Newelski's analysis of Lascar strong types*

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September 15, 2002

1 Introduction

Definition 1 E_L is the finest bounded invariant equivalence relation and E_{KP} is the finest bounded type-definable equivalence relation. A *Lascar strong type* is an E_L-equivalence class and $\mathrm{Lstp}(a) = a/E_L$.

Definition 2 A formula $\theta(x, y)$ is *thick* if it is reflexive and symmetric and for some $n < \omega$ there are no a_i ($i < n$) such that $\models \neg\theta(a_i, a_j)$ for all $i < j < n$. $\mathrm{nc}(x, y)$ is the set of all thick formulas $\theta(x, y)$ and $\mathrm{nc}^n(x, y)$ is its composition:

1. $\mathrm{nc}^1(x, y) = \mathrm{nc}(x, y)$
2. $\mathrm{nc}^{n+1}(x, y) = \exists z(\mathrm{nc}^n(x, z) \wedge \mathrm{nc}(z, y))$.

The distance $d(a, b)$ is defined in a such a way that $d(a, a) = 0$ and for different a, b, $d(a, b)$ is the least $n < \omega$ (if there is some) for which there are $a_0, \ldots, a_n$ such that $a = a_0$, $b = a_n$ and for all $i < n$, a_i, a_{i+1} start an infinite indiscernible sequence. If there is no such n we put $d(a, b) = \infty$.

Facts 1.1 *1.* $\mathrm{nc}(x, y)$ *is a type and for all* a, b, $\mathrm{nc}(a, b)$ *if and only if* $a = b$ *or there is an infinite indiscernible sequence starting with* a, b.

2. $d(a, b) \leq n$ *if and only if* $\models \mathrm{nc}^n(a, b)$.

3. $E_L(a, b)$ *if and only if* $d(a, b) < \infty$ *if and only if* $\models \bigvee_{n<\omega} \mathrm{nc}^n(a, b)$.

4. $E_L = E_{KP}$ *if and only if* E_L *is type-definable.*

5. For any type $p(x) \in S(\emptyset)$, E_{KP} *is finer than any type-definable bounded equivalence relation on realizations of* $p(x)$.

Definition 3 The *diameter* of a set X is the supremum of all distances $d(a, b)$ of elements $a, b \in X$.

*Notes from the Research Seminar in Model Theory organized by T. Scanlon. Sessions from September 6, 2002 and September 20, 2002 dedicated to discuss Newelski's preprint *The diameter of a Lascar strong type.* Revised in December 10, 2002.

2 Newelski's derivative

Definition 4 Let K be an arbitrary topological space and let $\mathcal{A}$ be a family of subsets of K covering K. The *open analysis of K with respect to $\mathcal{A}$* is the family $(Z_\alpha : \alpha \in \mathrm{On})$ of open sets Z_α defined as follows:

1. $Z_0 = \bigcup_{A\in\mathcal{A}} \mathrm{int}(A)$
2. $Z_{\alpha+1} = \bigcup_{A\in\mathcal{A}} \mathrm{int}(A \cup Z_\alpha)$
3. $Z_\beta = \bigcup_{\alpha<\beta} Z_\alpha$ for limit β.

Clearly there is an ordinal β such that $Z_\beta = Z_{\beta+1}$. The least such β is called the *height* of the analysis. The *core* of the analysis is the set $C = K \smallsetminus \bigcup_{\alpha\in\mathrm{On}} Z_\alpha$. We say that K is $\mathcal{A}$-*analyzable* if the core is empty, i.e., if $K = \bigcup_{\alpha\in\mathrm{On}} Z_\alpha$. We define the *rank* of an element $a \in K \smallsetminus C$ as the least ordinal $\alpha = \mathrm{rk}(a)$ such that $a \in Z_\alpha$ and the rank of a non-empty $X \subseteq K \smallsetminus C$ as $\mathrm{rk}(A) = \min_{a\in X} \mathrm{rk}(a)$. Clearly $\mathrm{rk}(a)$ (and $\mathrm{rk}(A)$) is always zero or a successor ordinal.

Remark 2.1 *The Cantor-Bendixson derivative is a particular case of $\mathcal{A}$-analysis, namely, it is the analysis with respect to the family of all singletons $\mathcal{A} = \{\{a\} : a \in K\}$.*

Definition 5 Let $\mathcal{A}$ and $\mathcal{A}'$ be two families covering the topological space K. We say that $\mathcal{A}'$ is *finer* than $\mathcal{A}$ if every member of $\mathcal{A}'$ is contained in some member of $\mathcal{A}$.

