Ordering protoalgebraic logics

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**Abstract**

This paper is a first step in the study of the order structure of the set of all protoalgebraic logics over a fixed (but arbitrary) language. In particular, it is shown herein that the set is a join-complete semilattice, that it has no minimum, and that it is not a meet-semilattice. One of the key points in this study is the discovery of a large family of rather weak protoalgebraic logics, from which an infinity of denumerable sequences of protoalgebraic logics of strictly decreasing strength and with no lower protoalgebraic bound is constructed. Other properties of these logics are also studied, such as their classification in the Frege hierarchy and in the Leibniz hierarchy, and several common metalogical properties (conjunction, disjunction, deduction theorems, etc.; in fact, it turns out that they do not possess any of these properties). These logics provide examples of a kind of protoalgebraic logics, few of which have been provided in the literature to date. The paper ends by discussing the issue of how to extend the results to the class of all protoalgebraic logics over different languages, ordered under the expansion relation. This paper is an example of the use of combinatorial techniques within the field of abstract algebraic logic.

**Keywords:** Abstract algebraic logic, protoalgebraic logics, lattices of logics, iteration construction, coherent set, Leibniz hierarchy, Frege hierarchy.

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1 Introduction

The study of lattices of logics is a classic topic in abstract algebraic logic. The dual isomorphism between the lattice of all extensions of any algebraizable logic and the lattice of sub-quasivarieties of its equivalent algebraic semantics can be regarded as the generalization of several isomorphisms found in previous studies of relations between lattices of modal logics and lattices of varieties of modal algebras [1], or of relations between superintuitionistic logics and varieties of Heyting algebras [13]; in fact, both the cited papers played a rôle in the development of the central notion of algebraizability. In much weaker classes of logics in the Leibniz hierarchy (such as equivalential logics or protoalgebraic logics), there is no one-to-one relation between logics and classes of algebras, but the logics are still ordered under the extension relation (for logics over the same language) or the expansion relation (for logics over different languages), and the study of the structure of the ordered sets or classes may be of some interest.

Protoalgebraic logics are one of the most basic classes of logics in the Leibniz hierarchy, and hence most abstract algebraic logic is concerned with their study. As shown in [4], the protoalgebraicity of a logic is witnessed by a set $\Delta(x, y)$ of formulas in at most two variables, that satisfies two very elementary syntactic properties (see Definition 2.2). Therefore, it is clear that the weakest non-trivial protoalgebraic logic in a given language for which a given set $\Delta(x, y) \neq \emptyset$ witnesses protoalgebraicity exists; in the finite case it is the logic axiomatized by the two properties mentioned. Thus, arguably, the simplest (non-trivial) protoalgebraic logic is the logic axiomatized in this way for the simplest possible language, i.e., the language consisting of just one binary connective $\rightarrow$, and where the set $\Delta(x, y)$ consists of just the simplest binary formula $\{x \rightarrow y\}$. Thus, the logic $\mathcal{I}$ defined by the axiom $x \rightarrow x$ and the rule $x, x \rightarrow y \vdash y$ was extensively studied in [9, 10], where it is presented as “the simplest protoalgebraic logic”. However, it was later realized that the requirement for simplicity happens to break the argument for weakness, and that $\mathcal{I}$ is far from being the weakest protoalgebraic logic in the language $\{\rightarrow\}$; in fact, a strictly decreasing denumerable sequence of protoalgebraic logics weaker than $\mathcal{I}$ and with no protoalgebraic lower bound was found (i.e., constructed). Further work on this issue revealed that most of the techniques in the papers could be generalized to an arbitrary language (provided it contains at least one connective of arity 2 or greater) and led to the introduction of some novel constructions, such as the “iteration process” of Section 4.2. The aforementioned examples in the language $\{\rightarrow\}$ will therefore appear here as particular cases.

This paper is an initial contribution to the study of the order structure of the set of all protoalgebraic logics over a fixed but arbitrary language. It is hoped that some of the techniques and results herein will carry over to more restricted and interesting classes, such as those of equivalential logics or of regularly algebraizable logics. This paper is also an example of the use of purely combinatorial techniques within the field of abstract algebraic logic (here I am using “combinatorial” in the sense that its main results are obtained by working on the grammatical structure of formulas), which seems to be novel within the subject; the only exception is Proposition 5.11, where a bridge theorem is used, although in a way that brings its essentially semantical character close to its syntactical character.

Familiarity with the notions and notations of abstract algebraic logic as given in [2, 7, 11, 12, 17] is assumed; a summary of indispensable items is given in Section 2. Then, Section 3 contains the basic study of the family of logics that will be used to construct the counterexamples needed in later sections. The logics arise as generalizations of the logic $\mathcal{I}$ mentioned above, with the rôle of the simple formula $x \rightarrow y$ played by a finite set of formulas in two variables, $x, y$, that has the property of coherence (Definition 3.2), which roughly speaking means that all the formulas in the set are alphabetical variants of each other. For each such set $\Delta(x, y)$, a logic, $\mathcal{I}\Delta$, is defined, its theorems are determined (Proposition 3.4), and some relevant properties of its consequence relation are derived. These logics have the peculiar feature that the only non-theorems that can follow from a given set of assumptions are their subformulas (Proposition 3.6). This has a number of important consequences, and when the set is unitary, this

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1 Actually, the class of protoalgebraic logics contains all the other classes in the hierarchy except the class of truth-equational logics [14].

2 Some results of the study of the set of finitary extensions of a fixed equivalental logic are Theorems 3.2.3 and 3.4.9 of [7].
property can be considerably strengthened (Propositions 3.12 and 3.13).

Section 4 is the central section of the paper, and contains all the results concerning the order structure of the set Prot of all protoalgebraic logics over a given language. It is established that Prot is a join-complete sub-semilattice of the lattice of all logics (Theorem 4.2); that it has no lowest element (Theorem 4.3); and that it is not a meet-semilattice (Theorem 4.4). Indeed, an infinity of pairs of logics in Prot that do not have a common lower bound in the set are constructed. Then the “iteration process” is introduced (Definition 4.5) and with its help it is shown, among other properties, that there are infinitely many strictly decreasing denumerable sequences of logics in Prot that have no lower bound in the set (Theorem 4.10). In particular, no logic whose protoalgebraicity is witnessed by a single formula can be minimal, but in fact has denumerably many protoalgebraic logics weaker than it; this covers the vast majority of specific protoalgebraic logics found in the literature. The section ends with the determination of the (Boolean) order structure of the set of all logics of the form \( I \Delta \) where \( \Delta(x, y) \) varies over all the subsets of a given coherent set (Theorem 4.11), and finally by showing that all the logics \( I \Delta \) are different (Theorem 4.14). Since these logics are finitary, some of the results hold for the subset of all finitary protoalgebraic logics over the same language.

Section 5 studies the logics \( I \Delta \) in the more typical context of abstract algebraic logic. First, I determine a set of congruence formulas with parameters for \( I \Delta \) (Lemma 5.1). Then, with the help of this, some surprising results are shown: among them, that the Lindenbaum-Tarski algebra of the logic is isomorphic to the formula algebra; that its intrinsic variety consists of all algebras of the similarity type; and that its algebraic counterpart satisfies no non-trivial identity (Proposition 5.3). As for the classification of the logics in the hierarchies of abstract algebraic logic, it is shown that they are never selfextensional (Corollary 5.4), thus they do not belong to the Frege hierarchy; and also that they do not belong to any class in the Leibniz hierarchy other than that of protoalgebraic logics (Proposition 5.5, Corollary 5.6 and Proposition 5.7). This last result is interesting because there are few examples of this kind in the literature. In Subsection 5.3 it is shown that they do not have any of the metalogical properties that are commonly found in abstract algebraic logic (conjunction, disjunction, several kinds of deduction theorems, etc.).

Finally, Section 6 discusses the feasibility of studying similar issues in the much larger class that contains all protoalgebraic logics over all languages. My conclusion is that this would therefore not make much sense, as one would not obtain essentially better results, because the constructions already performed in Prot would be valid in the larger class as well.

2 Preliminary material

2.1 Languages and formulas

In this paper I consider logics over the same arbitrary but fixed sentential language (except in Section 6, where different languages are mixed; special notations will be introduced there). Thus, an initial general warning is in order: expressions such as “all logics” or “all protoalgebraic logics” actually mean “... over the same language” (except, as mentioned, in Section 6).

The language is assumed to have at least one connective of arity 2 or greater (because no protoalgebraic logics exist for languages with only constants and unary connectives, other than the trivial ones, so the whole issue would become pointless). This will be justified after Definition 2.2, and leads to a second similar warning: expressions such as “arbitrary language” or “all languages” actually mean “... having at least one connective of arity 2 or greater”.

To begin with, some points on the grammatical structure of formulas should be highlighted.

- The set of all formulas of the language is denoted by \( Fm \); as usual, the logical connectives are identified with operations on this set, and the resulting algebra of formulas is denoted by \( Fm \). Non-boldface lower-case Greek letters except \( \sigma \), perhaps with primes or subindices, denote formulas. Non-boldface upper-case Greek letters, which also admit primes or subindices, denote sets of formulas. Note that numerical and other superindices will have special meanings.
The set of atomic formulas or **variables** is denoted by \( \text{Var} \); this set is assumed to be countably infinite. For any formula \( \varphi \), the set of variables effectively occurring in it is denoted by \( \text{var} \varphi \). If \( \Gamma \) is a set of formulas, then \( \text{var} \Gamma := \bigcup \{ \text{var} : \varphi \in \Gamma \} \). The letters \( x, y, z, t \) denote variables. Variables represented by different letters are different, unless the contrary is explicitly stated.

As usual, writing \( \varphi(x, y, z) \) means that \( \text{var} \varphi \subseteq \{x, y, z\} \) and so on. Then \( \varphi(a, \beta, \gamma) \) denotes the result of simultaneously replacing the variables \( x, y, z \) in \( \varphi \) by the formulas \( a, \beta, \gamma \), respectively; that is, \( \varphi(a, \beta, \gamma) = \sigma \varphi \), where \( \sigma \) is any substitution such that \( \sigma x = a, \sigma y = \beta \), and \( \sigma z = \gamma \). Substitutions, technically, are the endomorphisms of the formula algebra.

The same notational device is used for sets: when I write \( \Delta(x, y) \), I am referring to a set \( \Delta \) of formulas where at most the variables \( x, y \) occur. Notice that the two notations \( \Delta \) and \( \Delta(x, y) \) refer to the same set; the latter only serves the purpose of emphasizing the variables occurring in it.

For any natural number \( n \), an \( n \)-**variable formula** is a formula in which exactly \( n \) variables occur; the set of all such formulas will be denoted by \( \text{Fm}(n) \). Note that \( \text{Fm}(2) \neq \emptyset \) if and only if the language contains at least one connective of arity 2 or greater (the restriction assumed herein), and that if I write \( \Delta(x, y) \subseteq \text{Fm}(2) \), then this means that all formulas in the set \( \Delta \) contain exactly the variables \( x \) and \( y \).

The following totally general property, which is routinely proved by induction on the complexity of the formula, will play a key rôle in some of the main proofs.

**Lemma 2.1.** Let \( \varphi \) be a formula, and let \( \sigma, \sigma' \) be substitutions. Then \( \sigma \varphi = \sigma' \varphi \) if and only if \( \sigma x = \sigma' x \) for all \( x \in \text{var} \varphi \).

Textbooks usually only pay attention to the implication from right to left, as an initial illustration of the absolutely free character of the formula algebra. However, it is the implication from left to right that is of interest here; I will often apply it when \( \varphi \) is a 2-variable formula \( \delta(x, y) \), in which case in informal notation it reads:

\[
\delta(a, \beta) = \delta(\epsilon, \gamma) \implies a = \epsilon \quad \text{and} \quad \beta = \gamma.
\]

**2.2 Logics and their order structure**

- A **logic** \( \mathcal{L} \) is identified by its consequence relation, which is a substitution-invariant closure relation \( \vdash_{\mathcal{L}} \) in the set of formulas; that is, \( \vdash_{\mathcal{L}} \subseteq \mathcal{P}(\text{Fm}) \times \text{Fm} \). The associated closure operator is denoted by \( C_{\mathcal{L}} \); thus, the **theory** generated by a set, \( \Gamma \), of formulas is denoted by \( C_{\mathcal{L}}\Gamma \). The set of theories of the logic \( \mathcal{L} \) is denoted by \( \mathcal{Th}_{\mathcal{L}} \); its minimum element, the smallest theory, is the set of **theorems** of \( \mathcal{L} \), which is \( C_{\mathcal{L}}\emptyset \); I write \( \vdash_{\mathcal{L}} \varphi \) instead of \( \emptyset \vdash_{\mathcal{L}} \varphi \).

- A logic \( \mathcal{L} \) is **trivial** when it satisfies \( x \vdash_{\mathcal{L}} y \) for two distinct variables \( x, y \); equivalently, when \( a \vdash_{\mathcal{L}} b \) for any two formulas \( a, b \). In each language, there are exactly two trivial logics: the inconsistent logic and the almost inconsistent logic, which are the logics that have only one theory (the set of all formulas) and two different theories (the empty set and the set of all formulas), respectively.

