Esakia spaces $X$ and $Y$ and the generalized Esakia morphism $R \subseteq X \times Y$ shown in Fig.6, where $Y_0 = Y - \{z_1, z_2\}$, the elements of $Y_0$ are isolated points of $Y$, $z_1$ is the limit point of $\{x_1, x_2, \ldots\}$, $z_2$ is the limit point of $\{y_1, y_2, \ldots\}$, $R[r] = \uparrow z_1 \cup \uparrow z_2$, and $R[w_i] = \{u_i\}$ for $i = 1, 2, 3$. Then $\text{dom}(f_R) = \{w_1, w_2, w_3\}$ and $R_{fR}[r] = \{u_1, u_2, u_3\}$. Thus, $R[r] \neq R_{fR}[r]$, so $R \neq R_{fR}$, and so Corollary 10.13 does not extend to generalized Esakia spaces.

11. Duality at work

In this section we show how the duality developed in the previous sections works by establishing dual descriptions of a number of algebraic concepts that play an important role in the theory of distributive meet semi-lattices and implicative meet semi-lattices.
11.1. Dual description of Frink ideals, ideals, and filters. We start by recalling that for a bounded distributive lattice (resp. Heyting algebra) $L$ and its dual Priestley space (resp. Esakia space) $X$, there is a lattice isomorphism between the lattice of ideals of $L$ and the lattice of open upsets of $X$, and the lattice of filters of $L$ (ordered by $\supseteq$) and the lattice of closed upsets of $X$. These isomorphisms are obtained as follows. If $I$ is an ideal of $L$, then $U(I) = \bigcup \{ \varphi(a) : a \in I \}$ is the open upset of $X$ corresponding to $I$, and if $U$ is an open upset of $X$, then $I(U) = \{ a \in L : \varphi(a) \subseteq U \}$ is the ideal of $L$ corresponding to $U$; if $F$ is a filter of $L$, then $C(F) = \bigcap \{ \varphi(a) : a \in F \}$ is the closed upset of $X$ corresponding to $F$, and if $C$ is a closed upset of $X$, then $F(C) = \{ a \in L : C \subseteq \varphi(a) \}$ is the filter of $L$ corresponding to $C$. Then we have $I \subseteq J$ iff $U(I) \subseteq U(J)$, $I = I(U(I))$, and $U(I(U)) = U$; and $F \supseteq G$ iff $C(F) \subseteq C(G)$, $F = F(C(F))$, and $C(F(C)) = C$. Now we show how these correspondences work for Frink ideals, ideals, and filters of bounded distributive meet semi-lattices and bounded implicative meet semi-lattices.

Let $L$ be a bounded distributive meet semi-lattice and let $D(L)$ be its distributive envelope. Let also $X = (X, \tau, \leq, X_0)$ be the generalized Priestley space of $L$. We know that $(X, \tau, \leq)$ is order-isomorphic and homeomorphic to the Priestley space of $D(L)$. Since the lattice of Frink ideals of $L$ is isomorphic to the lattice of ideals of $D(L)$, we immediately obtain from the above:

**Proposition 11.1.** Let $L$ be a bounded distributive meet semi-lattice and let $X$ be its generalized Priestley space. Then the maps $I \mapsto U(I)$ and $U \mapsto I(U)$ set an isomorphism of the lattice of Frink ideals of $L$ with the lattice of open upsets of $X$.

In particular, prime $F$-ideals of $L$ correspond to the open upsets of $X$ of the form $(\downarrow x)^c$ for $x \in X$. Now we give a dual description of ideals of $L$. Since each ideal of $L$ is an $F$-ideal, ideals correspond to special open upsets of $X$.

**Lemma 11.2.** Let $L$ be a bounded distributive meet semi-lattice and let $X$ be its generalized Priestley space. If $I$ is an ideal of $L$, then $X - U(I) = \downarrow (X_0 - U(I))$.

**Proof.** The inclusion $\downarrow (X_0 - U(I)) \subseteq X - U(I)$ is trivial. To prove the other inclusion, let $x \in X - U(I)$. Then $x \cap I = \emptyset$. By the prime filter lemma, there is a prime filter $y$ of $L$ such that $x \subseteq y$ and $y \cap I = \emptyset$. Thus, $y \in X_0 - U(I)$, and so $x \in \downarrow (X_0 - U(I))$. 

**Lemma 11.3.** Let $L$ be a bounded distributive meet semi-lattice and let $X$ be its generalized Priestley space. If $U$ is an open upset of $X$ such that $X - U = \downarrow (X_0 - U)$, then $I(U)$ is an ideal of $L$.

**Proof.** Since $U$ is an open upset of $X$, it follows from Proposition 11.1 that $I(U)$ is an $F$-ideal of $L$. Let $a, b \in I(U)$ with $\uparrow a \cap \uparrow b \cap I(U) = \emptyset$. By the optimal
filter lemma, there exists \( x \in X \) such that \( \uparrow a \cap \uparrow b \subseteq x \) and \( x \cap I(U) = \emptyset \). Therefore, \( x \notin U \), so \( x \in X - U \), and so there exists \( y \in X_0 - U \) such that \( x \leq y \). It follows that \( \uparrow a \cap \uparrow b \subseteq y \), and as \( y \) is a prime filter, we have \( \uparrow a \subseteq y \) or \( \uparrow b \subseteq y \). Thus, \( a \in y \) or \( b \in y \), which is a contradiction because \( a, b \in I(U) \).

Consequently, \( \uparrow a \cap \uparrow b \cap I(U) \neq \emptyset \), and so \( I(U) \) is an ideal of \( L \).

Putting Proposition 11.1 and Lemmas 11.2 and 11.3 together, we obtain:

**Theorem 11.4.** Let \( L \) be a bounded distributive meet semi-lattice and let \( X \) be its generalized Priestley space. Then the maps \( I \mapsto U(I) \) and \( U \mapsto I(U) \) set an isomorphism of the ordered set of ideals of \( L \) with the ordered set of open upsets \( U \) of \( X \) such that \( X - U = \downarrow (X_0 - U) \).

**Remark 11.5.** Since for an open upset \( U \) of \( X \), \( X - U \) is closed in \( X \), and so for each \( x \in X - U \) there is \( y \in \text{max}(X - U) \) with \( x \leq y \), the condition \( X - U = \downarrow (X_0 - U) \) is obviously equivalent to the condition \( \text{max}(X - U) \subseteq X_0 \).

Our next task is to give a dual description of prime ideals of \( L \).

**Lemma 11.6.** Let \( L \) be a bounded distributive meet semi-lattice and let \( X \) be its generalized Priestley space. If \( I \) is a prime ideal of \( L \), then \( U(I) = (\downarrow x)^c \) for some \( x \in X_0 \).

**Proof.** Let \( I \) be a prime ideal of \( L \) and let \( x = L - I \). By Proposition 3.7, \( x \in X_0 \). Moreover, we have \( y \in U(I) \) iff \( y \cap I \neq \emptyset \) iff \( y \not\subseteq x \) iff \( y \in (\downarrow x)^c \). Thus, \( U(I) = (\downarrow x)^c \).

**Lemma 11.7.** Let \( L \) be a bounded distributive meet semi-lattice and let \( X \) be its generalized Priestley space. If \( U = (\downarrow x)^c \) for \( x \in X_0 \), then \( I(U) \) is a prime ideal of \( L \).

**Proof.** Clearly \( I(U) \) is an ideal of \( L \) because \( \text{max}(U^c) = \text{max}(\downarrow x) = \{x\} \subseteq X_0 \). We show that it is prime. Let \( a \land b \in I(U) \). Then \( \varphi(a) \cap \varphi(b) = \varphi(a \land b) \subseteq U \).

So \( \varphi(a) \cap \varphi(b) \subseteq (\downarrow x)^c \), and so \( \downarrow x \subseteq \varphi(a)^c \cup \varphi(b)^c \). Therefore, \( x \in \varphi(a)^c \cup \varphi(b)^c \), which implies that \( x \in \varphi(a)^c \) or \( x \in \varphi(b)^c \). Thus, \( \downarrow x \subseteq \varphi(a)^c \) or \( \downarrow x \subseteq \varphi(b)^c \), so \( \varphi(a) \subseteq (\downarrow x)^c \) or \( \varphi(b) \subseteq (\downarrow x)^c \). It follows that \( \varphi(a) \subseteq U \) or \( \varphi(b) \subseteq U \), so \( a \in I(U) \) or \( b \in I(U) \), and so \( I(U) \) is a prime ideal.

Putting Theorem 11.4 and Lemmas 11.6 and 11.7 together, we obtain:

**Proposition 11.8.** Let \( L \) be a bounded distributive meet semi-lattice and let \( X \) be its generalized Priestley space. Then the maps \( I \mapsto U(I) \) and \( U \mapsto I(U) \) set an isomorphism of the ordered set of prime ideals of \( L \) with the ordered set of open upsets of \( X \) of the form \( (\downarrow x)^c \) for some \( x \in X_0 \).

Now we give a dual description of filters of \( L \). Since there are less filters in \( L \) than in \( D(L) \), not every closed upset of \( X \) corresponds to a filter of \( L \).
Lemma 11.9. Let $L$ be a bounded distributive meet semi-lattice and let $X$ be its generalized Priestley space. If $F$ is a filter of $L$, then $X - C(F) = \downarrow (X_0 - C(F))$.

Proof. The inclusion $\downarrow (X_0 - C(F)) \subseteq X - C(F)$ is trivial. For the other inclusion, let $x \in X - C(F)$. Then $x \notin C(F)$, and so there exists $a \in F$ such that $a \nsubseteq x$. By the prime filter lemma, there is $y \in X_0$ such that $x \subseteq y$ and $a \nsubseteq y$. Thus, $y \notin C(F)$, so $y \in X_0 - C(F)$, and so $x \in \downarrow (X_0 - C(F))$. \qed

Lemma 11.10. Let $L$ be a bounded distributive meet semi-lattice and let $X$ be its generalized Priestley space. If $C$ is a closed upset of $X$, then $F(C)$ is a filter of $L$, and $C = C(F(C))$ if and only if $X - C = \downarrow (X_0 - C)$.