Lemma 2.2 *If $\mathcal{A}'$ is finer than $\mathcal{A}$ then the core of an $\mathcal{A}$-analysis of K is contained in the core of an $\mathcal{A}'$-analysis of K. In particular, if K is $\mathcal{A}'$-analyzable, then K is $\mathcal{A}$-analyzable.*

Proof. Let $(Z_\alpha : \alpha \in \mathrm{On})$ the $\mathcal{A}$-analysis of K and let $(Z'_\alpha : \alpha \in \mathrm{On})$ be its $\mathcal{A}'$-analysis. Is easy to see by induction that $Z'_\alpha \subseteq Z_\alpha$.

Remark 2.3 *It follows from the previous lemma that the core of the Cantor-Bendixson analysis contains any other core coming from an analysis. A scattered space is analyzable with respect to any family.*

Lemma 2.4 *Let $(Z_\alpha : \alpha \in On)$ be the $\mathcal{A}$-analysis of K and let C be its core. Let $X \subseteq K$ a subspace with the induced topology, let $\mathcal{A}_X = \{A \cap X : A \in \mathcal{A}\}$, let $(Z^X_\alpha : \alpha \in \mathrm{On})$ be the $\mathcal{A}_X$-analysis of X and let C_X be its core. Then*

1. *$C_X \subseteq C$*
2. *If $C \subseteq X$, then $C_X = C$.*
3. *If $X \cap C = \emptyset$ then $C_X = \emptyset$ and X is $\mathcal{A}_X$-analyzable.*

Proof. *1.* By induction on α we show that $X \cap Z_\alpha \subseteq Z^X_\alpha$. Consider the case $\alpha + 1$. Assume $a \in X \cap Z_{\alpha+1}$. Then for some $A \in \mathcal{A}$ and some open $U \subseteq K$, $a \in U \subseteq A \cup Z_\alpha$. By inductive hypothesis $a \in U \cap X \subseteq (A \cap X) \cup Z^X_\alpha$. Hence $a \in \mathrm{int}_X((A \cap X) \cup Z^X_\alpha)$. Since $A \cap X \in \mathcal{A}_X$, we conclude that $a \in Z^X_{\alpha+1}$. The case $\alpha = 0$ is similar and the case α limit is clear.

2. Assume $C \subseteq X$ and assume also $C \smallsetminus C_X \neq \emptyset$. Let $a \in C \cap \bigcup_{\alpha \in \mathrm{On}} Z^X_\alpha$ be an element of minimal rank $\mathrm{rk}(a) = \beta = \mathrm{rk}(C)$ in the $\mathcal{A}_X$-analysis of X. Assume $\beta = \alpha + 1$ (the case $\beta = 0$ is similar). Then $a \in Z^X_{\alpha+1} \cap C$ and $C \cap Z^X_\alpha = \emptyset$. There is an open $U \subseteq K$ and some $A \in \mathcal{A}$ such that $a \in U \cap X \subseteq A \cup Z^X_\alpha$. Then $\emptyset \neq U \cap C \subseteq A$, which implies $C \cap \bigcup_{\alpha \in \mathrm{On}} Z_\alpha \neq \emptyset$, a contradiction.

3 is clear by *1*.

Lemma 2.5 *Let* $(Z_\alpha : \alpha \in \mathrm{On})$ *be an* $\mathcal{A}$*-analysis of* K*. Let* $Z = \bigcup_{\alpha \in \mathrm{On}} Z_\alpha$.

1. Z_0 *is dense in* Z.
2. $Z_{\alpha+1} \smallsetminus Z_\alpha$ *is dense in* $Z \smallsetminus Z_\alpha$ *for all* α.

Proof. We prove *1*. The proof of *2* is similar. We show that for any open set O, if $Z \cap O \neq \emptyset$ then $Z \cap O$ has rank zero. Assume $\mathrm{rk}(Z \cap O) = \alpha + 1$. Choose $a \in O \cap Z$ of minimal rank $\alpha + 1$ and choose $U \subseteq K$ open and $A \in \mathcal{A}$ such that $a \in U \subseteq A \cup Z_\alpha$. Then $O \cap U$ is open and $a \in O \cap U \subseteq A$. Hence $a \in Z_0$ and $\mathrm{rk}(a) = 0$, a contradiction.