- A logic \( \mathcal{L} \) is **weaker** than a logic \( \mathcal{L}' \) (in symbols, \( \mathcal{L} \subseteq \mathcal{L}' \)) when \( \vdash_{\mathcal{L}} \subseteq \vdash_{\mathcal{L}'} \); that is, when \( \Gamma \vdash_{\mathcal{L}} \varphi \) implies \( \Gamma \vdash_{\mathcal{L}'} \varphi \) for all \( \Gamma \) and all \( \varphi \). Then \( \mathcal{L}' \) is **stronger** than or an **extension** of \( \mathcal{L} \). \( \mathcal{L} \) is **strictly weaker** than \( \mathcal{L}' \) (in symbols, \( \mathcal{L} \subsetneq \mathcal{L}' \)) when it is weaker than \( \mathcal{L}' \) and different from it. The **extension relation** \( \subseteq \) is an order.

- The set \( \text{Log} \) of all logics, ordered under the relation \( \subseteq \), is a complete lattice \([17, \S 1.5]\). The meet operation is given by the intersection of the consequence relations; that is, \( \mathcal{L} = \bigwedge_{i \in I} \mathcal{L}_i \) when \( \Gamma \vdash_{\mathcal{L}} \varphi \) if and only if for all \( i \in I \), \( \Gamma \vdash_{\mathcal{L}_i} \varphi \). The join is best described in general by its set of theories: \( \mathcal{L} = \bigvee_{i \in I} \mathcal{L}_i \) when \( \mathcal{Th}_{\mathcal{L}} = \bigcap_{i \in I} \mathcal{Th}_{\mathcal{L}_i} \); but if an axiomatic system for each of the \( \mathcal{L}_i \) is known, then it is more useful to know that \( \mathcal{L} \) is axiomatized by putting together all the axioms and all the rules of the \( \mathcal{L}_i \).

- The ordered set \( \text{Log} \) has a maximum, the inconsistent logic, and a minimum, the “identity logic” \( \mathcal{L} \) defined as: \( C_{\mathcal{L}}\Gamma = \Gamma \) for all \( \Gamma \subseteq \text{Fm} \). The almost inconsistent logic is a maximal point in \( \text{Log} \), as it
is only weaker than the inconsistent logic; moreover, observe that only logics without theorems are weaker than the almost inconsistent logic.

- A logic $\mathcal{L}$ is finitary when $\Gamma \vdash_{\mathcal{L}} \phi$ implies $\Gamma_0 \vdash_{\mathcal{L}} \phi$ for some finite $\Gamma_0 \subseteq \Gamma$. The set $\text{FinLog}$ of all finitary logics is a sublattice of $\text{Log}$ and also a complete lattice in itself. More precisely, it is a join-complete sub-semilattice of $\text{Log}$, so that the join in $\text{FinLog}$ is the same as that in $\text{Log}$ and is also denoted as $\lor$. However, it is not a meet-complete sub-semilattice of $\text{Log}$: while the meet in $\text{FinLog}$ of finitely many finitary logics is the same as that in $\text{Log}$, the meet of an infinite family may be different.

Since many authors prefer to include the property of finitarity in the definition of what a logic is, in this paper the relevant results will be given in two versions, a general one and another concerning only finitary logics.

### 2.3 Protoalgebraic logics

One of the most interesting features of protoalgebraic logics is that they can be defined or characterized in a variety of ways using quite different approaches (syntactical, model-theoretical, lattice-theoretical, etc.). Here it is convenient to focus on the following syntactical characterization, due to Blok and Pigozzi [4], which can be taken as the working definition for the present paper:

**Definition 2.2.** A logic $\mathcal{L}$ is protoalgebraic when there is a set $\Delta(x,y)$ of formulas, in at most two distinct variables $x,y$, that satisfies the following properties:

\[
\begin{align*}
\vdash_{\mathcal{L}} & \Delta(x,x) \\
x, \Delta(x,y) & \vdash_{\mathcal{L}} y
\end{align*}
\]

(InA) (MPA)

In this case, it is said that the set $\Delta$ witnesses the protoalgebraicity of $\mathcal{L}$. The set of all protoalgebraic logics (over the same language) is denoted by $\text{Prot}$, and that of all finitary protoalgebraic logics by $\text{FinProt}$.

Since $\text{Prot} \subseteq \text{Log}$ and $\text{FinProt} \subseteq \text{FinLog}$, the two sets just defined are ordered under the extension relation $\leq$. Note that $\text{FinProt} = \text{Prot} \cap \text{FinLog}$.

Note that witnessing sets need not be unique. When the logic $\mathcal{L}$ is finitary, the witnessing set can always be taken as finite. The cases where $\Delta = \emptyset$ or $\Delta$ is a set of 1-variable formulas are in principle not excluded, but in both cases the above conditions lead to $x \vdash_{\mathcal{L}} y$, so that the only protoalgebraic logics witnessed by such sets are the trivial ones. Moreover, any 1-variable formula appearing in $\Delta(x,y)$ can safely be removed, as it will be a theorem of $\mathcal{L}$. So we may assume $\Delta(x,y)$ is either empty or made entirely of 2-variable formulas, i.e., $\Delta(x,y) \subseteq \text{Fm}(2)$. If there is no connective of arity 2 or greater in the language, then there are no 2-variable formulas, so the only protoalgebraic logics in such languages are again the trivial ones; this justifies the assumption regarding the language made at the beginning of Section 2.1. It also follows from these considerations that in any language the only protoalgebraic logic without theorems is the almost inconsistent logic.

Protoalgebraic logics (but not the order structure of $\text{Prot}$ or $\text{FinProt}$) are studied in depth in [7]. Their properties will be recalled when needed.

### 3 A peculiar family of logics

Several results in this paper exploit the following family of logics as counterexamples. In order not to interrupt the flow of the paper, their general properties are presented in this section, although each will be used in the appropriate place.

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3In the literature, the term “protoimplication set” has also been used.
DEFINITION 3.1. For each non-empty finite set \( \Delta(x,y) \subseteq \text{Fm}(2) \) the logic \( \mathcal{I}\Delta \) is the logic defined by the axiomatic system with:

\[
\begin{align*}
\text{the axioms} & \quad \Delta(x,x) \\
\text{and the rule} & \quad x, \Delta(x,y) \vdash y.
\end{align*}
\]

(I\(_\Delta\))

(MP\(_\Delta\))

NOTATION. For simplicity, the consequence relation of the logic \( \mathcal{I}\Delta \) is denoted by \( \vdash_{\Delta} \), and its associated closure operator by \( C_\delta \). When \( \Delta = \{ \delta \} \), the braces are suppressed; thus, for each 2-variable formula \( \delta(x,y) \), Definition 3.1 yields the logic \( \mathcal{I}\delta \) defined by the single axiom (I\(_\delta\)) and the single rule (MP\(_\delta\)), with the consequence relation denoted by \( \vdash_\delta \) and the associated closure operator \( C_\delta \).

For an arbitrary \( \Delta(x,y) \), the weakest (i.e., the minimum) of all protoalgebraic logics witnessed by the set \( \Delta \) exists, because all these logics satisfy the conditions (I\(_\Delta\)) and (MP\(_\Delta\)), and hence the meet of the family in \( \text{Log} \) (which is the intersection) also satisfies them and by Definition 2.2 belongs to the family and so it will trivially be its minimum. For a finite \( \Delta(x,y) \), this minimum logic is the logic \( \mathcal{I}\Delta \) just defined; actually, for a finite \( \Delta(x,y) \), an arbitrary logic \( \mathcal{L} \) is protoalgebraic, and its protoalgebraicity is witnessed by \( \Delta \), if and only if \( \mathcal{I}\Delta \subseteq \mathcal{L} \). The simplest case of this construction, the logic \( \mathcal{I}(x \to y) \) in the language that has only a binary connective of implication \( \to \), is the logic \( \mathcal{I} \) studied in [9, 10]. More generally, one may consider the logic \( \mathcal{I}(x \to y) \) in any language that contains \( \to \).

3.1 Coherent sets

Although the definition given above makes sense for an arbitrary non-empty finite set \( \Delta(x,y) \), we are interested in the particular cases where the set is constituted by formulas (in the variables \( x \) and \( y \)) which are alphabetical variants of each other:

DEFINITION 3.2. A non-empty set \( \Delta(x,y) \subseteq \text{Fm}(2) \) is coherent when \( \delta(x,x) = \delta'(x,x) \) for all \( \delta, \delta' \in \Delta(x,y) \). If \( \Delta(x,y), \Delta'(x,y) \subseteq \text{Fm}(2) \) are non-empty, then \( \Delta \) is coherent with \( \Delta' \) when the set \( \Delta \cup \Delta' \) is coherent.

Equivalently, \( \Delta(x,y) \) is coherent with \( \Delta'(x,y) \) when \( \delta(x,x) = \delta'(x,x) \) for all \( \delta \in \Delta \) and all \( \delta' \in \Delta' \). The particular usages of these terms for unitary sets are the obvious ones; thus, for instance, \( \delta(x,y) \) is coherent with \( \delta'(x,y) \) when \( \delta(x,x) = \delta'(x,x) \), and \( \Delta(x,y) \) is coherent with \( \delta(x,y) \) when \( \delta'(x,x) = \delta(x,x) \) for all \( \delta' \in \Delta \). Note that if \( \Delta(x,y) \) is coherent, then it is coherent with any of its members, but it can also be coherent with a formula outside it; in several results this is relevant.

Since only the variables \( x \) and \( y \) are assumed to occur in the formulas of \( \Delta(x,y) \), necessarily a coherent set must be finite; note that the empty set is excluded by definition.\(^4\) There are infinitely many coherent sets: each 2-variable formula \( \delta(x,y) \) produces a finite number of sets that are coherent with it, and one is the largest among them. As a simple example, in a language with \( \to \), the sets coherent with \( x \to y \) are \( \{ x \to y \}, \{ y \to x \} \) and \( \{ x \to y, y \to x \} \).

Informally speaking, two formulas in the variables \( x, y \) are coherent with each other when they differ only in the positions of the two variables (assuming both variables occur in them). This is reflected in the following lemma, which will be needed later on.

LEMMA 3.3. Let \( \delta_1(x,y), \delta_2(x,y) \in \text{Fm}(2) \) be different and coherent with each other. If \( \delta_1(\alpha, \beta) = \delta_2(\gamma, \epsilon) \) then \( \alpha = \epsilon \) or \( \beta = \gamma \). In particular: if \( \delta_1(\alpha, \beta) = \delta_2(\alpha, \beta) \), then \( \alpha = \beta \).

PROOF. If \( \delta_1(x,x) = \delta_2(x,x) \), there is some \( \delta^*(x,y,z,t) \in \text{Fm} \) such that \( \delta_1(x,y) = \delta^*(x,y,z,t) \) and \( \delta_2(x,y) = \delta^*(x,y,y,x) \). Intuitively, \( \delta^* \) has \( x \) where both \( \delta_1 \) and \( \delta_2 \) have \( x \), and the same for \( y \); moreover, it has \( z \) where \( \delta_1 \) has \( x \) and \( \delta_2 \) has \( y \); and it has \( t \) where \( \delta_1 \) has \( y \) and \( \delta_2 \) has \( x \) (see the examples after this proof). If \( \delta_1 \neq \delta_2 \), at least \( z \) or \( t \) must be present in this \( \delta^* \). Now the assumption entails that

\(^4\)The reason for this is that the logic defined by the axiom (I\(_\Delta\)) and the rule (MP\(_\Delta\)) when \( \Delta = \emptyset \) is the almost inconsistent logic, which is outside the intended range of applications.
\(\delta^*(a, \beta, \alpha, \beta) = \delta^*(\gamma, \epsilon, \epsilon, \gamma)\). Applying Lemma 2.1 to the cases where \(z\) or \(t\) are present, we obtain that \(a = \epsilon\) or \(\beta = \gamma\), respectively, as stated. The particular case where \(\gamma = a\) and \(\epsilon = \beta\) results in simply \(a = \beta\), again as stated.

This result tells us that the only substitution instances that two different coherent formulas may have in common arise when we interchange the values of the two variables, or when the two replacement values coincide. As an example to visualize this, consider the formulas \(\Delta\), \(\Gamma\), \(\Phi\), and \(\Psi\). For instance, if we have \(\Delta\) and \(\Gamma\), then the substitution instance \(\Delta\) implies that \(\Gamma\) if and only if \(\Delta\) and \(\Gamma\) are coherent with each other; a common substitution instance will be of the form \(\alpha \rightarrow \beta = \delta_1(\alpha, \beta) = \delta_2(\beta, \alpha)\). In this case, the formula \(\delta^*\) in the proof would be \(z \rightarrow t\). As another less simple example, take \(\delta_1(x, y) := (x \rightarrow y) \rightarrow (y \rightarrow x)\) and \(\delta_2(x, y) := (x \rightarrow y) \rightarrow (x \rightarrow y)\). In this case, \(\delta^* = (x \rightarrow z) \rightarrow (t \rightarrow y)\), and all the possibilities mentioned in the construction of \(\delta^*\) appear. Here, the only common instance is one of the form \((\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha)\) for some \(\alpha\).

