The recent history of model theory

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The history of Model Theory can be traced back to the work of Charles Sanders Peirce and Ernst Schröder, when semantics started playing a role in Logic. But the first outcome dates from 1915. It appears in the paper *Über Möglichkeiten im Relativkalkül* (Math. Ann. 76, 445-470) by Leopold Löwenheim. The period from 1915 to 1935 is, in words of R.L. Vaught, extraordinary. The method of elimination of quantifiers is developed and applied to give decision methods for the theories of $(\mathbb{Q}, <)$ (C.H. Langford), of $(\omega, +)$ and $(\mathbb{Z}, +, <)$ (M. Presburger) and, finally, of the field of complex numbers and of the ordered field of real numbers (A. Tarski). Also the completeness theorem of Kurt Gödel (1930) has to be considered among the earliest results of Model Theory.

The influence of Alfred Tarski was decisive in this early stage and in the successive years. This is due not only to his discovery of unquestionable definitions of the notions of truth and definability in a structure, but also to his founding of the basic notions of the theory, such as elementary equivalence and elementary extension. In the fifties and sixties Jerry Łoš introduced the ultraproducts, Ronald Fraïssé developed the back-and-forth methods and investigated amalgamation properties, and Abraham Robinson started his voluminous contribution to Model Theory, including his celebrated non-standard analysis. Robinson's non-standard analysis attracted the attention of mathematicians and philosophers. But at that time a feeling of exhaustion started pervading the whole theory. Daniel Lascar describes the situation as "un temps d'arrêt, comme si la machinerie, prête à tourner, ne savait quelle direction prendre." At this point Michael Morley appeared in the scene, causing what can be called a second birth of Model Theory.

A theory is said to be categorical at κ if it has only one model of cardinality κ up to isomorphism. In 1954 J. Łoš had asked whether, for every (countable) theory, categoricity at one uncountable cardinal implies categoricity at every other uncountable cardinal. In 1965 M. Morley publishes *Categoricity in power* (Transactions of the American Math. Society 114, 514-538) where he solves the problem in the affirmative. He introduces the (topological) spaces of types and defines a rank on types and formulas, now called Morley rank. A theory is called totally trascendental if all types have ordinal Morley rank. M. Morley shows that for countable theories, this is just ω-stability: over a countable set there are at most countably many complete types. He also proves that any theory categorical in an uncountable cardinality is ω-stable. In the proofs he uses heavily constructions with indiscernible sequences which had been studied a few years ago by Andrzej Ehrenfeucht and Andrzej Mostowski. The methods were partly combinatorial, based on Ramsey's theorem, and partly topological.

The importance of the results, methods and notions of M. Morley was recognized very soon. M. Morley investigated also the structure of countable models of uncountably categorical theories, but the main theorems on this topic are due to John T. Baldwin and Alistair H. Lachlan. In 1971 they published *On strongly minimal sets* (The Journal of Symbolic Logic 36, 79-96). Their analysis complements Morley's results, giving a clear picture of the structure of models of any countable uncountably categorical theory. Moreover it will act as a paradigm for further investigations.

In 1973 appeared the book *Model Theory* by C.C. Chang and H. J. Keisler. It has been the book of reference for decades. It contains an exposition of Morley's theorem. In the seventies several branches of Model Theory started having some independent life, like infinitary logics, admissible structures, Topological Model Theory, Recursive Model Theory, Abstract Model Theory and Model Theory of Peano Arithmetic. Some of them disappeared after a few years, some of them are still active. In pure Model Theory the dependence on hypotheses and methods from Set Theory was growing alarmingly. It took some time to extend Morley's ideas to a broader setting eliminating this dependence on Set Theory. This was done mainly by Saharon Shelah.

S. Shelah has led the research in pure Model Theory for twenty years. At the same time he was also leading Set Theory. In his Ph. D. Thesis he generalized Morley's theorem to uncountable theories. His first paper appeared in 1969. By now he has written more than seven hundred. He defined stable theories, a class of theories extending the ω-stable ones. Stable theories share some properties with ω-stable, but there are important differences. Shelah's ideology was finding a dividing line between theories where a classification of models by some kind of invariants can be done and theories with too many models to try to classify them. If the theory has 2^κ many non-isomorphic models of cardinality κ for all κ big enough, a classification by invariants is impossible. S. Shelah proved that unstable theories had that big number of models. Later he defined superstable theories, a kind of theories comprised between ω-stable and stable ones, and proved that also non-superstable theories had too many models. On the other hand he started developing a theory of independence and dimension intended to help analyzing and describing the way models are constructed from its basic parts. The theory needs sometimes the assumption of stability or superstability, but in some aspects it is completely general. It is based on the notion of "forking". Forking is defined combinatorily. There are intrinsic difficulties in understanding the very first steps of the theory and Shelah's particular way of writing definitions, theorems and proofs makes the whole thing really harder. Bruno Poizat explains the impact of Shelah with the following words: "L'intrusion de Shelah dans ce domain a découragé un certain nombre de bons mathématiciens, qui en sont sortis; d'autres on voulu y survivre; d'autres encore — les inconscients — ont voulu y entrer, et, à défaut de concurrencer Shelah dans son domain favori, du moins en traiter des aspects que Shelah n'avait pas considéré."

