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GLS based unit root tests for bounded processes*

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Abstract

We show that the use of generalized least squares (GLS) detrending procedures leads to important empirical power gains compared to ordinary least squares (OLS) detrending method when testing the null hypothesis of unit root for bounded processes. The non-centrality parameter that is used in the GLS-detrending depends on the bounds, so that improvements on the statistical inference are to be expected if a case-specific parameter is used. This initial hypothesis is supported by the simulation experiment that has been conducted.

Keywords: Unit root, bounded process, quasi GLS-detrending

JEL codes: C12, C22

1 Introduction

There are some cases where standard unit root tests applied to bounded variables would not lead to good inference results. Cavaliere (2005) and Cavaliere and Xu (2012) address this issue adapting different unit root tests that consider the bounded nature that exhibits some time series. These authors show that the narrower the bounds, the higher the bias towards rejecting the null hypothesis of unit root when unit root tests that do not account for the bounds are used. To overcome this issue, Cavaliere and Xu (2012) propose the application of

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modified unit root tests à la Ng and Perron (2001) when analyzing the stochastic properties of bounded time series. Their proposal bases on the use of OLS-detrending procedure, although they also mention that GLS-detrending could also be used.

In this note we provide the elements that are needed to implement the GLS-detrending when testing the unit root hypothesis on bounded time series. We show that the non-centrality parameter that GLS-detrending requires depends on the values of the bounds that affect the time series, being smaller the narrower the range defined by the bounds. A small Monte Carlo simulation experiment is conducted to evaluate the performance of the OLS and GLS-detrending procedures and the potential improvement that should be expected if a bounds-specific non-centrality parameter is used instead of applying the non-centrality parameter that ignores the bounded nature of the stochastic process in Elliott et al. (1996).

2 The model

Let x_t be a stochastic process with data generating process (DGP) given by:

$$x_t = \mu + y_t \quad (1)$$

$$y_t = \alpha y_{t-1} + u_t, \quad (2)$$

$t = 1, \ldots, T$, where $x_t \in \left[\underline{b}, \overline{b}\right]$ almost surely for all t and $y_0 = O_p(1)$. The presence of bounds requires that Δx_t lies within the interval $\left[\underline{b} - x_{t-1}, \overline{b} - x_{t-1}\right]$, where $\left[\underline{b}, \overline{b}\right]$ denote the boundaries that affect the time series. The disturbance term u_t is assumed to decompose as:

$$u_t = \varepsilon_t + \underline{\xi_t} - \overline{\xi_t}, \quad (3)$$

with $\varepsilon_t = C(L)\upsilon_t$, where $C(L) = \sum_{i=0}^{\infty} c_i L^i$ with $\sum_{i=0}^{\infty} i\,|c_i| < \infty$, and υ_t is a martingale difference sequence adapted to the filtration $F_t = \sigma - field\{\upsilon_{t-i}; i \geq 0\}$. The long-run variance (LRV) of ε_t is given by $\sigma^2 = \lim_{T\to\infty} E[T^{-1}(\sum_{t=1}^{T} \varepsilon_t)^2]$. The variables $\underline{\xi_t}$ and $\overline{\xi_t}$ are non-negative processes (regulators) such that $\underline{\xi_t} > 0$ if and only if $y_{t-1} + \varepsilon_t < \underline{b} - \mu$ and $\overline{\xi_t} > 0$ if and only if $y_{t-1} + \varepsilon_t > \overline{b} - \mu$. The stochastic processes involved in (3) satisfy the Assumptions A and B in Cavaliere and Xu (2012), so that $(\underline{b} - \mu) = \underline{c}\sigma T^{1/2}$ and $\left(\overline{b} - \mu\right) = \overline{c}\sigma T^{1/2}$, with $\underline{c} \leq 0 \leq \overline{c}$, $\underline{c} \neq \overline{c}$. This representation can be particularized to the cases of stochastic processes that are only limited below – i.e., $x_t \in [\underline{b}, \infty]$ – or only limited above – i.e., $x_t \in \left[-\infty, \overline{b}\right]$ – but also covers the case of unbounded processes – i.e., $x_t \in [-\infty, \infty]$.

The GLS-detrended unit root test statistics that we analyze in this paper are based on the use of the quasi-differenced variables $x_t^{\bar{\alpha}}$ and $z_t^{\bar{\alpha}}$, defined as $x_1^{\bar{\alpha}} = x_1$ and $z_1^{\bar{\alpha}} = z_1$, and

$$x_t^{\bar{\alpha}} = (1 - \bar{\alpha} L)\, x_t \quad \text{and} \quad z_t^{\bar{\alpha}} = (1 - \bar{\alpha} L)\, z_t,$$

for $t = 2, \ldots, T$, with z_t the vector collecting the deterministic regressors – $z_t = 1$ in our case. The autoregressive parameter that is used in the GLS-detrending is written in a local-to-unity representation:

$$\bar{\alpha} = 1 + \frac{\bar{\kappa}\,(\underline{c}, \bar{c})}{T}, \tag{4}$$

with the non-centrality parameter $\bar{\kappa}\,(\underline{c}, \bar{c})$ to be defined below. We can estimate μ in (1) minimizing the following objective function:

$$S^*\,(\mu, \bar{\alpha}) = \sum_{t=1}^{T} \left(x_t^{\bar{\alpha}} - \mu z_t^{\bar{\alpha}}\right)^2, \tag{5}$$

where the minimum of this objective function is denoted as $S\,(\bar{\alpha})$.