Coherent sets are only used to consider the logics \(\mathcal{I}\Delta\) associated with them. Therefore, by substitution invariance, it does not matter which particular variables \(x, y\) are used in them (provided they are different), so that writing simply \(\Delta\) instead of \(\Delta(x, y)\) cannot lead to any misunderstanding.

When \(\Delta\) is coherent, the logic \(\mathcal{I}\Delta\) actually has only one axiom: \((I_\delta)\), i.e., the formula \(\delta(x, x)\) for any \(\delta \in \Delta\); and only one rule: \((MP_\delta)\). In contrast, if \(\Delta\) is not coherent, then \(\mathcal{I}\Delta\) has more than one axiom although still only a single rule.

Observe that a unitary set \(\{\delta\}\) is trivially coherent; so, the logics \(\mathcal{I}\delta\) are particular cases of the logics \(\mathcal{I}\Delta\)s with a coherent \(\Delta\), and therefore the results regarding these hold for all the logics of the form \(\mathcal{I}\delta\).

In Theorem 4.14, it is proved that the logics of the form \(\mathcal{I}\Delta\) for different coherent sets \(\Delta\) are all different. Some of them, however, can have the same set of theorems (Proposition 3.5). In fact, the theorems of each of these logics can be determined; although this can be easily proved, it is the key to several of the main results in this paper.

**Proposition 3.4.** If \(\Delta(x, y) \subseteq \text{Fm}(2)\) is coherent with \(\delta(x, y)\), then the theorems of the logic \(\mathcal{I}\Delta\) are all the substitution instances of \(\delta(x, x)\); that is, \(C_\delta \circ \Delta \subseteq \{\delta(\varphi, \varphi) : \varphi \in \text{Fm}\}\). In particular:
- For any coherent \(\Delta(x, y) \subseteq \text{Fm}(2)\), \(C_\delta \circ \Delta \subseteq \{\delta(\varphi, \varphi) : \varphi \in \text{Fm}\}\), for any \(\delta \in \Delta\).
- For any \(\delta(x, y) \in \text{Fm}(2)\), \(C_\delta \circ \Delta = \{\delta(\varphi, \varphi) : \varphi \in \text{Fm}\}\).

**Proof.** It is enough to prove that the set \(T_\delta = \{\delta(\varphi, \varphi) : \varphi \in \text{Fm}\}\), which actually consists of all substitution instances of the only axiom \((I_\delta)\) of \(\mathcal{I}\Delta\), is closed under the rule \((MP_\delta)\). So let \(\alpha \in T_\delta\) and \(\Delta(\alpha, \beta) \subseteq T_\delta\). In particular, for an arbitrary \(\delta \in \Delta\), \(\delta(\alpha, \beta) = \delta(\varphi, \varphi)\) for some \(\varphi\), and by property (1) this implies \(\alpha = \varphi\) and \(\beta = \varphi\); that is, \(\alpha = \beta\), so that \(\beta \in T_\delta\) as well. Therefore, \(T_\delta\) is the smallest theory of the logic; that is, it is the set of theorems of the logic.

Observe that any 1-variable formula \(\varphi(x)\) that contains at least one occurrence of a connective of arity 2 or greater is a theorem of one of the logics \(\mathcal{I}\Delta\) (actually, of several; but finitely many of them). This is because, for such a formula, one can always find a 2-variable formula, \(\delta(x, x)\), such that \(\varphi(x) = \delta(x, x)\), so that the theorems of the logic \(\mathcal{I}\delta\) are all the substitution instances of \(\varphi(x)\) (there is another, trivial way to obtain a logic with this property: to take \(\varphi(x)\) as the only axiom, and no rules of inference at all; but this logic, besides being rather silly, is non-protologalgebraic).

**Proposition 3.5.** Let \(\Delta(x, y), \Delta'(x, y) \subseteq \text{Fm}(2)\) be two coherent sets. The logics \(\mathcal{I}\Delta\) and \(\mathcal{I}\Delta'\) have the same theorems if and only if \(\Delta\) and \(\Delta'\) are coherent with each other; that is, if and only if \(\delta(x, x) = \delta'(x, x)\) for any \(\delta \in \Delta\) and any \(\delta' \in \Delta'\).

**Proof.** Assume the two logics have the same theorems, and take any \(\delta \in \Delta\) and any \(\delta' \in \Delta'\). In particular, \(\delta(x, x)\) should be a theorem of \(\mathcal{I}\Delta'\) and \(\delta'(x, x)\) a theorem of \(\mathcal{I}\Delta\). Thus, by Proposition 3.4, \(\delta(x, x) = \delta'(\varphi(x), \varphi(x))\) and \(\delta'(x, x) = \delta(\psi(x), \psi(x))\) for some formulas \(\varphi(x)\) and \(\psi(x)\). Therefore \(\delta(x, x) = \delta(\psi(\varphi(x)), \psi(\varphi(x)))\), which by Lemma 2.1 applied to \(\delta(x, x)\) viewed as a 1-variable formula, implies \(x = \psi(\varphi(x))\). This is only possible when \(\varphi(x) = \psi(x) = x\), which implies that \(\delta(x, x) = \delta'(x, x)\). The converse follows directly from Proposition 3.4.
One can think of many cases of this situation: for each \( \delta(x, y) \) there is a finite number of formulas \( \delta'(x, y) \) such that \( \delta'(x, x) = \delta(x, x) \), hence there is a finite number of logics \( \mathcal{I} \Delta' \) that have the same theorems as \( \mathcal{I} \delta \). The simplest example, in a language with implication, is that of formulas \( x \rightarrow y \) and \( y \rightarrow x \).

Unlike the theorems, the theories of the logics \( \mathcal{I} \Delta \) are not so neatly characterized, but still they have a striking characteristic:

**Proposition 3.6.** Let \( \Delta \) be a coherent set. If \( \Gamma \vdash_\Delta \beta \), then \( \beta \) is a theorem or \( \beta \) is a subformula of some formula in \( \Gamma \), and in this latter case \( \text{var} \beta \subseteq \text{var} \Gamma \).

**Proof.** By induction on the length of the proof of \( \beta \) from \( \Gamma \). If \( \beta \in \Gamma \) or \( \beta \) is an instance of the axiom (hence, a theorem), then it satisfies the property. Now assume there is some \( \alpha \), such that \( \alpha \) and all the \( \delta(a, \beta) \), for \( \delta \in \Delta \), have shorter proofs from \( \Gamma \) and hence, by the induction hypothesis, satisfy the property, and consider any \( \delta \in \Delta \). If \( \delta(a, \beta) \) is a theorem, then by Proposition 3.4 \( \delta(a, \beta) = \delta(\varphi, \varphi) \) for some \( \varphi \), and this by property (1) implies that \( a = \beta \); hence, \( \beta \) also satisfies the property. If \( \delta(a, \beta) \) is a subformula of some formula in \( \Gamma \), then \( \text{a fortiori} \beta \) is also one.

Thus, in these logics, from a given set of assumptions, besides the theorems, it is only possible to derive subformulas of the assumptions! These subformulas are not arbitrary (that is, the converse of Proposition 3.6 does not hold): in the case of 1-element \( \Delta \), i.e., of the logics \( \mathcal{I} \delta \), the form of the relevant subformulas will be determined in Proposition 3.13, for 1-element \( \Gamma \), and in Proposition 3.12 for arbitrary \( \Gamma \).

Some straightforward consequences of Proposition 3.6 that will be needed in Section 5 can already be obtained:

**Corollary 3.7.** For each coherent set \( \Delta \):
1. If \( \Gamma \) is a set of variables, then \( \mathcal{C}_A \Gamma = \Gamma \cup \mathcal{C}_A \emptyset \).
2. If \( x, y \) are two distinct variables, then \( \mathcal{C}_A \{x\} \cap \mathcal{C}_A \{y\} = \mathcal{C}_A \emptyset \).
3. A finitely generated theory contains only a finite number of non-theorems.
4. No finite set is inconsistent; in particular, no inconsistent formula exists.
5. The interderivability relation of the logic \( \mathcal{I} \Delta \) is: \( \alpha \vdash_\Delta \beta \) if and only if \( \alpha \) and \( \beta \) are both theorems, or \( \alpha = \beta \).

**Proof.** The first four points are straightforward. As for the proof of 5: if two formulas are interderivable, then they are either both theorems, or both non-theorems; but in the latter case, by Proposition 3.6, each should be a subformula of the other, which implies they coincide.

In general, inference in a logic, \( \mathcal{I} \Delta \), with a coherent \( \Delta \) is dramatically trivial unless we have enough premises. Intuitively, the reason is obvious: if \( \Delta = \{ \delta_1, \ldots, \delta_k \} \) with all the \( \delta_i \) different, then non-trivial applications of the rule (MP\( \Delta \)) have the form \( \alpha, \delta_1(a, \beta), \ldots, \delta_k(a, \beta) \vdash_\Delta \beta \) with \( a \neq \beta \); by Lemma 3.3 the formulas \( \delta_i(a, \beta) \) are all different, and by Proposition 3.4 they are not theorems. So effective applications of this rule, even in the simpler case where \( \alpha \) is a theorem, require us to have at least as many real premises as elements in \( \Delta \).

**Proposition 3.8.** Let \( \Delta(x, y) \subseteq \text{Fm}(2) \) be coherent, and let \( \Gamma \subseteq \text{Fm} \) be a finite set with fewer elements than \( \Delta \). Then \( \Gamma \vdash_\Delta \beta \) if and only if \( \beta \) is a theorem or \( \beta \in \Gamma \); that is, \( \mathcal{C}_A \Gamma = \Gamma \cup \mathcal{C}_A \emptyset \).

**Proof.** As before, \( (\subseteq) \) is obvious, and \( (\supseteq) \) is also obvious when \( \Delta \) has one element, since then \( \Gamma = \emptyset \) and the result becomes trivial. Now assume \( \Delta \) has more than one element. The property is proved by induction on the length of a proof of \( \beta \) from \( \Gamma \); the only thing to be checked is that the property that applies to \( \beta \) is preserved under (MP\( \Delta \)). Let \( \Delta = \{ \delta_1, \ldots, \delta_k \} \) and assume that \( \gamma, \delta_1(\gamma, \varepsilon), \ldots, \delta_k(\gamma, \varepsilon) \) satisfy it, and prove that \( \varepsilon \) satisfies it. If one of the \( \delta_i(\gamma, \varepsilon) \) is a theorem, by Proposition 3.4 \( \gamma = \varepsilon \), therefore \( \varepsilon \) satisfies the property. Otherwise, we must have \( \delta_1(\gamma, \varepsilon), \ldots, \delta_k(\gamma, \varepsilon) \in \Gamma \). Since by assumption, \( \Gamma \) has fewer than \( k \) members, at least two of these formulas must coincide: \( \delta_i(\gamma, \varepsilon) = \delta_j(\gamma, \varepsilon) \). But by Lemma 3.3, this is only possible if \( \gamma = \varepsilon \); so again, \( \varepsilon \) must satisfy the property. 

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In particular, whenever \( \Delta \) has more than one element, \( C_\Delta \{ a \} = \{ a \} \cup C_\Delta \emptyset \) for any \( a \in Fm \). There are other sets \( \Gamma \), even infinite ones, such that \( C_\Delta \varphi = \Gamma \cup C_\emptyset \), as Corollary 3.7.1 shows.

### 3.2 More logics associated with a given coherent \( \Delta \)

The previous results can be extended to other logics defined in a similar way; these results will only be needed at the end of Section 4. In passing, I show that they have (almost) all the properties seen so far for the logics \( \mathcal{I} \Delta \).

**Proposition 3.9.** Let \( \Delta(x, y) \subseteq Fm(2) \) be coherent, and let \( \mathcal{L} \) be a logic axiomatized by the single axiom \( (I_\Delta) \) and any number of rules of the form \( (MP_\Delta) \) for non-empty \( \Delta' \subseteq \Delta \) (there is a finite number of possibilities).

1. The theorems of \( \mathcal{L} \) are the same as those of \( \mathcal{I} \Delta : C_\emptyset = C_\emptyset \).
2. If \( \Gamma \vdash_\mathcal{L} \beta \) then \( \beta \) is a theorem or \( \beta \) is a subformula of some formula in \( \Gamma \) (in which case \( \text{var} \beta \subseteq \text{var} \Gamma \)).
3. The logic \( \mathcal{L} \) satisfies all the properties stated for \( \mathcal{I} \Delta \) in Corollary 3.7.
4. If \( \Gamma \subseteq Fm \) is a finite set with fewer elements than all the \( \Delta' \subseteq \Delta \) such that \( (MP_\Delta) \) is one of the axioms of \( \mathcal{L} \), then \( C_\emptyset \Gamma = \Gamma \cup C_\emptyset \).