Lemma 2.6 *Let* $(Z_\alpha : \alpha \in \mathrm{On})$ *be an* $\mathcal{A}$*-analysis of* K*, let* $Z = \bigcup_{\alpha \in \mathrm{On}} Z_\alpha$ *and assume* $\mathcal{A}$ *is closed under finite unions.*

1. *If for some* $A \in \mathcal{A}$*,* $Z_\beta \subseteq A$*, then* $Z_\beta = Z$.
2. *If* U *is open and for some* $A \in \mathcal{A}$*,* $U \cap (Z_{\alpha+1} \smallsetminus Z_\alpha) \subseteq A$*, then* $U \cap Z \subseteq Z_{\alpha+1}$.

Proof. *1.* We show that $Z_{\beta+1} = Z_\beta$. Let $a \in Z_{\beta+1} \smallsetminus Z_\beta$. For some $B \in \mathcal{A}$ and some open set U, $a \in U \subseteq B \cup Z_\beta$. Since $Z_\beta \subseteq A$ and $A \cup B \in \mathcal{A}$, a is in the interior of some element of $\mathcal{A}$ and therefore $a \in Z_0 \subseteq Z_\beta$.

2. We first show that $U \cap Z_{\alpha+2} \subseteq Z_{\alpha+1}$. Let $a \in U \cap Z_{\alpha+2}$. For some open W and some $B \in \mathcal{A}$, $a \in W \subseteq B \cup Z_{\alpha+1}$. Then $a \in U \cap W \subseteq B \cup A \cup Z_\alpha$ and $B \cup A \in \mathcal{A}$. Therefore $a \in Z_{\alpha+1}$. Now by lemma 2.5 we know that $Z_{\alpha+2} \smallsetminus Z_{\alpha+1}$ is dense in $Z \smallsetminus Z_{\alpha+1}$. Therefore if the open set $U \cap Z$ has elements in $Z \smallsetminus Z_{\alpha+1}$ then it has elements in $Z_{\alpha+2} \smallsetminus Z_{\alpha+1}$, which is impossible, since $U \cap Z_{\alpha+2} \subseteq Z_{\alpha+1}$.

Lemma 2.7 *If a compact space* K *is* $\mathcal{A}$*-analyzable, its height* β *is zero or a successor ordinal and its last level* $Z_\beta \smallsetminus \bigcup_{\alpha<\beta} Z_\alpha$ *is covered by finitely many elements of* $\mathcal{A}$.

Proof. The first point follows from the fact that $\{Z_0\} \cup \{Z_{\alpha+1} : \alpha < \beta\}$ is an open cover of K. For the second, let $Z_{\beta-1} = \bigcup_{\alpha<\beta} Z_\alpha$. Since $K = Z_\beta$, for any $a \in K$ there is an open U and some $A \in \mathcal{A}$ such that $a \in U \subseteq A \cup Z_{\beta-1}$. By compactness, for some $k < \omega$ there are open sets U_i and $A_i \in \mathcal{A}$ $(i < k)$ such that $U_i \subseteq A_i \cup Z_{\beta-1}$ and $K \subseteq \bigcup_{i<k} U_i$. Hence $Z_\beta \setminus Z_{\beta-1} \subseteq \bigcup_{i<k} A_i$.

Proposition 2.8 *Let $f : K' \to K$ be a surjective continuous mapping between the topological spaces K, K'. Let $\mathcal{A}$, $\mathcal{A}'$ be families covering K and K', let $(Z_\alpha : \alpha \in \mathrm{On})$, $(Z'_\alpha : \alpha \in \mathrm{On})$ be its analysis and let C, C' be its cores.*

1. *If $\{f^{-1}[A] : A \in \mathcal{A}\}$ is finer than $\mathcal{A}'$ then $f[C'] \subseteq C$.*
2. *Assume K, K' are compact Hausdorff spaces and $\mathcal{A}' = \{f^{-1}[A] : A \in \mathcal{A}\}$. Then $f[C'] = C$. In particular, K' is $\mathcal{A}'$-analyzable if and only if K is $\mathcal{A}$-analyzable.*
3. *Assume K, K' are compact Hausdorff spaces and $\mathcal{A} = \{f[A] : A \in \mathcal{A}'\}$. If K' is $\mathcal{A}'$-analyzable, then K is $\mathcal{A}$-analyzable.*