Let us specify the null hypothesis of bounded I(1) process (BI(1)) and the alternative hypothesis of bounded I(0) process (BI(0)) as:

$$\begin{cases} H_0: & \alpha = 1 & \equiv x_t \sim BI(1) \\ H_1: & \alpha = \bar{\alpha} < 1 & \equiv x_t \sim BI(0) \end{cases},$$

and, following Elliott et al. (1996), define the following feasible point optimal statistic to test the null hypothesis against the alternative hypothesis:

$$P_T = \left[S\,(\bar{\alpha}) - \bar{\alpha} S\,(1)\right] / s^2, \tag{6}$$

where s^2 denotes a consistent (parametric or non-parametric) estimate of σ^2 to be defined below. Note that the null hypothesis of BI(1) is obtained when $\kappa\,(\underline{c}, \bar{c}) = 0$ and the alternative hypothesis of BI(0) with $\kappa\,(\underline{c}, \bar{c}) < 0$. As stated in Remarks 3.4 and 3.5 in Cavaliere and Xu (2012), the computation of the modified unit root tests for bounded processes that they propose can be based on GLS-detrending, although in this case the non-centrality parameter $\kappa\,(\underline{c}, \bar{c})$ will depend on the bounds. In order to obtain the non-centrality parameter we need to derive the limiting distribution of the P_T test statistic.

Theorem 1 *Let $\{x_t\}_{t=1}^{T}$ be the stochastic process with DGP given in (1) and (2) with $\alpha = 1 + \kappa\,(\underline{c}, \bar{c})\,/T$ and s^2 be a consistent estimate of σ^2. Then, as $T \to \infty$*

$$P_T \Rightarrow \bar{\kappa}\,(\underline{c}, \bar{c})^2 \int_0^1 W^2_{\kappa(\underline{c},\bar{c})}\,(r)\,dr - \bar{\kappa}\,(\underline{c}, \bar{c})\, W^2_{\kappa(\underline{c},\bar{c})}\,(1)\,,$$

where $\Rightarrow$ denotes weak convergence in distribution and $W_{\kappa(\underline{c},\overline{c})}(r)$ is a regulated Ornstein-Uhlenbeck process.

The limiting distribution in Theorem 1 allows us to obtain the power envelope of the P_T test statistic. The power function helps to define the "optimal" non-centrality parameter $\kappa(\underline{c},\overline{c})$ that Elliott et al. (1996) recommended choosing as the value that yields a P_T test statistic with a 50% asymptotic power.

The value of $\kappa(\underline{c},\overline{c})$ for each case needs to be approximated by means of numerical simulation following the procedure described in Elliott et al. (1996). To be specific, the DGP that is defined to generate the bounded processes is given by (1) and (2) with $\mu = 0$, $y_0 = 0$, $\varepsilon_t \sim iid\ N(0,1)$ using 1,000 steps to approximate the limiting distribution of the P_T test statistic in Theorem 1. In the generation of the bounded processes we have implemented the algorithm detailed in Cavaliere (2005) with the pairs of bounds $(\underline{c},\overline{c})$ that can be obtained using all possible combinations of values of $-\underline{c}$ and $\overline{c}$ in the set $C = \{0.1, 0.15, 0.20, 0.25, 0.3, 0.35, 0.40, 0.45, 0.5, 0.55, 0.60, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 1, 1.05\}$ satisfying $\underline{c} \leq 0 \leq \overline{c}$, $\underline{c} \neq \overline{c}$. Note that we cover both the symmetric and asymmetric bounds cases. For each pair of values, the power envelope for the P_T test statistic is obtained using 10,000 replications. Finally, the $\bar{\kappa}(\underline{c},\overline{c})$ parameter is chosen so that the P_T test statistic has a 50% asymptotic power as mentioned above.

Figure 1 illustrates the dependence of $\bar{\kappa}(\underline{c},\overline{c})$ on the bounds for the symmetric case where $\underline{c} = -\overline{c}$. As can be seen, there is a range of values of the (symmetric) bound parameter for which the non-centrality parameter is far away from the value that is used for the unbounded stochastic processes – i.e., the $\bar{\kappa}(-\infty,\infty) = -7$ computed in Elliott et al. (1996), which is represented with the horizontal line in Figure 1. Therefore, we can conclude that the narrower the band defined by the bounds, the farthest is $\bar{\kappa}(\underline{c},\overline{c})$ from -7.

