What is (abstract) algebraic logic?

The study of the **connections** between **logics** and **algebra-based semantics**.

- **Describe** these connections: Since BOOLE (1847)
  Completeness theorems and others

- **Exploit** these connections: Use the strength of algebra
  Bridge theorems

- **Explain** these connections: General theories
  General notion of a logic …
What is an algebra-based semantics?

- The structures taken as models are **algebras** $A = \langle A, L^A \rangle$ of the same similarity type $L$ as the language of the logic, sometimes endowed with some **additional structure** (matrices, generalized matrices, ordered algebras, ...)

- **Valuations** or **interpretations** are homomorphisms $h: Fm \to A$, $\forall \alpha \in Fm, \; h\alpha \in A$, and is computed from $\{hx : x \in Var\alpha\}$ using exclusively the algebraic structure,

  i.e., an algebra-based semantics is always a **truth-functional** semantics.

- **Truth of $\alpha$ in the model** is defined by conditions involving $h\alpha$ and the algebraic structure, and perhaps the additional structure.
What is a logic?

- Let $L$ be a sentential language, or algebraic similarity type.

- $Fm = \langle Fm, L \rangle$ is the **formula algebra** or algebra of terms of type $L$. It is freely generated by some set $\text{Var}$ of variables or atomic formulas.

- Our logics $L = \langle Fm, \vdash_L \rangle$ are **substitution-invariant consequence relations** over $Fm$; i.e., relations $\vdash_L \subseteq P(Fm) \times Fm$ such that:
  
  (I) $\varphi \in \Gamma$ implies $\varphi \vdash_L \varphi$.

  (M) $\Gamma \vdash_L \varphi$ implies $\Delta \vdash_L \varphi$ whenever $\Gamma \subseteq \Delta$.

  (C) $\Gamma \vdash_L \varphi$ implies $\Delta \vdash_L \varphi$ whenever $\Delta \vdash_L \psi$ for each $\psi \in \Gamma$.

  (S) $\Gamma \vdash_L \varphi$ implies $\sigma \Gamma \vdash_L \sigma \varphi$ for any substitution $\sigma$.

- The **theorems** of $L$ are the $\alpha \in Fm$ such that $\emptyset \vdash_L \alpha$.

- Logics **not** conceived as set of formulas, but as consequence relations,

- Notion independent of the way how $\vdash_L$ is defined, either proof-theoretically or semantically (i.e., using “$\vdash$” does not mean we assume any syntactical presentation!).
The typical algebraic completeness of a logic $\mathcal{L}$ with respect to a class $\mathbf{K}$ of algebras

\begin{align}
\Gamma \vdash_{\mathcal{L}} \varphi & \iff \forall A \in \mathbf{K}, \forall h \in \text{Hom}(\mathit{Fm}, A), \\
& \text{if } h \gamma = 1 \forall \gamma \in \Gamma \text{ then } h \varphi = 1.
\end{align}

**Proof.** ($\Rightarrow$) "Routine checking"; $\mathbf{K}$ and 1 wisely chosen, so that $\{1\}$ is an $\mathcal{L}$-filter for each $A \in \mathbf{K}$.

($\Leftarrow$) Assume $\Gamma \not\vdash_{\mathcal{L}} \varphi$, and **construct** some $A \in \mathbf{K}$ and $h: \mathit{Fm} \to A$ such that $h \gamma = 1 \forall \gamma \in \Gamma$, while $h \varphi \neq 1$.

**Heavily semantical-dependent constructions:**

**EXAMPLE:** If one has another semantics for $\mathcal{L}$:

$\Gamma \not\vdash_{\mathcal{L}} \varphi \Rightarrow \exists M, \bar{a} \text{ such that } M \models \gamma \llbracket \bar{a} \rrbracket \forall \gamma \in \Gamma$, but $M \not\models \varphi \llbracket \bar{a} \rrbracket$.

Then **construct** some $A \in \mathbf{K}$ and $h \in \text{Hom}(\mathit{Fm}, A)$ such that for all $\alpha \in \mathit{Fm}$,

$M \models \alpha \llbracket \bar{a} \rrbracket \iff h \alpha = 1$. 
A syntactic construction: the LINDENBAUM-TARSKI process
(the case of classical logic $C\mathcal{P}C$

Assume $\Gamma \not\vdash_{C\mathcal{P}C} \varphi$. Let $\Gamma'$ be the $C\mathcal{P}C$-theory generated by $\Gamma$.