Proof. *1.* We show by induction on α that $f^{-1}[Z_\alpha] \subseteq Z'_\alpha$ for all α. From this it follows immediately that $f[C'] \subseteq C$. We consider the case $\alpha + 1$. The case $\alpha = 0$ is similar and the case α limit is clear. Using the inductive hypothesis and all other hypotheses we see that $f^{-1}[Z_{\alpha+1}] = f^{-1}[\bigcup_{A\in\mathcal{A}} \mathrm{int}(A \cup Z_\alpha)] = \bigcup_{A\in\mathcal{A}} f^{-1}[\mathrm{int}(A \cup Z_\alpha)] \subseteq \bigcup_{A\in\mathcal{A}} \mathrm{int}(f^{-1}[A \cup Z_\alpha]) = \bigcup_{A\in\mathcal{A}} \mathrm{int}(f^{-1}[A] \cup f^{-1}[Z_\alpha]) \subseteq \bigcup_{A\in\mathcal{A}} \mathrm{int}(f^{-1}[A] \cup Z'_\alpha) \subseteq \bigcup_{A\in\mathcal{A}'} \mathrm{int}(A \cup Z'_\alpha) = Z'_{\alpha+1}$.

2. By *1* we know that $f[C'] \subseteq C$. Assume $C \setminus f[C'] \neq \emptyset$. We will reach a contradiction. First observe that we may assume $K = C$. This follows from lemma 2.4 since we may restrict to the subspace C of K and the subspace $f^{-1}[C]$ of K'. Both are compact Hausdorff, its cores are C and C' and the corresponding restricted covering families $\mathcal{A}_C$ and $\mathcal{A}'_{f^{-1}[C]}$ still verify that $\mathcal{A}'_{f^{-1}[C]} = \{f^{-1}[A] : A \in \mathcal{A}_C\}$. We will use frequently the fact that in a compact Hausdorff space if U is open and $a \in U$ there is a closed set F such that $a \in \mathrm{int}(F) \subseteq F \subseteq U$. Since $f[C']$ is closed (because f is continuous, K' is compact and K is Hausdorff) and it is a proper subset of K, there is a closed set $F \subseteq K$ such that $\mathrm{int}(F) \neq \emptyset$ and $F \cap f[C'] = \emptyset$. Clearly $f^{-1}[F] \subseteq K' \setminus C' = \bigcup_{\alpha\in\mathrm{On}} Z'_\alpha$ and from this it follows that $f^{-1}[F] \subseteq Z'_\alpha$ for some ordinal α. We assume we have chosen α minimal with the property that there is a closed set F in K such that

$$\mathrm{int}(F) \neq \emptyset,\ F \cap f[C'] = \emptyset \text{ and } f^{-1}[F] \subseteq Z'_\alpha.$$

By compactness, $\alpha = 0$ or is a successor ordinal. Let $Z'_{\alpha-1} = \bigcup_{\beta<\alpha} Z'_\beta$. Hence $Z'_{\alpha-1} = \emptyset$ if $\alpha = 0$ and $Z'_{\alpha-1} = Z'_\beta$ if $\beta+1 = \alpha$. In any case, for every $a \in f^{-1}[F]$ there is some $A \in \mathcal{A}'$ such that $a \in \mathrm{int}(A \cup Z'_{\alpha-1})$ and therefore there is a closed set G such that $a \in \mathrm{int}(G) \subseteq G \subseteq \mathrm{int}(A \cup Z'_{\alpha-1})$. The open sets $\mathrm{int}(G)$ cover the closed set $f^{-1}[F]$. By compactness finitely many of them suffice. Hence for some $k < \omega$ there are $A_0, \ldots, A_{k-1} \in \mathcal{A}'$ and closed sets $G_0, \ldots, G_{k-1}$