3 The modified unit root test statistics

The analysis that has been conducted evidences that some potential gains in terms of performance of the modified unit root tests – henceforth, M-type test statistics – are to be expected if they are computed using GLS-detrending methods with a bounds-specific non-centrality parameter, especially in those cases where the bounds define a narrow band of values for x_t.

The M-type test statistics are defined as:

$$MZ_{\alpha}^{GLS} = \frac{T^{-1}\hat{y}_T^2 - T^{-1}\hat{y}_0^2 - s^2}{2T^{-2}\sum_{t=1}^{T}\hat{y}_{t-1}^2} \tag{7}$$

$$MSB^{GLS} = \left(T^{-2}\sum_{t=1}^{T}\hat{y}_{t-1}^2/s^2\right)^{1/2} \tag{8}$$

$$MZ_t^{GLS} = MZ_{\alpha}^{GLS} * MSB^{GLS}, \tag{9}$$

where $\hat{y}_t = x_t - \hat{\mu}_{GLS} z_t$ with $\hat{\mu}_{GLS}$ the parameter that minimizes the objective function (5). As for the estimation of s^2, two different approaches are available in the literature. First, s^2 can be estimated using the non-parametric estimator (s_{NP}^2) given by:

$$s_{NP}^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 + 2T^{-1}\sum_{j=1}^{l} w(j,l) \sum_{t=j+1}^{T}\hat{u}_t\hat{u}_{t-j},$$

where $\hat{u}_t$ denotes the OLS residuals from regressing $\hat{y}_t$ against $\hat{y}_{t-1}$, and $w(j,l)$ indicates the spectral window – i.e., the Bartlett or quadratic spectral windows are popular choices. The second approach estimates σ^2 in a parametric way and bases on the OLS estimation of the following augmented Dickey-Fuller (ADF) regression equation:

$$\Delta\hat{y}_t = \beta_0\hat{y}_{t-1} + \sum_{j=1}^{k}\beta_j\Delta\hat{y}_{t-j} + e_t, \tag{10}$$

from which the parametric estimation of σ^2 is obtained:

$$s_{AR}^2 = \frac{\hat{\sigma}_e^2}{\left(1-\hat{\beta}(1)\right)^2},$$

with $\hat{\sigma}_e^2 = (T-k)\sum_{t=k+1}^{T}\hat{e}_t^2$ and $\hat{\beta}(1) = \sum_{j=1}^{k}\hat{\beta}_j$.

The limiting distribution of the M-type test statistics in (7)-(9) can be found in Cavaliere and Xu (2012), from which it can be seen that they depend on the bounds and the non-centrality parameter that is used in the GLS-detrending. It should be noticed that the computation of the M-type test statistics can also be done using OLS-detrending if we define $\hat{y}_t = x_t - \bar{x}$. In this case, the test statistics computed as in (7)-(9) but using the OLS detrended time series are denoted by MZ_{α}^{OLS}, MSB^{OLS} and MZ_t^{OLS}, and their limiting distribution is derived in Cavaliere and Xu (2012). In all these cases, the statistical inference is performed on the left tail of the distribution and the asymptotic critical values for these test statistics can be obtained by means of Monte Carlo simulation using the procedure described above for the P_T test statistic.

The empirical implementation of these statistics based on the GLS-detrending that relies on a bound-specific non-centrality parameter requires following an iterative estimation procedure. Following Cavaliere and Xu (2012), the estimation of $(\underline{c}, \overline{c})$ is achieved by $\widehat{\underline{c}} = s^{-1}T^{-1/2}\left(\underline{b} - \hat{y}_1\right)$ and $\widehat{\overline{c}} = s^{-1}T^{-1/2}\left(\overline{b} - \hat{y}_1\right)$, which requires an estimation of σ^2 that, in turn, is based on the detrended variable.[1] Consequently, to perform the GLS-detrending we need to estimate $\bar{\kappa}\left(\underline{c}, \overline{c}\right)$, which requires $(\widehat{\underline{c}}, \widehat{\overline{c}})$. To obtain $(\widehat{\underline{c}}, \widehat{\overline{c}})$ we need the GLS detrended variable, which requires $\bar{\kappa}\left(\underline{c}, \overline{c}\right)$. This estimation problem is solved in an iterative fashion. First, we obtain an initial consistent estimation of σ^2 using the OLS-detrending procedure. Second, using this educated estimation of σ^2, we compute $(\widehat{\underline{c}}, \widehat{\overline{c}})^1$ and obtain $\bar{\kappa}(\widehat{\underline{c}}, \widehat{\overline{c}})^1$, where the superscript indicates that it is the initial estimation. Third, we proceed to perform the GLS-detrending using $\bar{\kappa}(\widehat{\underline{c}}, \widehat{\overline{c}})^1$, then re-estimate σ^2 based on the GLS detrended data and obtain the final estimation of $(\underline{c}, \overline{c})$ and $\bar{\kappa}\left(\underline{c}, \overline{c}\right)$. These updated estimates are used to obtain the final GLS detrended data that is used to compute the unit root test statistics.

