Newelski’s analysis of Lascar strong types

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1 Introduction

Definition 1 \( E_L \) is the finest bounded invariant equivalence relation and \( E_{KP} \) is the finest bounded type-definable equivalence relation. A Lascar strong type is an \( E_L \)-equivalence class and \( \text{Lstp}(a) = a/E_L \).

Definition 2 A formula \( \theta(x, y) \) is thick if it is reflexive and symmetric and for some \( n < \omega \) there are no \( a_i \) (\( i < n \)) such that \( |= \neg \theta(a_i, a_j) \) for all \( i < j < n \). \( \text{nc}(x, y) \) is the set of all thick formulas \( \theta(x, y) \) and \( \text{nc}^n(x, y) \) is its composition:

1. \( \text{nc}^1(x, y) = \text{nc}(x, y) \)
2. \( \text{nc}^{n+1}(x, y) = \exists z(\text{nc}^n(x, z) \land \text{nc}(z, y)) \).

The distance \( d(a, b) \) is defined in such a way that \( d(a, a) = 0 \) and for different \( a, b \), \( d(a, b) \) is the least \( n < \omega \) (if there is some) for which there are \( a_0, \ldots, a_n \) such that \( a = a_0, b = a_n \) and for all \( i < n, a_i, a_{i+1} \) start an infinite indiscernible sequence. If there is no such \( n \) we put \( d(a, b) = \infty \).

Facts 1.1

1. \( \text{nc}(x, y) \) is a type and for all \( a, b \), \( \text{nc}(a, b) \) if and only if \( a = b \) or there is an infinite indiscernible sequence starting with \( a, b \).
2. \( d(a, b) \leq n \) if and only if \( |= \text{nc}^n(a, b) \).
3. \( E_L(a, b) \) if and only if \( d(a, b) < \infty \) if and only if \( |= \bigvee_{n<\omega} \text{nc}^n(a, b) \).
4. \( E_L = E_{KP} \) if and only if \( E_L \) is type-definable.
5. For any type \( p(x) \in S(\emptyset) \), \( E_{KP} \) is finer than any type-definable bounded equivalence relation on realizations of \( p(x) \).

Definition 3 The diameter of a set \( X \) is the supremum of all distances \( d(a, b) \) of elements \( a, b \in X \).

2 Newelski’s derivative

Definition 4 Let $K$ be an arbitrary topological space and let $\mathcal{A}$ be a family of subsets of $K$ covering $K$. The open analysis of $K$ with respect to $\mathcal{A}$ is the family $(Z_\alpha : \alpha \in \text{On})$ of open sets $Z_\alpha$ defined as follows:

1. $Z_0 = \bigcup_{A \in \mathcal{A}} \text{int}(A)$
2. $Z_{\alpha + 1} = \bigcup_{A \in \mathcal{A}} \text{int}(A \cup Z_\alpha)$
3. $Z_\beta = \bigcup_{\alpha < \beta} Z_\alpha$ for limit $\beta$.

Clearly there is an ordinal $\beta$ such that $Z_\beta = Z_{\beta + 1}$. The least such $\beta$ is called the height of the analysis. The core of the analysis is the set $C = K \setminus \bigcup_{\alpha \in \text{On}} Z_\alpha$. We say that $K$ is $\mathcal{A}$-analyzable if the core is empty, i.e., if $K = \bigcup_{\alpha \in \text{On}} Z_\alpha$. We define the rank of an element $a \in K \setminus C$ as $\text{rk}(a)$ and the rank of a non-empty $X \subseteq K \setminus C$ as $\text{rk}(A) = \min_{a \in X} \text{rk}(a)$. Clearly $\text{rk}(a)$ (and $\text{rk}(A)$) is always zero or a successor ordinal.

Remark 2.1 The Cantor-Bendixson derivative is a particular case of $\mathcal{A}$-analysis, namely, it is the analysis with respect to the family of all singletons $\mathcal{A} = \{\{a\} : a \in K\}$.

Definition 5 Let $\mathcal{A}$ and $\mathcal{A}'$ be two families covering the topological space $K$. We say that $\mathcal{A}'$ is finer than $\mathcal{A}$ if every member of $\mathcal{A}'$ is contained in some member of $\mathcal{A}$.