such that $f^{-1}[F] \subseteq \bigcup_{i<k} \mathrm{int}(G_i)$ and for all $i < k$, $G_i \subseteq \mathrm{int}(A_i \cup Z'_{\alpha-1})$. Let $G = f[\bigcup_{i<k} G_i \smallsetminus Z'_{\alpha-1}]$. It is a closed subset of K. We will see now that G is nowhere dense, i.e., $\mathrm{int}(G) = \emptyset$. Since the union of two nowhere dense sets is again nowhere dense and $G = \bigcup_{i<k} f[G_i \smallsetminus Z'_{\alpha-1}]$, if G is not nowhere dense, for some $i < k$ some $f[G_i \smallsetminus Z'_{\alpha-1}]$ has non-empty interior. Since $G_i \subseteq A_i \cup Z'_{\alpha-1}$, it follows that $f[G_i \smallsetminus Z'_{\alpha-1}] \subseteq f[A_i]$. But $f[A_i] \in \mathcal{A}$, so we have find an element of $\mathcal{A}$ with non-empty interior, which contradicts our first assumption that $C = K$. Hence we have to admit that G is nowhere dense. Clearly there are points in $\mathrm{int}(F) \smallsetminus G$ and we can separate any of them from G by disjoint open sets. Therefore we can find $H \subseteq F$ closed such that $\mathrm{int}(H) \neq \emptyset$ and $H \cap G = \emptyset$. Now $H \cap f[C'] = \emptyset$ (because $H \subseteq F$) and $f^{-1}[H] \subseteq Z'_{\alpha-1}$ (because $f^{-1}[H]$ is contained in $\bigcup_{i<k} G_i$ and H is disjoint to G), and this shows that $\alpha > 0$ and contradicts its minimality.

3. Let $\mathcal{A} = \{f[A] : A \in \mathcal{A}'\}$. Observe that $\mathcal{A}'$ refines $\{f^{-1}[A] : A \in \mathcal{A}\}$ and use lemma 2.2 and point *2*.

3 Lascar strong types

Theorem 3.1 *Let X be a union of Lascar strong types of infinite diameter. Assume that all elements of X have the same type over the empty set and that X is type-definable over some parameters. Let $\bar{a} = (a_i : i \in I)$ be a sequence of representatives of the different Lascar strong types in X. Then*

1. *If $\mathcal{A} = \{Y_i^n : i \in I, n < \omega\}$ where for any $i \in I$ and any $n < \omega$, $Y_i^n = \{\mathrm{tp}(b/\bar{a}) : d(b, a_i) \leq n\}$ then $Y = \{\mathrm{tp}(b/\bar{a}) : b \in X\}$ is not $\mathcal{A}$-analyzable.*

2. *There is a $X' \subseteq X$ type-definable over $\bar{a}$ such that for every formula $\varphi(x)$ over $\bar{a}$, if some element of X' realizes $\varphi(x)$ then there are at least two realizations of $\varphi(x)$ in X' with different Lascar strong type.*

Proof. *1.* Since X is $\bar{a}$-invariant and it is type-definable over some set of parameters, it is also type-definable over $\bar{a}$. Let $Y_i = \{tp(b/\bar{a}) : \mathrm{Lstp}(b) = \mathrm{Lstp}(a_i)\}$. Hence $Y_i = \bigcup_{n<\omega} Y_i^n$ and $Y = \bigcup_{i \in I} Y_i^n$. Y and every Y_i^n are closed subsets of the Stone space $S(\bar{a})$. Fix some $i \in I$ and consider the restriction mapping $f : S(\bar{a}) \to S(a_i)$ defined by $f(p) = p \restriction a_i$. It is a continuous surjection. Let $U = f(Y) = \{\mathrm{tp}(b/a_i) : b \in X\}$, let $U_i = f[Y_i] = \{\mathrm{tp}(b/a_i) : \mathrm{Lstp}(b) = \mathrm{Lstp}(a_i\}$ and let $U_i^n = f[Y_i^n] = \{\mathrm{tp}(b/a_i) : d(b, a_i) \leq n\}$. U is a closed subspace of $S(a_i)$, U_i^n is a closed subspace of U and U_i is a subspace not necessarily closed $U_i^n \subseteq U_i \subseteq U$. If Y is $\mathcal{A}$-analizable then, by lemma 2.2, it is also analyzable with respect to $\{Y_i^n : n < \omega\} \cup \{\bigcup_{j \neq i} Y_j\}$ and by proposition 2.8 U is analyzable with respect to $\{U_i^n : n < \omega\} \cup \{\{\mathrm{tp}(b/a_i) : b \in X \text{ and } \mathrm{Lstp}(b) \neq \mathrm{Lstp}(a_i)\}\}$. By lemma 2.4 U_i is analyzable with respect to $\{U_i^n : n < \omega\}$. By isomorphism, for any $b \in X$ the space $U_b = \{\mathrm{tp}(c/b) : \mathrm{Lstp}(c) = \mathrm{Lstp}(b)\}$ is analyzable with respect to $\{U_b^n : n < \omega\}$ where $U_b^n = \{\mathrm{tp}(c/b) : d(c, b) \leq n\}$. On the other hand, by lemma 2.2, if $Y^n = \bigcup_{i \in I} Y_i^n$ then Y is also analyzable with respect to