**Proof.**
1. Note that the logic \( \mathcal{L} \) has the same axiom as the logic \( \mathcal{I} \Delta \). The proof of Proposition 3.4 actually shows that the set \( C_\emptyset = \{ \delta(\varphi, \varphi) : \varphi \in Fm \} \) contains the substitution instances of the axiom and is closed under any rule of the form \( (MP_\Delta) \) for some non-empty \( \Delta' \subseteq \Delta \). This shows that this set is also the set of theorems of \( \mathcal{L} \).
2. The proof of Proposition 3.6 actually shows that the property of being a theorem or a subformula of some formula in \( \Gamma \) is preserved by the rule \( (MP_\Delta) \) for any non-empty \( \Delta' \subseteq \Delta \). Therefore, all consequences of \( \Gamma \) satisfy it.
3. Follows directly from 2.
4. The proof of Proposition 3.8 actually shows that the set \( \Gamma \cup C_\emptyset \), which by point 1 coincides with \( \Gamma \cup C_\emptyset \), is closed under any rule of the form \( (MP_\Delta) \) for a non-empty \( \Delta' \subseteq \Delta \) provided that \( \Delta' \) has strictly more elements than \( \Gamma \); by assumption, this means that this set is closed under all the rules of \( \mathcal{L} \), and therefore it is the theory generated by \( \Gamma \). \( \Box \)

### 3.3 Characterizing consequence in the logics \( \mathcal{I} \delta \)

In order to make Proposition 3.6 more precise in the case of a 1-element \( \Delta \), the following notational tricks are needed.

**Notation.**
- Vectors such as \( \vec{z} \) or \( \vec{\varepsilon} \) denote (finite or infinite) sequences of variables or formulas, respectively; this is indicated by writing simply \( \vec{z} \in \text{Var} \) or \( \vec{\varepsilon} \in Fm \), leaving the length of the sequence unspecified unless needed. For convenience, the members of non-empty finite sequences will be indexed starting from 1. Sequences of length 1 are identified with their only member.
- If \( \vec{a} = \langle a_1, \ldots, a_n \rangle \) is a finite, non-empty sequence of formulas, then its final segments are denoted by \( \vec{a}_{\langle k \rangle} := \langle a_k, \ldots, a_n \rangle \); we also need to set \( \vec{a}_{\langle n+1 \rangle} := \emptyset \) for some applications. Observe that \( \vec{a}_{\langle 1 \rangle} = \vec{a} \) and \( \vec{a}_{\langle n \rangle} = \langle a_n \rangle = a_n \).
- Let \( \delta \in Fm(2) \), let \( \vec{a} \) be a finite sequence of formulas, and let \( \beta \) be a formula. Then \( \vec{\delta}(\vec{a}, \beta) \) denotes the formula defined inductively by the following clauses:

\[
\begin{align*}
\vec{\delta}(\emptyset, \beta) & := \beta \\
\vec{\delta}(\langle a \rangle, \beta) & := \delta(a, \beta) \\
\vec{\delta}(\vec{a}, \beta) & := \delta(\langle a_1, \vec{\delta}(\vec{a}_{\langle 2 \rangle}, \beta) \rangle) \quad \text{for } \vec{a} \text{ of length } > 1.
\end{align*}
\]
This is a rather natural construction. For instance, if \( \delta(x, y) = x \rightarrow y \), then \( \tilde{\delta}(\langle a_1, a_2, a_3 \rangle, \beta) = a_1 \rightarrow (a_2 \rightarrow (a_3 \rightarrow \beta)) \). If all the members of the sequence are equal, this coincides with the construction denoted as \( \alpha^n \rightarrow \beta \) or \( \alpha \rightarrow^n \beta \) in the area of BCK logic and many-valued logic.

The notation \( \tilde{\delta}(\tilde{a}, \beta) \), while sufficiently self-explanatory, is also potentially confusing, because it may suggest that \( \tilde{\delta} \) is a 2-variable formula with arguments \( \tilde{a} \) and \( \beta \); which is not the case. In order to avoid misunderstandings, it may be useful to formulate the following property, which is not hard to prove by induction (using Lemma 2.1 in an essential way):

**Lemma 3.10.** Let \( \tilde{a} = \langle a_1, \ldots, a_n \rangle \) and \( \tilde{\gamma} = \langle \gamma_1, \ldots, \gamma_m \rangle \) be finite, non-empty sequences of formulas, and let \( \beta, \varepsilon \) be formulas. If \( \tilde{\delta}(\tilde{a}, \beta) = \tilde{\delta}(\tilde{\gamma}, \varepsilon) \), then one of the following mutually exclusive situations holds:

- \( n = m \); that is, \( \tilde{a} \) and \( \tilde{\gamma} \) have the same length, and \( a_i = \gamma_i \) for all \( i \in \{1, \ldots, n\} \) and \( \beta = \varepsilon \).
- \( n < m \); that is, \( \tilde{a} \) is strictly shorter than \( \tilde{\gamma} \), and \( a_i = \gamma_i \) for all \( i \in \{1, \ldots, n\} \) and \( \beta = \tilde{\delta}(\tilde{\gamma}[n+1], \varepsilon) \).
- \( n > m \); that is, \( \tilde{a} \) is strictly longer than \( \tilde{\gamma} \), and \( a_i = \gamma_i \) for all \( i \in \{1, \ldots, m\} \) and \( \varepsilon = \tilde{\delta}(\tilde{a}[m+1], \beta) \).

The obvious usage of this notation is to implement the following iterated form of rule (MP\( _3 \)):

**Lemma 3.11.** For each \( \delta \in Fm(2) \) if \( \tilde{\varepsilon} = \langle \varepsilon_1, \ldots, \varepsilon_n \rangle \), then \( \varepsilon_1, \ldots, \varepsilon_n, \tilde{\delta}(\tilde{\varepsilon}, \beta) \vdash_\delta \beta \).

Thanks to this notation, we can see that the subformulas that appear in Proposition 3.6 cannot be arbitrary, but have a well-determined structure (incidentally, this shows why its exact converse does not hold). Notice how the second property in Proposition 3.6 has been split here into conditions (b) and (c); these could be merged into one by deleting the “non-empty” proviso in (c), but it is more practical the way it is.

**Proposition 3.12.** If \( \Gamma \vdash_\delta \beta \) then \( \beta \) satisfies one of the following conditions:

(a) \( \beta \) is a theorem (i.e., an instance of the axiom of \( \exists \delta \)).

(b) \( \beta \in \Gamma \).

(c) There is a finite non-empty sequence \( \tilde{\varepsilon} \) of formulas, each satisfying one of the three conditions (a), (b) or (c), such that \( \tilde{\delta}(\tilde{\varepsilon}, \beta) \in \Gamma \).

**Proof.** By induction on the length of a proof of \( \beta \) from \( \Gamma \). The only thing to be checked is that the property that applies to \( \beta \) is preserved under (MP\( _3 \)), so we assume that \( \gamma, \delta(\gamma, \eta) \) satisfy it, and prove that \( \eta \) satisfies it. We examine the three cases concerning \( \delta(\gamma, \eta) \):

(a) If \( \delta(\gamma, \eta) \) is a theorem, then \( \gamma = \eta \). Since by assumption \( \gamma \) satisfies the property, \( \eta \) satisfies it.

(b) If \( \delta(\gamma, \eta) \in \Gamma \), then \( \eta \) directly satisfies (c) for the sequence \( \langle \gamma \rangle \), because by assumption \( \gamma \) satisfies one of the three conditions.

(c) If \( \tilde{\delta}(\tilde{\varepsilon}, \delta(\gamma, \eta)) \in \Gamma \) for some non-empty sequence \( \tilde{\varepsilon} \) of formulas such that each of them satisfies one of the three conditions, then as before \( \eta \) satisfies property (c) for the sequence \( \langle \tilde{\varepsilon}, \gamma \rangle \).

If the self-referential character of condition (c) makes you uncomfortable, you can remove the phrase “each satisfying one of the three conditions (a), (b) or (c)”': the resulting, weaker property is still true and sufficient for the applications of this result in the next section.

Note that Proposition 3.6, in the case of the logics \( \exists \delta \), is a corollary to Proposition 3.12.

In the case where the assumption set \( \Gamma \) is unitary, Proposition 3.12 can be refined and turned into a complete characterization:

**Proposition 3.13.** \( \alpha \vdash_\delta \beta \) if and only if \( \beta \) satisfies one of the following conditions:

(a) \( \beta \) is a theorem.

(b) \( \beta = \alpha \).

(c) \( \alpha = \tilde{\delta}(\tau, \beta) \) for some finite non-empty sequence, \( \tau \), of theorems of \( \exists \delta \).
PROOF. \((\Leftarrow)\) is obvious (using Lemma 3.11), so let us prove \((\Rightarrow)\) by induction on the length of a proof of \(\beta\) from \(\alpha\). The only thing to be checked is that the property that applies to \(\beta\) is preserved under \((\text{MP})\), so we assume that \(\gamma, \delta(\gamma, \varepsilon)\) satisfy it, and prove that \(\varepsilon\) satisfies it. We first examine the three cases concerning \(\delta(\gamma, \varepsilon)\).

(a) If \(\delta(\gamma, \varepsilon)\) is a theorem, then \(\gamma = \varepsilon\). Since by assumption \(\gamma\) satisfies the property, \(\varepsilon\) satisfies it.

(b) If \(\delta(\gamma, \varepsilon) = \alpha\) then we show, by looking at the conditions satisfied by \(\gamma\), that \(\gamma\) must be a theorem, so that \(\varepsilon\) satisfies (c). If \(\gamma\) is a theorem by (a), then \(\varepsilon\) directly satisfies (c). Since \(\gamma \neq \alpha\), condition (b) is not possible. Finally, in the case of (c), \(\alpha = \delta(\bar{\tau}, \gamma)\) for some finite, non-empty sequence \(\bar{\tau}\) of theorems; then \(\alpha = \delta(\gamma, \varepsilon) = \delta(\bar{\tau}, \gamma)\) which by 3.10 implies that \(\gamma = \tau_i\), again a theorem.

(c) If \(\alpha = \delta(\bar{\rho}, \delta(\gamma, \varepsilon))\) for some sequence \(\bar{\rho}\) of theorems, then as before we show, by looking at the conditions satisfied by \(\gamma\), that \(\gamma\) must be a theorem, so that \(\varepsilon\) satisfies (c). If \(\gamma\) is a theorem by (a), then again \(\varepsilon\) directly satisfies (c). Since \(\gamma \neq \alpha\), condition (b) is not possible. Finally, in case (c), \(\alpha = \delta(\bar{\tau}, \gamma)\) for some finite, non-empty sequence \(\bar{\tau}\) of theorems; then \(\delta(\bar{\tau}, \gamma) = \delta(\bar{\rho}, (\gamma, \varepsilon))\). Looking at 3.10, we see that \(\bar{\tau}\) and \(\bar{\rho}\) cannot have the same length, for this would imply that \(\gamma = \delta(\gamma, \varepsilon)\), which is absurd; and \(\bar{\rho}\) cannot be strictly longer than \(\bar{\tau}\), for then \(\gamma = \delta(\bar{\rho}[k], (\gamma, \varepsilon))\) for some \(k\), which is absurd as well. Thus, \(\bar{\rho}\) must be strictly shorter than \(\bar{\tau}\) and \(\delta(\gamma, \varepsilon) = \delta(\bar{\tau}[k], \gamma)\) for some \(k\), and this implies that \(\gamma = \tau_k\), a theorem.

4 Main results regarding order in Prot and in FinProt

4.1 General results

As recalled in Section 2, the sets Prot and FinProt, as subsets of Log, are posets. Their join structure is easily determined, due to the following fact:

**Lemma 4.1.** If \(\mathcal{L} \in \text{Prot}\) and \(\mathcal{L} \subseteq \mathcal{L}' \in \text{Prot}\) then \(\mathcal{L}' \in \text{Prot}\) and has the same witnessing set as \(\mathcal{L}\). Thus, Prot is an up-set of the lattice Log, and FinProt is an up-set of the lattice FinLog.

**Proof.** This is obvious from the syntactical characterization (Definition 2.2): if \(\mathcal{L} \in \text{Prot}\) with witnessing set \(\Delta(x, y)\), then it satisfies \((\text{I}A)\) and \((\text{MP})\); but this implies that any logic \(\mathcal{L}'\) with \(\mathcal{L} \subseteq \mathcal{L}'\) will satisfy them, that is, it will be protoalgebraic as well, and have the same witnessing set.

Therefore:

**Theorem 4.2.** If \(\{\mathcal{L}_i : \ i \in I\} \subseteq \text{Prot}\) is non-empty, then \(\bigvee_{i \in I} \mathcal{L}_i \in \text{Prot}\) and has any of the witnessing sets of the \(\mathcal{L}_i\) as its witnessing set. Thus, the poset Prot is a join-complete sub-semilattice of Log, and FinProt is a join-complete sub-semilattice of Prot, of FinLog and of Log.

In this result, the notion of a join-complete semilattice is understood as concerning only joins of non-empty families (otherwise, this would be the same as the notion of a complete lattice). Thus, the (infinite) join of non-empty families in any of these posets is simply \(\bigvee\), the join operation in Log.

As for the meet operation, we are going to see that it does not exist in general in Prot or in FinProt. Note that, since these sets are join-complete semilattices, a subset has a meet if and only if it has some lower bound (then the meet of the set is the join of all its lower bounds).

**Theorem 4.3.** There is no weakest protoalgebraic logic and no weakest finitary protoalgebraic logic; that is, the posets Prot and FinProt have no minimum.

