Institut de Recerca en Economia Aplicada Regional i Pública
Research Institute of Applied Economics

Document de Treball 2013/11 43 pàg.
Working Paper 2013/11 43 pag.

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Indicators for the characterization of discrete Choquet integrals

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Abstract

Ordered weighted averaging (OWA) operators and their extensions are powerful tools used in numerous decision-making problems. This class of operator belongs to a more general family of aggregation operators, understood as discrete Choquet integrals. Aggregation operators are usually characterized by indicators. In this article four indicators usually associated with the OWA operator are extended to discrete Choquet integrals: namely, the degree of balance, the divergence, the variance indicator and Rényi entropies. All of these indicators are considered from a local and a global perspective. Linearity of indicators for linear combinations of capacities is investigated and, to illustrate the application of results, indicators of the probabilistic ordered weighted averaging (POWA) operator are derived. Finally, an example is provided to show the application to a specific context.
Keywords: Orness, Divergence, Entropy, Choquet integral, OWA, POWA

1. Introduction

Aggregation operators are very useful tools for summarizing information and have been widely used in recent decades [1, 10, 31]. In this context, the Choquet integral [4], a class of integral linked to non-additive measures, has taken a leading role. Integrals are used to aggregate values of functions, and as such can be understood as aggregation operators. The Choquet integral includes a wide range of the aggregation operators as specific

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cases. Over the last few years, the Choquet integral has received much attention from researchers, and this has generated new extensions and generalizations of this class of integral. For instance, Greco et al. [12] proposed an extension of Choquet integrals in which the capacities depend on the values to be aggregated. Similarly, Yager [39] presented new induced aggregation operators inspired by Choquet integrals and Xu [34] introduced some intuitionistic fuzzy aggregation functions also based on the Choquet integral. Klement et al. [13] presented a universal integral that covers the Choquet and the Sugeno integral for non-negative functions, while Torra and Narukawa [32] studied a generalization of the Choquet integral inspired by the Losonczi mean. Bolton et al. [3] connected the Choquet integral with distance metrics and, more recently, Torra and Narukawa [33] introduced an operator that generalizes the Choquet integral and the Mahalanobis distance.

Two specific cases of Choquet integral are the weighted arithmetic mean (WAM) and the ordered weighted averaging (OWA) operator [35]. Several authors have turned their attention to the study of the OWA operator [41], since it serves to provide a parameterized family of aggregation operators between the minimum and the maximum. In order to assess OWA operators appropriately, indicators for characterizing the weighting vector are required. Initially, Yager [35] introduced the orness/andness indicators and the entropy of dispersion for just this purpose. Later, he propose complementary indicators, including the balance indicator [36] and the divergence [38], for use in exceptional situations. Meanwhile, Fullér and Majlender [8] suggested the use of a variance indicator and Majlender [16] introduced the Rényi entropy [26] as a generalization of the Shannon entropy [27]. Some of these indicators have been extended for the Choquet integral. For example, Marichal [18] and Grabisch et al. [10] presented several types of degree of orness indicators: the former author specifically for Choquet integrals, the latter for general aggregation functions. Likewise, Yager [37], Marichal [17] and Kojadinovic et al. [14] studied the entropy of dispersion in the framework of the Choquet integral. However, to the best of our knowledge, some of these indicators have yet to be defined at the Choquet aggregation level.

The aim of this article is to further enrich the present characterization of the Choquet integral, by incorporating new indicators to earlier contributions and by presenting an unified compilation of indicators for its characterization. Four indicators commonly used for the OWA operator -that is, the degree of balance, the divergence, the variance indicator and Rényi entropies- are extended to the discrete Choquet integral. The advantage of incorporating these additional indicators is that they can help to cover a wide range of situations, including exceptional types of aggregation that cannot be correctly characterized with the degree of orness or the entropy of dispersion. Two different perspectives are considered so as to allow both local and global indicators to be defined.

The linearity of indicators is investigated when dealing with linear combinations of capacities. Indicators are presented for characterizing the probabilistic OWA (POWA) operator [19, 20], which deals with a linear combination of two particular cases of Choquet integrals (the OWA and the WAM) in order to obtain more complex aggregations. The importance of these two aggregation operators is determined by the particular weight assigned to them in the linear combination.

Finally, an example is presented to show the application of our results in a specific context, namely a hypothetical customer online satisfaction assessment conducted using survey analysis and Choquet aggregation. The main advantage of using Choquet integrals is that a wide range of scenarios and attitudes can be considered and the one in closest accordance with our interests can then be selected. The example includes the estimation of indicators that characterize different Choquet integrals.

The rest of this paper is organized as follows. In section 2 some basic preliminaries regarding the OWA operator and the Choquet integral are briefly reviewed. In section 3 the main indicators for characterizing the OWA operator and the existing indicators for the Choquet integral are compiled. New indicators for the Choquet integral, the degree of balance, the divergence, the variance indicator and Rényi entropies are presented in section 4. A concise analysis of the linearity of indicators with respect to linear combinations of capacities is presented in section 5. In addition, the indicators inherited by the POWA operator, understood as a Choquet integral, are also provided in this section. In section 6 an illustrative example is given and in section 7 the main conclusions of the article are summarized.

2. OWA operators and Choquet integrals

2.1. OWA operators

Ordered weighted averaging (OWA) operators were first introduced by Yager [35]. Let $\vec{w} = (w_1, w_2, ..., w_n) \in [0,1]^n$ be such that $\sum_{i=1}^{n} w_i = 1$. The OWA operator with respect to $\vec{w}$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ defined by $\mathrm{OWA}_{\vec{w}}(x_1, x_2, ..., x_n) := \sum_{i=1}^{n} x_{\sigma(i)} \cdot w_i$, where σ is a permutation of $(1, 2, ..., n)$ such that $x_{\sigma(1)} \le x_{\sigma(2)} \le ... \le x_{\sigma(n)}$, i.e. $x_{\sigma(i)}$ is the i-th smallest value of $x_1, x_2, ..., x_n$.

OWA operators generalize the concept of the weighted arithmetic mean (WAM)[1] by requiring the components of $\vec{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ to be ordered before the aggregation is computed. For convenience, we consider the components of $\vec{x}$ in ascending as opposed to descending order. The OWA operator has been widely developed in the literature [41]. For example, Yager [40] proposed the use of generalized means to extend the OWA operator. A further interesting generalization of the OWA operator is the Quasi-OWA operator, in which quasi-arithmetic means[2] are used [7]. A Quasi-OWA operator is defined by $\text{Quasi-OWA}_{\vec{w}}(x_1, x_2, ..., x_n) := g^{-1}\left(\sum_{i=1}^{n} g\left(x_{\sigma(i)}\right) \cdot w_i\right)$, where $g : \mathbb{R} \to \mathbb{R}$ is a strictly continuous monotonic function.

2.2. Choquet integrals

In order to analyze the Choquet integral the concept of capacity must first be defined. Let $N = \{m_1, ..., m_n\}$ be a finite set and $2^N = \wp(N)$ be the set of all subsets of N. A capacity or a fuzzy measure on N is a mapping from 2^N to $[0,1]$ which satisfies that $\mu(\emptyset) = 0$ and that if $A \subseteq B$ then $\mu(A) \le \mu(B)$, for any $A, B \in 2^N$ (monotonicity).

[1] Note that the WAM with respect to $\vec{w}$ is an aggregation operator defined as $\mathrm{WAM}_{\vec{w}}(x_1, x_2, ..., x_n) := \sum_{i=1}^{n} x_i \cdot w_i$. It is an aggregation operator on $\mathbb{R}^n$.

[2] Merigó and Gil-Lafuente [22] presented similar generalizations when dealing with induced aggregation operators.

If μ is a capacity such that $\mu(N) = 1$, then we say that μ satisfies normalization. A capacity μ is additive if $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for any $A, B \subseteq N$. A capacity μ is symmetric if $\mu(A) = \mu(B)$ for all A, B with the same cardinality (i.e., $|A| = |B|$).

Let μ be a capacity on N, and $f : N \to [0, +\infty)$ be a function. Let σ be a permutation of $(1, ..., n)$, such that $f\left(m_{\sigma(1)}\right) \leq f\left(m_{\sigma(2)}\right) \leq ... \leq f\left(m_{\sigma(n)}\right)$ and $A_{\sigma,i} = \left\{m_{\sigma(i)}, ..., m_{\sigma(n)}\right\}$, with $A_{\sigma,n+1} = \emptyset$. The Choquet integral of f with respect to μ is defined as

$$\mathcal{C}_{\mu}(f) := \sum_{i=1}^{n} f\left(m_{\sigma(i)}\right)\left(\mu\left(A_{\sigma,i}\right) - \mu\left(A_{\sigma,i+1}\right)\right). \tag{2.1}$$

Extensions of the Choquet integral of real functions (and not just positive-real functions) can be found in the literature [25]. In this last reference, the asymmetric extension is formulated as in expression (2.1) but taking into account that the domain of f is $(-\infty, +\infty)$.

Choquet integrals can be generalized to obtain Choquet-like integrals [24, 13]. We consider particular Choquet-like integrals which, inspired by quasi-arithmetic means, are referred to here as Quasi-Choquet integrals[3]. These integrals are defined as follows. Given a strictly continuous monotonic function g from $\mathbb{R}$ to $\mathbb{R}$, the Quasi-Choquet integrals are defined by

$$\mathcal{QC}_{\mu}(f) := g^{-1}\left(\sum_{i=1}^{n} g\left(f\left(m_{\sigma(i)}\right)\right)\left(\mu\left(A_{\sigma,i}\right) - \mu\left(A_{\sigma,i+1}\right)\right)\right). \tag{2.2}$$

2.3. Relationship between OWA operators and Choquet integrals

Let $N = \{m_1, ..., m_n\}$ be a finite set and $\vec{w}$ and $\vec{p}$ be two vectors with components belonging to $[0, 1]$ and such that $\sum_{i=1}^{n} w_i = 1$ and $\sum_{i=1}^{n} p_i = 1$. Consider the aggregation operators $\mathrm{OWA}_{\vec{w}}$ and $\mathrm{WAM}_{\vec{p}}$ defined on N.