(2) Define $\alpha \equiv \beta (\Omega \Gamma') \iff \alpha \leftrightarrow \beta \in \Gamma'$ $(\iff \Gamma \vdash_{C\mathcal{P}C} \alpha \leftrightarrow \beta)$

Show:

(3) $\Omega \Gamma'$ is a congruence of the formula algebra $Fm$.

(4) The quotient algebra $Fm/\Omega \Gamma' \in BA$.

(5) $\alpha \in \Gamma' \iff \alpha/\Omega \Gamma' = 1$, for all $\alpha \in Fm$.

Take $A := Fm/\Omega \Gamma'$ and $h\alpha := \alpha/\Omega \Gamma'$ $\forall \alpha \in Fm$.

and then $h\gamma = 1$ $\forall \gamma \in \Gamma$ and $h\varphi \neq 1$.

Note: Condition (5) actually gathers three different facts:

(5a) $\alpha, \beta \in \Gamma' \Rightarrow \alpha \equiv \beta (\Omega \Gamma')$.

(5b) $\alpha \in \Gamma', \alpha \equiv \beta (\Omega \Gamma') \Rightarrow \beta \in \Gamma'$.

(5c) $\Gamma'/\Omega \Gamma' = 1$. 

Josep Maria Font (University of Barcelona) Logic and Algebra Abstract Algebraic Logic (I)
The first generalization of the process

- ~ 1930  **LINDENBAUM** and **TARSKI**
- ~ 1950  **HENKIN**; **SIKORSKI** and **RASIOWA**: sufficient conditions for logics to which the process can be applied without any changes

**RASIOWA** [1974]: **implicative logics** \( \mathcal{L} \): Have a binary \( \to \) such that:

(2): \( \alpha \equiv \beta (\mathcal{O}T) \iff \alpha \to \beta, \beta \to \alpha \in T \)

(3): \( \mathcal{O}T \) is a congruence: \( \vdash_{\mathcal{L}} \alpha \to \alpha \); \( \alpha \to \beta, \beta \to \gamma \vdash_{\mathcal{L}} \alpha \to \gamma \)

\[
\left\{ \begin{array}{l}
\alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n \\
\beta_1 \to \alpha_1, \ldots, \beta_n \to \alpha_n 
\end{array} \right\} \vdash_{\mathcal{L}} \lambda \alpha_1 \ldots \alpha_n \to \lambda \beta_1 \ldots \beta_n \ \forall \lambda \in \mathcal{L}
\]

(5b): Modus Ponens  \( \alpha, \alpha \to \beta \vdash_{\mathcal{L}} \beta \)

(5a) and (5c): Rule K  \( \alpha \vdash_{\mathcal{L}} \beta \to \alpha \)

(4): \( A \in \text{Alg}^{\ast \mathcal{L}} \) \( \overset{\text{def}}{\iff} \) \( \exists 1 \in A \) such that part \( (\Rightarrow) \) of (1) works

i.e., such that \( \langle A, \{1\} \rangle \) is a **model** of \( \mathcal{L} \),

and \( A \models x \to y \approx 1 \& y \to x \approx 1 \implies x \approx y \).
Some classical examples

Classical logic $\leftrightarrow$ Boolean algebras
Intuitionistic logic $\leftrightarrow$ Heyting algebras
Positive implicative logic $\leftrightarrow$ Hilbert algebras
(Gödel-)Dummett’s logic $\leftrightarrow$ “linear” Heyting algebras
Logic of constructive negation $\leftrightarrow$ Nelson’s algebras
Normal modal logics $\leftrightarrow$ Boolean algebras with operators
(Global consequences) (modal algebras, etc.)
Łukasiewicz’s $\mathcal{L}_\infty$ $\leftrightarrow$ Wajsberg algebras (MV-algebras)
Post’s many-valued logics $\leftrightarrow$ Post algebras
Generalizations by inessential changes

Change the truth condition \( h \alpha = 1 \), i.e., the truth set \( F = \{1\} \)

- In many substructural logics (linear, fuzzy, relevance logic \( \mathcal{RM} \)):
  Replace “\( h \alpha = 1 \)” by “\( h \alpha \geq 1 \)” everywhere
  Take as truth set \( F = \{a \in A : a \geq 1\} \)

- Other relevance logics (\( \mathcal{R} \)):
  Replace “\( h \alpha = 1 \)” by “\( h \alpha \geq h \alpha \rightarrow h \alpha \)” everywhere
  Take as truth set \( F = \{a \in A : a \geq a \rightarrow a\} \)

What is essential here?