**Proof.** Suppose a logic \(\mathcal{L}\) is the minimum of Prot. Then \(\mathcal{L} \leq \mathcal{I}\delta\) for all \(\delta(x, y)\), and in particular a theorem \(\varphi\) of \(\mathcal{L}\) would be a theorem of all the logics \(\mathcal{I}\delta\). However, by Proposition 3.4, the theorems of \(\mathcal{I}\delta\) are all substitution instances of the formula \(\delta(x, x)\) and hence have a complexity (length) that is at least equal to that of \(\delta(x, y)\). Since there are formulas \(\delta(x, y)\) of as large complexity as desired, there can be no such formula \(\varphi\). Thus, \(\mathcal{L}\) cannot have theorems. But the only protoalgebraic logic without
The formula \( \Delta \) procedure is to set interesting to observe that, in the notation of Section 3.3, arbitrary formulas with exactly 2 variables may, however, have other applications. I now introduce it with more generality: for an arbitrary set of to use \( \Box \) I used to produce the counterexample in Theorem 4.4 are not coherent with each other; the (nicer) order that do not even have a common lower bound in Prot \( \delta \) is inconsistent, \( \mathcal{L} \), does not exist. Since the logics \( \mathcal{I} \delta \) are finitary, the same argument shows that \( \text{FinProt} \) has no minimum either.

Theorem 4.3 already implies that the posets \( \text{Prot} \) and \( \text{FinProt} \) are not meet-complete semilattices, and hence they are not complete lattices. But there is more:

**Theorem 4.4.** The posets \( \text{Prot} \) and \( \text{FinProt} \) are not meet-semilattices, and they are not filters of the lattices \( \text{Log} \) and \( \text{FinLog} \) respectively.

**Proof.** It is enough to produce two logics in \( \text{FinProt} \) that have no common lower bound at all in \( \text{Prot} \). The argument is basically the same as in Theorem 4.3: the theorems of any such logic would be theorems of both logics, but the two logics can be chosen in such a way that their sets of theorems have an empty intersection, so that any common lower bound will have no theorems and hence cannot be protoalgebraic. To produce a particular example, take any 2-variable formula \( \phi \), \( \psi \) be protoalgebraic. To produce a particular example, take any 2-variable formula that have no common lower bound at all in \( \text{FinProt} \) has no minimum either. Since the logics \( \mathcal{I} \delta \) are inconsistent, \( \mathcal{L} \), does not exist. Since the logics \( \mathcal{I} \delta \) are finitary, the same argument shows that \( \text{FinProt} \) has no minimum either.

The proof shows a stronger result than that stated: there are infinitely many pairs of logics in \( \text{FinProt} \) that do not even have a common lower bound in \( \text{Prot} \). Theorem 4.10 will show that there are infinitely many strictly decreasing sequences of logics in the same situation. Note that the formulas \( \delta \) and \( \delta' \) used to produce the counterexample in Theorem 4.4 are not coherent with each other; the (nicer) order structure of the family of the logics \( \mathcal{I} \Delta \) where \( \Delta \) ranges over all sets that are coherent with each other, is determined in Theorem 4.11.

### 4.2 The iteration process

There is a standard way of iterating a unary connective: for instance, it is usual in the modal world to use \( \Box^0 a := a \) and \( \Box^{n+1} a := \Box \Box^n a \). For a binary connective such as \( \rightarrow \), a fairly standard iteration procedure is to set \( a \rightarrow^0 \beta := \beta \) and \( a \rightarrow^{n+1} \beta := (a \rightarrow (a \rightarrow^\beta)) \). Here I propose another, essentially different (and I believe new) way to iterate binary connectives, with a specific purpose; the construction may, however, have other applications. I now introduce it with more generality: for an arbitrary set of arbitrary formulas with exactly 2 variables.

**Definition 4.5.** For each 2-variable formula \( \delta(x, y) \in \text{Fm}(2) \), the formula \( \delta^1(x, y) \) is defined as:

\[
\delta^1(x, y) := \delta(\delta(x, x), \delta(x, y)),
\]

and for each set \( \Delta(x, y) \subseteq \text{Fm}(2) \), the set \( \Delta^1(x, y) \) is defined as:

\[
\Delta^1(x, y) := \{\delta'(\delta(x, x), \delta(x, y)) : \delta, \delta' \in \Delta\}.
\]

The formula \( \delta^1 \) and the set \( \Delta^1 \) are called the iteration of \( \delta \) and of \( \Delta \), respectively.

Observe that if \( \Delta = \{\delta\} \), then \( \Delta^1 = \{\delta^1\} \); and that if \( \Delta \) is finite, then \( \Delta^1 \) is finite as well. It may also be interesting to observe that, in the notation of Section 3.3, \( \delta(x, y) = \tilde{\delta}(\delta(x, x), x, y) \).

These constructions can in turn be iterated, and the complexity of the formulas grows exponentially, as does the cardinality of the sets. As examples, in a language containing \( \rightarrow \):

\[
(x \rightarrow y)^1 = (x \rightarrow x) \rightarrow (x \rightarrow y)
\]

and

\[
(((x \rightarrow y)^1)^1)^1 = \left(((x \rightarrow x) \rightarrow (x \rightarrow x)) \rightarrow ((x \rightarrow x) \rightarrow (x \rightarrow x))\right) \rightarrow \\
\rightarrow \left(((x \rightarrow x) \rightarrow (x \rightarrow x)) \rightarrow ((x \rightarrow x) \rightarrow (x \rightarrow y))\right).
\]
and for $\Delta(x, y) = \{ x \to y, y \to x \}$,
\[
\Delta^i(x, y) = \{ (x \to x) \to (x \to y), (x \to x) \to (y \to x),
(\neg y \to \neg x) \to (x \to y), (y \to x) \to (x \to y) \}.
\]

It is straightforward to see that:

**Lemma 4.6.** If $\Delta(x, y)$ is coherent, then $\Delta^i(x, y)$ is coherent as well; more precisely, if $\Delta(x, y)$ is coherent with $\delta(x, y)$, then $\Delta^i(x, y)$ is coherent with $\delta(x, y)$.

Thus, for each logic $\mathcal{I}\Delta$ with a coherent $\Delta$ we have another logic, $\mathcal{I}\Delta^i$, that belongs to the same family, and Proposition 3.4 applies to it: the theorems of $\mathcal{I}\Delta^i$ are the formulas of the form $\delta^i(\varphi, \varphi)$ for any formula $\varphi$ and any $\delta \in \Delta$; that is, of the form $\delta(\delta(\varphi, \varphi), \delta(\varphi, \varphi))$.

**Proposition 4.7.** For each coherent set $\Delta(x, y) \subseteq \text{Fm}(2)$, $\mathcal{I}\Delta^i < \mathcal{I}\Delta$. In particular, for any formula $\delta(x, y) \in \text{Fm}(2)$, $\mathcal{I}\Delta^i < \mathcal{I}\delta$.

**Proof.** I first show that $\mathcal{I}\Delta^i \leq \mathcal{I}\Delta$. The axiom of $\mathcal{I}\Delta^i$ is the formula $\delta^i(x, x) = \delta(\delta(x, x), \delta(x, x))$ for any $\delta \in \Delta$, and by Proposition 3.4 it is a theorem of $\mathcal{I}\Delta$. To see that the rule (MP$_\delta$) of $\mathcal{I}\Delta^i$ holds in $\mathcal{I}\Delta$, it is enough to show that $\Delta^i(x, y) \supseteq \Delta(x, y)$. To see this, take any $\delta(x, y) \in \Delta(x, y)$. Making a suitable substitution in the rule (MP$_\delta$) of $\mathcal{I}\Delta$, we know that $\delta(x, x), \Delta(\delta(x, x), \delta(x, y)) \vdash \Delta(x, y)$. But $\delta(x, x)$ is a theorem of $\mathcal{I}\Delta$, and $\Delta(\delta(x, x), \delta(x, y)) \subseteq \Delta^i(x, y)$, therefore the claim is established. This shows that $\mathcal{I}\Delta^i$ is weaker than $\mathcal{I}\Delta$, and in particular all the theorems of $\mathcal{I}\Delta^i$ are theorems of $\mathcal{I}\Delta$. To see that $\mathcal{I}\Delta^i \leq \mathcal{I}\Delta$, it is enough to see that it has strictly fewer theorems; for each $\delta \in \Delta$, the formula $\delta(x, x)$, which is a theorem of $\mathcal{I}\Delta$, cannot be a substitution instance of $\delta(\delta(x, x), \delta(x, x))$, therefore it is not a theorem of $\mathcal{I}\Delta^i$.

As a particular case, this result shows that $\mathcal{I}\delta^i < \mathcal{I}\delta$ because $\mathcal{I}\delta^i$ has strictly fewer theorems than $\mathcal{I}\delta$. As an alternative proof, it is interesting to see that it also has strictly fewer rules:

**Proposition 4.8.** For each $\delta(x, y) \in \text{Fm}(2)$, the rule (MP$_\delta$) does not hold in the logic $\mathcal{I}\delta^i$.

**Proof.** If we had $x, \delta(x, y) \vdash \varphi$ since $y$ is not a theorem and $y \neq x$, by Proposition 3.12 there should be some finite, non-empty sequence of formulas, $\vec{\varepsilon}$, such that $\delta^i(\vec{\varepsilon}, y) = \delta(x, y)$. But this is absurd; since $\vec{\varepsilon}$ is non-empty, the complexity of the formula $\delta^i(\vec{\varepsilon}, y)$ is at least that of $\delta^i(x, y)$, which in turn is strictly greater than the complexity of $\delta(x, y)$. Therefore, the previous equality never holds and (MP$_\delta$) cannot hold in $\mathcal{I}\delta^i$.

**Theorem 4.9.** If the protoalgebraicity of a logic $\mathcal{L} \in \text{Prot}$ is witnessed by a coherent set $\Delta(x, y)$, in particular if $\mathcal{L} = \mathcal{I}\Delta$ for a coherent $\Delta(x, y)$, then $\mathcal{L}$ is not a minimal element of $\text{Prot}$. If the logic is finitary, then $\mathcal{L}$ is not minimal in $\text{FinProt}$ either.

**Proof.** That the $\mathcal{I}\Delta$ for coherent $\Delta$ are not minimal follows directly from Proposition 4.7. In general, if a finite $\Delta$ witnesses the protoalgebraicity of $\mathcal{L}$, then $\mathcal{I}\Delta \leq \mathcal{L}$; therefore, if $\Delta$ is coherent, then $\mathcal{L}$ cannot be minimal either. The second part follows because $\mathcal{I}\Delta^i$ is finitary.

In particular, no protoalgebraic logic (whether finitary or not) whose protoalgebraicity is witnessed by a single formula, is a minimal element of $\text{Prot}$. This covers the vast majority of protoalgebraic logics found in the literature, whose protoalgebraicity is witnessed by the single formula $x \to y$: their language includes $\to$, they have the formula $x \to x$ as a theorem, and they satisfy the rule of modus ponens $x, x \to y \vdash \varphi$. By Theorem 4.9, none of these logics is minimal in $\text{Prot}$; and, if the logic is finitary, it is not minimal in $\text{FinProt}$ either. The equivalence fragments of classical and intuitionistic logic, whose protoalgebraicity is witnessed by the formula $x \leftrightarrow y$, also fall under the scope of this result.

It is clear that from any of the logics $\mathcal{I}\Delta$ for a coherent $\Delta(x, y)$, a strictly decreasing sequence of finitary protoalgebraic logics appears by defining the family of sets $\Delta^n := \Delta$, and $\Delta^{n+1} := (\Delta^n)^*$ for all $n \in \omega$. The formulas $\delta^n$ are defined in a similar way.
Theorem 4.10. The poset $\text{FinProt}$ (and hence the poset $\text{Prot}$) possesses infinitely many strictly decreasing sequences having no lower bound in $\text{Prot}$ (hence, no lower bound in $\text{FinProt}$; specifically, for each coherent set $\Delta(x, y)$, the sequence $(\mathcal{I}\Delta^n : n \in \omega)$.

Proof. That the sequence $(\mathcal{I}\Delta^n : n \in \omega)$ is strictly decreasing is a consequence of Proposition 4.7. That it has no lower bound at all in $\text{Prot}$ is proved by the same reasoning as that of Theorem 4.3: by considering the (exponentially growing) complexity of the theorems of the logics $\mathcal{I}\Delta^n$.

4.3 Ordering the logics $\mathcal{I}\Delta$

Theorem 4.11. Let $\Delta(x, y) \subseteq \text{Fm}(2)$ be a coherent set, and let $\Delta', \Delta'' \subseteq \Delta$ be non-empty. Then $\Delta' \subseteq \Delta''$ if and only if $\mathcal{I}\Delta' \subseteq \mathcal{I}\Delta''$.

Proof. $(\Rightarrow)$ If $\Delta', \Delta'' \subseteq \Delta$ with $\Delta' \subseteq \Delta''$, then the logics $\mathcal{I}\Delta'$ and $\mathcal{I}\Delta''$ have the same axiom (as the two sets are subsets of the same coherent $\Delta$) and the rule $(\text{MP}_{\varphi'})$ is a consequence of the rule $(\text{MP}_{\varphi''})$ simply by weakening; therefore the rule $(\text{MP}_{\varphi'})$ holds in $\mathcal{I}\Delta'$. This shows that $\mathcal{I}\Delta' \subseteq \mathcal{I}\Delta''$.