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[3] Alternative generalizations of the Choquet integral can be found in the literature. For instance, Yager [40] generalized Choquet integrals inspired by generalized means. Tan and Chen [28] extended Yager's approach by using induced generalized aggregation operators. Merigó and Casanovas [21] extended these models for environments with imprecise information that can be represented with interval numbers.

The representation of OWA and WAM operators as Choquet integrals has been shown in the literature [9, 11]. Propositions 10(v) and proposition 10(vi) in Grabisch et al. [11] imply that OWA and WAM operators can be understood as Choquet integrals with respect to normalized capacities μ and $\mathcal{P}$ respectively, $\text{OWA}_{\vec{w}} = \mathcal{C}_\mu$ and $\text{WAM}_{\vec{p}} = \mathcal{C}_{\mathcal{P}}$. These capacities are such that:

- $\mu(A) = \sum_{j=0}^{i-1} w_{n-j}$, for all $A \in N$ with cardinality i ($|A| = i$), $i = 1, ..., n$. Because of $|A_{\sigma,i}| = n-i+1$, then $\mu(A_{\sigma,i}) = \sum_{j=i}^{n} w_j$ for all $i = 1, ..., n$, being σ a permutation as in the definition of $\text{OWA}_{\vec{w}}$;
- $\mathcal{P}(\{m_i\}) = p_i$ for all $i = 1, ..., n$, being $\mathcal{P}$ additive. That is, the probability $\mathcal{P}$ understood as an additive capacity on N.

Remark 2.1. *The definitions of Quasi-OWA operators and Quasi-Choquet integrals considered in this section are such that neither the weights of Quasi-OWA operators nor the capacity of Quasi-Choquet integrals are affected by functions g or g^{-1}. Consequently, the relationship between Quasi-OWA and Quasi-WAM operators and Quasi-Choquet integrals can be established in a similar way as the relationship between OWA and WAM operators and Choquet integrals. This reflects the fact that these relationships are based only on the way in which the weights and capacities are linked.*

3. Indicators for aggregation operators

3.1. Indicators associated with OWA operators

Various indicators associated with OWA operators can be found in the literature and the main ones are briefly explained here. A summary of these indicators, their analytical expressions and references are shown in Table 3.1.

Degree of orness

The degree of orness of an $\text{OWA}_{\vec{w}}$ operator was defined in Yager [35] as representing the level of aggregation preference between the minimum and the maximum operators given by $\vec{w} \in [0, 1]^n$. The degree of orness of $\text{OWA}_{\vec{w}}$ can be understood as the value

Table 3.1: Summary of indicators associated with OWA operators

Indicator	Analytical expression	Reference
Degree of orness	$\boldsymbol{\omega}(\vec{w}) = \sum_{i=1}^{n} \left(\frac{i-1}{n-1}\right) \cdot w_i$	Yager [35]
Dispersion (Shannon entropy)[1]	$Disp(\vec{w}) = -\sum_{i=1}^{n} \ln(w_i) \cdot w_i$	Yager [35]
Degree of balance	$Bal(\vec{w}) = \sum_{i=1}^{n} \left(\frac{2i-(n+1)}{n-1}\right) \cdot w_i$	Yager [36]
Divergence	$Div(\vec{w}) = \sum_{i=1}^{n} \left(\frac{i-1}{n-1} - \boldsymbol{\omega}(\vec{w})\right)^2 \cdot w_i$	Yager [38]
Variance indicator	$D^2(\vec{w}) = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2}$	Fullér and Majlender [8]
Rényi entropy ($\alpha \neq 1$)	$H_\alpha(\vec{w}) = \frac{1}{1-\alpha}\log_2\left(\sum_{i=1}^{n} w_i^\alpha\right)$	Majlender [16]

[1] If the Shannon entropy of $\vec{w}$ is denoted by $H_S(\vec{w}) = -\sum_{i=1}^{n} \log_2(w_i) \cdot w_i$, then $Disp(\vec{w}) = \ln(2) \cdot H_S(\vec{w})$.

that the $\text{OWA}_{\vec{w}}$ operator returns when it is applied to $\vec{x^*} = \left(\frac{0}{n-1}, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, \frac{n-1}{n-1}\right)$ or, alternatively, as the value of $\text{WAM}_{\vec{w}}\left(\vec{x^*}\right)$.

Dispersion (Shannon entropy)

The dispersion indicator of an $\text{OWA}_{\vec{w}}$ operator was introduced by Yager [35] to measure the amount of information given by $\vec{x}$ that is used when $\text{OWA}_{\vec{w}}(\vec{x})$ is computed. This indicator provides the same information as the entropy introduced by Shannon [27] but at a different scale, as shown in Table 3.1.

Degree of balance

The concept of degree of balance of an $\text{OWA}_{\vec{w}}$ operator was introduced by Yager [36] and is closely related to the degree of orness, providing the same information but at a different scale. The degree of orness of an $\text{OWA}_{\vec{w}}$ operator is in the range $[0,1]$, while the degree of balance of the same operator is in the range $[-1,1]$. However, both indicators measure the degree to which the lower- or higher-valued elements are favored when weights $\vec{w}$ are applied. The degree of balance of an $\text{OWA}_{\vec{w}}$ can be understood as the value of the $\text{OWA}_{\vec{w}}$ operator applied to $\vec{y} = (y_1, y_2, ..., y_n)$, with $y_i = \frac{2i-(n+1)}{n-1}$ for all $i = 1, ..., n$. Note that the permutation $\sigma = id$ on $(1, ..., n)$ satisfies that $y_{\sigma(i)} \leq y_{\sigma(j)}$ if $i \leq j$, and thus $Bal(\vec{w}) = \text{OWA}_{\vec{w}}(\vec{y})$ and also $Bal(\vec{w}) = \text{WAM}_{\vec{w}}(\vec{y})$.

Divergence indicator

The divergence indicator of an $\text{OWA}_{\vec{w}}$ operator was introduced by Yager [38] and is understood to be the value of the $\text{WAM}_{\vec{w}}$ applied to $\vec{z} = (z_1, z_2, ..., z_n)$ where $z_i = \left(\frac{i-1}{n-1} - \boldsymbol{\omega}(\vec{w})\right)^2$ for all $i = 1, ..., n$. In general, $z_i \leq z_j$ does not hold if $i \leq j$. Therefore, divergence indicator $Div(\vec{w})$ cannot be expressed as $\text{OWA}_{\vec{w}}(\vec{z})$.

From a statistical viewpoint, if the random variable X^* is considered with $x_i^* = \frac{i-1}{n-1}$ and the probabilities $\mathcal{P}(X^* = x_i^*)$ are equal to w_i for all $i = 1, ..., n$, then $Div(\vec{w})$ is just the variance[4] of the random variable X^* with respect to the probabilities $\vec{p}$ when the latter are equal to the weights $\vec{w}$, i.e. $\vec{p} = \vec{w}$. In other words, we can understand the divergence indicator as $Div(\vec{w}) = Var_{\vec{w}}(X^*) = \mathbb{E}_{\vec{w}}\left[(X^*)^2\right] - (\mathbb{E}_{\vec{w}}[X^*])^2$.

The main advantage of the divergence indicator is that it complements the degree of orness indicator, especially in situations where the degree of orness and the dispersion indicator are insufficient for characterizing a weighting vector $\vec{w}$. As Yager [38] claimed when analyzing the OWA operator, such situations emerge for weighting vectors that provide the same results for the degree of orness and for the dispersion indicator. For example, let us consider two vectors in $\mathbb{R}^9$, $\vec{w} = (0, 0.5, 0, 0, 0, 0, 0, 0.5, 0)$ and $\vec{w^*} = (0, 0, 0, 0.5, 0, 0.5, 0, 0, 0)$. An analysis of the degree of orness and the dispersion of $\text{OWA}_{\vec{w}}$ and $\text{OWA}_{\vec{w^*}}$ provides the same results: $\boldsymbol{\omega}(\vec{w}) = \boldsymbol{\omega}\left(\vec{w^*}\right) = 0.5$ and $Disp(\vec{w}) = Disp\left(\vec{w^*}\right) = 0.693$. Thus, in order to distinguish between $\text{OWA}_{\vec{w}}$ and $\text{OWA}_{\vec{w^*}}$ operators, an additional measure is required. By using the divergence indicator (Table 3.1), such a distinction can be achieved. In this particular example, $Div(\vec{w}) = 0.140625$

[4]The variance of a random variable X with respect to a probability $\mathcal{P}$ is $Var_{\mathcal{P}}(X) := \mathbb{E}_{\mathcal{P}}\left[(X - \mathbb{E}_{\mathcal{P}}(X))^2\right]$, where $\mathbb{E}_{\mathcal{P}}(X)$ denotes the mathematical expectation of random variable X with respect to probability $\mathcal{P}$. In the discrete and finite case, $\mathbb{E}_{\mathcal{P}}(X) = \sum_{i=1}^{n} x_i \cdot p_i$ and $Var_{\mathcal{P}}(X) = \sum_{i=1}^{n}\left(x_i - \sum_{i=1}^{n} x_i \cdot p_i\right)^2 \cdot p_i$.

and $Div\left(\vec{w^*}\right) = 0.015625$. Thus, although $\text{OWA}_{\vec{w}}$ and $\text{OWA}_{\vec{w^*}}$ presents identical degrees of orness and dispersion, the latter has a lower divergence than the former.

Variance indicator

A further approach that might be adopted in analyzing the features of $\text{OWA}_{\vec{w}}$ operators is the computation of the variance of the weighting vector $\vec{w}$ where each component is considered equally probable. This indicator is defined as $D^2\left(\vec{w}\right) = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2}$ and, for instance, has been used in [8] to determine the analytical expression of a minimum variability $\text{OWA}_{\vec{w}}$ operator.

