Abstract Algebraic Logic
An overview (II)

Josep Maria Font

Department of Probability, Logic and Statistics
University of Barcelona

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Matrix semantics

A logical matrix is $\langle A, F \rangle$ with $F \subseteq A$. $F$ is the filter/truth set of the matrix.

Want a class $M$ of matrices complete for $\mathcal{L}$; i.e., such that

$$\Gamma \vDash_{\mathcal{L}} \varphi \iff \forall \langle A, F \rangle \in M, \forall h \in \text{Hom}(Fm, A),$$

$$\text{if } h\gamma \in F \forall \gamma \in \Gamma \text{ then } h\varphi \in F.$$

Definition

$\langle A, F \rangle$ is a model of $\mathcal{L}$ when it satisfies ($\Rightarrow$) in (1), i.e.,

$$\Gamma \vDash_{\mathcal{L}} \varphi \implies \forall h \in \text{Hom}(Fm, A),$$

$$\text{if } h\gamma \in F \forall \gamma \in \Gamma \text{ then } h\varphi \in F.$$

$F$ is a filter of $\mathcal{L}$ when $\langle A, F \rangle$ is a model of $\mathcal{L}$.

$\text{Mod}_{\mathcal{L}} = \{\text{models of } \mathcal{L}\}$.

For each $A$, $\mathcal{F}_{i\mathcal{L}} A = \{F \subseteq A : F \text{ is a filter of } \mathcal{L} \text{ over } A\}$. 

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Matrix semantics

**Theorem (LINDENBAUM, WÓJCICKI)**

For every logic \( \mathcal{L} \), the class \( \text{Mod}\mathcal{L} \) is a complete matrix semantics for \( \mathcal{L} \).

**Proof:** LINDENBAUM models \( \{\langle Fm, \Gamma \rangle : \Gamma \text{ a theory of } \mathcal{L} \} \subseteq \text{Mod}\mathcal{L} \).

- More meaningful solutions: Reduced matrices

\[ \forall \langle A, F \rangle \exists \Omega^A F := \max \{ \theta \in \text{Co}A : a \in F, \ a \equiv b (\theta) \Rightarrow b \in F \} \]

the **Leibniz congruence** of \( \langle A, F \rangle \)

(it is an **intrinsic**, algebraic property of \( A \) and \( F \))

**Reduced models:** \( \text{Mod}^*\mathcal{L} = \{ \langle A, F \rangle \in \text{Mod}\mathcal{L} : \Omega^A F = \text{Id}_A \} \)

**\( \mathcal{L} \)-algebras:** \( \text{Alg}^*\mathcal{L} = \{ A : \exists F \subseteq A \text{ with } \langle A, F \rangle \in \text{Mod}^*\mathcal{L} \} \)

\[ \langle A, F \rangle \in \text{Mod}\mathcal{L} \quad \mapsto \quad \langle A / \Omega^A F, F / \Omega^A F \rangle \in \text{Mod}^*\mathcal{L} \]

\[ A / \Omega^A F \in \text{Alg}^*\mathcal{L} \]
The LINDENBAUM-TARSKI process, generalized

Assume $\Gamma \not\models_L \varphi$.

Let $\Gamma'$ be the $\mathcal{L}$-theory generated by $\Gamma$. Then $\langle Fm, \Gamma' \rangle \in \text{Mod}_L$.

(2) Take $\Omega \Gamma'$.

We know that:

(3) $\Omega \Gamma'$ is a congruence of the formula algebra $Fm$.

(4) The quotient matrix $\langle Fm/\Omega \Gamma', \Gamma'/\Omega \Gamma' \rangle \in \text{Mod}^*L$.

(5) $\alpha \in \Gamma' \iff \alpha/\Omega \Gamma' \in \Gamma'/\Omega \Gamma'$, for all $\alpha \in Fm$.

Take $A := Fm/\Omega \Gamma'$, $F := \Gamma'/\Omega \Gamma'$, $h\alpha := \alpha/\Omega \Gamma'$ $\forall \alpha \in Fm$.

This shows:

**Theorem (WÓJCICKI)**

*For every logic $\mathcal{L}$, the class $\text{Mod}^*L$ is a complete matrix semantics for $\mathcal{L}$.***
The Leibniz operator

- On each algebra \( A \), we could consider \( \Omega^A : P(A) \rightarrow \text{Co}A \)
  \[ F \mapsto \Omega^A F \]

Let \( K \) be a class of algebras and let \( A \) be any algebra.