$(\Leftarrow)$ Let $\Delta', \Delta'' \subseteq \Delta$ be such that $\mathcal{I}\Delta' \subseteq \mathcal{I}\Delta''$, and take any $\delta \in \Delta'$. By the first part of the Theorem, $\mathcal{I}\Delta' \subseteq \mathcal{I}\delta$, therefore $\mathcal{I}\Delta'' \subseteq \mathcal{I}\delta$. In particular, the rule $(\text{MP}_{\varphi'})$ holds in $\mathcal{I}\delta$: $x, \Delta''(x, y) \vdash y$. By Proposition 3.12, since $x \neq y$ and $y$ is not a theorem, there must be some $\delta' \in \Delta''$ and some finite, non-empty sequence of formulas, $\tilde{\varepsilon}(x, y)$, such that $\delta'(x, y) = \delta(\tilde{\varepsilon}(x, y), y)$. Replacing $y$ by $x$, we get $\delta'(x, x) = \delta(\tilde{\varepsilon}(x, x), x)$. But since $\delta, \delta' \in \Delta$, the two formulas are coherent with each other, so that this is $\delta(x, x) = \tilde{\delta}(\tilde{\varepsilon}(x, x), x)$. By Lemma 3.10, this implies that $x = \varepsilon_1(x, x)$ and $x = \tilde{\delta}(\tilde{\varepsilon}_2(x, x), x)$; the second fact implies $\tilde{\varepsilon}_2(x, x) = \emptyset$, so that $\tilde{\varepsilon}$ is of length 1, with $\tilde{\varepsilon}(x, x) = \varepsilon_1(x, x) = x$, and then $\delta'(x, y) = \varepsilon_1(x, y)$. From the equality $\varepsilon_1(x, y) = x$, it follows that either $\varepsilon_1(x, y) = x$ or $\varepsilon_1(x, y) = y$. The second case is not possible because it implies that $\delta'(x, y) = \delta(y, y)$, which is against the assumption that $\delta' \in \text{Fm}(2)$. So the only possibility is the first case, which implies that $\delta'(x, y) = \delta(x, y)$, and thus shows that actually $\delta \in \Delta''$. Thus, I have proved that $\Delta' \subseteq \Delta''$.

This result could be paraphrased by saying that the map $\Delta \mapsto \mathcal{I}\Delta$ establishes a dual-order isomorphism between the power set of $\Delta$ minus the empty set, and the set of logics $\{ \mathcal{I}\Delta' : \Delta' \subseteq \Delta, \Delta' \neq \emptyset \}$; the latter set therefore has the structure of a finite Boolean lattice without the top. It could easily be turned into a real Boolean lattice just by adding the inconsistent logic, which is strictly above all these logics and would play the rô1e of $\mathcal{I}\emptyset$, which is not defined by convention. Since we can find coherent sets $\Delta$ of any finite cardinality in a convenient language (in fact, in the simplest language with just $\rightarrow$), it follows:

Corollary 4.12. Every finite Boolean lattice is isomorphic to a lattice of (finitary and protoalgebraic) logics.

In particular, the logics $\mathcal{I}\delta$ for $\delta \in \Delta$ are pairwise incomparable (assuming $\Delta$ has more than one element, of course) and are maximal in this set. Note that they are not maximal in the larger poset $\text{FinProt}$: in fact, if $\delta, \delta' \in \Delta$, then the join $\mathcal{I}\delta \vee \mathcal{I}\delta'$ in $\text{FinProt}$, or equivalently in $\text{Log}$, exists, and is the logic axiomatized by the common axiom $\delta(x, x)$ (by coherence) and the two rules $(\text{MP}_2)$ and $(\text{MP}_\varphi)$. Since the two logics are incomparable and maximal in the set, the join must be strictly above them and outside that set; but it is not the inconsistent logic, because its theorems are also the same (by Proposition 3.9).

A similar situation presents itself when we consider any two logics of the form $\mathcal{I}\Delta'$ and $\mathcal{I}\Delta''$ for non-empty and incomparable $\Delta', \Delta'' \subseteq \Delta$: we know that the join $\mathcal{I}\Delta' \vee \mathcal{I}\Delta''$ in $\text{FinProt}$, or equivalently in $\text{Log}$, exists, and is axiomatized by the common axiom $\delta(x, x)$ (by coherence) and the two rules $(\text{MP}_2)$ and $(\text{MP}_\varphi)$. By Proposition 3.9, the join has the same theorems as the other two logics, and it is not the inconsistent logic. If $\Delta' \cap \Delta'' = \emptyset$, then the logics $\mathcal{I}\Delta'$ and $\mathcal{I}\Delta''$ cannot have a common upper bound in

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5Note that applying Definition 3.1 to $\Delta = \emptyset$ one would obtain the almost inconsistent logic, which is not comparable with any of the logics considered here.

6This observation was first made by Ramon Jansana.
the family, by Theorem 4.11. If \( \Delta' \cap \Delta'' \neq \emptyset \), then by Theorem 4.11, the join of \( \mathcal{I}\Delta \) and \( \mathcal{I}\Delta' \) in the family is the logic \( \mathcal{I}(\Delta' \cap \Delta'') \); but the two joins do not coincide:

**Proposition 4.13.** Let \( \Delta(x, y) \subseteq Fm(2) \) be a coherent set, and let \( \Delta', \Delta'' \subseteq \Delta \) be non-empty, incomparable and such that \( \Delta' \cap \Delta'' \neq \emptyset \). Then \( \mathcal{I}\Delta' \cup \mathcal{I}\Delta'' < \mathcal{I}(\Delta' \cap \Delta'') \).

**Proof.** Since \( \Delta' \cap \Delta'' \subseteq \Delta' \), the logic \( \mathcal{I}(\Delta' \cap \Delta'') \) is a common upper bound of the two logics by Theorem 4.11; therefore, \( \mathcal{I}\Delta' \cup \mathcal{I}\Delta'' \leq \mathcal{I}(\Delta' \cap \Delta'') \). If \( \mathcal{I}\Delta' \cup \mathcal{I}\Delta'' = \mathcal{I}(\Delta' \cap \Delta'') \), the rule \( \text{MP}_{\Delta' \cap \Delta''} \) should be probable in \( \mathcal{I}\Delta' \cup \mathcal{I}\Delta'' \). In particular, and putting \( \mathcal{L} := \mathcal{I}\Delta' \cup \mathcal{I}\Delta'' \) and \( \Delta := (\Delta' \cap \Delta'') \), to simplify the notation, by structurality and this rule we should have that \( \Gamma \vdash \mathcal{L} \beta \) for all \( \beta \in Fm \). The assumption that the sets are incomparable and have a non-empty intersection implies that \( \Gamma \) has strictly fewer elements than both \( \Delta' \) and \( \Delta'' \), so that all the assumptions of Proposition 3.9 are satisfied; therefore, \( C\mathcal{L} \Gamma = \Gamma \cup C\Delta \emptyset \), and hence it is not possible that \( \beta \in C\mathcal{L} \Gamma \) for all \( \beta \in Fm \).

We can now prove that all the logics we have been considering so far are different:

**Theorem 4.14.** The logics in the family \( \{ \mathcal{I} \Delta : \Delta(x, y) \subseteq Fm(2) \text{ coherent} \} \) are all different.

**Proof.** Let \( \Delta'(x, y), \Delta''(x, y) \subseteq Fm(2) \) be two different coherent sets. If the logics \( \mathcal{I}\Delta' \) and \( \mathcal{I}\Delta'' \) have different theorems, then they are obviously different. If they have the same theorems, then by Proposition 3.5 the sets are coherent with each other, and as a consequence the set \( \Delta := \Delta' \cup \Delta'' \) is a coherent set. Then \( \Delta', \Delta'' \subseteq \Delta \) and Theorem 4.11 applies: if \( \Delta', \Delta'' \) are different, then the logics \( \mathcal{I}\Delta' \) and \( \mathcal{I}\Delta'' \) are also different.

In particular, the logics in the family \( \{ \mathcal{I}\delta : \delta \in Fm(2) \} \) are all different. By Theorem 4.11 applied to the set \( \{ \delta, \delta' \} \), two logics \( \mathcal{I}\delta \) and \( \mathcal{I}\delta' \) are pairwise incomparable when \( \delta \) and \( \delta' \) are different but coherent with each other (in which case they have the same theorems). On the other hand, Proposition 4.7 shows that in other cases they can be comparable, for instance when one is an iteration of the other, and in that case one of the logics has both fewer theorems and, by Proposition 4.8, fewer rules than the other.

## 5 Further properties of the logics \( \mathcal{I} \Delta \)

### 5.1 Sets of congruence formulas with parameters

**Notation.** In abstract algebraic logic, some points (such as the definition used here of a protoalgebraic logic in terms of a set of formulas \( \Delta(x, y) \)) require us to focus on two particular, arbitrary but fixed, distinct variables: \( x \) and \( y \). In this context, variables other than these two that occur in other formulas are called parameters. In particular, one considers arbitrary formulas and sets of formulas where such parameters can appear, which are indicated by the notations \( \varphi(x, y, \bar{z}) \) and \( \Delta_0(x, y, \bar{z}) \). Obviously, only a finite number of parameters can occur in a particular formula, but all possible values might effectively occur in an infinite set of formulas; thus, in these notations the sequence \( \bar{z} \) of parameters may be infinite, sometimes even consisting of all the variables except \( x \) and \( y \). In many applications these parameters are replaced by arbitrary formulas, which in this context will be represented by a sequence \( \bar{c} \) of the required length. When interpreting the formulas in some algebra \( A \), the value of the formula \( \varphi(x, y, \bar{z}) \), when \( x, y, \bar{z} \) are given the values \( a, b, \bar{c} \) respectively, is denoted by \( \varphi^A(a, b, \bar{c}) \).

A set of congruence formulas with parameters for a logic \( \mathcal{L} \) is a set \( \Delta_0(x, y, \bar{z}) \) of formulas containing the variables \( x, y \) and possibly parameters \( \bar{z} \) such that for all theories \( \Gamma \) of \( \mathcal{L} \) and all \( a, \beta \in Fm \):

\[
a \equiv \beta \ (\Omega \Gamma) \quad \text{if and only if} \quad \Delta_0(a, \beta, \bar{c}) \subseteq \Gamma \quad \text{for all} \ \bar{c} \in Fm,
\]

where \( \Omega \Gamma \) is the Leibniz congruence of theory \( \Gamma \), which is the largest congruence of the formula algebra that is compatible with \( \Gamma \) (that is, that does not identify formulas in \( \Gamma \) with formulas outside it). Similarly, if \( (A, F) \) is a matrix model of the logic over an arbitrary algebra \( A \), its Leibniz congruence, \( \Omega^AF \), is
the largest congruence of $A$ that does not identify elements in $F$ with elements outside it, and when (2) holds, then for any $a, b \in A$:

$$a \equiv b \ (\Omega AF) \text{ if and only if } \Delta^A_0(a, b, \bar{c}) \subseteq F \text{ for all } \bar{c} \in A. \quad (3)$$

A matrix is reduced when its Leibniz congruence $\Omega AF$ is the identity relation.

Determining a set of congruence formulas with parameters usually allows us to obtain some deeper insight into the properties of a logic. From [3, Theorem 13.5] and [10, Proposition 2.1] we know the following:

**Lemma 5.1.** Let $\mathcal{L}$ be an arbitrary protoalgebraic logic with witnessing set $\Delta(x, y)$. Then the set

$$\Delta_0(x, y, z) := \bigcup \{ \Delta(\psi(x, \bar{z}), \psi(y, \bar{z})) : \psi(x, \bar{z}) \in Fm \}.$$

is a set of congruence formulas with parameters for $\mathcal{L}$. 

Observe that $\psi(x, \bar{z})$ ranges over all possible formulas, with or without parameters; thus, $\Delta(x, y) \subseteq \Delta_0(x, y, z)$ by taking $\psi(x, \bar{z}) = x$.

A word of warning is in order before I continue: all the following properties hold for the logics considered in Proposition 3.9, i.e., any logic axiomatized by the axiom ($I_\Delta$), for a coherent $\Delta$, and any number of rules (not just one) of the form (MP$_x$) for several non-empty $\Lambda' \subseteq \Delta$. These logics share with the logics $I\Delta$ all the properties necessary to prove all the results in this section. One has to take into account that the protoalgebraicity of these logics is witnessed by any of the $\Lambda'$ such that (MP$_x$) is among their rules. For simplicity, however, the properties are only stated for the logics $I\Delta$.

In particular, Lemma 5.1 applies to the logics $I\Delta$ for a coherent $\Delta$. The important fact to bear in mind is that the set $\Delta_0$ is obtained by using all formulas $\psi(x, \bar{z})$; therefore all variables occur in this set. This fact has some immediate consequences.

**Proposition 5.2.** If $\Delta$ is a coherent set and $\Gamma$ is a set of formulas in which not all the variables occur, then $\Omega C_\Delta\Gamma$ is the identity relation; i.e., the logical matrix $(Fm, C_\Delta\Gamma)$ is a reduced model of $I\Delta$.