Rényi entropies

Entropy measures other than dispersion can be used to characterize the weighting vector. Generalizations of the Shannon entropy that could be used include Rényi entropies [16, 26]. Recall that the Rényi entropy of $\vec{w} \in \mathbb{R}^n$ with degree $\alpha \in \mathbb{R}\backslash\{1\}$ is defined as $H_\alpha\left(\vec{w}\right) = \frac{1}{1-\alpha}\log_2\left(\sum_{i=1}^{n} w_i^\alpha\right)$. Thus, given the $\text{OWA}_{\vec{w}}$, $H_\alpha\left(\vec{w}\right)$ can be considered as the Rényi entropy of degree α of this OWA operator.

3.2. Existing indicators extended to Choquet integrals

Some of the indicators described above have already been generalized for discrete Choquet integrals, the case for example of the degree of orness and the dispersion indicator (Shannon entropy). The purpose of this article is to propose indicators that have not yet been defined for the Choquet integral. However, we describe the existing indicators here in order to provide a complete compilation of indicators for characterizing Choquet integrals. Hereinafter, the indicators are considered from two perspectives -the global and the local. Broadly speaking, a global indicator does not depend on the input values to be aggregated while a local one does. This terminology is taken from Dujmović [5], who proposes a classification of orness indicators by means of a three-letter code[5] X/Y/Z,

[5]This terminology is also adopted by Kolesárová and Mesiar [15] who provide an elegant explanation of the two perspectives and, in addition, introduce a generalized characterization which they refer to as the 'mixed approach'.

where X ∈ { L,G,M }, Y ∈ { D,I,S } and Z ∈ { N,C } (see Table 3.2). We extend this categorization to all of the indicators. Indeed, here the categories G/D/N and L/D/N are seen as global and local, respectively, and denoted as G and L. Note that we are focused on the level of aggregation of points in the input space. The last two letters of the codes are common to both categories, which means that we only consider direct indicators that depend on the number of variables. This being the case, the categories can be determined solely by the first letter in the suggested classification.

Table 3.2: Codes for the classification of indicators

Letter	Type of indicator
L=	Local indicator that has a specific value in each point of the input space $\mathbb{R}^n$.
G=	Global indicator that has an aggregated value that characterizes GCD in all points of the input space $\mathbb{R}^n$.
M=	Mean value indicator obtained as the mean value of a local andness/orness indicator.
D=	Direct indicator obtained by processing directly the GCD function in all points of $\mathbb{R}^n$.
I=	Indirect indicator obtained from the related features of the GCD function (e.g. from the properties of the generator function of quasi-arithmetic means).
S=	Statistical indicator (e.g. various forms of distribution of local andness/orness inside $\mathbb{R}^n$).
N=	An indicator that is a function of the number of variables n.
C=	A constant indicator that is independent of n.

Source: Dujmović [5]. GCD stands for Generalized Conjunction/Disjunction function.
Interval $[0,1]^n$ has been substituted by $\mathbb{R}^n$.

Degree of orness for Choquet integrals

A generalization of the global degree of orness for Choquet integrals has been proposed by Marichal [18]. As shown in expression (2.1), if $\mathcal{C}_\mu$ is the Choquet integral with respect to μ, then

$$\boldsymbol{\omega}_G\left(\mathcal{C}_\mu\right) = \frac{1}{n-1}\sum_{i=1}^{n-1}\left[\binom{n}{i}^{-1}\cdot\sum_{\substack{A\subseteq N\\|A|=i}}\mu\left(A\right)\right]. \tag{3.1}$$

Likewise, a local degree of orness for a Choquet integral has been suggested by Belles-Sampera et al. [2], who propose the following local degree of orness,

$$\boldsymbol{\omega}_L\left(\mathcal{C}_\mu\right) = \sum_{i=1}^{n}\left(\frac{i-1}{n-1}\right)\cdot\left(\mu\left(A_{id,i}\right)-\mu\left(A_{id,i+1}\right)\right). \tag{3.2}$$

The idea underpinning this local generalization is to transfer to the Choquet integral the fact that the degree of orness of $\text{OWA}_{\vec{w}}$ can be understood as the value that the $\text{OWA}_{\vec{w}}$ operator returns when it is applied to $\vec{x^*} = \left(\frac{0}{n-1}, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, \frac{n-1}{n-1}\right)$ and, at the same time, as the value of $\text{WAM}_{\vec{w}}\left(\vec{x^*}\right)$. When considering a Choquet integral with respect to a normalized symmetric capacity μ (that is, when dealing with OWA operators), the local and global degrees of orness are equal, i.e. $\boldsymbol{\omega}_G\left(\mathcal{C}_\mu\right) = \boldsymbol{\omega}_L\left(\mathcal{C}_\mu\right)$. On the other hand, if μ is normalized and additive ($\mathcal{C}_\mu = \text{WAM}_{\vec{p}}$ with $p_i = \mu(\{m_i\})$ for all $i = 1, ..., n$), it is straightforward to prove that $\boldsymbol{\omega}_G\left(\mathcal{C}_\mu\right) \neq \boldsymbol{\omega}_L\left(\mathcal{C}_\mu\right)$. The difference derives from the fact that $\boldsymbol{\omega}_L\left(\mathcal{C}_\mu\right)$ only takes into account one of the $n!$ feasible permutations of $(1, 2, ..., n)$ - the identity permutation - while $\boldsymbol{\omega}_G\left(\mathcal{C}_\mu\right)$ considers them all. In order to simplify the notation, hereinafter, $\boldsymbol{\omega}_L(\mu)$ and $\boldsymbol{\omega}_G(\mu)$ will be used instead of $\boldsymbol{\omega}_L\left(\mathcal{C}_\mu\right)$ and $\boldsymbol{\omega}_G\left(\mathcal{C}_\mu\right)$, respectively.

Alternative generalizations of the degree of orness for the Choquet integral and other aggregation functions can be found in Grabisch et al. [10].

Dispersion (Shannon entropy) for Choquet integrals

The dispersion indicator (Shannon entropy) associated with the OWA operator has been analyzed and generalized in several studies [37, 6, 17, 14]. Unlike the degree of orness, the Shannon entropy is always a global indicator because the value of $Disp(\vec{w})$ is not modified if $w_{\sigma(i)}$ instead of w_i is used for all i (see Table 3.1). The analytical expression of the generalization proposed in Yager [37] is shown in Table 3.3.

4. New indicators extended to Choquet integrals

Generalizations of the degree of balance, the divergence, the variance indicator and Rényi entropies for Choquet integrals are proposed in this section. Each of these generalizations satisfies the following property: when the capacity μ linked to the Choquet integral $\mathcal{C}_\mu$ is symmetric and normalized (implying that a weighting vector $\vec{w}$ exists such that $\mathcal{C}_\mu = \text{OWA}_{\vec{w}}$), then the indicators for $\mathcal{C}_\mu$ coincide with the respective indicators for $\text{OWA}_{\vec{w}}$.

Table 3.3: Summary of existing indicators extended to Choquet integrals

Indicator	Analytical expression	Reference
Global degree of orness[1]	$\boldsymbol{\omega}_G(\mu) = \frac{1}{n-1}\sum_{i=1}^{n-1}\left[\binom{n}{i}^{-1} \cdot \sum_{\substack{A\subseteq N\\ \lvert A\rvert=i}} \mu(A)\right]$	Marichal [18]
Local degree of orness	$\boldsymbol{\omega}_L(\mu) = \sum_{i=1}^{n}\left(\frac{i-1}{n-1}\right) \cdot \left(\mu\left(A_{id,i}\right) - \mu\left(A_{id,i+1}\right)\right)$	Belles-Sampera et al. [2]
Dispersion (Yager's Shannon entropy)[2,⋆]	$H_Y(\mu) = -\sum_{i=1}^{n} \phi_i(\mu) \ln\left[\phi_i(\mu)\right]$	Yager [37]

[1] Other degrees of orness can be found in Grabisch et al. [10].

[2] Following notation used in Kojadinovic et al. [14], where $\phi_i(\mu)$ stands for the i-th component of the Shapley value of μ.

⋆ Alternative entropy measures can be found in Dukhovny [6], Marichal [17] and Kojadinovic et al. [14].

Degree of balance for Choquet integrals

We propose expressions (4.1) for the global and local degrees of balance indicators associated with Choquet integrals. Note that the degree of balance introduced by Yager [36] was in the range $[-1, 1]$, where values of the degree of orness from $[0, 1]$ were rescaled. Here, the degree of balance is defined for any interval $[a, b] \subseteq \mathbb{R}$ where $b > a$.

$$\begin{aligned} Bal_{G,[a,b]}\left(\mathcal{C}_\mu\right) &:= (b-a) \cdot \boldsymbol{\omega}_G(\mu) + \mu(N) \cdot a, \\ Bal_{L,[a,b]}\left(\mathcal{C}_\mu\right) &:= (b-a) \cdot \boldsymbol{\omega}_L(\mu) + \mu(N) \cdot a. \end{aligned} \tag{4.1}$$

Note that definitions (4.1) are linear transformations of the degree of orness. If μ is not normalized, the values of the degree of balance belong to the interval $[a \cdot \mu(N), b - a \cdot (1 - \mu(N))]$. These definitions fulfill linearity conditions with respect to capacities, as shown in section 5.

It is straightforward to check that when μ is symmetric $Bal_{L,[a,b]}\left(\mathcal{C}_\mu\right) = Bal_{G,[a,b]}\left(\mathcal{C}_\mu\right)$. If, in addition, μ is normalized and $a = -1$ and $b = 1$ then $Bal(\vec{w}) = Bal_{L,[-1,1]}\left(\mathcal{C}_\mu\right) = Bal_{G,[-1,1]}\left(\mathcal{C}_\mu\right)$.

As in the case of the degree of orness, if μ is additive and normalized, then in general $Bal_{L,[a,b]}\left(\mathcal{C}_\mu\right) \neq Bal_{G,[a,b]}\left(\mathcal{C}_\mu\right)$. In particular, $Bal_{G,[a,b]}\left(\mathcal{C}_\mu\right) = \frac{a+b}{2}$ and $Bal_{L,[a,b]}\left(\mathcal{C}_\mu\right) = b + a \cdot \sum_{i=1}^{n}\left(\frac{n-i}{n-1}\right) \cdot p_i$ are both satisfied.