Lemma 2.2 If $\mathcal{A}'$ is finer than $\mathcal{A}$ then the core of an $\mathcal{A}$-analysis of $K$ is contained in the core of an $\mathcal{A}'$-analysis of $K$. In particular, if $K$ is $\mathcal{A}'$-analyzable, then $K$ is $\mathcal{A}$-analyzable.

Proof. Let $(Z_\alpha : \alpha \in \text{On})$ the $\mathcal{A}$-analysis of $K$ and let $(Z'_\alpha : \alpha \in \text{On})$ be its $\mathcal{A}'$-analysis. It is easy to see by induction that $Z'_\alpha \subseteq Z_\alpha$.

Remark 2.3 It follows from the previous lemma that the core of the Cantor-Bendixson analysis contains any other core coming from an analysis. A scattered space is analyzable with respect to any family.

Lemma 2.4 Let $(Z_\alpha : \alpha \in \text{On})$ be the $\mathcal{A}$-analysis of $K$ and let $C$ be its core. Let $X \subseteq K$ a subspace with the induced topology, let $\mathcal{A}_X = \{A \cap X : A \in \mathcal{A}\}$, let $(Z_{\alpha X} : \alpha \in \text{On})$ be the $\mathcal{A}_X$-analysis of $X$ and let $C_X$ be its core. Then

1. $C_X \subseteq C$
2. If $C \subseteq X$, then $C_X = C$.
3. If $X \cap C = \emptyset$ then $C_X = \emptyset$ and $X$ is $\mathcal{A}_X$-analyzable.
Lemma 2.6 Let \( Z_\alpha : \alpha \in \text{On} \) be an \( \mathcal{A} \)-analysis of \( K \). Let \( Z = \bigcup_{\alpha \in \text{On}} Z_\alpha \).

1. \( Z_0 \) is dense in \( Z \).
2. \( Z_{\alpha+1} \setminus Z_\alpha \) is dense in \( Z \setminus Z_\alpha \) for all \( \alpha \).

Proof. We prove 1. The proof of 2 is similar. We show that for any open set \( O \), if \( Z \cap O \neq \emptyset \) then \( Z \cap O \) has rank zero. Assume \( \text{rk}(Z \cap O) = \alpha + 1 \). Choose \( a \in O \cap Z \) of minimal rank \( \alpha + 1 \) and choose \( U \subseteq K \) open and \( A \subseteq \mathcal{A} \) such that \( a \in U \subseteq A \cup Z_\alpha \). Then \( \emptyset \neq U \cap Z \subseteq A \). Hence \( a \in Z_0 \) and \( \text{rk}(a) = 0 \), a contradiction.

Lemma 2.7 If a compact space \( K \) is \( \mathcal{A} \)-analyzable, its height \( \beta \) is zero or a successor ordinal and its last level \( Z_\beta \) is \( \bigcup_{\alpha < \beta} Z_\alpha \) is covered by finitely many elements of \( \mathcal{A} \).
Proof. The first point follows from the fact that \( \{Z_0\} \cup \{Z_{\alpha+1} : \alpha < \beta\} \) is an open cover of \( K \). For the second, let \( Z_{\beta-1} = \bigcup_{\alpha<\beta} Z_\alpha \). Since \( K = Z_\beta \), for any \( a \in K \) there is an open \( U \) and some \( A \in \mathcal{A} \) such that \( a \in U \subset A \cup Z_{\beta-1} \). By compactness, for some \( k < \omega \) there are open sets \( U_i \) and \( A_i \in \mathcal{A} \) \( (i < k) \) such that \( U_i \subset A_i \cup Z_{\beta-1} \) and \( K \subset \bigcup_{i<k} U_i \). Hence \( Z_\beta \setminus Z_{\beta-1} \subset \bigcup_{i<k} A_i \).

Proposition 2.8 Let \( f : K' \to K \) be a surjective continuous mapping between the topological spaces \( K, K' \). Let \( \mathcal{A}, \mathcal{A}' \) be families covering \( K \) and \( K' \), let \( (Z_\alpha : \alpha \in \text{On}) \), \( (Z'_\alpha : \alpha \in \text{On}) \) be its analysis and let \( C, C' \) be its cores.