Proof. If $C$ is a closed upset, it is routine to see that $F(C)$ is a filter of $L$. Suppose that $C = C(F(C))$. Since $F(C)$ is a filter of $L$, Lemma 11.9 implies that $X - C(F(C)) = \downarrow (X_0 - C(F(C)))$. From $C = C(F(C))$ and the last equality we get $X - C = \downarrow (X_0 - C)$. Conversely, suppose that $X - C = \downarrow (X_0 - C)$. We show that $C = C(F(C))$. Since $C(F(C)) = \bigcap \{ \varphi(a) : C \subseteq \varphi(a) \}$, it is obvious that $C \subseteq C(F(C))$. For the converse, suppose that $x \notin C$. Then there exists $y \in X_0 - C$ such that $x \leq y$. Since $C$ is a closed upset of $X$, $C$ is the intersection of clopen upsets of $X$ containing $C$. Therefore, from $y \notin C$ it follows that there is a clopen upset $U$ of $X$ such that $C \subseteq U$ and $y \notin U$. As each clopen upset of $X$ is a finite union of elements of $\varphi[L]$, there exist $a_1, \ldots, a_n \in L$ such that $U = \varphi(a_1) \cup \ldots \cup \varphi(a_n)$. Thus, $y \notin \varphi(a_1) \cup \ldots \cup \varphi(a_n)$, and so $a_1, \ldots, a_n \notin y$. Since $y$ is a prime filter of $L$, we have $\bigcap_{i=1}^n a_i \nsubseteq y$. Therefore, there exists $a \in \bigcap_{i=1}^n a_i$ such that $a \nsubseteq y$. But then $\varphi(a) \supseteq \varphi(a_1) \cup \ldots \cup \varphi(a_n) = U \supseteq C$ and $y \notin \varphi(a)$. Consequently, $C \subseteq \varphi(a)$ and $x \notin \varphi(a)$, implying that $x \notin C(F(C))$. \qed

Putting Lemmas 11.9 and 11.10 together, we obtain:

Theorem 11.11. Let $L$ be a bounded distributive meet semi-lattice and let $X$ be its generalized Priestley space. Then the maps $F \mapsto C(F)$ and $C \mapsto F(C)$ set an isomorphism of the lattice of filters of $L$ (ordered by reverse inclusion) and the lattice of closed upsets $C$ of $X$ satisfying the condition $X - C = \downarrow (X_0 - C)$.

In particular, since there is a 1-1 correspondence between prime filters and prime ideals of $L$, we obtain that prime filters of $L$ correspond to closed upsets of $X$ of the form $\uparrow x$ for $x \in X_0$. Also, since there is a 1-1 correspondence between optimal filters and prime $F$-ideals of $L$, optimal filters of $L$ correspond to closed upsets of $X$ of the form $\uparrow x$ for $x \in X$. In the next table we gather together the dual descriptions of different notions of filters and ideals of a bounded distributive meet semi-lattice $L$ and its distributive envelope $D(L)$. 

<table>
<thead>
<tr>
<th>Filters</th>
<th>Ideals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\downarrow (X_0 - C(F))$</td>
<td>$\varphi(a)$</td>
</tr>
<tr>
<td>$X - C(F) = \downarrow (X_0 - C(F))$</td>
<td>$\cup \varphi(a_i)$</td>
</tr>
<tr>
<td>$C(F(C))$</td>
<td>$\cap C(F(C))$</td>
</tr>
<tr>
<td>$C = C(F(C))$</td>
<td>$C \subseteq \varphi(a)$</td>
</tr>
<tr>
<td>$X - C = \downarrow (X_0 - C)$</td>
<td>$x \notin \varphi(a)$</td>
</tr>
</tbody>
</table>

Putting Lemmas 11.9 and 11.10 together, we obtain:
Dual description of filters of \(L\) and \(D(L)\)

<table>
<thead>
<tr>
<th>Filters of (D(L))</th>
<th>Closed upsets of (L_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Filters of (L)</td>
<td>Closed upsets (C) of (L_s) such that (L_s - C = \downarrow(L_s - C))</td>
</tr>
<tr>
<td>Optimal filters of (L = \text{Prime filters of } D(L))</td>
<td>(\uparrow x, x \in L_s)</td>
</tr>
<tr>
<td>Prime filters of (L)</td>
<td>(\uparrow x, x \in L_+)</td>
</tr>
</tbody>
</table>

Dual description of ideals of \(L\) and \(D(L)\)

<table>
<thead>
<tr>
<th>F-ideals of (L = \text{Ideals of } D(L))</th>
<th>Open upsets of (L_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prime F-ideals of (L = \text{Prime ideals of } D(L))</td>
<td>((\downarrow x)^c, x \in L_s)</td>
</tr>
<tr>
<td>Ideals of (L)</td>
<td>Open upsets (U) of (L_s) such that (L_s - U = \downarrow(L_+-U))</td>
</tr>
<tr>
<td>Prime ideals of (L)</td>
<td>((\downarrow x)^c, x \in L_+)</td>
</tr>
</tbody>
</table>

The picture remains unchanged for bounded implicative meet semi-lattices.

11.2. **Dual description of 1-1 and onto homomorphisms.** Our next task is to give a dual description of 1-1 and onto homomorphisms.

**Lemma 11.12.** Let \(X\) and \(Y\) be generalized Priestley spaces and let \(R \subseteq X \times Y\) be a generalized Priestley morphism.

1. If \(F\) is a closed subset of \(X\), then \(R[F]\) is a closed upset of \(Y\).
2. If \(G\) is a closed subset of \(Y\), then \(R^{-1}[G]\) is a closed downset of \(X\).

**Proof.** (1) Suppose that \(F\) is a closed subset of \(X\). It follows from condition (2) of Definition 8.2 that \(R[F]\) is an upset of \(Y\). We show that \(R[F]\) is closed in \(Y\). Let \(y \not\in R[F]\). Then for each \(x \in F\) we have \(xRy\). By condition (3) of Definition 8.2, there is \(U_x \in Y^*\) such that \(R[x] \subseteq U_x\) and \(y \not\in U_x\). Thus, \(x \in \square_R U_x\) and by condition (4) of Definition 8.2, \(\square_R U_x \subseteq X^*\), so \(\square_R U_x\) is clopen. Then we have \(F \subseteq \bigcup \{\square_R U_x : x \in F\}\). Since \(F\) is a closed subset of a compact space, \(F\) is compact. Therefore, there are \(x_1, \ldots, x_n \in F\) such that \(F \subseteq \bigcup_{i=1}^n \square_R U_{x_i}\). We claim that \(U_{x_1}^c \cap \ldots \cap U_{x_n}^c \cap R[F] = \emptyset\). If not, then there exists \(z \in U_{x_1}^c \cap \ldots \cap U_{x_n}^c \cap R[F]\). Thus, there is \(u \in F\) such that \(uRz\). But then \(u \in \square_R U_{x_i}\), so \(z \in U_{x_i}\) for some \(i \leq n\), which is a contradiction. It follows that there is an open neighborhood \(U_{x_1}^c \cap \ldots \cap U_{x_n}^c\) of \(y\) missing \(R[F]\), so \(R[F]\) is closed in \(Y\).
(2) Suppose that $G$ is a closed subset of $Y$. It follows from condition (1) of Definition 8.2 that $R^{-1}[G]$ is a downset of $X$. We show that $R^{-1}[G]$ is closed in $X$. Let $x \notin R^{-1}[G]$. Then for each $y \in G$ we have $x \mathbin{R} y$. So, by condition (3) of Definition 8.2, there is $U_y \in Y^*$ such that $x \in \Box_R U_y$ and $y \notin U_y$. Therefore, $G \subseteq \bigcup_i U_{y_i}^c$ and as $G$ is compact, there are $y_1, \ldots, y_n \in G$ such that

$$G \subseteq \bigcup_{i=1}^n U_{y_i}^c.$$  

We claim that $\Box_R U_{y_1} \cap \cdots \cap \Box_R U_{y_n} \cap R^{-1}[G] = \emptyset$. If not, then there is $z \in \Box_R U_{y_1} \cap \cdots \cap \Box_R U_{y_n} \cap R^{-1}[G]$. So $R[z] \subseteq U_{y_1} \cap \cdots \cap U_{y_n}$ and $z \in R^{-1}[G]$. Thus, there is $u \in G$ such that $z \mathbin{R} u$. But then $u \in U_{y_1} \cap \cdots \cap U_{y_n} \cap G$, which is a contradiction. Consequently, there is an open neighborhood $\Box_R U_{y_1} \cap \cdots \cap \Box_R U_{y_n}$ of $x$ missing $R^{-1}[G]$, so $R^{-1}[G]$ is closed in $X$. \hfill \Box

**Definition 11.13.** Let $X$ and $Y$ be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism.

1. We call $R$ onto if for each $y \in Y$ there is $x \in X$ such that $R[x] = \uparrow y$.
2. We call $R$ 1-1 if for each $x \in X$ and $U \in X^*$ with $x \notin U$, there is $V \in Y^*$ such that $R[U] \subseteq V$ and $R[x] \nsubseteq V$.

Let $X$ and $Y$ be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism. We observe that if $R$ is 1-1, then $R$ is total. Indeed, if $R$ is not total, then there exists $x \in X$ such that $R[x] = \emptyset$. Therefore, for each $V \in Y^*$ we have $R[x] \subseteq V$. Thus, $R$ can not be 1-1. We also observe that using condition (5) of Definition 6.10, it is easy to verify that if a generalized Priestley morphism $R$ is 1-1, then $x \nsubseteq y$ implies $R[y] \nsubseteq R[x]$, and $x \notin U$ implies $R[x] \nsubseteq R[U]$ for each $x, y \in X$ and $U \in X^*$. However, these two conditions do not imply that $R$ is 1-1 as the following example shows.

**Example 11.14.** Let $X$ and $Y$ be the generalized Esakia spaces and $R \subseteq X \times Y$ be the generalized Esakia morphism shown in Fig.7. We have that $Y_0 = Y - \{x_1\}$, the elements of $Y_0$ are isolated points of $Y$, $x_1$ is the only limit point of $Y$, $R[x] = \uparrow x_1$, $R[y] = \{y_1\}$, and $R[z] = \{z_1\}$. Then it is easy to verify that for each $x, y \in X$ and $U \in X^*$, we have $x \nsubseteq y$ implies $R[y] \nsubseteq R[x]$, and $x \notin U$ implies $R[x] \nsubseteq R[U]$. On the other hand, for $U = \{y, z\} \in X^*$ we have $x \notin U$, $R[x] = \uparrow x_1$, and $R[U] = \{y_1, z_1\}$. Now $R[U] \notin Y^*$ and for each $V \in Y^*$ with $R[U] \subseteq V$, we have $x_1 \in V$, and so $R[x] \subseteq V$. Thus, there is no $V \in Y^*$ such that $R[U] \subseteq V$ and $R[x] \nsubseteq V$, and so $R$ is not 1-1. The bounded implicative meet semi-lattices $X^*$ and $Y^*$ corresponding to $X$ and $Y$ together with the bounded implicative meet semi-lattice homomorphism $h_R : Y^* \to X^*$ corresponding to $R$ are shown in Fig.7. Clearly $h_R$ is not onto as $h_R^{-1}(\{y, z\}) = \emptyset$. 

Lemma 11.15. Let $L$ and $K$ be bounded distributive meet semi-lattices and let $h : L \rightarrow K$ be a meet semi-lattice homomorphism preserving top. Then for $x \in K_*$ and $y \in L_*$ we have $R_h[x] = \uparrow y$ iff $h^{-1}(x) = y$.