Divergence indicator for Choquet integrals

Extensions of the divergence indicator to the Choquet integral level are provided in this section. As mentioned previously in the context of $\text{OWA}_{\vec{w}}$ operators, situations exist in which the degree of orness and the dispersion indicator are insufficient for characterizing a weighting vector $\vec{w}$. In such instances, a supplementary measure providing additional information is required. The divergence indicator is a good candidate to fill this gap.

The divergence indicator of a Choquet integral is defined from a global and a local perspective. In order to introduce the global divergence indicator $Div_G(\mathcal{C}_\mu)$, we must first define the ascending quadratic weighted additive (AQWA) capacity.

Definition 4.1 (AQWA capacity)**.** *Let μ be a capacity on $N = \{m_1, ..., m_n\}$. The ascending quadratic weighted additive (AQWA) capacity linked to μ is an additive capacity η on N defined by*

(i) $$\eta(\{m_j\}) := \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\binom{n}{n-j+1}^{-1} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) - \binom{n}{n-j}^{-1} \sum_{\substack{A \subseteq N \\ |A|=n-j}} \mu(A) \right],$$

for all $j = 1, .., n$;

(ii) $\eta(A) := \sum_{m_k \in A} \eta(\{m_k\})$*; and* $\eta(\emptyset) := 0$.

Proof that η is a capacity on N is provided in Appendix A. Two specific cases of AQWA capacities are those linked to symmetric capacities and those linked to additive capacities:

- If μ is symmetric, then for all $j = 1, ..., n$

$$\eta(\{m_j\}) = \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\frac{\sum\limits_{\substack{A \subseteq N \\ |A|=n-j+1}} \sum\limits_{k=j}^{n} w_k}{\binom{n}{n-j+1}} - \frac{\sum\limits_{\substack{A \subseteq N \\ |A|=n-j}} \sum\limits_{k=j+1}^{n} w_k}{\binom{n}{n-j}} \right] =$$
$$= \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\sum_{k=j}^{n} w_k - \sum_{k=j+1}^{n} w_k \right] = \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot w_j. \tag{4.2}$$

- If μ is additive, then for all $j = 1, ..., n$

$$\begin{aligned}
\eta(\{m_j\}) = & \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\frac{\sum\limits_{\substack{A \subseteq N \\ |A|=n-j+1}} \sum\limits_{m_k \in A} \mu(\{m_k\})}{\binom{n}{n-j+1}} - \frac{\sum\limits_{\substack{A \subseteq N \\ |A|=n-j}} \sum\limits_{m_k \in A} \mu(\{m_k\})}{\binom{n}{n-j}} \right] = \\
= & \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\sum_{k=1}^{n} \frac{\binom{n-1}{n-j}}{\binom{n}{n-j+1}} w_k - \sum_{k=1}^{n} \frac{\binom{n-1}{n-j-1}}{\binom{n}{n-j}} w_k \right] = \\
= & \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\sum_{k=1}^{n} \frac{n-j+1}{n} w_k - \sum_{k=1}^{n} \frac{n-j}{n} w_k \right] = \\
= & \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \frac{1}{n} \sum_{k=1}^{n} w_k.
\end{aligned} \tag{4.3}$$

The definition of the global divergence indicator for a discrete Choquet integral is as follows:

$$Div_G(\mathcal{C}_\mu) := \frac{n(2n-1)}{3(n-1)} \cdot \boldsymbol{\omega}_G(\eta) - [2 - \mu(N)] \cdot \boldsymbol{\omega}_G^2(\mu). \tag{4.4}$$

We should point out that the divergence indicator of the OWA operator was interpreted as a variance, $Div(\vec{w}) = \mathbb{E}_{\vec{w}}\left[(X^*)^2\right] - (\mathbb{E}_{\vec{w}}[X^*])^2$. The parallelism between this interpretation and expression 4.4 is direct[6]. To some extent, $Div(\vec{w})$ could be considered as a mean variance over permutations of the components $x_i^* = \frac{i-1}{n-1}$ for all $i = 1, ..., n$. Hence, the global divergence is given a mean variability around the degree of orness of any input value to be aggregated. In other words, the value of the global divergence is associated with the scattering of the aggregation function around the global degree of orness. This means that as the global divergence increases, the global degree of orness becomes less absorbent in the aggregation process.

[6]The role that the mathematical expectation was playing for $Div(\vec{w})$ is now the role of the global degree of orness, and η replaces X^{*2} while μ maps to X^*. Factor $\dfrac{n(2n-1)}{3(n-1)}$ emerges to guarantee that η is a capacity $(\eta(N) \leq 1)$ and factor $[2 - \mu(N)]$ ensures that not only normalized capacities μ are considered.

Remark 4.1. *It can be proved that definition (4.4) is equivalent to*

$$Div_G\left(\mathcal{C}_\mu\right) = \sum_{i=1}^{n}\left(\frac{i-1}{n-1} - \boldsymbol{\omega}_G(\mu)\right)^2 \cdot \left[\frac{\sum\limits_{\substack{A\subseteq N\\|A|=n-i+1}} \mu\left(A\right)}{\binom{n}{n-i+1}} - \frac{\sum\limits_{\substack{A\subseteq N\\|A|=n-i}} \mu\left(A\right)}{\binom{n}{n-i}}\right]. \tag{4.5}$$

The definition of the local divergence indicator is as follows:

$$Div_L\left(\mathcal{C}_\mu\right) := \sum_{i=1}^{n}\left(\frac{i-1}{n-1} - \boldsymbol{\omega}_L(\mu)\right)^2 \cdot \left(\mu\left(A_{id,i}\right) - \mu\left(A_{id,i+1}\right)\right). \tag{4.6}$$

This definition is inspired by the fact that $Div\left(\vec{w}\right) = \text{WAM}_{\vec{w}}\left(\vec{z}\right)$ in the case of OWA operators (see section 3.1) and corresponds to the local perspective.

When μ is symmetric and normalized, $Div_L\left(\mathcal{C}_\mu\right) = Div_G\left(\mathcal{C}_\mu\right) = Div\left(\vec{w}\right)$ is satisfied. The proof is as follows. Note that the global degree of orness of the AQWA capacity linked to a symmetric capacity is equal to,

$$\begin{aligned}
\boldsymbol{\omega}_G(\eta) = {} & \frac{1}{(n-1)}\sum_{i=1}^{n-1}\binom{n}{i}^{-1}\sum_{\substack{A\subseteq N\\|A|=i}}\eta\left(A\right) = \\
= {} & \frac{1}{(n-1)}\sum_{i=1}^{n-1}\binom{n}{i}^{-1}\sum_{\substack{A\subseteq N\\|A|=i}}\sum_{m_j\in A}\frac{6(j-1)^2}{(n-1)n(2n-1)}\cdot w_j = \\
= {} & \frac{1}{(n-1)}\sum_{i=1}^{n-1}\binom{n}{i}^{-1}\sum_{j=1}^{n}\binom{n-1}{i-1}\frac{6(j-1)^2}{(n-1)n(2n-1)}\cdot w_j = \\
= {} & \frac{1}{(n-1)}\sum_{i=1}^{n-1}\frac{i}{n}\sum_{j=1}^{n}\frac{6(j-1)^2}{(n-1)n(2n-1)}\cdot w_j = \\
= {} & \frac{1}{(n-1)}\frac{6}{(n-1)n(2n-1)}\frac{n-1}{2}\sum_{j=1}^{n}(j-1)^2\cdot w_j = \\
= {} & \frac{3(n-1)}{n(2n-1)}\sum_{j=1}^{n}\left(\frac{j-1}{n-1}\right)^2\cdot w_j.
\end{aligned} \tag{4.7}$$

If μ is symmetric, $\boldsymbol{\omega}_G(\eta)$ in expression (4.4) may be replaced by (4.7) and it holds:

$$Div_G\left(\mathcal{C}_\mu\right) = \sum_{j=1}^{n}\left(\frac{j-1}{n-1}\right)^2\cdot w_j - \left[2-\mu\left(N\right)\right]\cdot\boldsymbol{\omega}_G^2(\mu), \tag{4.8}$$

),

Taking into account that when μ is symmetric $\boldsymbol{\omega}_G^2(\mu) = \boldsymbol{\omega}_L^2(\mu)$, expression (4.8) is equivalent to $Div_L(\mathcal{C}_\mu)$ as follows,

$$\begin{aligned} Div_L(\mathcal{C}_\mu) = & \sum_{i=1}^{n}\left[\left(\frac{i-1}{n-1}\right)^2 - 2\cdot\left(\frac{i-1}{n-1}\right)\cdot\boldsymbol{\omega}_L(\mu) + \boldsymbol{\omega}_L^2(\mu)\right]\cdot\left(\mu\left(A_{id,i}\right) - \mu\left(A_{id,i+1}\right)\right) = \\ = & \sum_{i=1}^{n}\left(\frac{i-1}{n-1}\right)^2\cdot w_i - \left[2-\mu(N)\right]\cdot\boldsymbol{\omega}_L^2(\mu). \end{aligned} \tag{4.9}$$

When μ is additive and normalized, expression (4.4) can be simplified. Note that η is additive but not normalized. If μ is additive and normalized then from expression (4.3) $\eta(N) = \frac{1}{n}$. Furthermore, we know that

$$\begin{aligned} \boldsymbol{\omega}_G(\eta) = & \frac{1}{n-1}\sum_{i=1}^{n-1}\left[\binom{n}{i}^{-1}\cdot\sum_{\substack{A\subseteq N\\|A|=i}}\sum_{m_j\in A}\eta(\{m_j\})\right] = \\ = & \frac{1}{n-1}\sum_{i=1}^{n-1}\left[\binom{n}{i}^{-1}\cdot\sum_{j=1}^{n}\binom{n-1}{i-1}\eta(\{m_j\})\right] = \\ = & \frac{1}{n-1}\sum_{j=1}^{n}\sum_{i=1}^{n-1}\left(\frac{i}{n}\right)\frac{6(j-1)^2}{(n-1)n(2n-1)}\cdot\frac{1}{n} = \\ = & \frac{1}{n-1}\sum_{j=1}^{n}\left[\frac{n-1}{2}\cdot\frac{6(j-1)^2}{(n-1)n(2n-1)}\cdot\frac{1}{n}\right] = \frac{1}{2n}. \end{aligned}$$

As μ is additive and normalized $\boldsymbol{\omega}_G(\mu) = \frac{1}{2}$, and hence expression (4.4) in this situation becomes

$$Div_G(\mathcal{C}_\mu) = \frac{n(2n-1)}{3(n-1)}\frac{1}{2n} - (2-1)\frac{1}{4} = \frac{1}{12}\cdot\frac{n+1}{n-1}. \tag{4.10}$$

In general, it is easy to observe that $Div_G(\mathcal{C}_\mu) \neq Div_L(\mathcal{C}_\mu)$ when μ is additive and normalized.