1. If \( \{f^{-1}[A] : A \in \mathcal{A}\} \) is finer than \( \mathcal{A}' \) then \( f[C'] \subset C \).

2. Assume \( K, K' \) are compact Hausdorff spaces and \( \mathcal{A}' = \{f^{-1}[A] : A \in \mathcal{A}\} \).

Then \( f[C'] = C \). In particular, \( K' \) is \( \mathcal{A}' \)-analyzable if and only if \( K \) is \( \mathcal{A} \)-analyzable.

3. Assume \( K, K' \) are compact Hausdorff spaces and \( \mathcal{A} = \{f[A] : A \in \mathcal{A}'\} \). If \( K' \) is \( \mathcal{A}' \)-analyzable, then \( K \) is \( \mathcal{A} \)-analyzable.

Proof. 1. We show by induction on \( \alpha \) that \( f^{-1}[Z_\alpha] \subset Z'_\alpha \) for all \( \alpha \). From this it follows immediately that \( f[C'] \subset C \). We consider the case \( \alpha + 1 \). The case \( \alpha = 0 \) is similar and the case \( \alpha \) limit is clear. Using the inductive hypothesis and all other hypotheses we see that \( f^{-1}[Z_{\alpha+1}] = f^{-1}[\bigcup_{A \in \mathcal{A}} \text{int}(A \cup Z_\alpha)] = \bigcup_{A \in \mathcal{A}} f^{-1}\text{int}(A \cup Z_\alpha) \subset \bigcup_{A \in \mathcal{A}} \text{int}(f^{-1}[A \cup Z_\alpha]) = \bigcup_{A \in \mathcal{A}} \text{int}(f^{-1}[A] \cup f^{-1}[Z_\alpha]) \subset \bigcup_{A \in \mathcal{A}} \text{int}(A \cup Z'_\alpha) = Z_{\alpha+1} \).

2. By 1 we know that \( f[C'] \subset C \). Assume \( C \setminus f[C'] \neq \emptyset \). We will reach a contradiction. First observe that we may assume \( K = C \). This follows from Lemma 2.4 since we may restrict to the subspace \( C \) of \( K \) and the subspace \( f^{-1}[C] \) of \( K' \). Both are compact Hausdorff, its cores are \( C \) and \( C' \) and the corresponding restricted covering families \( \mathcal{A}_C \) and \( \mathcal{A}'_{f^{-1}[C]} \) still verify that \( \mathcal{A}'_{f^{-1}[C]} = \{f^{-1}[A] : A \in \mathcal{A}_C\} \). We will use frequently the fact that in a compact Hausdorff space if \( U \) is open and \( a \in U \) there is a closed set \( F \) such that \( a \in \text{int}(F) \subset F \subset U \). Since \( f[C'] \) is closed (because \( f \) is continuous, \( K' \) is compact and \( K \) is Hausdorff) and it is a proper subset of \( K \), there is a closed set \( F \subset K \) such that \( \text{int}(F) \neq \emptyset \) and \( F \cap f[C'] = \emptyset \). Clearly \( f^{-1}[F] \subset K' \setminus C' = \bigcup_{\alpha \in \text{On}} Z'_\alpha \) and from this it follows that \( f^{-1}[F] \subset Z'_\alpha \) for some ordinal \( \alpha \). We assume we have chosen \( \alpha \) minimal with the property that there is a closed set \( F \) in \( K \) such that

\[ \text{int}(F) \neq \emptyset, F \cap f[C'] = \emptyset \text{ and } f^{-1}[F] \subset Z'_\alpha. \]