Proof. First suppose that $R_h[x] = \uparrow y$. Then $xR_h y$, so $h^{-1}(x) \subseteq y$, and so $h^{-1}(x)$ is a proper filter of $L$. Thus, by the optimal filter lemma, $h^{-1}(x) = \bigcap \{ z \in L_* : h^{-1}(x) \subseteq z \} = \bigcap \{ z \in L_* : xR_h z \} = \bigcap \{ y \in y = y \}$.

Now suppose that $h^{-1}(x) = y$. Then $R_h[x] = \{ z \in L_* : xR_h z \} = \{ z \in L_* : h^{-1}(x) \subseteq z \} = \{ z \in L_* : y \subseteq z \} = \uparrow y$. □

Theorem 11.16. Let $L$ and $K$ be bounded distributive meet semi-lattices and let $h : L \rightarrow K$ be a meet semi-lattice homomorphism preserving top.

1. $h$ is 1-1 iff $R_h$ is onto.
2. $h$ is onto iff $R_h$ is 1-1.

Proof. (1) Suppose that $h$ is 1-1. We show that $R_h$ is onto. Let $y \in L_*$. Since $h$ preserves $\land$, we have $\uparrow h[y]$ is a filter of $K$. Let $J$ be the F-ideal generated by $h[L - y]$. We claim that $\uparrow h[y] \cap J = \emptyset$. If $\uparrow h[y] \cap J \neq \emptyset$, then there exist $a \in y$, $a_1, \ldots, a_n \in L - y$, and $b \in K$ such that $h(a) \leq b$ and $\bigcap_{i=1}^n \uparrow h(a_i) \subseteq \uparrow b$. Therefore, $\bigcap_{i=1}^n \uparrow h(a_i) \subseteq \uparrow h(a)$. Since $h$ is 1-1, we have $\bigcap_{i=1}^n \uparrow a_i \subseteq \uparrow a$. As $y$ is an optimal filter of $L$, we have $L - y$ is an F-ideal of $L$, so $a \in L - y$, a contradiction. Thus, $\uparrow h[y] \cap J = \emptyset$, and by the optimal filter lemma, there is
Proposition 11.17. Let \( \phi \) be a generalized Priestley morphism. Theorem 11.18. Let \( \psi \) be a generalized Priestley morphism.

Proof. By Proposition 8.6, \( \Box_{R_h}(\varphi(a)) = \varphi(b) \). So \( R_h[\varphi(b)] \subseteq \varphi(a) \) and \( R_h[x] \not\subseteq \varphi(a) \). Thus, \( R_h \) is 1-1.

Now suppose that \( R_h \) is 1-1. Let \( b \in K \). For each \( x \in K \) such that \( b \not\in x \), we have \( x \not\notin \varphi(b) \). Since \( R_h \) is 1-1, there exists \( a_x \in L \) such that \( R_h[\varphi(b)] \subseteq \varphi(a_x) \) and \( R_h[x] \not\subseteq \varphi(a_x) \). Then \( \varphi(b) \subseteq \Box_{R_h}\varphi(a_x) \) and \( x \not\in \Box_{R_h}\varphi(a_x) \). Therefore, \( \bigcap\{\Box_{R_h}\varphi(a_x) : x \not\in \varphi(b)\} \cap \varphi(b)^c = \emptyset \). Since \( X \) is compact, there exist \( x_1, \ldots, x_n \notin \varphi(b) \) such that \( \bigcap_{i=1}^n \Box_{R_h}\varphi(a_{x_i}) \cap \varphi(b)^c = \emptyset \).

Thus, \( \Box_{R_h}\varphi(a_{x_1} \land \ldots \land a_{x_n}) \cap \varphi(b)^c = \emptyset \), so \( \varphi(b) = \Box_{R_h}\varphi(a_{x_1} \land \ldots \land a_{x_n}) = \varphi(h(a_{x_1} \land \ldots \land a_{x_n})) \), and so \( b = h(a_{x_1} \land \ldots \land a_{x_n}) \). It follows that \( h \) is onto. \( \square \)

Proposition 11.17. Let \( X \) and \( Y \) be generalized Priestley spaces and let \( R \subseteq X \times Y \) be a generalized Priestley morphism.

1. \( R \subseteq X \times Y \) is onto iff \( R_{h_R} \subseteq X^*_s \times Y^*_s \) is onto.
2. \( R \subseteq X \times Y \) is 1-1 iff \( R_{h_R} \subseteq X^*_s \times Y^*_s \) is 1-1.

Proof. Apply Theorem 6.21 and Propositions 8.6 and 8.7. \( \square \)

Theorem 11.18. Let \( X \) and \( Y \) be generalized Priestley spaces and let \( R \subseteq X \times Y \) be a generalized Priestley morphism.

1. \( R \) is 1-1 iff \( h_R \) is onto.
2. \( R \) is onto iff \( h_R \) is 1-1.

Proof. It follows from Theorem 11.16 and Proposition 11.17. \( \square \)

Thus, we obtain that 1-1 homomorphisms of bounded distributive meet semi-lattices preserving top correspond to onto generalized Priestley morphisms, and that bounded 1-1 homomorphisms correspond to total onto generalized Priestley morphisms. Moreover, onto homomorphisms of bounded distributive meet semi-lattices preserving top coincide with bounded onto homomorphisms (which is easy to see either algebraically or by recalling that each 1-1 generalized Priestley morphism is total) and correspond to 1-1 generalized Priestley morphisms.
As an immediate consequence of the bounded distributive meet semi-lattice case, we obtain that 1-1 homomorphisms of bounded implicative meet semi-lattices correspond to onto generalized Esakia morphisms, that bounded 1-1 homomorphisms correspond to total onto generalized Esakia morphisms, and that onto homomorphisms are the same as bounded onto homomorphisms and correspond to 1-1 generalized Esakia morphisms.

11.2.1. Dual description of 1-1 and onto homomorphisms for bounded distributive lattices and Heyting algebras. Now we show how our results above imply the well-known dual description of 1-1 and onto homomorphisms of bounded distributive lattices and Heyting algebras.

Lemma 11.19. Let $X$ and $Y$ be Priestley spaces and let $R \subseteq X \times Y$ be a functional generalized Priestley morphism.

1. $R$ is 1-1 iff $f^R$ is an embedding.
2. $R$ is onto iff $f^R$ is onto.

Proof. (1) Suppose that $R$ is 1-1. We show that $f^R$ is an embedding. Let $x \leq y$. Then $R[y] \subseteq R[x]$. Since $R$ is a functional generalized Priestley morphism, both $R[x]$ and $R[y]$ have least elements. Let $l_x$ be the least element of $R[x]$ and $l_y$ be the least element of $R[y]$. Then $R[y] \subseteq R[x]$ implies $l_x \leq l_y$. Thus, $f^R(x) \leq f^R(y)$. Now let $x \not\leq y$. Since $R$ is 1-1, we have $R[y] \not\subseteq R[x]$. Therefore, $l_x \not\leq l_y$, so $f^R(x) \not\leq f^R(y)$, and so $f^R$ is an embedding. Conversely, suppose that $f^R$ is an embedding, $x \in X$, $U \in \mathcal{U}(X)$, and $x \notin U$. Since $f^R$ is an embedding, we have $f^R(x) \notin \uparrow f^R(U)$. So there exists $V \in \mathcal{U}(Y)$ such that $f^R(x) \notin V$ and $\uparrow f^R(U) \subseteq V$. Thus, $R[x] = \uparrow f^R(x) \not\subseteq U$ and $R[U] = \uparrow f^R(U) \subseteq V$, and so $R$ is 1-1.

(2) Suppose that $R$ is onto and $y \in Y$. Then there exists $x \in X$ such that $R[x] = \uparrow y$. Thus, $f^R(x) = y$, and so $f^R$ is onto. Now suppose that $f^R$ is onto and $y \in Y$. Then there exists $x \in X$ such that $f^R(x) = y$. Thus, $R[x] = \uparrow f^R(x) = \uparrow y$, and so $R$ is onto. □

Corollary 11.20. Let $X$ and $Y$ be Priestley spaces and let $f : X \to Y$ be a Priestley morphism.

1. $f$ is an embedding iff $R^f$ is 1-1.
2. $f$ is onto iff $R^f$ is onto.

Proof. Apply Lemmas 9.4 and 11.19. □

Let $L$ and $K$ be bounded distributive lattices and let $h : L \to K$ be a bounded lattice homomorphism. Define $f_h : K_+ \to L_+$ by $f_h(x) = h^{-1}(x)$ for each $x \in K_+$. It is well-known (and follows from Lemmas 10.1 and 9.2) that $f_h$ is a well-defined Priestley morphism. The next proposition is a well-known
consequence of the Priestley duality. We show that it as an easy consequence of Theorem 11.16 and Lemma 11.19.

**Lemma 11.21.** Let $L$ and $K$ be bounded distributive lattices and let $h : L \to K$ be a bounded lattice homomorphism.

1. $h$ is 1-1 iff $f_h$ is onto.
2. $h$ is onto iff $f_h$ is an embedding.

**Proof.** For $x \in K_+$ and $y \in L_+$ we have $xRhy$ iff $h^{-1}(x) \subseteq y$ iff $f_h(x) \subseteq y$. Therefore, $R_h[x] = \uparrow f_h(x)$, so $f_h = f_{R_h}$ (and $R_h = R_{f_h}$). Thus, by Theorem 11.16 and Lemma 11.19, $h$ is 1-1 iff $R_h$ is onto iff $f_h$ is onto, and $h$ is onto iff $R_h$ is 1-1 iff $f_h$ is an embedding.

The above results immediately imply that for a Heyting algebra homomorphism $h$, we have that $h$ is 1-1 iff $f_h$ is onto, and that $h$ is onto iff $f_h$ is 1-1. Another way to see this is through partial Heyting functions. But first we characterize 1-1 and onto $(\wedge, \to)$ and $(\wedge, \to, \bot)$-homomorphisms of Heyting algebras in terms of onto and 1-1 partial Esakia functions.

**Definition 11.22.** Let $X$ and $Y$ be Esakia spaces and let $f : X \to Y$ be a partial Esakia function.

1. We call $f$ onto if the restriction of $f$ to $\text{dom}(f)$ is an onto function.
2. We call $f$ 1-1 if for each $x \in X$ and $U \subseteq X^*$, from $x \notin U$ it follows that there exists $V \in Y^*$ such that $U \subseteq (\downarrow f^{-1}(V^c))^c$ and $f[\uparrow x] \subseteq V$.

Let $X$ and $Y$ be Esakia spaces and let $f : X \to Y$ be a partial Esakia function. We observe that if $f$ is 1-1, then $f$ is well. Indeed, if $f$ is not well, then there exists $x \in \text{max}X$ such that $x \notin \text{dom}(f)$. Therefore, $f[\uparrow x] = \emptyset$, and so $f$ can not be 1-1.