Variance indicator and Rényi entropies for Choquet integrals

In section 3.1 above two additional indicators for OWA operators were shown, namely the variance indicator of the weighting vector and the Rényi entropy of degree α. The generalized definitions of the global indicators for Choquet integrals can be provided but the local perspective for this indicators is not considered. The reason for this being that the two indicators are only defined in terms of the weighting vector in the case of OWA operators, but not in terms of $\dfrac{i-1}{n-1}$ or $\dfrac{2i-(n+1)}{n-1}$.

The global variance indicator of a capacity linked to a Choquet integral may be defined as

$$D_G^2\left(\mathcal{C}_\mu\right)=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\sum\limits_{\substack{A\subseteq N\\|A|=n-i+1}}\mu\left(A\right)}{\begin{pmatrix}n\\n-i+1\end{pmatrix}}-\frac{\sum\limits_{\substack{A\subseteq N\\|A|=n-i}}\mu\left(A\right)}{\begin{pmatrix}n\\n-i\end{pmatrix}}\right]^2-\frac{\mu(N)^2}{n^2}\quad. \tag{4.11}$$

The global Rényi entropies of degree $\alpha\in\mathbb{R}\backslash\{1\}$ for a Choquet integral with respect to μ may be defined as

$$H_{G,\alpha}\left(\mathcal{C}_\mu\right)=\frac{1}{1-\alpha}\log_2\left(\sum_{i=1}^{n}\left[\frac{\sum\limits_{\substack{A\subseteq N\\|A|=n-i+1}}\mu\left(A\right)}{\begin{pmatrix}n\\n-i+1\end{pmatrix}}-\frac{\sum\limits_{\substack{A\subseteq N\\|A|=n-i}}\mu\left(A\right)}{\begin{pmatrix}n\\n-i\end{pmatrix}}\right]^\alpha\right)\quad. \tag{4.12}$$

Table 4.1: Summary of new indicators extended to Choquet integrals.

Indicator	Analytical expression
Global degree of balance	$Bal_{G,[a,b]}\left(\mathcal{C}_\mu\right)=(b-a)\cdot\boldsymbol{\omega}_G\left(\mu\right)+\mu\left(N\right)\cdot a$
Local degree of balance	$Bal_{L,[a,b]}\left(\mathcal{C}_\mu\right)=(b-a)\cdot\boldsymbol{\omega}_L\left(\mu\right)+\mu\left(N\right)\cdot a$
Global divergence	$Div_G\left(\mathcal{C}_\mu\right)=\frac{n(2n-1)}{3(n-1)}\cdot\boldsymbol{\omega}_G(\eta)-\left[2-\mu\left(N\right)\right]\cdot\boldsymbol{\omega}_G^2(\mu)$
Local divergence	$Div_L\left(\mathcal{C}_\mu\right)=\sum_{i=1}^{n}\left(\frac{i-1}{n-1}-\boldsymbol{\omega}_L(\mu)\right)^2\cdot\left(\mu\left(A_{id,i}\right)-\mu\left(A_{id,i+1}\right)\right)$
Variance indicator	$D_G^2\left(\mathcal{C}_\mu\right)=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\sum_{A\subseteq N,\,\lvert A\rvert=n-i+1}\mu\left(A\right)}{\binom{n}{n-i+1}}-\frac{\sum_{A\subseteq N,\,\lvert A\rvert=n-i}\mu\left(A\right)}{\binom{n}{n-i}}\right]^2-\frac{\mu(N)^2}{n^2}$
Rényi entropy ($\alpha\neq1$)	$H_{G,\alpha}\left(\mathcal{C}_\mu\right)=\frac{1}{1-\alpha}\log_2\left(\sum_{i=1}^{n}\left[\frac{\sum_{A\subseteq N,\,\lvert A\rvert=n-i+1}\mu\left(A\right)}{\binom{n}{n-i+1}}-\frac{\sum_{A\subseteq N,\,\lvert A\rvert=n-i}\mu\left(A\right)}{\binom{n}{n-i}}\right]^\alpha\right)$

5. Indicators with respect to a linear combination of capacities

5.1. Linearity features of the extended indicators

Let us denote any global or local indicator associated with a Choquet integral with respect to a capacity μ as $F(\mu)$. We want to assess the expressions of $F(\lambda_1\mu_1 + \lambda_2\mu_2)$, where $\lambda_1, \lambda_2 \in [0,1]$ and μ_1, μ_2 are capacities defined on N. If the indicator is linear with respect to capacities then $F(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 F(\mu_1) + \lambda_2 F(\mu_2)$ must hold.

Linearity of the degree of orness and the degree of balance

The global and the local degrees of orness are both linear with respect to capacities. From expressions (3.1) and (3.2) with $\mu = \lambda_1\mu_1+\lambda_2\mu_2$, and noting that $(\lambda_1\mu_1 + \lambda_2\mu_2)(A) = \lambda_1\mu_1(A)+\lambda_2\mu_2(A)$ for any $A \in 2^N$, then it is deduced that $\boldsymbol{\omega}_G(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1\boldsymbol{\omega}_G(\mu_1)+\lambda_2\boldsymbol{\omega}_G(\mu_2)$ and $\boldsymbol{\omega}_L(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1\boldsymbol{\omega}_L(\mu_1) + \lambda_2\boldsymbol{\omega}_L(\mu_2)$.

The linearity of the degree of balance (global and local) with respect to capacities can be assessed using the above expressions and the fact that this indicator is a linear transformation of the degree of orness (as shown in section 4). The expression

$$\begin{aligned} Bal_{*,[a,b]}\left(\mathcal{C}_{\lambda_1\mu_1+\lambda_2\mu_2}\right) = & \; (b-a)\boldsymbol{\omega}_*(\lambda_1\mu_1 + \lambda_2\mu_2) + (\lambda_1\mu_1 + \lambda_2\mu_2)(N)\, a = \\ = & \; \lambda_1(b-a)\boldsymbol{\omega}_*(\mu_1) + \lambda_1\mu_1(N)\, a + \lambda_2(b-a)\boldsymbol{\omega}_*(\mu_2) + \lambda_2\mu_2(N)\, a = \\ = & \; \lambda_1 Bal_{*,[a,b]}\left(\mathcal{C}_{\mu_1}\right) + \lambda_2 Bal_{*,[a,b]}\left(\mathcal{C}_{\mu_2}\right), \end{aligned} \tag{5.1}$$

holds for global and local indicators (i.e., either if $* = G$ or $* = L$). Thus, the degree of balance is linear with respect to capacities.

Non-linearity of the divergence, the dispersion, the variance indicator and Rényi entropies

The divergence indicator is not linear with respect to capacities in the general case, as can be deduced from expressions (4.4) and (4.6). Nonetheless, a result that characterizes the geometric locus where the divergence indicator satisfies linearity is presented in Appendix B. Although not explicitly proved, the lack of linearity of the dispersion, the variance indicator and Rényi entropies is evident due to the lack of linearity (in the general case) of functions $\ln(x)$, x^2 and $\log_2(x)$, respectively.

5.2. Application: inherited indicators of POWA operators

Indicators for the probabilistic ordered weighted averaging (POWA) operator are derived. The POWA operator was introduced in Merigó [19], Merigó and Wei [23] and Merigó [20]. Let $\vec{w} = (w_1, w_2, ..., w_n) \in [0,1]^n$ be such that $\sum_{i=1}^{n} w_i = 1$ and let $\vec{p} = (p_1, p_2, ..., p_n) \in [0,1]^n$ be such that $\sum_{i=1}^{n} p_i = 1$. In addition, consider $\beta \in [0,1]$. The POWA operator with respect to $\vec{w}, \vec{p}$ and β is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\mathrm{POWA}_{\vec{w},\vec{p},\beta}(x_1, x_2, ..., x_n) := \beta \cdot \sum_{i=1}^{n} x_{\sigma(i)} \cdot w_i + (1-\beta) \cdot \sum_{i=1}^{n} x_{\sigma(i)} \cdot p_{\sigma(i)}, \tag{5.2}$$

where σ is a permutation of $(1, 2, ..., n)$ such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq ... \leq x_{\sigma(n)}$, i.e. $x_{\sigma(i)}$ is the i-th smallest value of $x_1, x_2, ..., x_n$.

An alternative expression to (5.2) is

$$\mathrm{POWA}_{\vec{w},\vec{p},\beta}(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_{\sigma(i)} \cdot v_{i,\sigma}, \tag{5.3}$$

where $v_{i,\sigma} = \beta \cdot w_i + (1-\beta) \cdot p_{\sigma(i)}$ for all $i = 1, ..n$. It is straightforward to see that $v_{i,\sigma} \in [0,1]$ and $\sum_{i=1}^{n} v_{i,\sigma} = 1$. Note that the POWA operator can be understood as a weighted average between an OWA operator and a WAM. When the random variable X that can take n different values denoted by $\{x_i\}_{i=1,...,n}$ is such that $\mathcal{P}(X = x_i) = p_i$ for all $i = 1, ..., n$, then the POWA operator can also be understood as a weighted average between an OWA operator and the mathematical expectation of the random variable X:

$$\mathrm{POWA}_{\vec{w},\vec{p},\beta}(\vec{x}) = \beta \cdot \mathrm{OWA}_{\vec{w}}(\vec{x}) + (1-\beta) \cdot \mathrm{WAM}_{\vec{p}}(\vec{x}) = \beta \cdot \mathrm{OWA}_{\vec{w}}(\vec{x}) + (1-\beta) \cdot \mathbb{E}(X). \tag{5.4}$$

Note that this implies two different levels of decision-maker preference. The first level concerns the introduction of an OWA operator as an additional way of evaluating the likeliness of the random events, and which differs from that provided by *real* risk information. The second concerns the degrees of plausibility given to the OWA operator introduced in the previous step, on the one hand, and to the WAM representing *real* risk information, on the other.