We consider the **relative congruences** \( \text{Co}_K A := \{ \theta \in \text{Co}A : A/\theta \in K \} \).

If \( F \in \mathcal{F}_L A \), then \( A/\Omega^A F \in \text{Alg}^*L \), i.e., \( \Omega^A F \in \text{Co}_{\text{Alg}^*L} A \).

- So we restrict \( \Omega^A : \mathcal{F}_L A \rightarrow \text{Co}_{\text{Alg}^*L} A \) (always surjective)
  \[ F \mapsto \Omega^A F \]

The behaviour of this operator on the filters of a logic \( L \) determines the “algebraic behaviour” of \( L \).
Theorem (main intrinsic characterization)

A logic \( \mathcal{L} \) is \emph{algebraizable} if and only if

the Leibniz operator on the formula algebra \( \Omega : \text{Th}\mathcal{L} \rightarrow \text{Co}_{\text{Alg}}^{\ast \mathcal{L}} \mathcal{Fm} \)

is monotone, injective and commutes with inverse substitutions.

(commutes with inverse substitutions: \( \Omega \sigma^{-1} \Gamma = \sigma^{-1} \Omega \Gamma \) for all \( \Gamma, \sigma \).)

(monotone + injective \( \implies \Omega : \text{Th}\mathcal{L} \cong \text{Co}_{\text{Alg}}^{\ast \mathcal{L}} \mathcal{Fm} \))

Theorem (the isomorphism theorem)

Let \( \mathcal{L} \) be a finitary logic, and let \( \mathcal{K} \) be a quasivariety.

Then \( \mathcal{L} \) is \emph{finitely algebraizable} and \( \mathcal{K} \) is its equivalent algebraic semantics

if and only if

for every algebra \( A \), \( \Omega^A : \mathcal{F}_{i\mathcal{L}}A \cong \text{Co}_{\mathcal{K}}A \).
Algebraizability is an **intrinsic** property of a logic.

Helps to show that a certain $\mathcal{L}$ is algebraizable, once we empirically know $\mathcal{K}$, and to confirm that $\mathcal{K}$ is the equivalent algebraic semantics.

- Anderson and Belnap’s relevance logic $\mathcal{R}$.
- Łukasiewicz’s $\mathcal{L}_\infty$ ··· Wajsberg algebras (MV-algebras).

Helps to show that a logic is **not** algebraizable, wrt to any $\mathcal{K}, E(x), \Delta(x, y)$.

- $\mathcal{IPC}^*$ ( intuitionistic logic without implication).
- Relevance implication $\mathcal{R}_\rightarrow$.
- Da Costa’s paraconsistent logic $\mathcal{C}_1$.
- The local consequences associated with normal modal logics.

Helps to show that some $\mathcal{K}$ is **not** the equivalent algebraic semantics of any algebraizable logic. (“$\mathcal{K}$ is not logifiable.”)

- \{distributive lattices\} and \{De Morgan algebras\} are not “logifiable”.

If a (quasi)-variety $\mathcal{K}$ satisfies some kind of **isomorphism theorems**, maybe there is some algebraizable $\mathcal{L}$ whose equivalent algebraic semantics is $\mathcal{K}$. 
**Applications (II)**

- $\mathcal{K}$ is an equivalent algebraic semantics of some algebraizable logic $\mathcal{L}$ if and only if there are $E(x) \subseteq Fm \times Fm$ and $\Delta(x, y) \subseteq Fm$ such that:

$$
\begin{align*}
\mathcal{K} &\models E(\Delta(x, x)) \\
E(\Delta(x, y)) &\models_\mathcal{K} x \approx y
\end{align*}
$$

(A4)

Actually, $\mathcal{L}$ is determined by the completeness condition (1), i.e., (A1):

$$
\Gamma \vdash_\mathcal{L} \varphi \iff \forall A \in \mathcal{K}, \forall \bar{a} \in \text{Hom}(Fm, A), \text{ if } A \models E(\gamma) \models_\mathcal{A} \forall \gamma \in \Gamma \text{ then } A \models E(\varphi) \models_\mathcal{A}.
$$

- If $\mathcal{K}$ is the (quasi-)variety generated by a single algebra, then it is enough to check (A4) for this algebra.
  - MOISIL’s “determination principle” in 3-valued Łukasiewicz algebras: If $a \to b = b \to a = \neg a \to \neg b = \neg b \to \neg a = 1$ then $a = b$.
  - SETTE's “maximal paraconsistent logic” $\mathcal{P}1$ is algebraizable, with $\Delta(x,y) = \{ x \to y, y \to x, \neg x \to \neg y, \neg y \to \neg x \}$ and $E(x) = \{ \top \to x \approx \top \}$. 