**Lemma 11.23.** Let $X$ and $Y$ be Esakia spaces and let $f : X \to Y$ be a partial Esakia function. Then $f$ is onto iff $R_f$ is onto, and $f$ is 1-1 iff $R_f$ is 1-1.

**Proof.** First suppose that $f$ is onto and $y \in Y$. Then there exists $x \in \text{dom}(f)$ such that $f(x) = y$. Therefore, $\uparrow f(x) = \uparrow y$. As $x \in \text{dom}(f)$, we have $R_f[x] = \uparrow f(x) = \uparrow y$, and so $R_f$ is onto. Conversely, suppose that $R_f$ is onto and $y \in Y$. Then there is $x \in X$ such that $R_f[x] = \uparrow y$. Therefore, $xR_fy$, and so there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $f(z) = y$. Thus, $f$ is onto.

Now suppose that $f$ is 1-1, $x \in X$, $U \subseteq X^*$, and $x \notin U$. Then there exists $V \in Y^*$ such that $U \subseteq (\downarrow f^{-1}(V^c))^c$ and $x \notin (\downarrow f^{-1}(V^c))^c$. Therefore, $U \subseteq \square_{R_f}(V)$ and $x \notin \square_{R_f}(V)$. Thus, $R_f[U] \subseteq V$ and $R_f[x] \not\subseteq V$, and so $R_f$ is 1-1. Conversely, suppose that $R_f$ is 1-1, $x \in X$, $U \subseteq X^*$, and $x \notin U$. Then there exists $V \in Y^*$ such that $R_f[U] \subseteq V$ and $R_f[x] \not\subseteq V$. Therefore, $U \subseteq \square_{R_f}(V)$ and $R_f[x] \not\subseteq V$. Since $\square_{R_f}(V) = (\downarrow f^{-1}(V^c))^c$ and $R_f[x] = f[\uparrow x]$, we obtain $U \subseteq (\downarrow f^{-1}(V^c))^c$ and $f[\uparrow x] \not\subseteq V$. Thus, $f$ is 1-1. 

\[\square\]
As a consequence, we obtain that 1-1 $(\land, \to)$-homomorphisms of Heyting algebras correspond to onto partial Esakia functions, that 1-1 $(\land, \to, \bot)$-homomorphisms correspond to well onto partial Esakia functions, that onto $(\land, \to)$-homomorphisms are the same as onto $(\land, \to, \bot)$-homomorphisms and correspond to 1-1 partial Esakia functions. Moreover, 1-1 Heyting algebra homomorphisms correspond to onto partial Heyting functions and onto Heyting algebra homomorphisms correspond to 1-1 partial Heyting functions. We show that a partial Heyting function $f$ is onto iff $g_f$ is an onto function and that $f$ is 1-1 iff $g_f$ is a 1-1 function.

Let $X$ and $Y$ be Esakia spaces and let $f : X \to Y$ be a partial Heyting function. First we show that $f$ is onto iff $g_f$ is an onto function. Let $f$ be onto and $y \in Y$. Then there exists $x \in \text{dom}(f)$ such that $f(x) = y$. Since $x \in \text{dom}(f)$, we have $g_f(x) = f(x)$. Thus, $g_f(x) = y$, and so $g_f$ is an onto function. Conversely, suppose that $g_f$ is an onto function and $y \in Y$. Then there exists $x \in X$ such that $g_f(x) = y$. Therefore, there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $g_f(x) = f(z)$. Thus, $f(z) = y$ and so $f$ is onto.

Next we show that $f$ is 1-1 iff $g_f$ is a 1-1 function. Let $f$ be 1-1. We show that $g_f$ is a 1-1 function. Let $x, y \in X$ with $x \neq y$. Without loss of generality we may assume that $x \not\leq y$. Then there exists a clopen upset $U$ of $X$ such that $x \in U$ and $y \notin U$. Since $f$ is 1-1, there exists a clopen upset $V$ of $Y$ such that $U \subseteq (\downarrow f^{-1}(V^c))^c$ and $f[\downarrow y] \not\subseteq V$. But $(\downarrow f^{-1}(V^c))^c = g_f^{-1}(V)$ and $f[\downarrow y] = \uparrow g_f(y)$. Therefore, $U \subseteq g_f^{-1}(V)$ and $g_f(y) \notin V$. Thus, $x \in U \subseteq g_f^{-1}(V)$ and $y \notin g_f^{-1}(V)$, implying that $g_f(x) \not\leq g_f(y)$. Consequently, $g_f$ is a 1-1 function. Conversely, suppose that $g_f$ is a 1-1 function. Let $x \in X$, $U$ be a clopen upset of $X$, and $x \notin U$. Since $g_f$ is a 1-1 function, we have $g_f(x) \notin g_f(U)$. As $g_f(U)$ is a closed upset of $Y$, there exists a clopen upset $V$ of $Y$ such that $g_f(U) \subseteq V$ and $g_f(x) \notin V$. Therefore, $U \subseteq g_f^{-1}(V) = (f^{-1}(V^c))^c$ and $f[\uparrow x] = \uparrow g_f(x) \not\subseteq V$. Thus, $U \subseteq (\downarrow f^{-1}(V^c))^c$ and $f[\uparrow x] \not\subseteq V$, implying that $f$ is 1-1.

Consequently, we obtain the well-known result that for a Heyting algebra homomorphism $h$ we have $h$ is 1-1 iff $f_h$ is onto, and that $h$ is onto iff $f_h$ is 1-1.

12. NON-BOUNDED CASE

The duality we have developed for bounded distributive meet semi-lattices can be modified accordingly to obtain duality for non-bounded distributive meet semi-lattices. In this section we discuss briefly the main ideas of the modification.

First we deal with the case of distributive meet semi-lattices with top but possibly without bottom. Let $L \in \text{DM}$. If $L$ does not have bottom, then we have to add $L$ to the set of optimal filters of $L$. As a result, $L$ is the greatest element of $L_\ast$, and so $\operatorname{max}(L_\ast)$ is not contained in $L_\ast$. Thus, we have to drop condition (3) of Definition 6.10. Moreover, for $x = L$ we have $\mathcal{I}_x = \emptyset$, so $\mathcal{I}_x$
is trivially directed, although \( L \notin L_+ \). Thus, we have to modify condition (4) of Definition 6.10 as follows: \( x \in L_+ \) iff \( I_x \) is nonempty and directed. This suggests the following modification of the definition of a generalized Priestley space.

**Definition 12.1.** A quadruple \( X = \langle X, \tau, \leq, X_0 \rangle \) is called a \( * \)-generalized Priestley space if (i) \( \langle X, \tau, \leq \rangle \) is a Priestley space, (ii) \( X_0 \) is a dense subset of \( X \), (iii) \( x \in X_0 \) iff \( I_x \) is nonempty and directed, and (iv) for all \( x, y \in X \), we have \( x \leq y \) iff \( (\forall U \in X^*)(x \in U \Rightarrow y \in U) \).

Clearly each generalized Priestley space is a \( * \)-generalized Priestley space, so the concept of \( * \)-generalized Priestley space generalizes that of generalized Priestley space. Moreover, a \( * \)-generalized Priestley space is a generalized Priestley space iff \( \max(X) \subseteq X_0 \) iff \( X^* \) has a bottom element. Therefore, a \( * \)-generalized Priestley space is a generalized Priestley space iff it satisfies condition (4) of Definition 6.10. The same generalization of generalized Esakia spaces yields \( * \)-generalized Esakia spaces. Let \( \text{GPS}^* \) denote the category of \( * \)-generalized Priestley spaces and generalized Priestley morphisms, and let \( \text{GES}^* \) denote the category of \( * \)-generalized Esakia spaces and generalized Esakia morphisms. Then we immediately obtain the following theorem, which generalizes the duality we obtained for \( \text{BDM} \) and \( \text{BIM} \) to \( \text{DM} \) and \( \text{IM} \), respectively.

**Theorem 12.2.**
1. \( \text{DM} \) is dually equivalent to \( \text{GPS}^* \).
2. \( \text{IM} \) is dually equivalent to \( \text{GES}^* \).

If \( L \) is a distributive meet-semilattice without top but with bottom, then two cases are possible: either \( D(L) \) has top or \( D(L) \) does not have top. If \( D(L) \) has top, then we obtain the dual of \( L \) in exactly the same way as in the bounded case. But in this case \( L \) will be realized as \( L^*_s = \{L_s\} \). If \( D(L) \) does not have top, then again we construct the dual of \( L \) as before, however in this case the space we obtain is locally compact but not compact. We can handle this as the case for distributive lattices [14, Section 10] by adding a new top to \( L \). If \( L^\top \) is the resulting meet semi-lattice, then the dual space of \( L^\top \) is the one-point compactification of the dual of \( L \). Moreover, the new point of \( (L^\top)_s \) is the smallest optimal filter \( \{\top\} \) of \( L^\top \), which is below every point of \( L_s \).

This way we can handle all possible situations; that is, when \( L \) has \( \top \), but lacks \( \bot \); when \( L \) has \( \bot \), but lacks \( \top \); or the most general case, when \( L \) lacks both \( \top \) and \( \bot \).

13. **Comparison with the relevant work**

In this final section we compare our duality with the existing dualities for distributive meet semi-lattices and implicational meet semi-lattices.
13.1. **Comparison with the work of Celani and Hansoul.** The first representation of distributive meet semi-lattices is already present in the pioneering work of Stone [15]. It was made more explicit in Grätzer [6], where with each join semi-lattice \( L \) with bottom is associated the space \( S(L) \) of prime ideals of \( L \). The topology on \( S(L) \) is generated by the basis consisting of the sets \( r(a) = \{ I \in S(L) : a \notin I \} \). The space \( S(L) \) is not in general Hausdorff, and it is compact iff \( L \) has a top. In [6] a purely topological characterization of such spectral-like spaces is given. We call a topological space \( \langle X, \tau \rangle \) spectral-like if (i) \( X \) is \( T_0 \), (ii) the compact open subsets of \( X \) form a basis for \( \tau \), and (iii) for each closed subset \( F \) of \( X \) and each nonempty down-directed family of compact open subsets \( \{ U_i : i \in I \} \) of \( X \), from \( F \cap U_i \neq \emptyset \) for each \( i \in I \), it follows that \( F \cap \bigcap \{ U_i : i \in I \} \neq \emptyset \). It was shown in [6] that \( S(L) \) is spectral-like for each \( L \), that the basis of compact open subsets of a spectral-like space forms a distributive join semi-lattice with bottom, and that this correspondence between distributive join semi-lattices with bottom and spectral-like spaces is 1-1. But there was no attempt made in [6] to obtain categorical duality between the category of distributive join semi-lattices with bottom and the category of spectral-like spaces.