The equational definability of the truth condition “\( h \alpha \in F \)”:

There is some set of equations \( E(x) \subseteq Fm \times Fm \) such that

\[
a \in F \iff \delta^A(a) = \varepsilon^A(a) \quad \forall \delta \approx \varepsilon \in E(x) \iff A \models E(x) [a]
\]
Algebraic semantics, the technical notion

**Definition (BLOK and PIGOZZI, 1989)**

A class $K$ of algebras is an **algebraic semantics** for a logic $\mathcal{L}$ when there is a set of equations $E(x) \subseteq Fm \times Fm$ such that

$$(1') \quad \Gamma \vdash_{\mathcal{L}} \varphi \iff \forall A \in K, \, \forall \bar{a} \in \text{Hom}(Fm, A),$$

if $A \models E(\gamma)[\bar{a}]$ $\forall \gamma \in \Gamma$ then $A \models E(\varphi)[\bar{a}]$.

(equations in $E(x)$ are called the **defining equations**)

$F = \{1\} \quad \iff \quad E(x) = \{x \approx 1\} \text{ or } \{x \approx x \to x\}$

$F = \{a \in A : a \geq 1\} \quad \iff \quad E(x) = \{x \lor 1 \approx x\} \text{ or } \{x \land 1 \approx 1\}$

$F = \{a \in A : a \geq a \to a\} \quad \iff \quad E(x) = \{x \land (x \to x) \approx x \to x\}$
Algebraic semantics, the technical notion

The **relative equational consequence** $\models_K$ intrinsically determined by a class of algebras $\mathcal{K}$

$$\{\alpha_i \approx \beta_i : i \in I\} \models_K \alpha \approx \beta \iff \forall A \in \mathcal{K}, \forall \bar{a} \in \text{Hom}(Fm, A),$$

if $A \models \alpha_i \approx \beta_i \, [\bar{a}] \; \forall i \in I$ then $A \models \alpha \approx \beta \, [\bar{a}]$.

$$\iff \mathcal{K} \models \forall \bar{x} \left( \&_{i \in I} \alpha_i \approx \beta_i \to \alpha \approx \beta \right)$$

Every set $E(x)$ determines a **transformer** $\tau : P(Fm) \to P(Fm \times Fm)$ by:

$$\tau(\alpha) := E(\alpha) \quad \forall \alpha \in Fm$$

$$\tau(\Gamma) := \bigcup \{\tau(\alpha) : \alpha \in \Gamma\} \quad \forall \Gamma \subseteq Fm$$

(1') amounts to:

$$\Gamma \vdash_L \varphi \iff \tau(\Gamma) \models_K \tau(\varphi)$$

($\tau$ is a **faithful interpretation** of $\vdash_L$ into $\models_K$)
The paradigm of the algebraization of a logic?

- Having an algebraic semantics is a rather **weak property**.
- **Not every logic** has an algebraic semantics.
- An algebraic semantics for a logic can be rather **weird**.
- The algebraic semantics for a logic **need not be unique**,
  e.g., for $\mathcal{CP}$ we have:
  - $\text{BA}$
  - $\{ \text{finite Boolean algebras} \}$
  - $\{2\}$

and **need not be natural**:
  - $\{ \text{dually complemented bounded distributive lattices} \}$
  - $\{ \text{Heyting algebras} \}$ with $E(x) = \{ \neg\neg x \approx 1 \}$ (GLIVENKO's theorem)

- How to characterize the “right” one?
What is essential in the properties of $\Omega T$?

(2) There is some set $\Delta(x, y) \subseteq Fm$ such that $\alpha \equiv \beta (\Omega T) \iff \Delta(\alpha, \beta) \subseteq T$ satisfies (3) and (5b).