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(a part of) the Leibniz (or protoalgebraic) hierarchy

protoalgebraic

weakly algebraizable

regularly weakly algebraizable

implicatively

finitely algebraizable

regularly finitely algebraizable

finitely equivalent

algebraizable

equivalent

implicatively

weakly algebraizable

protoalgebraic
Syntactic characterizations

- **\( \mathcal{L} \) is protoalgebraic** when \( \exists \Delta(x, y) \subseteq Fm \) such that
  \[
  (3a) \quad \vdash \mathcal{L} \Delta(\alpha, \alpha)
  
  (5b) \quad \alpha, \Delta(\alpha, \beta) \vdash \mathcal{L} \beta
  \]
  (e.g., all logics with \( \vdash \mathcal{L} \alpha \rightarrow \alpha \) and \( \alpha, \alpha \rightarrow \beta \vdash \mathcal{L} \beta \))
  (there is only one protoalgebraic logic without theorems)

- **\( \mathcal{L} \) is equivalential** when \( \exists \Delta(x, y) \subseteq Fm \) satisfying (3a), (3b) and (5b).

  \[
  (3b) \quad \Delta(\alpha_1, \beta_1) \cup \cdots \cup \Delta(\alpha_n, \beta_n) \vdash \mathcal{L} \Delta(\lambda \alpha_1 \ldots \alpha_n, \lambda \beta_1 \ldots \beta_n)
  \]
  for each primitive connective \( \lambda \), of arity \( n \).

- **\( \mathcal{L} \) is algebraizable** if and only if \( \exists \Delta(x, y) \subseteq Fm \) and \( \exists E(x) \subseteq Fm \times Fm \) such that conditions (3a), (3b), (5b) and (A3) are satisfied.

  \[
  (A3) \quad \Delta(E(\alpha)) \vdash \vdash \mathcal{L} \alpha
  \]

- **\( \mathcal{L} \) is regularly algebraizable** when \( \exists \Delta(x, y) \subseteq Fm \) such that conditions (3a), (3b), (5b) and (5a) are satisfied.

  \[
  (5a) \quad \alpha, \beta \vdash \mathcal{L} \Delta(\alpha, \beta)
  \]
Definability characterizations

- \( \mathcal{L} \) is **equivalential** when \( \exists \Delta(x,y) \subseteq Fm \) such that for every theory \( \Gamma \) of \( \mathcal{L} \), \( \Omega \Gamma \) is **definable** by \( \Delta(x,y) \), i.e., for every \( \alpha, \beta \in Fm \),

\[
(2) \quad \alpha \equiv \beta \left( \Omega \Gamma \right) \iff \Delta(\alpha, \beta) \subseteq \Gamma.
\]

or, equivalently,

\[
\forall \langle A, F \rangle \in \text{Mod}\mathcal{L}, \forall a, b \in A, \ a \equiv b \left( \Omega^A F \right) \iff \Delta^A(a,b) \subseteq F.
\]

- \( \mathcal{L} \) is **protoalgebraic** when \( \exists \Delta(x,y,\vec{z}) \subseteq Fm \) with parameters such that for every theory \( \Gamma \) of \( \mathcal{L} \) and every \( \alpha, \beta \in Fm \),

\[
(2') \quad \alpha \equiv \beta \left( \Omega \Gamma \right) \iff \Delta(\alpha, \beta, \vec{\gamma}) \subseteq \Gamma \ \text{for all} \ \vec{\gamma} \in Fm.
\]

or, equivalently,

\[
\forall \langle A, F \rangle \in \text{Mod}\mathcal{L}, \forall a, b \in A, \ a \equiv b \left( \Omega^A F \right) \iff \Delta^A(a,b,\vec{c}) \subseteq F \ \forall \vec{c} \in A.
\]