In [2] Celani filled in this gap. He chose to work with meet semi-lattices instead of join semi-lattices, hence his building blocks for the dual space were prime filters instead of prime ideals. To be more specific, let us recall that \( \text{DM} \) denotes the category of distributive meet semi-lattices with top as objects and meet semi-lattice homomorphisms preserving top as morphisms. Celani’s dual category has (in the terminology of [2]) DS-spaces as objects and meet-relations between two DS-spaces as morphisms. We recall from [2, Definition 14] that a **DS-space** is an ordered topological space \( \langle X, \tau, \leq \rangle \) such that (i) \( \langle X, \tau \rangle \) is a spectral-like space, (ii) each closed subset of \( X \) is an upset, and (iii) if \( x, y \in X \) with \( x \nleq y \), then there is a compact open subset \( U \) of \( X \) such that \( x \notin U \) and \( y \in U \).

**Remark 13.1.** For a topological space \( X \), we recall that the specialization order of \( X \) is defined by \( x \preceq y \) if \( x \in \{ y \} \), that \( \preceq \) is reflexive and transitive, and that it is a partial order iff \( X \) is \( T_0 \). Let \( X \) be a spectral-like space. Then it is easy to see that Celani’s order is nothing more than the dual of the specialization order of \( X \).

For a DS-space \( X \), let \( \mathcal{E}(X) \) denote the set of compact open subsets of \( X \), and let \( D_X = \{ U \subseteq X : U^c \in \mathcal{E}(X) \} \). Let \( X \) and \( Y \) be two DS-spaces and let \( R \subseteq X \times Y \) be a binary relation. Following [2, Definition 19], we call \( R \) a **meet-relation** if (i) \( U \in D_Y \) implies \( \Box_R U \in D_X \) and (ii) \( R[x] = \bigcap \{ U \in D_Y : R[x] \subseteq U \} \) for each \( x \in X \). Let \( \text{DS} \) denote the category of DS-spaces as objects and meet-relations as morphisms.
Celani defines two functors \((-\cdash)_+: \mathsf{DM} \to \mathsf{DS}\) and \((-\cdash)^+: \mathsf{DS} \to \mathsf{DM}\) as follows. If \(L \in \mathsf{DM}\), then \(L_+ = \langle L_+, \tau, \subseteq \rangle\), where \(L_+\) is the set of prime filters of \(L\) and \(\tau\) is the topology generated by the basis \(\{\sigma(a)^c : a \in L\}\); if \(h : L \to K\) is a meet semi-lattice homomorphism preserving \(\top\), then \(h_+ = R_h \subseteq K_+ \times L_+\) is defined by \(xR_hy\) iff \(h^{-1}(x) \subseteq y\). If \(X\) is a DS-space, then \(X^+ = \langle D_X, \cap, X \rangle\); if \(X\) and \(Y\) are DS-spaces and \(R \subseteq X \times Y\) is a meet-relation, then \(R^+ = h_R : D_Y \to D_X\) is defined by \(h_R(U) = \Box R U\). Then it follows from [2] that the functors \((-\cdash)_+\) and \((-\cdash)^+\) are well-defined, and that they establish dual equivalence of the categories \(\mathsf{DM}\) and \(\mathsf{DS}\).

**Remark 13.2.** Although not addressed in [2], the composition of two meet-relations is not the usual set-theoretic composition. Rather, similar to the case of \(\mathsf{GPS}\), we have that for DS-spaces \(X, Y\), and \(Z\) and meet-relations \(R \subseteq X \times Y\) and \(S \subseteq Y \times Z\), the composition \(S \circ R \subseteq X \times Z\) is given by

\[
x(S \circ R)z \text{ iff } (\forall U \in Z^+)(x \in (h_R \circ h_S)(U) \Rightarrow z \in U)
\]

for each \(x \in X\) and \(z \in Z\).

The bounded distributive meet semi-lattices are exactly the objects of \(\mathsf{DM}\) whose dual spaces are compact. Indeed, if \(L\) is a bounded distributive meet semi-lattice, then \(\sigma(\bot) = \emptyset\), so \(\sigma(a)^c = L_+\), and so \(L_+\) is compact as \(\{\sigma(a)^c : a \in L\} = \mathcal{E}(L_+)\). Conversely, if \(L_+\) is compact, then \(L_+ = \sigma(a)^c\) for some \(a \in L\), so \(a\) is the bottom of \(L\), and so \(L\) is bounded. It follows that the full subcategory \(\mathsf{BDM}\) of \(\mathsf{DM}\) whose objects are bounded distributive meet semi-lattices is dually equivalent to the full subcategory \(\mathsf{CDS}\) of \(\mathsf{DS}\) whose objects are compact DS-spaces. Now putting Celani’s duality together with ours, we obtain that \(\mathsf{CDS}\) is equivalent to the category of generalized Priestley spaces and generalized Priestley morphisms introduced in this paper. In order to make the connection easier to understand, we show how to construct compact DS-spaces from generalized Priestley spaces, and how to construct meet-relations from generalized Priestley morphisms.

**Proposition 13.3.** Let \(X = \langle X, \tau, \leq, X_0 \rangle\) be a generalized Priestley space. Then the space \(X_0 = \langle X_0, \tau_0, \leq_0 \rangle\) is a compact DS-space, where \(\leq_0\) is the restriction of \(\leq\) to \(X_0\) and \(\tau_0\) is the topology generated by the basis \(\{X_0 - U : U \in X^*\}\).

**Proof.** First we show that \(\{X_0 - U : U \in X^*\}\) is indeed a basis for \(\tau_0\). Let \(U, V \in X^*\) and \(x \in (X_0 - U) \cap (X_0 - V)\). Then \(x \in X_0\) and \(x \notin U, V\). By condition (4) of Definition 6.10, there exists \(W \in X^*\) such that \(U, V \subseteq W\) and \(x \notin W\). Thus, \(x \in X_0 - W \subseteq (X_0 - U) \cap (X_0 - V)\), and so \(\{X_0 - U : U \in X^*\}\) is a basis for \(\tau_0\). Next we show that compact open subsets of \(X_0\) are exactly of the form \(X_0 - U\) for \(U \in X^*\). Let \(U \in X^*\) and let \(X_0 - U \subseteq \bigcup_{i \in I} (X_0 - U_i),\) where...
\{U_i : i \in I\} \subseteq X^*$. Then \(\downarrow(X_0 - U) \subseteq \bigcup_{i \in I}(X_0 - U_i)\). Since for each \(V \in X^*\) we have \(X - V = \downarrow(X_0 - V)\), then \(X - U \subseteq \bigcup_{i \in I}(X - U_i)\). As \(X - U\) is closed, hence compact in \(X\), there is a finite \(J \subseteq I\) such that \(X - U \subseteq \bigcup_{i \in J}(X - U_i)\).

Hence, \(X_0 - U \subseteq \bigcup_{i \in J}(X_0 - U_i)\), and so \(X_0 - U\) is compact. Conversely, if \(V\) is a compact open subset of \(X_0\), then \(V\) is a finite union of the sets of the form \(X_0 - V_i\). Since \(X^*\) is closed under finite intersections, \(\{X_0 - U : U \in X^*\}\) is closed under finite unions. Thus, each compact open subset of \(X_0\) has the form \(X_0 - U\) for \(U \in X^*\), and so \(\{X_0 - U : U \in X^*\}\) is exactly the set of compact open subsets of \(X_0\). Moreover, as \(\emptyset \in X^*\), we have that \(X_0 = X_0 - \emptyset\) is compact open, and so \(X_0\) is compact. Now suppose that \(x, y \in X_0\) are such that \(x \not\leq y\). By condition (5) of Definition 6.10, there exists \(U \in X^*\) such that \(x \in U\) and \(y \not\in U\). Thus, \(x \not\in X_0 - U\) and \(y \in X_0 - U\), and so condition (iii) of the definition of a DS-space is satisfied. It also follows that \(X_0\) is a \(T_0\)-space. Since each \(U \in X^*\) is an upset of \(X\), each \(X_0 - U\) is a downset of \(X_0\). Thus, open sets of \(X_0\) are downsets, and so closed sets of \(X_0\) are upsets. Consequently, \(X_0\) satisfies condition (ii) of the definition of a DS-space. Finally, let \(F\) be a closed subset of \(X_0\) and let \(\{U_i : i \in I\}\) be a down-directed family of compact open subsets of \(X_0\) such that for each \(i \in I\) we have \(F \cap U_i \neq \emptyset\). Then \(F = \bigcap\{W_k : k \in K\}\) for some family \(\{W_k : k \in K\} \subseteq X^*\) and \(U_i = X_0 - V_i\) for some \(V_i \in X^*\). Since \(\{X_0 - V_i : i \in I\}\) is down-directed, so is \(\{\downarrow(X_0 - V_i) : i \in I\}\). But \(\downarrow(X_0 - V_i) = X - V_i\), so \(\{V_i : i \in I\} \subseteq X^*\) is directed. Therefore, \(\Delta = \{V_i : i \in I\}\) is an ideal of \(X^*\). Let \(\nabla\) be the filter of \(X^*\) generated by \(\{W_k : k \in K\}\). Clearly \(\nabla \cap \Delta = \emptyset\). Thus, there exists a prime filter \(P\) of \(X^*\) such that \(\nabla \subseteq P\) and \(P \cap \Delta = \emptyset\). Let \(x \in X_0\) be such that \(\varepsilon(x) = P\). Then \(x \in F\) and \(x \not\in V_i\) for each \(i \in I\). Consequently, \(x \in F \cap \bigcap_{i \in I}(X_0 - V_i)\), so \(X_0\) is a spectral-like space, so condition (i) of the definition of a DS-space is satisfied, and so \(X_0\) is a compact DS-space. \(\Box\)

**Proposition 13.4.** If \(X\) and \(Y\) are generalized Priestley spaces and \(R \subseteq X \times Y\) is a generalized Priestley morphism, then \(R_0 = R \cap (X_0 \times Y_0)\) is a meet-relation between the compact DS-spaces \(X_0\) and \(Y_0\).