- **Examples**

  $\Delta(x, y) = \{x \leftrightarrow y\}$  
  $\quad CPC, IPC, \ \text{global normal modal,}$  
  $\quad \text{many-valued } \mathcal{L}_\infty, \text{etc.}$

  $\Delta(x, y) = \{x \to y, y \to x\}$  
  $\quad \mathcal{IPC} \to, \text{etc.}$

  $\Delta(x, y) = \{x \leftrightarrow y, \Box(x \leftrightarrow y), \Box^2(x \leftrightarrow y), \ldots\}$  
  $\quad \text{local modal } \mathcal{K}, \mathcal{T}$

  $\Delta(x, y) = \{\Box(x \leftrightarrow y)\}$  
  $\quad \text{local modal } \mathcal{S}4, \mathcal{S}5$

  $\Delta(x, y) = \{x \to y, y \to x, \neg x \to \neg y, \neg y \to \neg x\}$  
  $\quad \mathcal{P}1, \mathcal{L}_3$
What is essential in the properties of $\Omega T$?

(2) There is some set $\Delta(x, y) \subseteq Fm$ such that $\alpha \equiv \beta (\Omega T) \iff \Delta(\alpha, \beta) \subseteq T$ satisfies (3) and (5b).

- Whenever such a $\Delta(x, y)$ exists such that this $\Omega T$ satisfies (3) and (5b), then it is the largest one (the Leibniz congruence of $T$).

- The formulas in $\Delta(x, y)$ are called equivalence formulas.

- Every set $\Delta(x, y)$ determines a transformer $\rho : P(Fm \times Fm) \rightarrow P(Fm)$ by: $\rho(\alpha \approx \beta) := \Delta(\alpha, \beta)$ $\forall \alpha \approx \beta \in Fm \times Fm$

$$\rho(\Theta) := \bigcup \{ \rho(\alpha \approx \beta) : \alpha \approx \beta \in \Theta \} \quad \forall \Theta \subseteq Fm \times Fm$$

and for $K = \text{Alg}^* \mathcal{L}$, $\rho$ is a faithful interpretation of $\models_K$ into $\vdash \mathcal{L}$, i.e., the dual of (1') holds:

$$\Theta \models_K \delta \approx \varepsilon \iff \rho(\Theta) \vdash \mathcal{L} \rho(\delta \approx \varepsilon)$$

- But there is more: $\tau$ and $\rho$ are mutually inverse …
The notion of an algebraizable logic

**Definition (BLOK and PIGOZZI, 1989)**

A logic \( \mathcal{L} \) is **algebraizable** when there exists a class of algebras \( \mathcal{K} \) and definable transformers \( \tau, \rho \) which are **mutually inverse faithful interpretations** of \( \vdash_{\mathcal{L}} \) into \( \models_{\mathcal{K}} \) and conversely, i.e., such that:

1. **(A1)** \( \Gamma \vdash_{\mathcal{L}} \varphi \iff \tau(\Gamma) \models_{\mathcal{K}} \tau(\varphi) \)
2. **(A2)** \( \Theta \models_{\mathcal{K}} \delta \approx \varepsilon \iff \rho(\Theta) \vdash_{\mathcal{L}} \rho(\delta \approx \varepsilon) \)
3. **(A3)** \( \varphi \vdash_{\mathcal{L}} \rho(\tau(\varphi)) \)
4. **(A4)** \( \delta \approx \varepsilon \models_{\mathcal{K}} \tau(\rho(\delta \approx \varepsilon)) \)

The class \( \mathcal{K} \) is called an **equivalent algebraic semantics** of \( \mathcal{L} \)

Actually, **(A1) + (A4) \iff (A2) + (A3)***
An equivalent algebraic semantics need not be unique. 
There is always the largest equivalent algebraic semantics: the algebraic counterpart of $\mathcal{L}$. 
For implicative logics, it coincides with $\text{Alg}^*\mathcal{L}$. 
If $\mathcal{L}$ is finitary and $\tau, \rho$ are finite then it coincides with $\mathcal{Q}(\mathcal{K})$. It is the only equivalent quasivariety. 
All implicative logics are algebraizable. 
The other logics treatable with “inessential changes”, are algebraizable. 
If $\mathcal{L}$ is algebraizable wrt $\mathcal{K}$ then any fragment of $\mathcal{L}$ having the connectives appearing in $E(x)$ and in $\Delta(x,y)$ is algebraizable, wrt \{subreducts of $\mathcal{K}$\}. 
If $\mathcal{L}$ is algebraizable wrt $\mathcal{K}$, then any extension of $\mathcal{L}$ is algebraizable, wrt a subclass of $\mathcal{K}$, and using the same $\tau, \rho$. 
If a finitary $\mathcal{L}$ is algebraizable wrt $\mathcal{K}$ with finite $\tau, \rho$, then any extension of $\mathcal{L}$ is algebraizable, wrt a subquasivariety of $\mathcal{K}$. 
If moreover $\mathcal{L}$ is algebraizable wrt a variety $\mathcal{K}$, then its axiomatic extensions are algebraizable and correspond to the subvarieties of $\mathcal{K}$. 
These correspondences are lattice isomorphisms.
Kinds of algebraizability