Finally, it is worth mentioning that a Matlab code that allows to implement the modified unit root test statistics with OLS and GLS-detrending and approximate the critical values is available upon request.

4 Monte Carlo simulations

We investigate the performance of the M-type test statistics using the DGP given by (1) and (2) with $\mu = 0$, $y_0 = 0$, $\varepsilon_t \sim iid\ N\left(0, 1\right)$ and α defined as a local-to-unit parameter by (4). The generation of bounded processes bases on the algorithm described in Cavaliere (2005) with symmetric bounds given by $\{-\underline{c}, \overline{c}\} \in C$, with C defined above. Under the null hypothesis of BI(1) $\kappa\left(\underline{c}, \overline{c}\right) = 0$, whereas the alternative hypothesis of BI(0) is defined using the $\kappa\left(\underline{c}, \overline{c}\right)$ values computed in the previous section:[2] (i) OLS-detrending as suggested in Cavaliere and Xu (2012), (ii) GLS-detrending proposed in Elliott et al. (1996) with $\kappa\left(-\infty, \infty\right) = -7$, i.e., ignoring the bounded nature of the stochastic processes (GLS-ERS), and (iii) GLS-detrending with the $\kappa\left(\underline{c}, \overline{c}\right)$ value obtained in this paper (GLS-BOUNDS). In all cases we estimate σ^2 that is required to obtain the estimated bounds $\widehat{\underline{c}} = s^{-1}T^{-1/2}\left(\underline{b} - \hat{y}_1\right)$

[1] It should be noticed that the constant is approximated by the initial condition, as suggested in Schmidt and Phillips (1992).

[2] Note that in this case the value of the empirical power should be close to 50% for the M-type test statistics that use $\kappa\left(\underline{c}, \overline{c}\right)$ when performing the GLS-detrending.

and $\widehat{\overline{c}} = s^{-1}T^{-1/2}\left(\overline{b} - \hat{y}_1\right)$ using both s^2_{NP} and s^2_{AR}.[3] The sample size is $T = 500$, the nominal size is set at the 5% level of significance and 10,000 replications are carried out using Matlab.

As for the critical values that are used in this section, we have simulated the empirical distribution of the M-type test statistics under the null hypothesis of BI(1) for the three detrending procedures – OLS, GLS-ERS and GLS-BOUNDS – using the DGP described in the previous paragraph for all possible combinations of symmetric bounds $\{-\underline{c}, \overline{c}\} \in C$ with $T = 500$ and 20,000 replications. From these empirical distributions we store the critical values corresponding to the 5% level of significance, which produces a look-up table of critical values for each test statistic and detrending method. As a result, we have six look-up tables of critical values for each test statistic, depending on the detrending procedure and the LRV estimate that is used. Note that when these critical values are obtained, the only set of parameters that is assumed to be known is $(-\underline{c}, \overline{c})$. When testing for the unit root hypothesis below, these parameters will be also estimated, which implies that $(\widehat{\underline{c}}, \widehat{\overline{c}})$ hardly ever will match one of the $(-\underline{c}, \overline{c})$ pairs that we have used to get the critical values. In order to address this issue, we have approximated the critical values for $(\widehat{\underline{c}}, \widehat{\overline{c}})$ interpolating with splines.[4]

Figure 2 presents the empirical size for the M-type test statistics. In general, we can observe that, regardless of the detrending procedure and LRV estimate that is used, the test statistics experience over-rejection problems for the smallest value of $\overline{c}$, being more important for the test statistics that use s^2_{AR} – provided that we are dealing with the symmetric bound case, we refer just to the $\overline{c}$ parameter to simplify the exposition.

It is worth mentioning that the use of OLS-detrending leads to less size distortion problems compared to GLS-detrending for small values of $\overline{c}$. However, mild size distortions persist for the OLS-detrending when $\overline{c} \in [0.5, 0.9]$ – for the statistics using s^2_{AR} – and $\overline{c} \in [0.5, 0.65]$ – for the statistics using s^2_{NP} – whereas the empirical size of the GLS-detrending test statistics are close to the nominal one for $\overline{c} \geq 0.5$ (using s^2_{AR}) and $\overline{c} \geq 0.3$ (using s^2_{NP}). Therefore and except for $\overline{c} < 0.3$, we can conclude that GLS-detrending outperforms OLS-detrending from an empirical size point of view. Finally and looking at the GLS-based results, we can observe that GLS-BOUNDS show less size distortions than GLS-ERS, although for large values of $\overline{c}$

[3] Throughout this section, the estimation of the long-run variance bases on the use of the quadratic spectral window, with the bandwidth l selected according to the proposal in Newey and West (1994) with $int\left[4(T/100)^{2/25}\right]$ initial lags. As for the parametric estimation of the long-run variance, the number of lags k is selected through the MAIC information criterion in Ng and Perron (2001) and Perron and Qu (2007) with the maximum number of lags set at $int\left[12(T/100)^{1/4}\right]$.