**Proof.** Suppose \(R \subseteq X \times Y\) is a generalized Priestley morphism between two generalized Priestley spaces \(X\) and \(Y\). We prove that \(R_0\) is a meet-relation between \(X_0\) and \(Y_0\). Let \(U \in D_{Y_0}\). Then \(Y_0 - U\) is compact open in \(Y_0\). Therefore, there is \(V \in Y^*\) such that \(Y_0 - U = Y_0 - V\). Thus, \(U = V \cap Y_0\) and \(\downarrow(Y_0 - U) = Y - V\). We show that \(\square_{R_0} U = \square_R V \cap X_0\). First suppose that
A tuple \( x \in \triangleleft_R V \cap X_0 \). Then \( x \in X_0 \) and \( R[x] \subseteq V \). Thus, \( R_0[x] \subseteq V \cap Y_0 = U \), and so \( x \in \triangleleft_{R_0} U \). Next suppose that \( x \in \triangleleft_{R_0} U \). If \( R[x] \not\subseteq V \), then there exists \( y \in Y \) such that \( xRy \) and \( y \not\in V \). Therefore, \( y \in Y - V = \downarrow(Y_0 - U) \). So there exists \( z \in Y_0 - U \) such that \( y \leq z \). Since \( xRy \) and \( y \leq z \), we have \( xRz \), so \( xR_0z \). Thus, \( z \in R_0[x] \) and \( z \not\in U \), a contradiction. We conclude that \( R[x] \subseteq V \), and so \( x \in \triangleleft_R V \cap X_0 \). It follows that \( \triangleleft_{R_0} U = \triangleleft_R V \cap X_0 \), so \( X_0 - \triangleleft_{R_0} U = X_0 - \triangleleft_R V \), and so \( \triangleleft_{R_0} U \in D_{X_0} \). Thus, condition (i) of the definition of a meet-relation is satisfied. Now we show that for each \( x \in X_0 \) we have \( R_0[x] = \bigcap\{U \in D_{Y_0} : R_0[x] \subseteq U\} \). Clearly \( R_0[x] \subseteq \bigcap\{U \in D_{Y_0} : R_0[x] \subseteq U\} \). Suppose that \( y \in Y_0 \) and \( xR_0y \). Then \( xRy \). By condition (3) of Definition 8.2, there exists \( V \in Y^* \) such that \( y \not\in V \) and \( R[x] \subseteq V \). Therefore, \( V \cap X_0 \in D_{Y_0}, \ R_0[x] \subseteq V \cap X_0 \), and \( y \not\in V \cap X_0 \). Thus, \( y \not\in \bigcap\{U \in D_{Y_0} : R_0[x] \subseteq U\} \), so \( R_0[x] = \bigcap\{U \in D_{Y_0} : R_0[x] \subseteq U\} \), and so condition (ii) of the definition of a meet-relation is satisfied. Consequently, \( R_0 \) is a meet-relation between the compact DS-spaces \( X_0 \) and \( Y_0 \). \( \square \)

Now we compare our work with that of Hansoul [7]. Like Grätzer, Hansoul prefers to work with distributive join semi-lattices. But unlike both Grätzer and Celani, he tries to build a Priestley-like dual of a bounded distributive join semi-lattice. Thus, his work is the closest to ours. We recall the main definition of [7]. For convenience, we call the spaces Hansoul calls Priestley structures simply Hansoul spaces.

**Definition 13.5.** A Hansoul space is a tuple \( X = \langle X, \tau, \leq, X_0 \rangle \), where:

1. \( \langle X, \tau, \leq \rangle \) is a Priestley space.
2. \( X_0 \) is a dense subset of \( X \).
3. If \( x, y \in X \) with \( x \not\leq y \), then there is \( z \in X_0 \) such that \( x \not\leq z \) and \( y \leq z \).
4. \( X_0 \) is the set of elements of \( X \) for which the family of clopen downsets \( U \) that contain \( x \) and have the property that \( U \cap X_0 \) is cofinal in \( U \) is a basis of clopen downset neighborhoods of \( x \).
5. For each \( x \in X \) there exists \( y \in X_0 \) such that \( x \leq y \).

Hansoul constructs the dual \( X \) of a bounded distributive join semi-lattice \( L \) by taking what he calls weakly prime ideals of \( L \) as points of \( X \) and by taking prime ideals of \( L \) as points of the dense subset \( X_0 \) of \( X \). The weakly prime ideals of \( L \) are exactly the optimal filters of the dual \( L^d \) of \( L \) (see Remark 4.19) and the prime ideals of \( L \) are exactly the prime filters of \( L^d \). Thus, Hansoul’s construction is dual to ours. In fact, we show that Hansoul spaces are the same as our generalized Priestley spaces.

**Proposition 13.6.** A tuple \( X = \langle X, \tau, \leq, X_0 \rangle \) is a Hansoul space iff it is a generalized Priestley space.
Proof. Since conditions (1) and (2) of Definition 6.10 are the same as conditions (1) and (2) of Definition 13.5 and condition (3) of Definition 6.10 is the same as condition (5) of Definition 13.5, it is sufficient to show that if $X$ is a Hansoul space, then $X$ satisfies conditions (4) and (5) of Definition 6.10, and that if $X$ is a generalized Priestley space, then $X$ satisfies conditions (3) and (4) of Definition 13.5.

Let $X$ be a Hansoul space. First we show that $X$ satisfies condition (5) of Definition 6.10. If $x, y \in X$ with $x \leq y$ then it is clear that for each $U \in X^*$ we have $x \in U$ implies $y \in U$ (because $U$ is an upset). Suppose that $x \not\leq y$. By condition (3) of Definition 13.5, there exists $z \in X_0$ such that $x \not\leq z$ and $y \leq z$. Since $x \not\leq z$ and $X$ is a Priestley space, there is a clopen downset $V$ of $X$ such that $z \in V$ and $x \not\in V$. By condition (4) of Definition 13.5, we can take $V$ such that $V \cap X_0$ is cofinal in $V$. Thus, there exists $U = V^c$ in $X^*$ such that $x \in U$ and $y \not\in U$, and so $X$ satisfies condition (5) of Definition 6.10. Now we show that $X$ satisfies condition (4) of Definition 6.10. Suppose that $x \in X_0$. We show that $\mathcal{I}_x$ is directed. Let $U, V \in \mathcal{I}_x$. Then $x \in (U \cup V)^c$, and so $(U \cup V)^c$ is a clopen downset neighborhood of $x$. By condition (4) of Definition 13.5, there is a clopen downset $W$ such that $W \cap X_0$ is cofinal in $W$ and $x \in W \subseteq (U \cup V)^c$. Therefore, $x \not\in W^c \in X^*$, so $W^c \in \mathcal{I}_x$, and $U \cup V \subseteq W^c$. Thus, $\mathcal{I}_x$ is directed. Now suppose that $\mathcal{I}_x$ is directed. In order to show that $x \in X_0$, let $V$ be a clopen downset neighborhood of $x$. Then $V^c$ is a clopen upset and $x \not\in V^c$. Since in the proof of Proposition 6.15 condition (4) of Definition 6.10 is not used, we can use Proposition 6.15 here, by which there exist $U_1, \ldots, U_n \in X^*$ such that $V^c = U_1 \cup \ldots \cup U_n$. Therefore, $U_1, \ldots, U_n \in \mathcal{I}_x$. Since $\mathcal{I}_x$ is directed, there exists $U \in \mathcal{I}_x$ such that $U_1 \cup \ldots \cup U_n \subseteq U$. Thus, $x \in U^c \subseteq V$ and $U^c$ is a clopen downset of $X$ with $U^c \cap X_0$ cofinal in $U^c$. This, by condition (4) of Definition 13.5, implies that $x \in X_0$. Consequently, $x \in X_0$ iff $\mathcal{I}_x$ is directed, and so $X$ satisfies condition (4) of Definition 6.10.

Now let $X$ be a generalized Priestley space. First we show that $X$ satisfies condition (3) of Definition 13.5. Suppose $x, y \in X$ are such that $x \not\leq y$. By condition (5) of definition 6.10, there is $U \in X^*$ such that $x \in U$ and $y \not\in U$. Since $U^c = \{X_0 - U\}$, there is $z \in X_0$ such that $y \leq z$ and $z \not\in U$. As $U$ is an upset, it follows that $x \not\leq z$. Thus, there is $z \in X_0$ such that $x \not\leq z$ and $y \leq z$, and so $X$ satisfies condition (3) of Definition 13.5. Finally, we show that $X$ satisfies condition (4) of Definition 13.5. First suppose that $x \in X_0$. By condition (4) of Definition 6.10, $\mathcal{I}_x$ is directed. Let $V$ be a clopen downset neighborhood of $x$. Then $x \not\in V^c$ and $V^c$ is a clopen upset of $X$. By Proposition 6.15, $V^c$ is a finite union of elements of $\mathcal{I}_x$. Since $\mathcal{I}_x$ is directed, there is $U \in \mathcal{I}_x$ such that $V^c \subseteq U$. Therefore, $x \in U^c \subseteq V$ and $U^c \cap X_0$ is cofinal in $U^c$. Thus, the family of clopen downsets $U$ that contain $x$ and have the property that $U \cap X_0$ is cofinal in $U$ form a basis of clopen downset neighborhoods of $x$. Conversely, suppose that for each clopen downset neighborhood $V$ of $x$ there
is a clopen downset $W$ of $X$ such that $W \cap X_0$ is cofinal in $W$ and $x \in W \subseteq V$. We show that $I_x$ is directed. Suppose that $U, W \in I_x$. Then $x \in (U \cup W)^c$ and $(U \cup W)^c$ is a clopen downset of $X$. Therefore, there is a clopen downset $V$ of $X$ such that $V \cap X_0$ is cofinal in $V$ and $x \in V \subseteq (U \cup W)^c$. Then $x \not\in V^c \in X^*$. Thus, $V^c \in I_x$ and $U \cup W \subseteq V^c$, so $I_x$ is directed. This, by condition (4) of Definition 6.10, implies that $x \in X_0$. It follows that $X$ satisfies condition (4) of Definition 13.5. □

Consequently, Hansoul spaces are exactly our generalized Priestley spaces. In [7] Hansoul provided duality for the category whose objects are bounded join semi-lattices and whose morphisms correspond to our sup-homomorphisms. However, he was unable to extend his duality to bounded join semi-lattice homomorphisms. Thus, our work can be viewed as a completion of Hansoul’s work.

13.2. Comparison with the work of Köhler. The first duality between finite implicative meet semi-lattices and finite partially ordered sets was developed by Köhler [9].

Let $\IM_{\text{fin}} = \BIM_{\text{fin}}$ denote the category of finite implicative meet semi-lattices (which are always bounded) and implicative meet semi-lattice homomorphisms. Köhler established that $\IM_{\text{fin}}$ is dually equivalent to the category of finite partially ordered sets and partial functions $f : X \to Y$ satisfying the following two conditions:

1. If $x, y \in \text{dom}(f)$ and $x < y$, then $f(x) < f(y)$.
2. If $x \in \text{dom}(f)$, $y \in Y$, and $f(x) < y$, then there is $z \in \text{dom}(f)$ such that $x < z$ and $f(z) = y$.

We call such functions partial Köhler functions, and denote the category of finite posets and partial Köhler functions by $\POS^K_{\text{fin}}$.

Now we show how the dual equivalence of $\HA \であり \to$ and $\ES_{\text{pf}}$ restricted to the finite case implies the Köhler duality. Let $\ES_{\text{pf}}^\text{fin}$ denote the category of finite Esakia spaces and partial Esakia functions. Since each finite implicative meet-semilattice is in fact a Heyting algebra, it follows from Corollary 10.14 that $\IM_{\text{fin}}$ is dually equivalent to $\ES_{\text{pf}}^\text{fin}$. But the objects of $\ES_{\text{pf}}^\text{fin}$ are simply finite partially ordered sets. For finite partially ordered sets $X$ and $Y$, a partial function $f : X \to Y$ is a partial Esakia function iff $f$ is a partial Köhler function. Therefore, the categories $\ES_{\text{pf}}^\text{fin}$ and $\POS^K_{\text{fin}}$ are isomorphic. Thus, Köhler’s theorem that $\IM_{\text{fin}}$ is dually equivalent to $\POS^K_{\text{fin}}$ is an easy consequence of the fact that $\HA \であり \to$ is dually equivalent to $\ES_{\text{pf}}$. Köhler was

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2Köhler called implicative meet semi-lattices Brouwerian semi-lattices.