$\{Y^n : n < \omega\}$. Let $(Z_\alpha : \alpha \in \mathrm{On})$ be this last analysis of Y and let $\alpha^* + 1$ be its height.

Now we claim that we can find $b \in X$, ordinals $\alpha < \beta \leq \alpha^*$, formulas $\varphi(x, \bar{y})$, $\psi(x, z)$ and natural numbers n, m such that

1. $\mathrm{tp}(b/\bar{a}) \in Z_{\beta+1} \setminus Z_\beta$
2. $\psi(x, b) \vdash \varphi(x, \bar{a})$
3. $\emptyset \neq U_b \cap [\psi(x, b)] \subseteq U_b^m$
4. $Y \cap [\varphi(x, \bar{a})] \subseteq Z_\alpha \cup Y^n$

We take $\beta = \alpha^*$ and choose $b \in X$ arbitrary with $\mathrm{tp}(b/\bar{a}) \in Z_{\beta+1} \setminus Z_\beta$. Since this is the last level of the analysis, by lemma 2.7 it is covered by just one Y^k. This means that for every $c \in X$ with $\mathrm{tp}(c/\bar{a}) \in Z^{\beta+1} \setminus Z_\beta$ there is $i \in I$ such that $d(c, a_i) \leq k$. Let $(Z_\alpha^b : \alpha \in \mathrm{On})$ be the analysis of U_b with respect to $\{U_b^n : n < \omega\}$. If there is a bound n on $d(c, b)$ for $c \in X$ such that $\mathrm{tp}(c/b) \in Z_0^b$ then, by lemma 2.6, $Z_0^b \subseteq U_b^n$ and the analysis stops in one step and $U_b = Z_0^b$. But in this case $\{c \in X : \mathrm{Lstp}(c) = \mathrm{Lstp}(b)\}$ has a diameter bounded by n, contrarily to the initial assumption. Therefore there is no such bound and we can find $c \in X$ such that $\mathrm{tp}(c/b) \in Z_0^b$ and $d(c, b) > 2k$. Choose $i \in I$ such that $\mathrm{Lstp}(b) = \mathrm{Lstp}(a_i)$. Since $\mathrm{tp}(b/\bar{a}) \in Z_{\beta+1} \setminus Z_\beta$, by choice of k, $d(b, a_i) \leq k$. It follows that $d(c, a_i) > k$ and therefore $\mathrm{tp}(c/\bar{a}) \in Z_\beta$. For the same reason, for any other $c' \models p(x) = \mathrm{tp}(c/b)$ we have $\mathrm{tp}(c'/\bar{a}) \in Z_\beta$. Now $Y \setminus Z_\beta$ is a closed subset of $S(\bar{a})$ and therefore it is the set of types in Y extending a partial type $\pi(x, \bar{a})$. We have seen that $\pi(x, \bar{a}) \cup p(x)$ is inconsistent. Hence there are $\psi(x, b) \in \mathrm{tp}(c/b)$ and $\varphi(x, \bar{a}) \in \pi(x, \bar{a})$ such that $\psi(x, b) \vdash \varphi(x, \bar{a})$ and $Y \cap [\varphi(x, a)] \subseteq Z_\beta$. Since $\mathrm{tp}(c/b) \in Z_0^b$, there is some open set W in $S(b)$ and some $m < \omega$ such that $\mathrm{tp}(c/b) \in W \cap U_b \subseteq U_b^m$. We may assume that W is a clopen set defined by $\psi(x, b)$. Hence $\emptyset \neq [\psi(x, b)] \cap U_b \subseteq U_b^m$. Now $Y \cap [\varphi(x, \bar{a})]$ is compact and it is contained in Z_β. If β is limit, clearly it is also contained in Z_α for some $\alpha < \beta$. In the case $\beta = \alpha + 1$ we apply the definition of the analysis and compactness to obtain some $n < \omega$ such that $Y \cap [\varphi(x, a)] \subseteq Z_\alpha \cup Y^n$. Therefore all conditions *1* to *4* are satisfied and the claim is proven.