[4] It should be stressed that practitioners willing to apply the test statistics in this paper do not need to carry out this interpolation, provided that the Matlab procedure that is available upon request computes the critical values for the specific values of $(\widehat{\underline{c}}, \widehat{\overline{c}})$ that are estimated in a particular empirical application. The interpolation issue is just a matter of the Monte Carlo simulations in this section and aims to speed up the computation of the empirical size and power analyses.

both procedures lead to the same results – this is to be expected provided that in the limit $\kappa(\underline{c}, \overline{c})$ tends to -7 and both approaches are then equivalent.

Figure 3 reports the empirical power. As can be seen, in all cases GLS-detrending outperforms OLS-detrending regardless of the long-run variance estimation that is used. Consequently, GLS-detrending is clearly superior to OLS-detrending in terms of empirical power. This result, together with the conclusion of the empirical size analysis, leads us to suggest the use of GLS-detrending when testing the null hypothesis of a unit root for bounded stochastic processes.

If we focus on the GLS-detrending results, in general, GLS-ERS produces more powerful test statistics than GLS-BOUNDS detrending method for $\overline{c} \in [0.2, 0.4]$, although this is the mere consequence of the size distortions pointed out above. This also explains the higher power of the test statistics that use s^2_{AR} instead of s^2_{NP}. The GLS-BOUNDS based statistics show an empirical power that is close to the asymptotic expected value of 50%. Only for the cases where $\overline{c} < 0.4$ the empirical power is below 50%, although the empirical power does not go below 40%.

To sum up, the simulation experiment that we have conducted indicates that GLS-detrending procedures render test statistics with better empirical size and power performance. The use of a specific non-centrality parameter to carry out the GLS-detrending gives test statistics with an empirical size close to the nominal size and with good power. Finally, the GLS-BOUNDS test statistics based on the use of s^2_{NP} outperform the ones that build upon s^2_{AR}.

5 Conclusions

We have shown that correctly sized and more powerful statistics to test the null hypothesis of unit root for bounded time series can be obtained if GLS-detrending is used instead of OLS-detrending. The paper stresses the idea that the non-centrality parameter in which GLS-detrending builds upon depends on the bounds affecting the time series. The simulation experiment that has been conducted reveals that improvements on the statistical inference are achieved if bounds-specific non-centrality parameters are used when carrying out the GLS-detrending. Finally, the non-parametric long-run variance estimate based test statistics are preferable to the parametric ones, so that this version of the test statistics should be used in practice.

A Appendix

Lemma 1 *Let* $\{y_t\}_{t=1}^T$ *be the near-integrated process generated by (2). Then, we have (a)* $T^{-1/2}y_t \Rightarrow \sigma W_{\kappa(\underline{c},\bar{c})}(r)$, *(b)* $T^{-3/2}\sum_{t=1}^T y_t \Rightarrow \sigma\int_0^1 W_{\kappa(\underline{c},\bar{c})}(r)\,dr$, *(c)* $T^{-2}\sum_{t=1}^T y_t^2 \Rightarrow \sigma^2\int_0^1 W_{\kappa(\underline{c},\bar{c})}^2(r)\,dr$, *(d)* $T^{-1}\sum_{t=1}^T y_{t-1}u_t \Rightarrow \sigma^2\left[\int_0^1 W_{\kappa(\underline{c},\bar{c})}(r)\,dW(r)+\gamma\right]$ *with* $\gamma=(\sigma^2-\sigma_u^2)/(2\sigma^2)$ *and* $\sigma_u^2 = p\lim T^{-1}u'u$, *where* $W_{\kappa(\underline{c},\bar{c})}(r)$ *denotes a regulated Ornstein-Uhlenbeck process.*