Taking into account the relationship between OWA operators and Choquet integrals (section 2.3), expression (5.4) may be formulated as $\mathrm{POWA}_{\vec{w},\vec{p},\beta} = \beta \cdot \mathcal{C}_{\mu} + (1-\beta) \cdot \mathcal{C}_{\mathcal{P}}$.

The capacities μ and $\mathcal{P}$ are normalized, where the former is symmetric and the latter a probability. This expression is a convex combination of two Choquet integrals that combines an OWA and a probabilistic perspective. Considering now the linearity of Choquet integrals with respect to the capacity (see Proposition 9(i) in [11]), the representation of the POWA operator as a Choquet integral is directly derived as

$$\text{POWA}_{\vec{w},\vec{p},\beta} = \mathcal{C}_{\beta\cdot\mu+(1-\beta)\cdot\mathcal{P}} \quad . \tag{5.5}$$

Therefore indicators for the POWA operator may be defined as follows:

$$\begin{aligned}
\boldsymbol{\omega}_*(\text{POWA}_{\vec{w},\vec{p},\beta}) &:= \boldsymbol{\omega}_*(\beta\cdot\mu+(1-\beta)\cdot\mathcal{P}) = \beta\cdot\boldsymbol{\omega}_*(\mu)+(1-\beta)\cdot\boldsymbol{\omega}_*(\mathcal{P}),\\
H_Y\left(\text{POWA}_{\vec{w},\vec{p},\beta}\right) &:= -\beta\cdot\sum_{i=1}^{n}\phi_i\left(\mu\right)\ln\left[\beta\cdot\phi_i\left(\mu\right)+(1-\beta)\cdot\phi_i\left(\mathcal{P}\right)\right]-\\
&\quad -(1-\beta)\cdot\sum_{i=1}^{n}\phi_i\left(\mathcal{P}\right)\ln\left[\beta\cdot\phi_i\left(\mu\right)+(1-\beta)\cdot\phi_i\left(\mathcal{P}\right)\right],\\
Bal_{*,[a,b]}(\text{POWA}_{\vec{w},\vec{p},\beta}) &:= Bal_{*,[a,b]}(\mathcal{C}_{\beta\cdot\mu+(1-\beta)\cdot\mathcal{P}}) = \beta\cdot Bal_{*,[a,b]}(\mathcal{C}_\mu)+(1-\beta)\cdot Bal_{*,[a,b]}(\mathcal{C}_\mathcal{P}),\\
Div_*(\text{POWA}_{\vec{w},\vec{p},\beta}) &:= Div_*(\mathcal{C}_{\beta\cdot\mu+(1-\beta)\cdot\mathcal{P}}),\\
D_G^2(\text{POWA}_{\vec{w},\vec{p},\beta}) &:= D_G^2(\mathcal{C}_{\beta\cdot\mu+(1-\beta)\cdot\mathcal{P}}),\\
H_{G,\alpha}(\text{POWA}_{\vec{w},\vec{p},\beta}) &:= H_{G,\alpha}(\mathcal{C}_{\beta\cdot\mu+(1-\beta)\cdot\mathcal{P}}).
\end{aligned}$$

Note that the linearity properties of the degree of orness and the degree of balance allow the degree of orness and the degree of balance indicators to be defined for the POWA operator as linear combinations of the indicators associated to the underlying OWA and WAM operators[7].

To conclude, we have derived indicators for the POWA operator. However, the POWA operator is only one of a set of possible examples. For instance, the weighted ordered weighted averaging (WOWA) operator introduced by Torra [29] might also be considered

[7]The dispersion, the divergence, the variance indicator and Rényi entropies of the POWA operator are not linear combinations of the dispersion, the divergence, the variance indicator and Rényi entropies of the underlying OWA and WAM operators. Only in special cases, such as those derived in Appendix B for the divergence, is linearity satisfied. Note that the dispersion and the divergence for the POWA operator introduced in [20] represents an alternative approach. Here, the author proposes a linear combination of these indicators for the underlying OWA and WAM operators.

and the inherited indicators shown, due to the relationship between the WOWA operator and the Choquet integral with respect to particular capacities (see Theorem 4 in Torra [30]).

6. Illustrative example

In section 3.1 above, an example of two different OWA operators with the same degree of orness and dispersion was presented as a means of introducing the divergence indicator for OWA operators. In the example that follows a hypothetical decision-making situation is described -involving customer on-line satisfaction assessment- in which the usefulness of the divergence indicator can be illustrated. Here, the two aggregation functions considered are Choquet integrals, both sharing the same global degree of orness but presenting different dispersion values. One of the goals of the decision maker is to hold an aggregation operator that does not return overly concentrated results. As such, what is sought is a large dispersion or a large divergence. The relevance of the global divergence indicator in this example is based on the fact that it leads to a quite distinct selection to that determined by the dispersion. In other words, the global divergence indicator does not provide the same kind of information as that provided by the dispersion indicator. A possible interpretation of this is that while the dispersion measures the amount of input information used in the aggregation process, the global divergence measures the distance separating the aggregated value from the value associated with the global degree of orness. Hence, the adoption of the latter indicator is more appropriate for fulfilling the objective of the decision maker in this context.

A company allows its customers to complete an on-line survey recording their satisfaction with the services provided by the company. This survey is divided in three sections: after-sales services (1), flexibility to satisfy the customers' requirements (2) and service quality (3). Each section contains several questions, and each question can be scored by the client with a value from $\{-3, -2, -1, 0, 1, 2, 3\}$, where -3 represents the lowest possible valuation and $+3$ the highest. Suppose that the average score in each section is assigned as the score of the section. The company's community manager is interested in ranking the answers to this survey based on the scores recorded for each of the three sections, named x_1, x_2 and x_3. The aim is to present the most favorable answers at the top of the ranking, and to employ a ranking system that will allow some variability among the ranking's values.

The community manager is provided by the company's IT team with two aggregation functions to rank the surveys. Both are Choquet integrals defined on $N = \{m_1, m_2, m_3\}$: one with respect to capacity κ and the other with respect to capacity μ, which are defined by:

- $\kappa(\emptyset) = 0$, $\kappa(\{m_1\}) = \kappa(\{m_2\}) = \kappa(\{m_3\}) = 0.3$, $\kappa(\{m_1, m_2\}) = \kappa(\{m_1, m_3\}) = \kappa(\{m_2, m_3\}) = 0.85$ and $\kappa(N) = 1$;
- $\mu(\emptyset) = 0$, $\mu(\{m_1\}) = 0.3$, $\mu(\{m_2\}) = 0.2$, $\mu(\{m_3\}) = 0.60625$, $\mu(\{m_1, m_2\}) = 0.54375$, $\mu(\{m_1, m_3\}) = 0.95$, $\mu(\{m_2, m_3\}) = 0.825$, and $\mu(N) = 1$.

The community manager must therefore select one of the two aggregation functions to build the ranking. To compare both aggregation operators, the community manager starts by considering the values returned when scores $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) = (1, -1.2, 0.5)$ and $\underline{x} = (\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-1.4, 0.85, -0.2)$ are taken into account. Expression (2.1) has to be used to compute these values.

The value that the first aggregation operator assigns to $\overline{x}$ is $\mathcal{C}_\kappa(\overline{x}) = -1.2 \times 0.15 + 0.5 \times 0.55 + 1 \times 0.3 = 0.395$ and the value that it assigns to $\underline{x}$ is $\mathcal{C}_\kappa(\underline{x}) = -1.4 \times 0.15 + (-0.2) \times 0.55 + 0.85 \times 0.3 = -0.06$. The value that the second aggregation operator assigns to $\overline{x}$ is $\mathcal{C}_\mu(\overline{x}) = -1.2 \times 0.45625 + 0.5 \times 0.34375 + 1 \times 0.2 = -0.175625$ and the value that it assigns to $\underline{x}$ is $\mathcal{C}_\mu(\overline{x}) = -1.4 \times 0.175 + (-0.2) \times 0.625 + 0.85 \times 0.2 = -0.2$. As such, both aggregation operators are ranking $\overline{x}$ and $\underline{x}$ in a similar way, although with different ranking values[8]

[8]Some abuse of notation has been implemented with Choquet integrals: $\overline{x}$ has been used instead of the function that has generated values $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) = (1, -1.2, 0.5)$. Something similar has occurred in the case of $\underline{x}$.

A fixed order in the components of the input data vector (x_1,x_2 and x_3) cannot be assumed. Consequently, the community manager prefers global to local indicators. Hence, the next element that the community manager assesses with regard to the aggregation operators is their global degree of orness. As can be shown, $\boldsymbol{\omega}_G(\kappa) = 0 \times 0.15 + 0.5 \times 0.55 + 1 \times 1 = 0.575$ and $\boldsymbol{\omega}_G(\mu) = 0 \times 0.227083 + 0.5 \times 0.395833 + 1 \times 0.377083 = 0.575$, so $\boldsymbol{\omega}_G(\kappa) = \boldsymbol{\omega}_G(\mu)$ and thus this indicator is not useful for selecting one or other of the aggregation operators.