S. Shelah's main work is contained in his book *Classification Theory* first published in 1978. In its second (revised) edition (1990) S. Shelah was able to accomplish his goals providing a list of properties satisfying the following: for any countable theory having all these properties we can introduce invariants which allows us to classify up to isomorphism all its uncountable models; if some of these properties fails, the theory has too many models. Superstability is one of these properties. Not many interesting mathematical structures are stable. Modules and algebraically closed fields are examples of stable structures. It is not surprising that some people tried to use Model Theory to understand structures, like ordered fields and valued fields, which laid outside Shelah's paradise of stable theories. Hence two different branches of Model Theory started developing quite independently: stability theory and model theory applied to algebra. One of the few results combining notions and methods from the two branches was obtained by Angus Macintyre in 1971. He proved that any infinite ω-stable field is algebraically closed. Later Gregory Cherlin and S.Shelah generalized this result to superstable division rings.

Daniel Lascar and Bruno Poizat found what seemed to be a more reasonable way of presenting stability theory, based on the notions of heir and coheir. This encouraged more people to do stability theory. Some specifics topics, like model theory of modules or model theory of groups of finite Morley rank, acquired some independence from the main theory. The beautiful theory of stable groups appeared, mainly due to the effort of B. Poizat. A lot of work concentrated also on Vaught's conjecture: a countable complete theory with uncountably many countable models must have continuum many countable models.

Around 1985 two new topics attracted the interest of model theoreticians: geometrical stability theory and o-minimality. Geometrical stability theory concentrates on the analysis of the combinatorial geometries arising in sets with a nice independence relation, like strongly minimal sets, and also on the kind of classical algebraic structures, like groups and fields, which are interpretable in them. Boris Zil'ber started the study of the fine structure of totally categorical theories, strongly motivated by analogies with algebraic geometry. He conjectured, roughly speaking, that a strongly minimal set in an uncountably categorical theory has to be similar to either an algebraically closed field, or a vector space or an infinite set. This was refuted by Ehud Hrushovski, the most relevant contributor to Model Theory in the last fifteen years. E. Hrushovski played a fundamental role in geometrical stability theory, introduced a plethora of new notions and methods an proved several outstanding theorems, solving problems that had remained open for many years.

O-minimality is a restriction on the definable sets in presence of a linear ordering: only finite unions of intervals and points are definable. The name and the definition parallel the situation in a strongly minimal set, only that the setting includes now an ordering of the universe, which implies that the theory is unstable. The general theory was created by Anand Pillay, although some ideas had been anticipated by Lou van den Dries. The interesting thing is that being a theory of unstable structures, it shares many concepts, methods and results with stability theory. There is, for instance, a certain notion of independence which originates a kind of dimension. Moreover there are mathematically interesting structures which happen to be o-minimal, like the real ordered field and some of its expansions.

Both geometrical stability theory and o-minimality have been extremely successful in showing the importance of Model Theory to mathematics. In 1996 E. Hrushovski published *The Mordell-Lang conjecture for function fields* (Journal of the American Math. Society 9, 667-690), where he gave for the first time a proof of the Mordell-Lang conjecture on all characteristics. This was a hard open problem in diophantine geometry and had been unexpectedly solved by model-theoretic methods. The second great result, this time in o-minimality, was due to Alex Wilkie. In 1996 appeared his paper *Model completeness results for expansions of the real field by restricted Pfaffian functions and the exponential* *function* (Journal of the American Math. Society 9, 1051-1094), where he proves that the real exponential field is o-minimal and model-complete. It is still open if it has decidable theory.

After this two successful applications still something happened that changed the whole aspect of the theory. It was the rediscovery of simple theories by Byunghan Kim and Anand Pillay. Simple theories had been defined by S. Shelah in 1980 as a generalization of stable theories where forking still had some of its nice properties. But Shelah was not able to develop the theory further, mainly because he didn't see how to prove symmetry and transitivity of independence in this context. B. Kim did it in his Ph. D. Thesis in 1996 under the direction of A. Pillay. They discovered that simple theories is the right setting for developing the tools that had been typical from stability theory. Many concepts had to be adapted to the new situation and many new and interesting problems arose. Moreover new examples were found of simple unstable theories with mathematical significance. In the last four years a lot of work has been done, many researchers have switched to this new field and many students have started doing his investigations on these problems. Moreover E. Hrushovski has been able to apply again this new model-theoretic notions and methods to problems in other parts of mathematics, this time to give a new proof of the Manin-Mumford conjecture.

The barrier between pure model theory (stability) and applied model theory is vanishing. Surprisingly, the most successful applications of the theory originated in the more pure and abstract part. Presently there is a sense of unity in Model Theory and there are great hopes that the analysis could be extended beyond simple theories and the methods could be applied again to obtain results in other parts of mathematics.

Before finishing this picture of the last developments in Model Theory one should say something about Finite Model Theory. It has been quite independent of the main streams we have talked about. It is strongly connected with computability theory, mainly via complexity theory and via data base theory. The main problem in this context is whether P=NP or not. R. Fagin in the seventies discovered that NP-problems correspond to classes of finite structures which can be captured by existential second-order sentences. Since then a lot of work has been done characterizing different complexity classes by means of some extensions of first-order logic.

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