Proof. *See Perron and Rodríguez (2003) and Cavaliere and Xu (2012).* ■

A.1 Proof of Theorem 1

Let us define the quadratic form:

$$M_T(\kappa(\underline{c},\bar{c}),\bar{\kappa}(\underline{c},\bar{c})) = (y^{\alpha\prime}z^{\alpha})(z^{\alpha\prime}z^{\alpha})^{-1}(z^{\alpha\prime}y^{\alpha}),$$

so that we have $S(\bar{\alpha}) = y^{\bar{\alpha}\prime}y^{\bar{\alpha}} - M_T(\kappa(\underline{c},\bar{c}),\bar{\kappa}(\underline{c},\bar{c}))$ and $S(1) = y^{1\prime}y^1 - M_T(\kappa(\underline{c},\bar{c}),0)$. Following Elliott et al. (1996) and Perron and Rodríguez (2003), we can express the P_T test statistic as:

$$\begin{aligned} s^2P_T &= (S(\bar{\alpha})-\bar{\alpha}S(1)) \\ &= \bar{\kappa}(\underline{c},\bar{c})T^{-2}\sum_{t=2}^T y_{t-1}^2 - \bar{\kappa}(\underline{c},\bar{c})T^{-1}y_T^2 + \bar{\kappa}(\underline{c},\bar{c})M_T(\kappa(\underline{c},\bar{c}),0) - M_T(\kappa(\underline{c},\bar{c}),\bar{\kappa}(\underline{c},\bar{c})). \end{aligned}$$

Note that $z^{\alpha} = (1,-\kappa(\underline{c},\bar{c})/T,-\kappa(\underline{c},\bar{c})/T,\ldots,-\kappa(\underline{c},\bar{c})/T)'$ so that in the limit as $T\to\infty$, $z^{\alpha\prime}z^{\alpha}\to 1$. Further and as shown in Perron and Rodríguez (2003), we have for $t=2,\ldots,T$

$$y_t^{\bar{\alpha}} = u_t + T^{-1}(\kappa(\underline{c},\bar{c})-\bar{\kappa}(\underline{c},\bar{c}))y_{t-1},$$

and $y_t^{\bar{\alpha}} = u_1$, so that $z^{\alpha\prime}y^{\alpha} = u_1 - \kappa(\underline{c},\bar{c})T^{-1}\sum_{t=2}^T[u_t + T^{-1}(\kappa(\underline{c},\bar{c})-\bar{\kappa}(\underline{c},\bar{c}))y_{t-1}] \Rightarrow u_1$. Consequently, $M_T(\kappa(\underline{c},\bar{c}),\bar{\kappa}(\underline{c},\bar{c})) \Rightarrow u_1$. Finally, $s^2\to\sigma^2$ as shown in Cavaliere (2005), for the non-parametric estimate of the long-run variance (s_{NP}^2), and in Cavaliere and Xu (2012), for the parametric estimate of the long-run variance (s_{AR}^2). Using these elements, we can see that the limiting distribution of the P_T test statistic is given by:

$$P_T \Rightarrow \bar{\kappa}(\underline{c},\bar{c})^2\int_0^1 W_{\kappa(\underline{c},\bar{c})}^2(r)\,dr - \bar{\kappa}(\underline{c},\bar{c})W_{\kappa(\underline{c},\bar{c})}^2(1).$$

References

[1] Cavaliere, G., 2005. Limited time series with a unit root. Econometric Theory 21, 907-945.

[2] Cavaliere, G., Xu, F., 2012. Testing for unit roots in bounded nonstationary time series. Journal of Econometrics, forthcoming.

[3] Elliott, G., Rothenberg, T.J., Stock, J.H., 1996. Efficient tests for an autoregressive unit root. Econometrica 64, 813-836.

[4] Newey, W.K., West, K.D., 1994. Automatic lag selection in covariance matrix estimation. Review of Economic Studies 61, 631-653.

[5] Ng, S., Perron,P., 2001. Lag length selection and the construction of unit root tests with good size and power. Econometrica 69, 1519-1554.

[6] Perron, P., Rodríguez, G., 2003. GLS-detrending, efficient unit root tests and structural change. Journal of Econometrics 115, 1-27.

[7] Perron, P., Qu, Z., 2007. A simple modification to improve the finite sample properties of Ng and Perron's unit root tests. Economics Letters 94, 12-19.

[8] Schmidt, R., Phillips, P.C.B., 1992. LM tests for a unit root in the presence of deterministic trends. Oxford Bulletin of Economics and Statistics 54, 257-287.