3Köhler developed his duality by working with downsets of posets. Since we prefer to work with upsets instead of downsets, we rephrased Köhler’s conditions using the dual of his order.
unable to extend his duality to the infinite case. Thus, our work can be viewed as a completion of Köhler’s work.

13.3. Comparison with the work of Celani and Vrancken-Mawet. In [1] Celani showed that a restricted version of his duality of [2] between DM and DS yields duality between the subcategory IM of DM of implicative meet semi-lattices and implicative meet semi-lattice homomorphisms and the subcategory IS of DS of IS-spaces and functional meet-relations. We recall from [1] that an IS-space is a DS-space $X$ such that for each $U, V \in D_X$ we have $\downarrow (U - V) c \in D_X$, and that a functional meet-relation is a meet-relation $R$ between two IS-spaces $X$ and $Y$ such that for each $x \in X$ and $y \in Y$, from $x R y$ it follows that there exists $z \in X$ with $x \leq z$ and $R[z] = \uparrow y$.

We consider the subcategory BIM of IM of bounded implicative meet semi-lattices, and the subcategory CIS of IS of compact IS-spaces. Then BIM is a subcategory of BDM, CIS is a subcategory of CDS, and BIM is dually equivalent to CIS. From this and our duality for BIM it follows that CIS is equivalent to GES. We give an explicit construction of a compact IS-space from a generalized Esakia space and of a functional meet-relation from a generalized Esakia morphism.

Proposition 13.7. Let $X = \langle X, \tau, \leq, X_0 \rangle$ be a generalized Esakia space and let $X_0 = \langle X_0, \tau_0, \leq_0 \rangle$ be the space constructed in Proposition 13.3. Then $X_0$ is a compact IS-space.

Proof. It follows from Proposition 13.3 that $X_0$ is a compact DS-space. It is left to be shown that if $U, V \in D_{X_0}$, then $X_0 - \downarrow_0 (U - V) \in D_{X_0}$. Since $U, V \in D_X$, there exist $U', V' \in X^*$ such that $U = U' \cap X_0$ and $V = V' \cap X_0$. We show that $X_0 - \downarrow_0 (U - V) = X_0 - \downarrow (U' - V')$. We have $x \in X_0 - \downarrow_0 (U - V)$ iff $(\forall y \in X_0)(x \leq_0 y$ and $y \in U$ $\Rightarrow y \in V$), and $x \in X_0 - \downarrow (U' - V')$ iff $(\forall y \in X)(x \leq y$ and $y \in U' \Rightarrow y \in V')$. Therefore, it is clear that $x \in X_0 - \downarrow (U' - V')$ implies $x \in X_0 - \downarrow_0 (U - V)$. Conversely, suppose that $x \in X_0 - \downarrow_0 (U - V)$. If there exists $y \in X$ such that $x \leq y, y \in U'$ and $y \notin V'$, then there exists $z \in \max (X - V')$ such that $y \leq z$. But $\max (X - V') \subseteq X_0$. Thus, there exists $z \in X_0$ such that $x \leq_0 z, z \in U$, and $z \notin V$, a contradiction. Consequently, $x \in X_0 - \downarrow (U' - V')$, and so $X_0 - \downarrow_0 (U - V) = X_0 - \downarrow (U' - V')$. Since $X$ is a generalized Esakia space, $X - \downarrow (U' - V') \in X^*$. Moreover, $X_0 - \downarrow_0 (U - V) = X_0 - \downarrow (U' - V') = (X - \downarrow (U' - V')) \cap X_0$. So $X_0 - \downarrow_0 (U - V) \in D_{X_0}$, and so $X_0$ is a compact IS-space.

Proposition 13.8. If $X$ and $Y$ are generalized Esakia spaces and $R \subseteq X \times Y$ is a generalized Esakia morphism, then $R_0 = R \cap (X_0 \times Y_0)$ is a functional meet-relation between the compact IS-spaces $X_0$ and $Y_0$.

\footnote{Warning: Do not confuse Celani’s notion of functional meet-relation with our notion of functional generalized Priestley morphism!}
Proof. It follows from Proposition 13.4 that \( R_0 \) is a meet-relation. We show that \( R_0 \) is functional. Let \( x \in X_0, \ y \in Y_0, \) and \( xR_0y. \) Then \( xRy \) and since \( y \in Y_0 \) and \( R \) is a generalized Esakia morphism, there exists \( z \in X_0 \) such that \( x \leq z \) and \( R[z] = \uparrow y. \) Thus, \( x \leq_0 z \) and \( R_0[z] = \uparrow_0 y, \) and so \( R_0 \) is a functional meet-relation. \( \square \)

Celani [1, Theorem 4.11] also claimed erroneously that functional meet-relations between IS-spaces \( X \) and \( Y \) are in a 1-1 correspondence with partial maps \( f : X \to Y, \) he calls IS-morphisms, that satisfy the following properties:

(i) If \( x, y \in \text{dom}(f) \) and \( x \leq y, \) then \( f(x) \leq f(y), \) (ii) if \( x \in \text{dom}(f), \ y \in Y, \) and \( f(x) \leq y, \) then there exists \( z \in \text{dom}(f) \) such that \( x \leq z \) and \( f(z) = y, \) and (iii) if \( U \in D_Y, \) then \( [\downarrow f^{-1}(U)]^c \in D_X. \)

For a functional meet-relation \( R \) between two IS-spaces \( X \) and \( Y, \) he constructed the IS-morphism \( f_R \) as follows. He set \( \text{dom}(f_R) = \{ z \in X : (\exists x \in X) (\exists y \in Y) (x \leq z \) and \( R[z] = \uparrow y \}, \) and for \( z \in \text{dom}(f_R) \) he set \( f_R(z) = y. \) For an IS-morphism \( f \) between two IS-spaces \( X \) and \( Y, \) he constructed the functional meet-relation \( R_f \subseteq X \times Y \) by \( xR_fy \iff (\exists z \in \text{dom}(f))(x \leq z \) and \( f(z) = y). \) However, his claim that the maps \( R \mapsto f_R \) and \( f \mapsto R_f \) establish a 1-1 correspondence between functional meet-relations and IS-morphisms between IS-spaces is false as the following simple example shows.

Example 13.9. Consider the IS-spaces \( X \) and \( Y \) and the IS-morphism \( f : X \to Y \) shown in Fig.8. The functional meet-relation \( R_f \subseteq X \times Y \) corresponding to \( f, \) together with the IS-morphism \( f_R : X \to Y \) corresponding to \( R_f \) are also shown in Fig.8. Clearly \( f \neq f_R, \) thus the maps \( R \mapsto f_R \) and \( f \mapsto R_f \) do not establish a 1-1 correspondence between functional meet-relations and IS-morphisms between IS-spaces.

In order to obtain a characterization of Celani’s functional meet-relations by means of partial functions, we need to strengthen the notion of an IS-morphism. For two IS-spaces \( X \) and \( Y, \) we call a map \( f : X \to Y \) a strong IS-morphism (SIS-morphism for short) if it is an IS-morphism and satisfies the two additional conditions:

\[ (*) \text{ For each } x \in X, \ x \in \text{dom}(f) \text{ iff there exists } y \in Y \text{ such that } f[\uparrow x] = \uparrow y. \]
(**) For each \( x \in X \) and each \( y \in Y \), if \( y \notin f[\downarrow x] \), then there is \( U \in D_Y \) such that \( f[\downarrow x] \subseteq U \) and \( y \notin U \).

We recall that \( IS \) is the category of IS-spaces and functional meet-relations. Let \( SIS \) denote the category of IS-spaces and SIS-morphisms. We show that \( IS \) is isomorphic to \( SIS \).

**Lemma 13.10.** Let \( X \) and \( Y \) be two DS-spaces and let \( R \subseteq X \times Y \) be a meet-relation. Then \( x \leq y \) and \( yRu \) imply \( xRu \), and \( xRu \) and \( u \leq v \) imply \( xRv \).

**Proof.** Let \( x \leq yRu \). If \( xRu \), then by condition (ii) of the definition of a meet-relation, there exists \( U \in D_Y \) such that \( R[x] \subseteq U \) and \( u \notin U \). Therefore, \( x \in \Box_R U \). By condition (i) of the definition of a meet-relation, \( \Box_R U \) is an upset. Thus, \( x \leq y \) implies \( y \in \Box_R U \). But then \( R[y] \subseteq U \), and since \( yRu \), it follows that \( u \in U \). The obtained contradiction proves that \( xRu \).

Let \( xRu \leq v \). If \( xRv \), then by condition (ii) of the definition of a meet-relation, there exists \( U \in D_Y \) such that \( R[x] \subseteq U \) and \( v \notin U \). From \( xRu \) it follows that \( u \in U \), and from \( u \leq v \), it follows that \( u \notin U \). The obtained contradiction proves that \( xRv \). \( \Box \)

Let \( X \) and \( Y \) be two IS-spaces and let \( R \subseteq X \times Y \) be a functional meet-relation. Similar to Celani [1, Theorem 4.11], we define a partial function \( f_R : X \to Y \) as follows. We set \( \text{dom}(f_R) = \{ x \in X : R[x] = \uparrow y \} \) and for \( x \in \text{dom}(f_R) \) we set \( f_R(x) = y \).

**Proposition 13.11.** Let \( X \) and \( Y \) be two IS-spaces and let \( R \subseteq X \times Y \) be a functional meet-relation. Then \( f_R \) is a SIS-morphism.