- Not exactly BLOK and PIGOZZI’s 1989 original definition.
- Extended by HERRMANN, CZELAKOWSKI, FONT and JANSANA.

**finitely algebraizable**: The transformer $\rho$ (i.e., the set $\Delta$) is finite.

**“BP-algebraizable”**: Finitary and finitely algebraizable.
(Then also $\tau$ (i.e., the set $E$) is finite).

**regularly algebraizable**: The truth set is unitary. Equivalently, condition (5a) holds, i.e., $\alpha, \beta \vdash_{L} \Delta(\alpha, \beta)$.

**strongly algebraizable**: The largest equivalent algebraic semantics $\text{Alg}^{*}L$ is a variety.

**weakly algebraizable**: Similar, weaker notion, where the equivalence formulas can contain parameters. Few proper examples; theoretical interest.
The algebraizable hierarchy

- Implicative
- Regularly finitely algebraizable
- Finitely algebraizable
- Algebraizable
- Regularly weakly algebraizable
- Weakly algebraizable
A syntactic intrinsic characterization

Theorem

A logic $\mathcal{L}$ is **algebraizable** if and only if

there are formulas $\Delta(x, y) \subseteq Fm$ and equations $E(x) \subseteq Fm \times Fm$ such that:

\begin{align*}
(3a) & \quad \vdash_{\mathcal{L}} \Delta(\alpha, \alpha) \\
(3b) & \quad \Delta(\alpha_1, \beta_1) \cup \cdots \cup \Delta(\alpha_n, \beta_n) \vdash_{\mathcal{L}} \Delta(\lambda \alpha_1 \ldots \alpha_n, \lambda \beta_1 \ldots \beta_n) \\
& \text{for each primitive connective } \lambda, \text{ of arity } n \\
(5b) & \quad \alpha, \Delta(\alpha, \beta) \vdash_{\mathcal{L}} \beta \\
(A3) & \quad \Delta(E(\alpha)) \not\vdash_{\mathcal{L}} \alpha
\end{align*}

Note: $(3a) + (3b) + (5b) \implies \left\{ \begin{array}{l}
(3c) \quad \Delta(\alpha, \beta) \vdash_{\mathcal{L}} \Delta(\beta, \alpha) \\
(3d) \quad \Delta(\alpha, \beta) \cup \Delta(\beta, \gamma) \vdash_{\mathcal{L}} \Delta(\alpha, \gamma)
\end{array} \right.$
Exploiting algebraizability: “bridge” theorems

If $\mathcal{L}$ is strongly BP-algebraizable, then:

$\mathcal{L}$ has property $P \iff \mathcal{K}$ has property $P'$

Examples

- finite axiomatizability $\iff$ finite presentation
- decidability $\iff$ decidability
- the deduction theorem $\iff$ having EDPC
- the local deduction theorem $\iff$ congruence extension property
- Craig’s interpolation theorem $\iff$ amalgamation
- Beth’s definability theorem $\iff$ epimorphisms are surjective
- regularity rule (5a) $\iff$ congruence-regularity
Importance of the notion of algebraizability

- It is a precise, mathematical formulation of a vague, informal idea.
- It has originated a rich and deep mathematical theory.
- It can be equivalently characterized from different points of view:
  - Syntactic (properties of $\vdash_L$, perhaps using $\Delta$ and $E$).
  - Semantic (properties of $K$, of the congruences of algebras in $K$, etc.)
  - Order-theoretic (properties of the mapping $T \mapsto \Omega T$ on the theories of $L$)
- Allows to show that a given logic $L$ is not algebraizable (wrt any class of algebras and any conceivable transformers).
- Allows to show that a given class $K$ of algebras is not “logifiable” (it cannot be the equivalent algebraic semantics of any algebraizable logic).
Finally,

What about **non-algebraizable** logics?

To be continued …