- \( \mathcal{L} \) is **weakly algebraizable** when it is protoalgebraic and \( \exists E(x) \subseteq Fm \times Fm \) such that the \( \mathcal{L} \)-filters in reduced models of \( \mathcal{L} \) are definable by \( E(x) \), i.e.,

if \( \langle A, F \rangle \in \text{Mod}^*\mathcal{L} \) then \( F = \{ a \in A : A \models E(x) \langle a \rangle \} \).
Lattice-theoretical characterizations

Conditions on $\Omega$: $\text{Th}\mathcal{L} \rightarrow \text{Co}_{\text{Alg}^*\mathcal{L}} Fm$,
(or equivalently on $\Omega^A$: $\text{Fi}_\mathcal{L} A \rightarrow \text{Co}_{\text{Alg}^*\mathcal{L}} A$ for arbitrary $A$)

<table>
<thead>
<tr>
<th>$\mathcal{L}$ is ...</th>
<th>iff</th>
<th>$\Omega$ is ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>protoalgebraic</td>
<td></td>
<td>monotone</td>
</tr>
<tr>
<td>equivalential</td>
<td></td>
<td>monotone and commutes with inverse substitutions</td>
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<tr>
<td>finitely equivalential</td>
<td></td>
<td>continuous</td>
</tr>
<tr>
<td>weakly algebraizable</td>
<td></td>
<td>an isomorphism (monotone and injective)</td>
</tr>
<tr>
<td>algebraizable</td>
<td></td>
<td>an isomorphism and commutes with inverse substitutions</td>
</tr>
<tr>
<td>finitely algebraizable</td>
<td></td>
<td>a continuous isomorphism</td>
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</tbody>
</table>
Model-theoretic characterizations

\[ \mathcal{L} \text{ is protoalgebraic } \iff \text{Mod}^*\mathcal{L} \text{ is closed under } \mathbb{P}_{SD}. \]

\[ \mathcal{L} \text{ is equivalential } \iff \text{Mod}^*\mathcal{L} \text{ is closed under } \mathbb{S} \text{ and } \mathbb{P}. \]

\[ \mathcal{L} \text{ is finitely equivalential } \iff \text{Mod}^*\mathcal{L} \text{ is closed under } \mathbb{S}, \mathbb{P} \text{ and } \mathbb{P}_U, \]
\hspace{1cm} \text{i.e., it is a quasivariety of matrices.} 

\[ \mathcal{L} \text{ is weakly algebraizable } \iff \text{Alg}^*\mathcal{L} \text{ is closed under } \mathbb{P}_{SD} \text{ and in } \text{Mod}^*\mathcal{L} \]
\hspace{1cm} \text{the filters are equationally definable.} 

\[ \mathcal{L} \text{ is algebraizable } \iff \text{Alg}^*\mathcal{L} \text{ is closed under } \mathbb{S} \text{ and } \mathbb{P} \text{ and in } \text{Mod}^*\mathcal{L} \text{ the filters are equationally definable.} \]
Protoalgebraic but neither equivalential nor weakly algebraizible

- The logic in the language $\langle \rightarrow \rangle$ axiomatized by $\alpha \rightarrow \alpha$ and Modus Ponens.

- The logic in the language $\langle \rightarrow \rangle$ defined by the Gentzen calculus with all structural rules and:

$$\frac{\alpha, \beta \triangleright \gamma}{\alpha \triangleright \beta \rightarrow \gamma} \quad (\text{DT1})$$

$$\frac{\Gamma \triangleright \alpha, \Gamma, \beta \triangleright \gamma}{\Gamma, \alpha \rightarrow \beta \triangleright \gamma} \quad (\text{MP})$$

- D\textsc{a Cost}a’s paraconsistent logic $C1$.

- The logics defined from the classical modal systems $E$ and $RE$ with Modus Ponens as the only rule of inference.
Classification of some protoalgebraic logics

Equivalential but non-algebraizable

- The logics defined from the *normal* modal systems $K$ and $T$ with Modus Ponens as only rule; i.e., the local consequences defined by the classes of all Kripke frames and of all reflexive Kripke frames. (Not finitely equivalential)

- PRATT’s dynamic logics. (Not finitely equivalential in general)

- The logics defined from the *normal* modal systems $S4$ and $S5$ with Modus Ponens as only rule; i.e., the local consequences defined by the classes of all reflexive and transitive Kripke frames and of all equivalence relations as Kripke frames. (Finitely equivalential)