**Proof.** If $a \equiv b$ $(\Omega C_\Delta\Gamma)$, then, by Lemma 5.1 and (2), for all $\psi(x, \bar{z})$ and all $\delta \in \Delta$, $\delta(\psi(a, \bar{z}), \psi(b, \bar{z})) \in C_\Delta\Gamma$. However, if $a \neq b$, by Proposition 3.4 these formulas are not theorems, so by Proposition 3.6 their variables should occur in $\Gamma$. But since $\psi(x, \bar{z})$ ranges over all formulas, with arbitrary variables, this would imply that all variables occur in $\Gamma$, contradicting the assumption.

In particular, the Leibniz congruence of all finitely generated theories of $I\Delta$ is the identity relation. Similar arguments can be used to show that the Leibniz congruence of other theories of this logic is also the identity relation; for instance, one can show that if $\Gamma$ is the set of all the variables, then $\Omega C_\Delta\Gamma$ is the identity relation as well.

Proposition 5.2 has further important consequences concerning other central notions of abstract algebraic logic [11, 12]: for a protoalgebraic logic $\mathcal{L}$, its algebraic counterpart, $\text{Alg}^*\mathcal{L}$, is the class of algebraic reducts of its reduced models, and its intrinsic variety, $\mathbb{V}\mathcal{L}$, is the variety generated by $\text{Alg}^*\mathcal{L}$; these two classes need not coincide because in general $\text{Alg}^*\mathcal{L}$ need not be a variety ($\text{BCK}$ logic being the best known example). The intrinsic variety can be obtained directly from the formula algebra by factoring it out through the Tarski congruence, $\Omega\mathcal{L}$, of the logic; this is defined as $\Omega\mathcal{L} := \bigcap \{ \Omega : \Gamma \in T\mathcal{L} \}$ and it is easy to see that $\Omega\mathcal{L}$ is the largest congruence compatible with all the theories of the logic. Then one can prove [11, Proposition 1.23] that $\mathbb{V}\mathcal{L}$ is the variety generated by the quotient $Fm/\Omega\mathcal{L}$; this quotient deserves to be called the Lindenbaum-Tarski algebra of $\mathcal{L}$. When $\mathcal{L}$ is protoalgebraic, the Tarski congruence of the logic $\Omega\mathcal{L}$ coincides with $\Omega C_{\mathbb{V}\mathcal{L}}$: the Leibniz congruence of the set of theorems of the logic [11, Proposition 3.1].

**Proposition 5.3.** Assume $\Delta$ is a coherent set. Then:

1. The Tarski congruence $\Omega I\Delta$ of the logic $I\Delta$ is the identity relation. Therefore, the Lindenbaum-Tarski algebra of the logic, $Fm/\Omega I\Delta$, is (isomorphic to) the formula algebra.
2. The intrinsic variety $\forall I\Delta$ of $I\Delta$ is the class of all algebras of the similarity type.
3. The algebraic counterpart $Alg^{*}I\Delta$ of $I\Delta$ satisfies no non-trivial identity.
4. The algebraic counterpart $Alg^{*}I\Delta$ of $I\Delta$ contains the formula algebras of the similarity type over sets of variables of all infinite cardinalities.

**Proof.** 1. Since $I\Delta$ is protolgebraic, its Tarski congruence equals the Leibniz congruence of its set $C_{\Delta}\emptyset$ of theorems, so that $\tilde{\Delta}I\Delta = \Omega C_{\Delta}\emptyset = \Omega C_{\Delta}(x,x)$. Clearly, not all variables occur in $\Delta(x,x)$, therefore Proposition 5.2 applies, showing that $\tilde{\Delta}I\Delta$ is the identity relation.

2. From the previous point it follows that the Lindenbaum-Tarski algebra $Fm/\tilde{\Delta}I\Delta$ is isomorphic to the formula algebra itself. Therefore, the variety it generates, which is $\forall I\Delta$, is the class of all formulas of the similarity type.

3. This point follows from the previous one and the fact that the variety generated by the class $Alg^{*}I\Delta$ is also the intrinsic variety $\forall I\Delta$.

4. By Proposition 5.2, $Alg^{*}I\Delta$ contains the formula algebra over a countably infinite set of variables. In the general case of a formula algebra $Fm_{\kappa}$ with an arbitrary infinite set of variables of cardinality $\kappa$, consider the set $F_{\kappa} = \{ \delta(\varphi,\varphi) : \varphi \in Fm_{\kappa} \}$ where $\delta \in \Delta$ is arbitrary. It is easily seen to be an $I\Delta$-filter, for it contains the images of the axioms and is closed under the rule $(MP\Delta)$. Now, its Leibniz congruence is the identity relation; to see this, let $\alpha, \beta \in Fm_{\kappa}$ with $\alpha \neq \beta$. Then, $\delta(\alpha,\beta) \notin F_{\kappa}$, which implies that $\Delta^{Fm_{\kappa}}(\alpha,\beta,\varepsilon) \notin F_{\kappa}$ for any $\varepsilon \in Fm_{\kappa}$. By Lemma 5.1 and (3), this means that $\alpha \neq \beta (\Omega^{Fm_{\kappa}}F_{\kappa})$. Therefore $\Omega^{Fm_{\kappa}}F_{\kappa}$ is the identity relation, and the matrix $(Fm_{\kappa},F_{\kappa})$ is a reduced model of $I\Delta$. This implies that $Fm_{\kappa} \in Alg^{*}I\Delta$.

**Corollary 5.4.** For each coherent set $\Delta$, the logic $I\Delta$ is not selfextensional.

**Proof.** The Tarski congruence (which is proved to be the identity relation in Proposition 5.3.1) is defined as the largest congruence of the formula algebra below the interderivability relation $\vdash_{\Delta}$. However, in Corollary 3.7.5, this has been shown not to be the identity relation; therefore it does not coincide with the Tarski congruence. This means that the interderivability relation is not a congruence. This is the definition of not being selfextensional.

Thus, the logics $I\Delta$ for coherent $\Delta$ are totally outside the Frege hierarchy, while in principle they belong to the bottom of the Leibniz hierarchy (as I confirm below).

Proposition 5.3 contains the few things known about the algebraic counterpart of the logics $I\Delta$ for coherent $\Delta$. In the particular case of $I(x \rightarrow y)$, some finite examples are presented in [10], showing for instance that it is not the class of all algebras of the type.

**5.2 Classification in the Leibniz hierarchy**

Having a set of congruence formulas with parameters allows us to prove many properties of the logic that depend on the Leibniz operator; most notably its classification inside the Leibniz hierarchy and other classifications (general references for this and the next section are [2, 7, 14]).

Although it does not belong to either hierarchy, the class of logics that have an algebraic semantics has its importance in abstract algebraic logic, as it represents a very natural form of relation between a logic and a class of algebras.

**Proposition 5.5.** For each coherent set $\Delta$, the logic $I\Delta$ does not have an algebraic semantics.

**Proof.** If $I\Delta$ had an algebraic semantics, then by Theorem 2.16 and Proposition 2.17 of [5], for each defining equation $\gamma \approx \varepsilon$ one should have $\gamma \equiv \varepsilon (\Omega C_{\Delta}\{x\})$. But by Proposition 5.2, $\Omega C_{\Delta}\{x\}$ is the identity relation; therefore $\gamma = \varepsilon$. Thus all defining equations have the form $\gamma \approx \gamma$. Since any algebra satisfies the equation $\gamma(\varphi) \approx \gamma(\varphi)$, it would follow that $\vdash_{\Delta} \varphi$ for all $\varphi$; that is, $I\Delta$ would be the inconsistent logic, which is not the case.
Concerning the Leibniz hierarchy, of course the logics $\mathcal{L}\Delta$ are protoalgebraic by their very definition, but we will see that they do not belong to any other class in the Leibniz hierarchy.

**Corollary 5.6.** For each coherent set $\Delta$, the logic $\mathcal{L}\Delta$ is not truth-equational. As a consequence, it does not belong to any class in the Leibniz hierarchy with “algebraizable” in its name: it is not weakly algebraizable nor algebraizable, etc. And it is not implicative.

**Proof.** The first statement is a corollary to Proposition 5.5, because all truth-equational logics have an algebraic semantics. But it is also a corollary to Proposition 5.2, because that proposition implies that the Leibniz operator is not injective on the theories of $\mathcal{L}\Delta$. However, by Theorem 28 of [14], the Leibniz operator of a truth-equational logic is always injective on the filters of the logic over any algebra, and in particular on the theories (which are the filters over the formula algebra). Therefore, $\mathcal{L}\Delta$ is not truth-equational. The rest follows because all the other classes mentioned are contained in the class of truth-equational logics.

**Proposition 5.7.** For each coherent set $\Delta$, the logic $\mathcal{L}\Delta$ is not equivalential.

**Proof.** Assume $\mathcal{L}\Delta$ is equivalential; that is, it has a parameter-free set $\Xi(x, y)$ of congruence formulas. It is well-known that all sets of congruence formulas for a logic are interderivable modulo the logic. Considering the set $\Delta_0$ of Lemma 5.1 in particular, we should have $\Delta_0(x, y, z) \subseteq C_\Xi(x, y)$. But by Proposition 5.6, this is not possible, because only the variables $x, y$ occur in the set $\Xi(x, y)$ while all the variables occur in $\Delta_0(x, y, z)$.

### 5.3 Metalogical properties

Now, after realizing that the logics $\mathcal{L}\Delta$ for a coherent $\Delta$ lie at the lowest level of the Leibniz hierarchy (and outside the Frege hierarchy), I now show that they have hardly any of the properties that are usually in the weak sense.

The same concept of bridge theorems, considered in the abstract algebraic logic when studying the Leibniz hierarchy. These properties are commonly called “metalogical” properties. Most of them concern the behaviour of the logical connectives (mainly that of conjunction, disjunction, and implication) in an abstract, generalized sense; in the case of implication this means considering several variants of the Deduction-Detachment Theorem (DDT) that have appeared to date in the literature.

A logic $\mathcal{L}$ is said to be **conjunctive** when there is a (binary) formula $\varphi^\wedge(x, y)$ such that $\{x, y\} \vdash_\mathcal{L} \varphi^\wedge(x, y)$ (such a formula is called a conjunction for $\mathcal{L}$ and is commonly denoted by $x \wedge y$).

**Proposition 5.8.** For each coherent set $\Delta$, the logic $\mathcal{L}\Delta$ is not conjunctive.

**Proof.** If $\{x, y\} \vdash_\mathcal{L} \varphi^\wedge(x, y)$ for some formula $\varphi^\wedge(x, y)$, then Corollary 3.7.1 implies that $\varphi^\wedge(x, y)$ must be either $x$ or $y$ (neither of which is possible because $x \not\vdash_\mathcal{L} y$ and $y \not\vdash_\mathcal{L} x$) or a theorem. In the latter case, however, the two variables would be theorems themselves, and this is not the case.

A set $\nabla(x, y, z) \subseteq Fm$ is a **parameterized disjunction** for a logic $\mathcal{L}$ when for all $\Gamma \cup \{\alpha, \beta\} \subseteq Fm$, $C_\mathcal{L}(\Gamma \cup \nabla(\alpha, \beta)) = C_\mathcal{L}(\Gamma, \alpha) \cap C_\mathcal{L}(\Gamma, \beta)$, where

$$\nabla(\alpha, \beta) := \bigcup\{\nabla(\alpha, \beta, \tilde{\gamma}) : \tilde{\gamma} \in Fm\}.$$ 

The same concept in the weak sense means requiring the property to hold just for $\Gamma = \emptyset$. Notice how the particular case where $\nabla$ reduces to a single formula without parameters, usually denoted as $x \vee y$, produces the property $C_\mathcal{L}(\Gamma, x \vee y) = C_\mathcal{L}(\Gamma, x) \cap C_\mathcal{L}(\Gamma, y)$: the common property of disjunction, which generalizes the proof by cases property.

**Proposition 5.9.** For each coherent set $\Delta$, the logic $\mathcal{L}\Delta$ does not have a parameterized disjunction, not even in the weak sense.

**Proof.** Assume a set $\nabla(x, y, z)$ exists that behaves as a parameterized disjunction in the weak sense for $\mathcal{L}\Delta$. In particular, for distinct variables $x$ and $y$, it should satisfy $C_\mathcal{L}\nabla(x, y) = C_\mathcal{L}\{x\} \cap C_\mathcal{L}\{y\} = C_\mathcal{L}\emptyset$ by
Corollary 3.7.2. Since $\forall (x, y, z) \subseteq \forall (x, y)$, all formulas in $\forall (x, y, z)$ should be theorems of $\mathcal{I}\Delta$; hence by structurality, all formulas in $\forall (a, b)$ as well, for all $a, b$. Then, for any formula $\alpha$, we would have $C_1(\alpha) = C_3(\alpha) \cap C_3(\alpha) = C_3 \forall (a, x) = C_3 \emptyset$; that is, every formula $\alpha$ would be a theorem of $\mathcal{I}\Delta$, which is not the case. Thus, a parameterized disjunction for $\mathcal{I}\Delta$ does not exist, not even in the weak sense. ☑

COROLLARY 5.10. For each coherent set $\Delta$:
1. The logic $\mathcal{I}\Delta$ is not filter-distributive.
2. The logic $\mathcal{I}\Delta$ does not satisfy any Contextual Deduction-Detachment Theorem (CDDT).
3. The logic $\mathcal{I}\Delta$ does not satisfy any Deduction-Detachment Theorem (DDT).