The decision maker is obliged therefore to compute an additional indicator, one that could help in the selection of a particular aggregation operator. Let us suppose that the global divergence indicator is calculated for $\mathcal{C}_\kappa$ and $\mathcal{C}_\mu$. AQWA capacities linked to κ and μ are determined by the following values: $\text{AQWA}_\kappa(\{m_1\}) = 0$, $\text{AQWA}_\kappa(\{m_2\}) = 0.11$ and $\text{AQWA}_\kappa(\{m_3\}) = 0.24$; and $\text{AQWA}_\mu(\{m_1\}) = 0$, $\text{AQWA}_\mu(\{m_2\}) = 0.07917$ and $\text{AQWA}_\mu(\{m_3\}) = 0.30167$. The global degrees of orness of these AQWA capacities are 0.175 and 0.19042, respectively, and thus $Div_G(\mathcal{C}_\kappa) = 0.106875$ and $Div_G(\mathcal{C}_\mu) = 0.145417$. As long as a greater divergence is preferred to build a more scattered ranking, the second aggregation operator $\mathcal{C}_\mu$ is selected.

Table 6.1: Values of the global indicators for the Choquet integrals in the example

Indicator	$\mathcal{C}_\kappa$	$\mathcal{C}_\mu$	$\text{OWA}_{\vec{w}}$	$\text{WAM}_{\vec{p}}$	AQWA - κ	AQWA - μ
Degree of orness	0.575	0.575	0.7	0.5	0.175	0.19042
Dispersion	1.098612	0.970458	1.098612	0.746033	0.585308	0.562310
Degree of balance [-1,1]	0.15	0.15	0.4	0	0	0
Divergence	0.106875	0.145417	0.085	0.166667	0.095302	0.099972
Variance indicator	0.027222	0.005703	0.040556	0	0	0
Rényi entropy ($\alpha = 1.5$)	1.331768	1.528260	1.170201	1.584963	6.128682	5.763268

$\text{OWA}_{\vec{w}}$ and $\text{WAM}_{\vec{p}}$ are such that $\mathcal{C}_\mu = \beta \cdot \text{OWA}_{\vec{w}} + (1-\beta) \cdot \text{WAM}_{\vec{p}}$ in the example, with $\beta = 0.375$.

Once the selection has been finalized, a number of remarks should be made. For instance, $\mathcal{C}_\kappa$ is equivalent to an OWA operator because κ is a normalized symmetric capacity. In fact, $\mathcal{C}_\kappa$ is equivalent to $\text{OWA}_{\vec{w}}$ with $\vec{w} = (w_1, w_2, w_3) = (0.15, 0.55, 0.3)$. It is easy to check that $\boldsymbol{\omega}(\vec{w}) = \boldsymbol{\omega}_G(\kappa)$. As for $\mathcal{C}_\mu$, it is equivalent to a $\text{POWA}_{\vec{w},\vec{p},\beta}$ operator.

The specific vectors $\vec{w}$, $\vec{p}$ and value β to obtain the equivalence are $\vec{w} = (0.05, 0.5, 0.45)$, $\vec{p} = (0.25, 0.05, 0.7)$ and $\beta = 0.375$. Note that in this situation, $\boldsymbol{\omega}(\vec{w}) = 0 \times 0.05 + 0.5 \times 0.5 + 1 \times 0.45 = 0.7$ and that the global degree of orness for the Choquet integral with respect to the capacity driven by $\vec{p}$ is 0.5, because it is a normalized additive capacity. Given these degrees of orness and recalling $\beta = 0.375$, it is straightforward to check the linearity of the global degree of orness for Choquet integrals in this particular case: $\boldsymbol{\omega}_G(\mu) = 0.575 = 0.375 \times 0.7 + 0.625 \times 0.5$. Additionally, the linearity with respect to capacities of the degree of balance and the lack of linearity of the dispersion, the divergence and the variance indicators, as well as the non-linearity of Rényi entropies can be checked in Table 6.1.

7. Discussion and conclusions

New indicators for characterizing the discrete Choquet integral have been presented with the aim of complementing those already available, so that a more complete formulation, covering a wider range of situations, might be provided. This need has arisen because at times the degree of orness and the entropy of dispersion may not be sufficient. This paper has, therefore, introduced the degree of balance, the divergence, the variance indicator and Rényi entropies as indicators within the framework of the Choquet integral. This paper has shown that these four indicators, which are commonly used for the OWA operator, can also be considered for the Choquet aggregation. It is our assertion that specific expressions of these indicators can be readily obtained for any aggregation operator that might be interpreted as a Choquet integral. We discuss the potential of the divergence indicator to provide supplementary information to decision makers in a fictitious example.

The paper has also undertaken an additional analysis of the linearity features of the indicators with respect to capacities. The conditions that capacities must satisfy in order to obtain linearity of the divergence indicator have likewise been investigated. The linearity analysis has been conducted to examine the inherited indicators for the POWA operator, an aggregation operator that can be understood as a Choquet integral with respect to a linear combination of capacities.

Appendix A. AQWA is a capacity defined on N

To prove that η is a capacity on N it is necessary to see that $\eta(A) \in [0,1]$ for all $A \subseteq N$, $\eta(\emptyset) = 0$ and that $\eta(A) \leq \eta(B)$ if $A \subseteq B$. By definition of AQWA capacity, $\eta(\emptyset) = 0$. If $\eta(\{m_k\}) \geq 0$ then $\eta(A) \leq \eta(B)$ if $A \subseteq B$ because $\eta(A) := \sum\limits_{m_k \in A} \eta(\{m_k\})$. So let us see that $\eta(\{m_k\}) \geq 0$. Recall that by definition 4.1, for all $j = 1, ..., n$

$$\eta(\{m_j\}) := \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\binom{n}{n-j+1}^{-1} \sum_{\substack{A \subseteq N \\ |A| = n-j+1}} \mu(A) - \binom{n}{n-j}^{-1} \sum_{\substack{A \subseteq N \\ |A| = n-j}} \mu(A) \right].$$

The first factor is less than or equal to 1, because $\frac{6(j-1)^2}{(n-1)n(2n-1)} = \frac{(j-1)^2}{\sum_{k=1}^{n}(k-1)^2} \leq 1$. For the second factor, for each j, two situations are considered: whether $s_{n-j+1} = \#\{A \quad s.t. \quad |A| = n-j+1\} = \binom{n}{n-j+1}$ is greater or less than $s_{n-j} = \#\{A \quad s.t. \quad |A| = n-j\} = \binom{n}{n-j}$. Once this notation is introduced, this second factor may be rewritten as $\frac{1}{s_{n-j+1}} \sum\limits_{\substack{A \subseteq N \\ |A| = n-j+1}} \mu(A) - \frac{1}{s_{n-j}} \sum\limits_{\substack{A \subseteq N \\ |A| = n-j}} \mu(A)$. So, supposing j is fixed:

- If $s_{n-j+1} \geq s_{n-j}$, then

$$\begin{aligned}
&\frac{1}{s_{n-j+1}} \sum_{\substack{A \subseteq N \\ |A| = n-j+1}} \mu(A) - \frac{1}{s_{n-j}} \sum_{\substack{A \subseteq N \\ |A| = n-j}} \mu(A) \geq^* \\
&\geq^* \frac{1}{s_{n-j+1}} \sum_{\substack{A \subseteq N \\ |A| = n-j}} \mu(A) - \frac{1}{s_{n-j}} \sum_{\substack{A \subseteq N \\ |A| = n-j}} \mu(A) = \\
&= \left(\frac{(n-j+1)!(j-1)!}{n!} - \frac{(n-j)!(j)!}{n!} \right) \cdot \sum_{\substack{A \subseteq N \\ |A| = n-j}} \mu(A) = \\
&= \frac{(j-1)!(n-j)!(n+1)}{n!} \cdot \sum_{\substack{A \subseteq N \\ |A| = n-j}} \mu(A) \geq 0.
\end{aligned}$$

The hypothesis is used to ensure that inequality $\geq^*$ holds, because $\sum\limits_{\substack{A \subseteq N \\ |A| = n-j+1}} \mu(A) \geq \sum\limits_{\substack{A \subseteq N \\ |A| = n-j}} \mu(A)$ under the hypothesis. This is true due to the fact that there are fewer summands on the right-hand side ($s_{n-j} \leq s_{n-j+1}$) and, in addition, each summand on the right is less than or equal to one on the left (μ is monotone);

- If $s_{n-j+1} < s_{n-j}$, then

$$\frac{1}{s_{n-j+1}} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) - \frac{1}{s_{n-j}} \sum_{\substack{A \subseteq N \\ |A|=n-j}} \mu(A) =$$

$$= \frac{1}{s_{n-j}} \left[\sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) + \frac{s_{n-j} - s_{n-j+1}}{s_{n-j+1}} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) \right] - \frac{1}{s_{n-j}} \sum_{\substack{A \subseteq N \\ |A|=n-j}} \mu(A) \geq^{**}$$

$$\geq^{**} \frac{1}{s_{n-j}} \left[\sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) + \frac{s_{n-j} - s_{n-j+1}}{s_{n-j+1}} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) \right] -$$

$$- \frac{1}{s_{n-j}} \sum_{\substack{B \subseteq N \\ |B|=n-j}} \left(\sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \frac{1}{s_{n-j+1}} \mu(A) \right) =$$

$$= \frac{1}{s_{n-j}} \left[\sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) + \frac{s_{n-j} - s_{n-j+1}}{s_{n-j+1}} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) - \frac{s_{n-j}}{s_{n-j+1}} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) \right] = 0.$$

In this case, the hypothesis is used to prove inequality $\geq^{**}$: for any $B \subseteq N$ such that $|B| = n - j$, $\mu(B) \leq \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \frac{1}{s_{n-j+1}} \mu(A)$ under the hypothesis. Otherwise, a contradiction with the fact that μ is monotone arises.