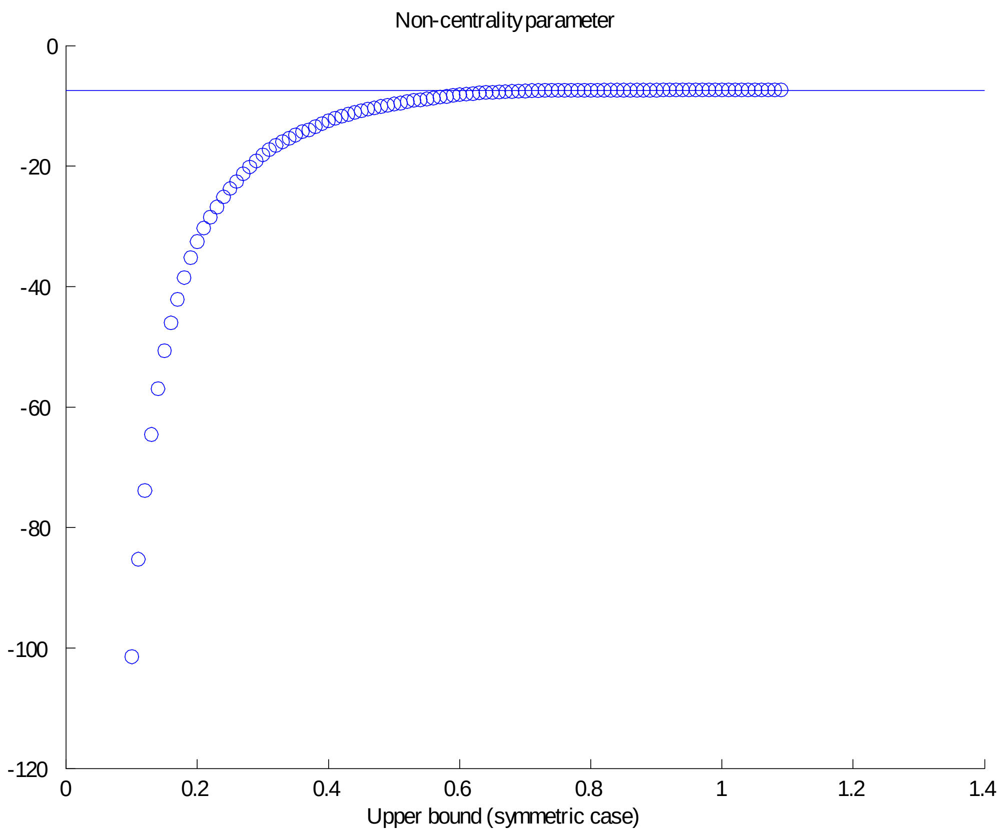

Figure 1: Non-centrality parameter for symmetric bounds

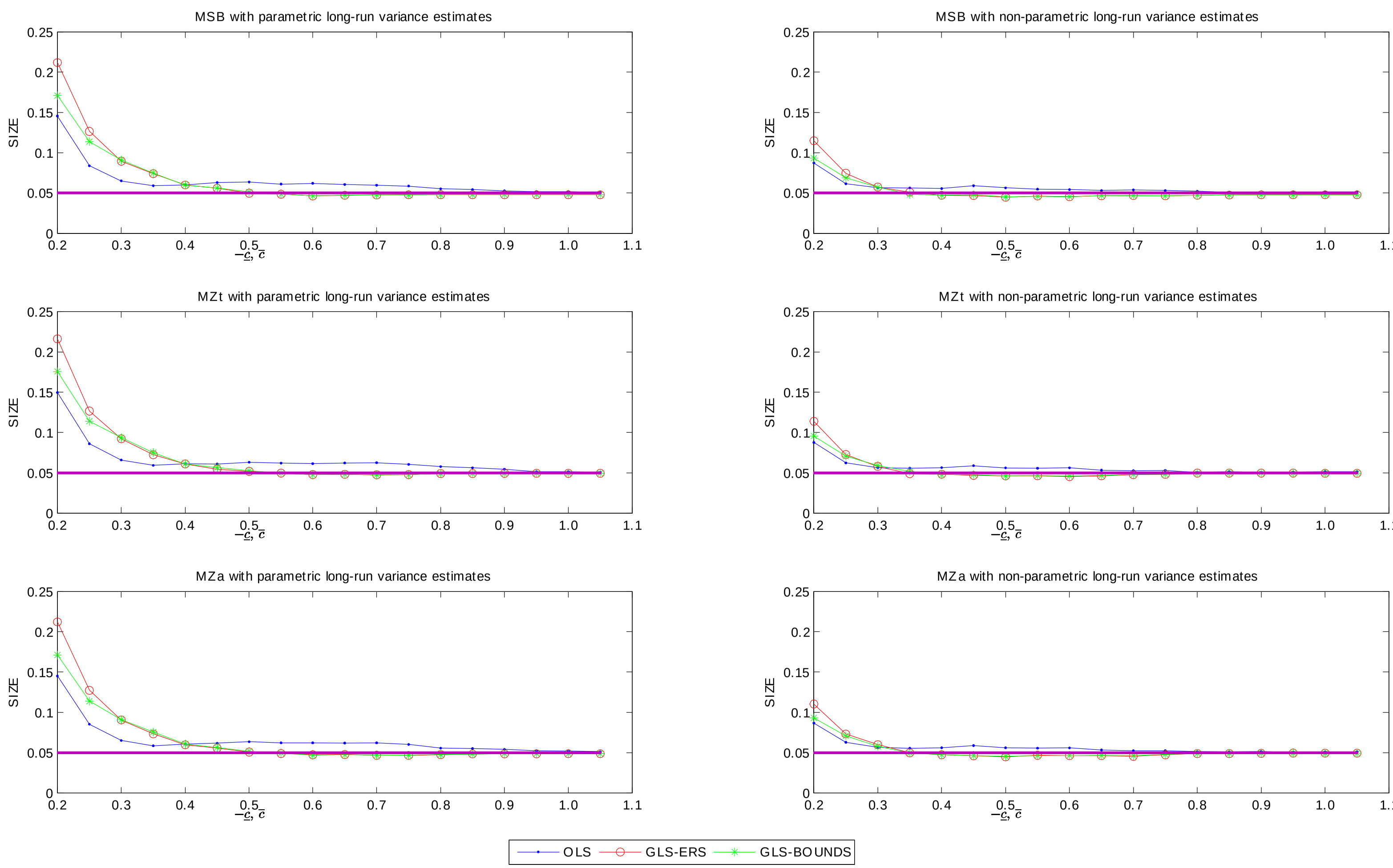

Figure 2: Empirical size for the M-tests using OLS and GLS detrending procedures