PROOF. For finitary protoalgebraic logics, being filter-distributive is equivalent to having a parameterized disjunction [7, Theorem 2.5.17]; and having a CDDT implies being filter-distributive [15, Theorem 6.8]. Finally, the DDT is a stronger form of the CDDT. ☑

The CDDT and the DDT are the strongest forms of the Deduction Theorem that have been studied so far in the literature, but other, weaker versions have also been considered. That research has had an important impact on the general theory of abstract algebraic logic. Another of the fundamental results of the theory of protoalgebraic logics [7, Theorem 2.1.5] is that a logic $\mathcal{L}$ is protoalgebraic if and only if it satisfies the Parameterized Local Deduction-Detachment Theorem (PLDDT): there is a family $\Phi(x, y, z)$ of sets of formulas in two variables and possibly parameters such that for all $\Gamma \cup \{a, b\} \subseteq \text{Fm}$,

$$\Gamma, a \vdash \Phi(x, y, z) \text{ if and only if there is some } \Sigma \in \Phi \text{ and there is } \delta \in \text{Fm} \text{ such that } \Gamma \vdash \Sigma(a, \beta, \delta). \quad \text{(PLDDT)}$$

The following stronger, non-local version of this property has been much less studied: a logic $\mathcal{L}$ satisfies the Parameterized Deduction-Detachment Theorem (PDDT) with respect to a set $\Sigma(x, y, z)$ when, for any $\Gamma \cup \{a, b\} \subseteq \text{Fm}$:

$$\Gamma, a \vdash \Phi(x, y, z) \text{ if and only if there is } \delta \in \text{Fm} \text{ such that } \Gamma \vdash \Sigma(a, \beta, \delta). \quad \text{(PDDT)}$$

Note that the right-to-left direction of this equivalence amounts to requiring that the set $\Sigma(x, y, z)$ satisfies the generalized form of modus ponens $x, \Sigma(x, y, z) \vdash y$.

The PDDT was introduced and characterized in the monograph [7]. By combining several results and comments in that book it is possible to see that not every protoalgebraic logic satisfies it; however, the first explicit example of a protoalgebraic logic that does not satisfy it that I know of appeared in [10]. Now we can see that one can find infinitely many such examples in any language:

PROPOSITION 5.11. For each coherent set $\Delta$, the logic $\mathcal{I}\Delta$ does not satisfy any Parameterized Deduction-Detachment Theorem (PDDT).

PROOF. I will use the bridge theorem of [7, Theorem 2.4.1], which states that a protoalgebraic logic satisfies a PDDT if and only if its class of matrix models has “factor-determined finitely generated filters on direct products”; i.e., any finitely generated filter of the logic on a product of two models is the product of two finitely generated filters on the factors. I show that $\mathcal{I}\Delta$ does not satisfy a PDDT by exhibiting a principal filter $F$ on the product of two models that is not the product of any pair of filters. Note that the filters involved in this property are filters on the model; that is, filters of the logic containing the base filter of the model.

I denote by $T_\delta$ the set of all theorems of $\mathcal{I}\Delta$, where $\delta \in \Delta$ is arbitrary by coherence (see Proposition 3.4). Then $(\text{Fm}, T_\delta)$ is a matrix model of $\mathcal{I}\Delta$. Consider the product $(\text{Fm} \times \text{Fm}, T_\delta \times T_\delta) = (\text{Fm}, T_\delta) \times (\text{Fm}, T_\delta)$, and the principal filter $F$ on this product generated by the element $(x, x)$ for a variable $x$. It is easy to see that actually $F = T_\delta \times T_\delta \cup \{(x, x)\}$, because this last set contains the base filter $T_\delta \times T_\delta$ (hence a fortiori it contains all images of the axioms); it contains the desired point $(x, x)$; and it is trivially closed under (MP$_\Delta$). But this set cannot be equal to the product of any two filters on $(\text{Fm}, T_\delta)$, i.e., to the
product of two theories: if it were, then the product would have to contain the point \( \langle x, x \rangle \), and then each of the theories would have to contain \( x \); but since they would also contain all theorems, their product would contain all pairs \( \langle \varphi, x \rangle \) and \( \langle x, \varphi \rangle \) for all theorems \( \varphi \), and none of these pairs belongs to \( F \). This shows that the class of matrix models of \( I \Delta \) does not satisfy the property of having “factor-determined finitely generated filters on direct products”; and by Theorem 2.4.1 of [7], this means that \( I \Delta \) does not satisfy the PDDT.

**Proposition 5.12.** For each coherent set \( \Delta \), the logic \( I \Delta \) satisfies neither the Inconsistency Lemma, nor the Classical Inconsistency Lemma, nor the Properties of Intuitionistic Reductio ad Absurdum or of plain Reductio ad Absurdum.

**Proof.** According to [16], a logic \( \mathcal{L} \) satisfies the Inconsistency Lemma if there is a sequence \( \langle \Psi_n : n \in \omega \rangle \) of finite sets of formulas, with \( \Psi_n \) having \( n \) variables, i.e., of the form \( \Psi_n(x_1, \ldots, x_n) \subseteq Fm(n) \), such that for all \( \Gamma \cup \{ a_1, \ldots, a_n \} \subseteq Fm: \)

\[
\Gamma \cup \{ a_1, \ldots, a_n \} \text{ is inconsistent if and only if } \Gamma \vdash_{\mathcal{L}} \Psi_n(a_1, \ldots, a_n).
\]

Note that if such a sequence exists, then for each \( n \) the set \( \Psi_n(x_1, \ldots, x_n) \cup \{ x_1, \ldots, x_n \} \) must be inconsistent. But this is a finite set, and by Corollary 3.7.4, we know that in the case of \( I \Delta \) no finite set is inconsistent. Therefore, \( I \Delta \) cannot satisfy any Inconsistency Lemma. The Classical Inconsistency Lemma includes the foregoing, so \( I \Delta \) cannot satisfy that either.

The properties called the Property of Intuitionistic Reductio ad Absurdum (PIRA) and the Property of Reductio ad Absurdum (PRA) in [11] are weaker, particular cases of the Inconsistency Lemma and the Classical Inconsistency Lemma considered above, limited to the case \( n = 1 \) and for a single formula which reflects the properties of intuitionistic (respectively, classical) negation, \( \neg \chi \); a similar argument shows that \( I \Delta \) cannot satisfy them.

Finally, I close this section, and the main body of the paper, with a property whose formulation does not explicitly involve the grammatical structure of formulas.

**Proposition 5.13.** For each coherent set \( \Delta \), the logic \( I \Delta \) is not “logically compact”, where this property means that every inconsistent set of formulas has a finite inconsistent subset.

**Proof.** This is a consequence of Corollary 3.7.4, which shows that no finite set is inconsistent; clearly there are infinite inconsistent sets, for instance the set of all formulas, or the set \( \{ \delta(\delta_0(x,x),\beta) : \delta \in \Delta, \beta \in Fm \} \) for one particular \( \delta_0 \in \Delta \).

### 6 Conclusions

The order structures of the set \( \text{Prot} \) of all protoalgebraic logics over a fixed arbitrary language and of the subset of its finitary members, \( \text{FinProt} \), have been examined. They have been found not to be lattices, in contrast to the sets of all logics and all finitary logics (over the given language), and also to display several unusual features. The latter include: not having a lowest element; the existence of infinitely many pairs of logics without any common lower bound in the set; and the existence of infinitely many strictly decreasing denumerable sequences, again without any lower bound in the set.

This study leaves many natural questions regarding the order structure of \( \text{Prot} \) unanswered though; for instance, whether there is some protoalgebraic logic that is minimal, and whether there are maximals\(^7\) other than the almost inconsistent logic.

Working with logics over the same language, which is a stipulation in this paper, might be perceived by some as a limitation; however, this allows us to obtain effective results. Nonetheless, to end the paper,

\(^7\) An anonymous referee suggests as a candidate logic a semantically defined one (in each of the languages under consideration) which could be proved to be maximal in this set; the proof would require purely universal algebraic techniques, however, and is rather removed from the framework of the present paper.
I feel a short discussion of the situation regarding the mixing of logics over different languages is in order.

Consider the family \( \text{Log}^* \) of all logics, irrespective of the language in which they are formulated. This set is still ordered under the expansion relation, \( \mathcal{L} \subseteq \mathcal{L}' \), which holds when (i) the language \( \mathcal{L}' \) of \( \mathcal{L}' \) includes the language \( \mathcal{L} \) of \( \mathcal{L} \); and (ii) for all \( \Gamma \cup \{ \varphi \} \subseteq \text{Fm}_\mathcal{L} \), if \( \Gamma \vdash_\mathcal{L} \varphi \) then \( \Gamma \vdash_{\mathcal{L}'} \varphi \). Observe that when the languages coincide, this relation is just the extension relation considered so far; thus there is no danger in denoting it the same way. The corresponding sets \(^8\) \( \text{FinLog}^* \), \( \text{Prot}^* \) and \( \text{FinProt}^* \) are also well-defined posets.

The first difficulty in this scenario is that the order structure of the total sets \( \text{Log}^* \) and \( \text{FinLog}^* \) has not been subject to scrutiny in the standard literature. It is clear, however, that these sets are not lattices, in contrast to the fixed-language ones \( \text{Log} \) and \( \text{FinLog} \); any two logics in disjoint languages will have no common lower bound at all; in particular \( \text{Log}^* \) and \( \text{FinLog}^* \) have no minimum.

The posets \( \text{Prot}^* \) and \( \text{FinProt}^* \) have no minimum either, and neither are they meet-semilattices, for the same reason: we can consider logics in disjoint languages. But it is not necessary to consider the disjoint language issue: the examples in Theorems 4.3 and 4.4 were produced by starting from a single, arbitrary language and constructing a family of logics such that any lower bound in \( \text{Log} \) could have no theorems, thereby preventing it from being in \( \text{Prot} \). Hence such logics would not be in \( \text{Prot}^* \) either: admitting smaller languages (since we are looking for a lower bound) would not solve the problem. The same holds for the sequences of strictly decreasing logics without lower bounds constructed in Theorem 4.10: they would have no lower bound at all in \( \text{Prot}^* \).

If \( \mathcal{L} \in \text{Prot}^* \) and \( \mathcal{L} \subseteq \mathcal{L}' \), then \( \mathcal{L}' \in \text{Prot}^* \) and any witnessing set for \( \mathcal{L} \) is a witnessing set for \( \mathcal{L}' \). Thus, \( \text{Prot}^* \) is an up-set of \( \text{Log}^* \), and \( \text{FinProt}^* \) is an up-set of \( \text{FinLog}^* \). Hence, the upper structure of \( \text{Prot}^* \) and \( \text{FinProt}^* \) will depend on that of \( \text{Log}^* \) and \( \text{FinLog}^* \); in particular, the issue of how to obtain the join of two logics over different languages is far from trivial (and this is not the place to discuss it).

Most of the basic constructions in the paper seem to work in the enlarged environment. Each finite, non-empty \( \Delta(x,y) \) is contained in a defined, finite language, and hence in all its expansions. So, there will be a corresponding logic, \( \mathcal{I}_\Delta \), in each language \( \mathcal{L} \) that expands the language of \( \Delta \). The notion of coherence concerns the replacement of variables by variables; therefore if \( \Delta \) is coherent in its own language, it will also be coherent when considered in any of its expansions. Conversely, if \( \Delta' \) is coherent with \( \Delta \), then they must belong to the same language. It is easy to see that the theorems of \( \mathcal{I}_\Delta \) will all be \( \mathcal{L} \)-substitution instances of \( \delta(x,x) \), for any \( \delta \in \Delta \), and that Proposition 3.6 still holds. And so on.

In contrast, some results, such as Theorem 4.14, do not make sense under a mixed language environment, as they concern only the order structure of the set of logics \( \{ \mathcal{I}_{\Delta'} : \Delta' \subseteq \Delta, \Delta \neq \emptyset \} \) for a fixed coherent \( \Delta \). As I remark above, it is not known whether the resulting lattice structure (Boolean without a top) coincides with that of the logics inside \( \text{Prot} \) or inside \( \text{Prot}^* \) (or their finitary counterparts).

Finally, the results in Section 5 establish properties of each individual logic \( \mathcal{I}_\Delta \) for a coherent \( \Delta \); they are not affected by allowing languages other than that of the logic under consideration.

It seems that this way of approaching the issue will not be particularly fruitful, and that only the limitation to a fixed, common language allows us to obtain interesting results. Another, more natural framework to deal with logics over different languages is the categorical one, where in recent years the technique of fibring [6] for combining logics has been developed. The exploration of this technique in relation to the Leibniz hierarchy is started in [8]. But then we move away from the study of the order structure of these sets of logics, and the task takes up quite a different story.

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\(^8\)Since arbitrary languages are considered, these families are actually proper classes, but I will not pay attention to this issue, and say “sets” just to simplify the terminology.
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