Let β be minimal for which there are $\alpha < \beta$, $b \in X$, ψ and φ with the properties 1-4. We will show that we still can find a smaller β, which is a contradiction and will finish the proof. We start choosing $\theta(z, \bar{a}) \in \mathrm{tp}(b/\bar{a})$ such that $\psi(x, z) \wedge \theta(z, \bar{a}) \vdash \varphi(x, \bar{a})$. For $\gamma < \beta$, $Y \cap [\theta(z, \bar{a})]$ is not contained in $Z_{\gamma+1}$ and therefore, by lemma 2.6, $[\theta(z, \bar{a})] \cap (Z_{\gamma+1} \setminus Z_\gamma)$ can not be covered by just one Y_k. This means that there is no bound on $d(c, a_i)$ for c and a_i such that $\models \theta(c, \bar{a})$, $\mathrm{Lstp}(c) = \mathrm{Lstp}(a_i)$ and $\mathrm{tp}(c/\bar{a}) \in Z_{\gamma+1} \setminus Z_\gamma$. In case β is a successor ordinal we take as β' the predecessor of β and in case β is a limit ordinal we choose β' such that $\alpha < \beta' < \beta$. Choose now some b' and $i \in I$ such that $\mathrm{tp}(b'/\bar{a}) \in Z_{\beta'+1} \setminus Z_\beta$, $\models \theta(b', \bar{a})$, $\mathrm{Lstp}(b') = \mathrm{Lstp}(a_i)$ and $d(b', a_i) > n + m$. Since $b \equiv b'$, we still have that $\emptyset \neq U_{b'} \cap [\psi(x, b')] \subseteq U_{b'}^m$. We claim that there is no c' such that $\mathrm{Lstp}(c') = \mathrm{Lstp}(b')$, $\models \psi(c', b')$ and $\mathrm{tp}(c'/\bar{a}) \notin Z_\alpha$. If there is such a c', we see that $\models \varphi(c', \bar{a})$ and hence $\mathrm{tp}(c'/\bar{a}) \in Y \cap [\varphi(x, \bar{a})] \setminus Z_\alpha$ and by point 4 $\text{tp}(c'/\bar{a}) \in Y^n$. But this means that $d(c', a_i) \leq n$ which contradicts the facts that $d(b', a_i) > n + m$ and $d(c', b') \leq m$. So, there is no such c'. Take some c' such that $\text{Lstp}(c') = \text{Lstp}(b')$ and $\models \psi(c', b')$, and let $p'(x) = \text{tp}(c'/b')$. If $\pi'(x, \bar{a})$ is a partial type characterizing the closed set $Y \setminus Z_\alpha$, then $p'(x) \cup \pi'(x, \bar{a})$ is inconsistent. As above, we find $\psi'(x, b') \in p'(x)$ and $\varphi'(x, \bar{a})$ such that $\psi'(x, b') \vdash \varphi'(x, \bar{a})$ and $Y \cap [\varphi'(x, \bar{a})] \subseteq Z_\alpha$. As in the initial situation from this follows that $Y \cap [\varphi(x, \bar{a})] \subseteq Z_{\alpha'} \cup Y^{n'}$ for some $\alpha' < \alpha$ and some $n' < \omega$. We may assume that $\psi'(x, b') \vdash \psi(x, b')$ and therefore $\emptyset \neq U_{b'} \cap [\psi'(x, b')] \subseteq U^m_{b'}$ is still true.

2. We know that Y is not $\mathcal{A}$-analyzable and therefore the core C is a non-empty closed subset of $S(\bar{a})$. Hence the set $X' = \{b \in X : \text{tp}(b/\bar{a}) \in C\}$ is type-definable over $\bar{a}$. Assume $\varphi(x)$ is a formula over $\bar{a}$ which is realized in X' but all whose realizations in X' have the same Lascar strong type over the empty set, say the same Lascar strong type as a_i. Then $\emptyset \neq C \cap [\varphi(x)] \subseteq Y_i = \bigcup_{n<\omega} Y_i^n$. Now we use the Baire category theorem in C. Since $C \cap [\varphi(x)]$ is a non-empty open set in C, it is not meager, i.e., it is not a countable union of nowhere dense sets. Hence for some n, $C \cap Y_i^n$ has non-empty interior in C, that is, there is some non-empty open set W in $S(\bar{a})$ such that $W \cap C \subseteq Y_i^n$. Then $W \cap Y \subseteq Y_i^n \cup (Y \setminus C)$, which means that the analysis continues beyond $Y \setminus C$, a contradiction.