As this result implies $\eta(A) \leq \eta(B)$ if $A \subseteq B$, if $\eta(N) \leq 1$ is shown then $\eta(A) \in [0,1]$ will hold for each $A \subseteq N$. To see that $\eta(N) \leq 1$, note first that for all $j = 1, ..., n$,

$$\begin{aligned}
\eta(\{m_j\}) := & \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\binom{n}{n-j+1}^{-1} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) - \binom{n}{n-j}^{-1} \sum_{\substack{A \subseteq N \\ |A|=n-j}} \mu(A) \right] \leq \\
\leq & \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\binom{n}{n-j+1}^{-1} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} \mu(A) \right] \leq \\
\leq & \frac{6(j-1)^2}{(n-1)n(2n-1)} \cdot \left[\binom{n}{n-j+1}^{-1} \sum_{\substack{A \subseteq N \\ |A|=n-j+1}} 1 \right] \leq \\
\leq & \frac{6(j-1)^2}{(n-1)n(2n-1)}.
\end{aligned}$$

Given the previous inequalities, $\eta(N) = \sum_{m_j \in N} \eta(\{m_j\}) \leq \sum_{j=1}^{n} \frac{6(j-1)^2}{(n-1)n(2n-1)} = 1$. Hence, the fact that η is a capacity has been proved.

Appendix B. Conditions for the linearity of the divergence indicator

A result that characterizes the geometric locus where the divergence indicator satisfies linearity is given here.

A first remark must be made. If the following notation is introduced,

$$\gamma_*(\mu) := \begin{cases} \sum_{i=1}^{n} \left(\frac{i-1}{n-1}\right)^2 [\mu(A_{id,i}) - \mu(A_{id,i+1})], & * = L \\ \frac{n(2n-1)}{3(n-1)} \cdot \boldsymbol{\omega}_G(\eta), & * = G, \end{cases} \tag{B.1}$$

then taking into account expressions (4.4) and (4.6), both global and local divergence indicators may be interpreted as the sum of two components: One linear with respect to capacities ($\gamma_*(\mu)$) and another one that is not ($-[2 - \mu(N)] \cdot \boldsymbol{\omega}_*^2(\mu)$). Or, in other words, if expression (B.2)

$$
\begin{aligned}
&Div_*\left(\mathcal{C}_{\lambda_1\cdot\mu_1+\lambda_2\cdot\mu_2}\right)= \\
&=\lambda_1\cdot\gamma_*(\mu_1)-\left[2-\lambda_1\cdot\mu_1\left(N\right)-\lambda_2\cdot\mu_2\left(N\right)\right]\cdot\lambda_1^2\cdot\boldsymbol{\omega}_*^2(\mu_1)+ \\
&+\lambda_2\cdot\gamma_*(\mu_2)-\left[2-\lambda_1\cdot\mu_1\left(N\right)-\lambda_2\cdot\mu_2\left(N\right)\right]\cdot\lambda_2^2\cdot\boldsymbol{\omega}_*^2(\mu_2)- \\
&-2\cdot\left[2-\lambda_1\cdot\mu_1\left(N\right)-\lambda_2\cdot\mu_2\left(N\right)\right]\cdot\lambda_1\cdot\lambda_2\cdot\boldsymbol{\omega}_*(\mu_1)\cdot\boldsymbol{\omega}_*(\mu_2)
\end{aligned}
\tag{B.2}
$$

is compared to

$$
\begin{aligned}
&\lambda_1\cdot Div_*\left(\mathcal{C}_{\mu_1}\right)+\lambda_2\cdot Div_*\left(\mathcal{C}_{\mu_2}\right)= \\
&=\lambda_1\cdot\gamma_*(\mu_1)-\left[2-\mu_1\left(N\right)\right]\cdot\lambda_1\cdot\boldsymbol{\omega}_*^2(\mu_1)+ \\
&+\lambda_2\cdot\gamma_*(\mu_2)-\left[2-\mu_2\left(N\right)\right]\cdot\lambda_2\cdot\boldsymbol{\omega}_*^2(\mu_2),
\end{aligned}
\tag{B.3}
$$

then the linearity condition of the divergence indicator is

$$
\begin{aligned}
&\lambda_1\cdot\left[2\cdot\lambda_1-\lambda_1^2\cdot\mu_1(N)-\lambda_1\cdot\lambda_2\cdot\mu_2(N)-2+\mu_1(N)\right]\cdot\boldsymbol{\omega}_*^2(\mu_1)+ \\
+\;&\lambda_2\cdot\left[2\cdot\lambda_2-\lambda_2^2\cdot\mu_2(N)-\lambda_1\cdot\lambda_2\cdot\mu_1(N)-2+\mu_2(N)\right]\cdot\boldsymbol{\omega}_*^2(\mu_2)+ \\
+\;&2\cdot\lambda_1\cdot\lambda_2\cdot\left[2-\lambda_1\cdot\mu_1\left(N\right)-\lambda_2\cdot\mu_2\left(N\right)\right]\cdot\boldsymbol{\omega}_*(\mu_1)\cdot\boldsymbol{\omega}_*(\mu_2)=0.
\end{aligned}
\tag{B.4}
$$

Note that if

$$A:=\lambda_1\cdot\left[2\cdot\lambda_1-\lambda_1^2\cdot\mu_1(N)-\lambda_1\cdot\lambda_2\cdot\mu_2(N)-2+\mu_1(N)\right],$$

$$B:=\lambda_2\cdot\left[2\cdot\lambda_2-\lambda_2^2\cdot\mu_2(N)-\lambda_1\cdot\lambda_2\cdot\mu_1(N)-2+\mu_2(N)\right],$$

$$C:=\lambda_1\cdot\lambda_2\cdot\left[2-\lambda_1\cdot\mu_1\left(N\right)-\lambda_2\cdot\mu_2\left(N\right)\right],$$

then condition (B.4) may be written as

$$
\begin{pmatrix}\boldsymbol{\omega}_*(\mu_1) & \boldsymbol{\omega}_*(\mu_2)\end{pmatrix}\cdot\begin{pmatrix}A & C\\ C & B\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{\omega}_*(\mu_1)\\ \boldsymbol{\omega}_*(\mu_2)\end{pmatrix}=0.
\tag{B.5}
$$

Interestingly, expression (B.5) is that of a degenerate conic section with respect to $x=\boldsymbol{\omega}_*(\mu_1)$ and $y=\boldsymbol{\omega}_*(\mu_2)$. So, given $\lambda_1,\lambda_2\in[0,1]$, the linear combinations $\lambda_1\cdot\mu_1+\lambda_2\cdot\mu_2$ such that $Div_*\left(\mathcal{C}_{\lambda_1\cdot\mu_1+\lambda_2\cdot\mu_2}\right)=\lambda_1\cdot Div_*\left(\mathcal{C}_{\mu_1}\right)+\lambda_2\cdot Div_*\left(\mathcal{C}_{\mu_2}\right)$ can only be found by choosing μ_1 and μ_2 among those capacities that satisfy that the pair $(\boldsymbol{\omega}_*(\mu_1),\boldsymbol{\omega}_*(\mu_2))$ belongs to the degenerate conic section given by expression (B.5). Assuming $\lambda_1\neq 0$, the particular degenerate conic section can be determined by $\Delta=C^2-A\cdot B$: If $\Delta<0$ then expression

(B.5) does not have any solutions in $\mathbb{R}$ (the conic section is two imaginary points); if $\Delta = 0$ then the solutions to (B.5) lies on the line $\boldsymbol{\omega}_*(\mu_2) = \frac{A}{-C} \cdot \boldsymbol{\omega}_*(\mu_1)$ (the conic section is a line counted twice); and if $\Delta > 0$ then solutions to expression (B.5) belongs to the line $\boldsymbol{\omega}_*(\mu_2) = \frac{A}{-C+\sqrt{\Delta}} \cdot \boldsymbol{\omega}_*(\mu_1)$ or to the line $\boldsymbol{\omega}_*(\mu_2) = \frac{A}{-C-\sqrt{\Delta}} \cdot \boldsymbol{\omega}_*(\mu_1)$ (the conic section is two different lines).

There are some specific cases of special interest:

- **Normalized capacities**. In the framework of decision making under risk and uncertainty, for example, $\mu_1(N) = \mu_2(N) = 1$ is usually required. In this case $A = \lambda_1 \cdot [\lambda_1 \cdot (2 - \lambda_1 - \lambda_2) - 1]$, $B = \lambda_2 \cdot [\lambda_2 \cdot (2 - \lambda_1 - \lambda_2) - 1]$, and $C = \lambda_1 \cdot \lambda_2 \cdot (2 - \lambda_1 - \lambda_2)$. So $\Delta = [2 - (\lambda_1 + \lambda_2)] \cdot (\lambda_1 + \lambda_2) - 1$. Note that $\lambda_1 + \lambda_2 \leq 1$ is a necessary condition to guarantee that $\lambda_1 \cdot \mu_1 + \lambda_2 \cdot \mu_2$ is a capacity on N. But the key remark is the following: $\Delta = -(\lambda_1 + \lambda_2 - 1)^2$, so $\Delta \leq 0$ always for normalized capacities.

- **Normalized capacities and $\boldsymbol{\lambda_1 + \lambda_2 = 1}$**. Without loosing any generality, we can write $\lambda_1 = \beta$ and $\lambda_2 = 1 - \beta$. Taking advantage of the previous item, $A = \beta \cdot (\beta - 1)$, $B = (\beta - 1) \cdot \beta$, and $C = \beta \cdot (1 - \beta)$. Therefore, condition (B.5) becomes

$$\beta \cdot (\beta - 1) \cdot \left[(\boldsymbol{\omega}_*(\mu_1) - \boldsymbol{\omega}_*(\mu_2))^2 \right] = 0. \tag{B.6}$$

  Expression (B.6) is fulfilled either if $\beta = 0$ or $\beta = 1$, or if $\boldsymbol{\omega}_*(\mu_1) = \boldsymbol{\omega}_*(\mu_2)$. Non-trivial cases are such that $\beta \in (0, 1)$, for which expression (B.6) is equivalent to $\boldsymbol{\omega}_*(\mu_1) = \boldsymbol{\omega}_*(\mu_2)$. In the case of the local divergence indicator, this is the line $x = y$ restricted to $x \in [0, 1]$. In the case of the global divergence indicator, this is the point $(x, y) = (\frac{1}{2}, \frac{1}{2})$, as long as $\boldsymbol{\omega}_G(\mu_2) = \frac{1}{2}$ because μ_2 is a probability on N.

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