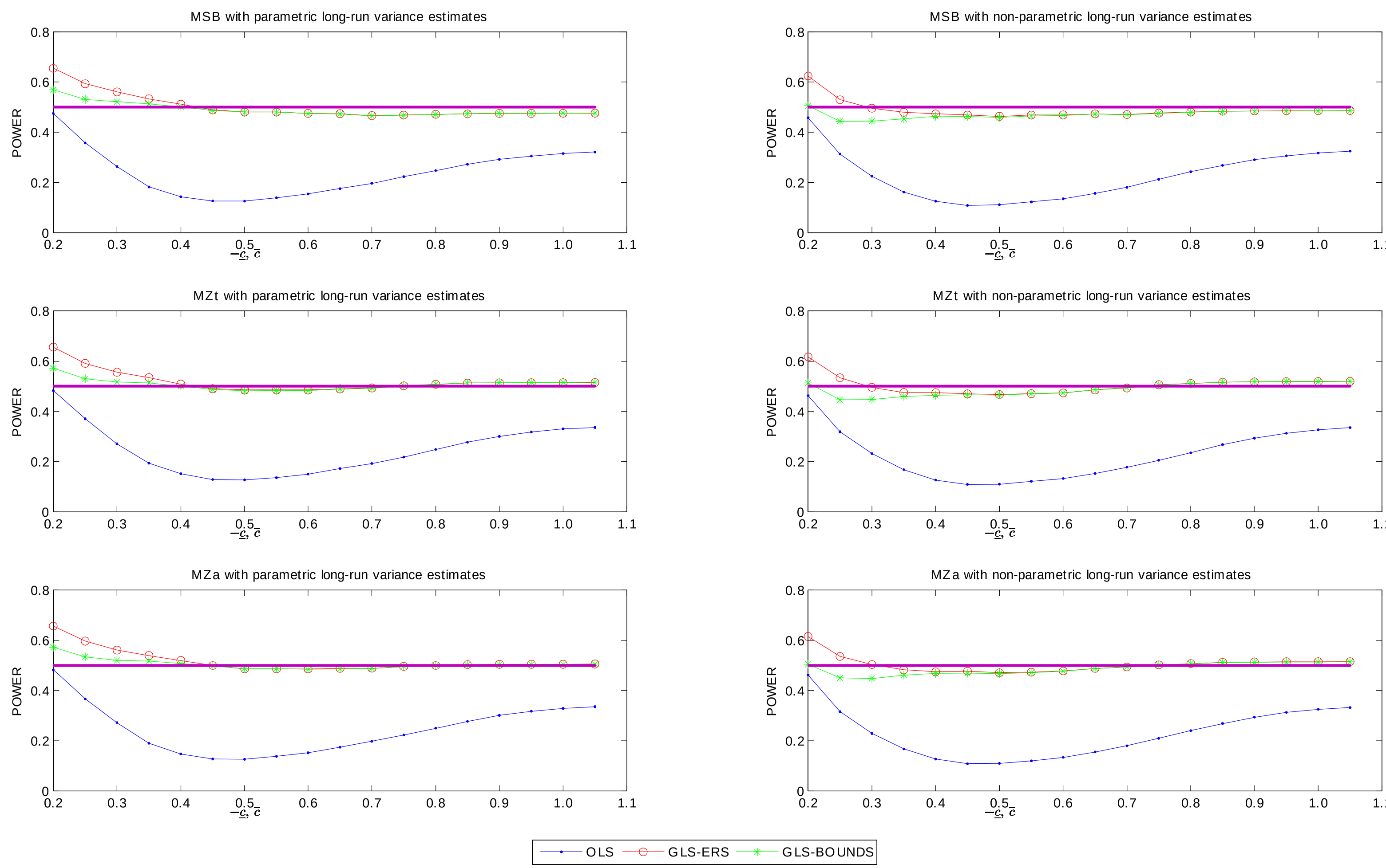

Figure 3: Empirical power for the M-tests using OLS and GLS detrending procedures

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