Theorem 3.2 *Let X be a union of Lascar strong types of infinite diameter. Assume that all elements of X have the same type over the empty set and that X is type-definable over some parameters. Then there are at least 2^ω Lascar strong types realized in X.*

Proof. Let X' as in point *2* of theorem 3.1. We first observe that whenever we have $a, b \in X'$ with different Lascar strong type, then for every $n < \omega$ we can find formulas $\varphi(x) \in \text{tp}(a/\bar{a})$ and $\psi(x) \in \text{tp}(b/\bar{a})$ such that $d(a', b') > n$ for all $a', b' \in X'$ such that $\models \varphi(a')$ and $\models \psi(b')$. This follows from the fact that if $p(x) = \text{tp}(a/\bar{a})$ and $q(x) = \text{tp}(b/\bar{a})$ then $\text{nc}^n(x, y) \cup p(x) \cup q(y)$ is inconsistent. Now this allows us to construct a tree of formulas $(\varphi_s(x) : s \in {}^{<\omega}2)$ such that

1. $\varphi_s(x)$ is a formula over $\bar{a}$ and it is realized in X'.
2. $\varphi_s(x) \vdash \varphi_t(x)$ if $t \subseteq s$.
3. If $s, t \in {}^n2$ are different then $d(a, b) \geq n$ for all $a, b \in X'$ such that $\models \varphi_s(a)$ and $\models \varphi_t(b)$.

Let $\Phi(x, \bar{a})$ a type over $\bar{a}$ defining X'. For all $\eta \in {}^\omega 2$, we have a type $p_\eta(x) = \Phi(x, \bar{a}) \cup \{\varphi_{\eta \restriction n}(x) : n < \omega\}$ and for different η, η' if $a \models p_\eta$ and $a' \models p'_\eta$, then $\text{Lstp}(a) \neq \text{Lstp}(a')$.

Corollary 3.3 *1. If a Lascar strong type is type-definable over some parameters, then it has finite diameter.*

2. *If a* E_{KP}*-class is not a Lascar strong type, then it splits into at least* 2^ω *Lascar strong types.*

3. *If for each* $n < \omega$ *there is a Lascar strong type of diameter at least* n*, then there is a Lascar strong type which is not type-definable, even with parameters, and therefore* $E_L \neq E_{KP}$*.*

4. *If* E_L *is type-definable over some parameters, then for some* n *it is defined by* $\mathrm{nc}^n(x, y)$*.*

Proof. *1* follows directly from theorem 3.2.

2. Consider a E_{KP}-class a/E_{KP} which is not a Lascar strong type. It is a union of Lascar strong types and it is type-definable over a. To apply theorem 3.2 we have to show that all its Lascar strong types have infinite diameter. Assume not. Let $p(x) = \mathrm{tp}(a)$ and let X be the set of all realizations of $p(x)$. Since X includes a/E_{KP} and all Lascar strong types contained in X are isomorphic, all have diameter bounded by n for some fixed n. Therefore in X the relation E_L of equality of Lascar strong types is type-definable by $\mathrm{nc}^n(x, y)$. Since E_L restricted to X is a bounded type-definable relation, it is refined by E_{KP} restricted to X. But this means that on X they coincide and therefore that a/E_{KP} is a Lascar strong type, which is a contradiction.

3. For each $n < \omega$ fix a sequence a_n whose Lascar strong type $\mathrm{Lstp}(a_n) = a_n/E_L$ has diameter at least n and consider $a = (a_n : n < \omega)$. It is easy to check that $\mathrm{Lstp}(a)$ has infinite diameter. By point *1* it is not type-definable.

4 Assume E_L is type-definable. By *3* there is a bound $n < \omega$ for the diameter of any Lascar strong type. Therefore $\mathrm{nc}^n(x, y)$ defines E_L.