- The implicational logics $BCI$, $E\rightarrow$ (implicational entailment) and $R\rightarrow$ (implicational relevance logic). (Finitely equivalential)
Classification of some protoalgebraic logics

Weakly algebraizable but non-equivalential

- The logic defined by $\{\langle A, \{1\} \rangle : A \text{ is an ortholattice} \}$.  
  (Regularly)

- ERNST’s logic, in $\langle \leftrightarrow \rangle$, axiomatized by $\alpha \leftrightarrow \alpha$, Modus Ponens for $\leftrightarrow$, and the infinite sets of rules:

  $\frac{\alpha}{\varphi(\alpha) \leftrightarrow \varphi(\alpha \leftrightarrow \alpha)}$  \quad  \forall \varphi  \quad  \frac{\alpha}{\varphi(\alpha \leftrightarrow \alpha) \leftrightarrow \varphi(\alpha)}$  \quad  \forall \varphi

  (Not regularly)
Classification of some protoalgebraic logics

Algebraizable but not finitely algebraizable

- **HERRMANN’s logic** $LJ$ (“Last Judgement”), in the full modal language, axiomatized by Modus Ponens as only rule, and the axioms:
  - All instances of theorems of $\mathcal{CPC}$.
  - $\Box^n \varphi$ for all theorems of $\mathcal{IPC}$ and all $n \geq 0$.
  - $\Box^n (\Box(\alpha \to \beta) \to (\Box \alpha \to \Box \beta))$ for all $n \geq 0$.
  - $(\alpha \to \beta) \to \Box^n (\neg \beta \to \neg \alpha)$ for all $n \geq 0$.

  (Not regularly)

- **DELLUNDE’s logic**, in the language $\langle \Box, \leftrightarrow \rangle$, axiomatized by $\alpha \leftrightarrow \alpha$, Modus Ponens for $\leftrightarrow$, and all the rules:

\[
\frac{\alpha \quad \beta}{\Box^n \alpha \leftrightarrow \Box^n \beta} \quad \forall n \geq 0
\]
\[
\frac{\alpha_1 \leftrightarrow \beta_1 \quad \alpha_2 \leftrightarrow \beta_2}{\Box^n (\alpha_1 \leftrightarrow \alpha_2) \leftrightarrow \Box^n (\beta_1 \leftrightarrow \beta_2)} \quad \forall n \geq 0
\]

(Regularly)
Classification of some protoalgebraic logics

Finitely algebraizable but not regularly

- Relevance logic $\mathcal{R}$.
- Some linear logics, some fuzzy logics.
- Substructural logics (Full Lambek calculus and extensions without weakening).

Regularly finitely algebraizable but not implicative

- $\mathcal{CP}C\leftrightarrow$ and $\mathcal{IPC}\leftrightarrow$, the equivalence fragments of $\mathcal{CP}C$ and of $\mathcal{IPC}$, resp.
- $\mathcal{IPC}\leftrightarrow,\neg$, the fragment of $\mathcal{IPC}$ with equivalence and negation.
Some bridge theorems

For an arbitrary $\mathcal{L}$:

- $\mathcal{L}$ is finitary $\iff$ $\text{Mod}\mathcal{L}$ is closed under ultraproducts.

For a protoalgebraic $\mathcal{L}$:

- $\mathcal{L}$ has the DDT $\iff$ $\text{Mod}\mathcal{L}$ has definable principal $\mathcal{L}$-filters.
  $\iff$ $\text{Mod}^*\mathcal{L}$ has definable principal $\mathcal{L}$-filters.
  $\iff$ For each $A$, the join-semilattice of the finitely generated $\mathcal{L}$-filters is dually Brouwerian.

- $\mathcal{L}$ has the LDT $\iff$ $\text{Mod}\mathcal{L}$ has the “$\mathcal{L}$-filter extension property”.
  $\iff$ $\text{Mod}^*\mathcal{L}$ has the “$\mathcal{L}$-filter extension property”.

For equivalential $\mathcal{L}$:

- $\mathcal{L}$ has Craig interpolation $\iff$ $\text{Mod}\mathcal{L}$ has amalgamation.
  $\iff$ $\text{Mod}^*\mathcal{L}$ has amalgamation.
Finally,

What about non-protoalgebraic logics?

To be continued …