**Proof.** First we show that \( f_R \) is an IS-morphism. Let \( x, y \in \text{dom}(f_R) \) and \( x \leq y \). Then \( R[x] = \uparrow f_R(x) \) and \( R[y] = \uparrow f_R(y) \). By Lemma 13.10, \( x \leq y \) implies \( R[y] \subseteq R[x] \). Therefore, \( f_R(x) \leq f_R(y) \), and so condition (i) of the definition of an IS-morphism is satisfied. To see that condition (ii) is also satisfied, let \( x \in \text{dom}(f_R) \), \( y \in Y \), and \( f_R(x) \leq y \). Then \( xRy \). Since \( R \) is a functional meet-relation, there is \( z \in X \) such that \( x \leq z \) and \( R[z] = \uparrow y \). Therefore, \( z \in \text{dom}(f_R) \), \( x \leq z \), and \( f_R(z) = y \). Finally, to see that condition (iii) of the definition of an IS-morphism is satisfied as well, let \( U \in D_Y \). We show that \( \Box_R U = [f_R^{-1}(U^c)]^c \). We have \( x \in \Box_R U \) iff \( R[x] \subseteq U \) and \( x \in [f_R^{-1}(U^c)]^c \) iff \( (\forall z \in \text{dom}(f_R))(x \leq z \Rightarrow f_R(z) \in U) \). Let \( x \in \Box_R U \), \( z \in \text{dom}(f_R) \), and \( x \leq z \). Then \( R[x] \subseteq U \), \( R[z] = \uparrow f_R(z) \), and \( R[z] \subseteq R[x] \). Therefore, \( \uparrow f_R(z) \subseteq U \), so \( f_R(z) \in U \), and so \( x \in [f_R^{-1}(U^c)]^c \). Now let \( x \in [f_R^{-1}(U^c)]^c \) and \( xRy \). Since \( R \) is a functional meet-relation, there exists \( z \in X \) such that \( x \leq z \) and \( R[z] = \uparrow y \). Then \( z \in \text{dom}(f_R) \), \( x \leq z \), and \( f_R(z) = y \). Therefore, \( f_R(z) \in U \), so \( y \in U \), and so \( x \in \Box_R U \). Thus, \( \Box_R U = [f_R^{-1}(U^c)]^c \). Now since \( R \) is a meet-relation, \( \Box_R U \subseteq D_X \), so \( [f_R^{-1}(U^c)]^c \subseteq D_X \), and so \( f_R \) is an IS-morphism.
Next we show that $f_R$ satisfies condition (*). Suppose that $x \in \text{dom}(f_R)$. Then $f_R[\uparrow x] = R[\uparrow x] = \uparrow f_R(x)$. Therefore, there exists $y \in Y$ ($y = f_R(x)$) such that $f_R[\uparrow x] = \uparrow y$. Now suppose that $f_R[\uparrow x] = \uparrow y$ for some $y \in Y$. Since $f_R[\uparrow x] = R[\uparrow x]$, we have $R[\uparrow x] = \uparrow y$. Therefore, $x \in \text{dom}(f_R)$ by the definition of $f_R$. Finally, we show that $f_R$ satisfies condition (**). Let $x \in X$, $y \in Y$, and $y \notin f_R[\uparrow x]$. Since $f_R[\uparrow x] = R[\uparrow x]$ and $R$ is a meet-relation, there is $U \in D_Y$ such that $R[\uparrow x] \subseteq U$ and $y \notin U$. This implies that $f_R[\uparrow x] \subseteq U$ and $y \notin U$. Consequently, $f_R$ is a SIS-morphism. 

Let $X$ and $Y$ be IS-spaces and $f : X \to Y$ be a SIS-morphism. Following Celani [1, Theorem 4.11], we define $R_f \subseteq X \times Y$ by

$$xR_f y \iff \text{there exists } z \in \text{dom}(f) \text{ such that } x \leq z \text{ and } f(z) = y.$$ 

**Proposition 13.12.** Let $X$ and $Y$ be two IS-spaces and let $f : X \to Y$ be a SIS-morphism. Then $R_f$ is a functional meet-relation.

**Proof.** First we show that $R_f$ is a meet-relation. Let $U \in D_Y$. We show that $\square_{R_f} U = [\downarrow f^{-1}(U^c)]^c$. We have $x \in \square_{R_f} U$ iff $R_f[x] \subseteq U$ and $x \in [\downarrow f^{-1}(U^c)]^c$ iff $(\forall z \in \text{dom}(f))(x \leq z \Rightarrow f(z) \in U)$. Let $x \in \square_{R_f} U$, $z \in \text{dom}(f)$, and $x \leq z$. Then $xR_f(z)$, so $f(z) \in U$, and so $x \in [\downarrow f^{-1}(U^c)]^c$. Now let $x \in [\downarrow f^{-1}(U^c)]^c$ and $xR_f y$. Then there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $f(z) = y$. Therefore, $f(z) \in U$, so $y \in U$, and so $x \in \square_{R_f} U$. Thus, $\square_{R_f} U = [\downarrow f^{-1}(U^c)]^c$.

By (iii) of the definition of an IS-morphism, $[\downarrow f^{-1}(U^c)]^c \subseteq D_X$. So $\square_{R_f} U \subseteq D_X$, and so $R_f$ satisfies condition (i) of the definition of a meet-relation. By the definition of $R_f$, we have that $R_f[x] = f[\uparrow x]$ for each $x \in X$. Therefore, by condition (**), $R_f$ satisfies condition (ii) of the definition of a meet-relation, and so $R_f$ is a meet-relation.

In order to show that $R_f$ is functional, we observe that if $z \in \text{dom}(f)$, then $R_f[z] = \uparrow f(z)$. To see this, let $zR_fu$. Then there exists $v \in \text{dom}(f)$ such that $z \leq v$ and $f(v) = u$. By condition (i) of the definition of an IS-morphism, $f(z) \leq f(v) = u$. Conversely, if $f(z) \leq u$, then by condition (ii) of the definition of an IS-morphism, there exists $v \in \text{dom}(f)$ such that $z \leq v$ and $f(v) = u$. Thus, $zR_fu$, and so $R_f[z] = \uparrow f(z)$. Now let $xR_f y$. Then there exists $z \in \text{dom}(f)$ such that $x \leq z$ and $f(z) = y$. Therefore, $R_f[z] = \uparrow f(z)$. Thus, $xR_f y$ implies that there exists $z \in X$ such that $x \leq z$ and $R_f[z] = \uparrow y$, and so $R_f$ is a functional meet-relation. 

Now we are in a position to prove that IS and SIS are isomorphic.

**Theorem 13.13.** The categories IS and SIS are isomorphic.

**Proof.** Define two functors $\Phi : \text{IS} \to \text{SIS}$ and $\Psi : \text{SIS} \to \text{IS}$ as follows. For an IS-space $X$ set $\Phi(X) = X = \Psi(X)$, for a functional meet relation $R$ set $\Phi(R) = f_R$, and for a SIS-morphism $f$ set $\Psi(f) = f_R$. Then it follows from
Propositions 13.11 and 13.12 that both \( \Phi \) and \( \Psi \) are well-defined. It is left to be shown that \( R = R_{f_{\Phi}} \) and \( f = f_{R_{\Psi}}. \)

First we show that \( R = R_{f_{\Phi}}. \) For \( x \in X \) and \( y \in Y, \) we have \( xRy \) iff there exists \( z \in X \) such that \( x \leq z \) and \( R[z] = \uparrow y. \) On the other hand, \( xR_{f_{\Phi}}y \) iff there exists \( z \in \text{dom}(f_{R_{\Psi}}) \) such that \( x \leq z \) and \( f_{R_{\Psi}}(z) = y. \) But \( R[z] = \uparrow f_{R_{\Psi}}(z), \) so \( R[z] = \uparrow y \) iff \( f_{R_{\Psi}}(z) = y. \) It follows that \( xR_{f_{\Phi}}y \) iff there exists \( z \in X \) such that \( x \leq z \) and \( R[z] = \uparrow y. \) Thus, \( xRy \) iff \( xR_{f_{\Phi}}y, \) and so \( R = R_{f_{\Phi}}. \) Next we show that \( f = f_{R_{\Psi}}. \) Suppose that \( x \in \text{dom}(f). \) Then \( R_{f_{\Psi}}[x] = \uparrow f(x). \) Therefore, \( f(x) \in \text{dom}(f_{R_{\Psi}}) \) and \( f_{R_{\Psi}}(x) = f(x). \) Now suppose that \( x \in \text{dom}(f_{R_{\Psi}}). \) Let \( f_{R_{\Psi}}(x) = y. \) Then \( R_{f_{\Psi}}[x] = \uparrow y. \) Since \( R_{f_{\Psi}}[x] = f(\uparrow x), \) we have \( f(\uparrow x) = \uparrow y. \) By condition \((*)\), \( x \in \text{dom}(f). \) Thus, \( f(x) = f_{R_{\Psi}}(x), \) so \( \text{dom}(f) = \text{dom}(f_{R_{\Psi}}) \) and whenever \( x \in \text{dom}(f) = \text{dom}(f_{R_{\Psi}}), \) then \( f(x) = f_{R_{\Psi}}(x). \) Consequently, \( f = f_{R_{\Psi}}, \) and so IS is isomorphic to \( \text{SIS}. \)

As a result, we obtain that \( \text{IM} \) is also dually equivalent to \( \text{SIS}. \) Thus, we can represent dually implicative meet semi-lattice homomorphisms by either functional meet-relations or by strong IS-morphisms between IS-spaces.

We conclude our comparison section by looking at the work of Vrancken-Mawet [16]. We recall that Vrancken-Mawet [16] constructed a dual category of \( \text{BIM}, \) which is similar to that of Celani [1]. The objects of the Vrancken-Mawet category are essentially Celani’s compact IS-spaces with the only difference that Vrancken-Mawet works with the specialization order, while Celani prefers to work with the dual of the specialization order. Because of this, Celani works with the bounded implicative meet semi-lattice of complements of compact open subsets of a compact IS-space \( X, \) while Vrancken-Mawet works with the bounded implicative meet semi-lattice of compact open subsets of \( X. \) Both these meet semi-lattices are isomorphic though because Celani’s order is dual to Vrancken-Mawet’s order. In order to simplify our comparison, we work with Celani’s compact IS-spaces instead of Vrancken-Mawet’s spaces and adjust the definition of Vrancken-Mawet morphisms accordingly. Note that unlike the case with objects, Vrancken-Mawet morphisms satisfy an extra condition that Celani’s morphisms do not satisfy. Let \( X \) and \( Y \) be two compact IS-spaces and let \( f : X \rightarrow Y \) be an IS-morphism. We call \( f \) a Vrancken-Mawet morphism if it satisfies the following additional condition:

\[
(\text{VM}) \quad x \in \text{dom}(f) \text{ iff } (\exists y \in Y)(\forall U \in D_{Y})(y \notin U \leftrightarrow (\exists z \in \text{dom}(f))(x \leq z \text{ and } f(z) \notin U)).
\]

Observe that in Example 13.9, the \( f \) we start with does not satisfy (VM), and so it is not a Vrancken-Mawet morphism. In fact, given two compact IS-spaces \( X \) and \( Y, \) it is relatively easy to verify that our conditions \((*)\) and \((**)\) imply (VM), and that (VM) implies \((*)\). It requires more work to show that (VM) also implies \((**)\). We skip the details and only mention that the upshot of these observations is that the category \( \text{CSIS} \) of compact IS-spaces
and SIS-morphisms is isomorphic to the category $\text{VM}$ of compact IS-spaces and Vrancken-Mawet morphisms. We feel that our conditions $(\ast)$ and $(\ast\ast)$ are more natural and easier to work with than the $(\text{VM})$ condition.

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