By compactness, \( \alpha = 0 \) or is a successor ordinal. Let \( Z'_0 = \bigcup_{\beta<\alpha} Z'_\beta \). Hence \( Z'_\alpha = \emptyset \) if \( \alpha = 0 \) and \( Z'_\alpha = Z_\beta \) if \( \beta + 1 = \alpha \). In any case, for every \( a \in f^{-1}[F] \) there is some \( A \in \mathcal{A} \) such that \( a \in \text{int}(A \cup Z'_\alpha) \) and therefore there is a closed set \( G \) such that \( a \in \text{int}(G) \subset G \subset \text{int}(A \cup Z'_\alpha) \). The open sets \( \text{int}(G) \) cover the closed set \( f^{-1}[F] \). By compactness finitely many of them suffice. Hence for some \( k < \omega \) there are \( A_0, \ldots, A_{k-1} \in \mathcal{A} \) and closed sets \( G_0, \ldots, G_{k-1} \).
such that \( f^{-1}[F] \subseteq \bigcup_{i<k} \text{int}(G_i) \) and for all \( i < k \), \( G_i \subseteq \text{int}(A_i \cup Z_{n-1}^i) \). Let \( G = f[\bigcup_{i<k} G_i \setminus Z_{n-1}^i] \). It is a closed subset of \( K \). We will see now that \( G \) is nowhere dense, i.e., \( \text{int}(G) = \emptyset \). Since the union of two nowhere dense sets is again nowhere dense and \( G = \bigcup_{i<k} f[G_i \setminus Z_{n-1}^i] \), if \( G \) is not nowhere dense, for some \( i < k \) some \( f[G_i \setminus Z_{n-1}^i] \) has non-empty interior. Since \( G_i \subseteq A_i \cup Z_{n-1}^i \), it follows that \( f[G_i \setminus Z_{n-1}^i] \subseteq f[A_i] \). But \( f[A_i] \in \mathcal{A} \), so we have an element of \( \mathcal{A} \) with non-empty interior, which contradicts our first assumption that \( C = K \). Hence we have to admit that \( G \) is nowhere dense. Clearly there are points in \( \text{int}(F) \setminus G \) and we can separate any of them from \( G \) by disjoint open sets. Therefore we can find \( H \subseteq F \) closed such that \( \text{int}(H) \neq \emptyset \) and \( H \cap G = \emptyset \). Now \( H \cap f[C'] = \emptyset \) (because \( H \subseteq F \) and \( f^{-1}[H] \subseteq Z_{n-1}^i \) (because \( f^{-1}[H] \) is contained in \( \bigcup_{i<k} G_i \) and \( H \) is disjoint to \( G \)), and this shows that \( \alpha > 0 \) and contradicts its minimality.

3. Let \( \mathcal{A} = \{ f[A] : A \in \mathcal{A}' \} \). Observe that \( \mathcal{A}' \) refines \( \{ f^{-1}[A] : A \in \mathcal{A} \} \) and use lemma 2.2 and point 2.

3 Lascar strong types

Theorem 3.1 Let \( X \) be a union of Lascar strong types of infinite diameter. Assume that all elements of \( X \) have the same type over the empty set and that \( X \) is type-definable over some parameters. Let \( \bar{a} = (a_i : i \in I) \) be a sequence of representatives of the different Lascar strong types in \( X \). Then

1. If \( \mathcal{A} = \{ Y^i_n : i \in I, n < \omega \} \) where for any \( i \in I \) and any \( n < \omega \), \( Y^i_n = \{ \text{tp}(b/a) : d(b, a_i) \leq n \} \) then \( Y = \{ \text{tp}(b/\bar{a}) : b \in X \} \) is not \( \mathcal{A} \)-analyzable.

2. There is a \( X' \subseteq X \) type-definable over \( \bar{a} \) such that for every formula \( \varphi(x) \) over \( \bar{a} \), if some element of \( X' \) realizes \( \varphi(x) \) then there are at least two realizations of \( \varphi(x) \) in \( X' \) with different Lascar strong type.

Proof. 1. Since \( X \) is \( \bar{a} \)-invariant and it is type-definable over some set of parameters, it is also type-definable over \( \bar{a} \). Let \( Y_i = \{ \text{tp}(b/a) : \text{Lstp}(b) = \text{Lstp}(a_i) \} \). Hence \( Y_i = \bigcup_{n<\omega} Y^i_n \) and \( Y = \bigcup_{i \in I} Y^i_n \). \( Y \) and every \( Y^i_n \) are closed subsets of the Stone space \( S(\bar{a}) \). Fix some \( i \in I \) and consider the restriction mapping \( f : S(\bar{a}) \to S(a_i) \) defined by \( f(p) = p \restriction a_i \). It is a continuous surjection. Let \( U = f[Y] = \{ \text{tp}(b/a_i) : b \in X \} \), let \( U_i = f[Y^i] = \{ \text{tp}(b/a_i) : \text{Lstp}(b) = \text{Lstp}(a_i) \} \) and let \( U^i_n = f[Y^i_n] = \{ \text{tp}(b/a_i) : d(b, a_i) \leq n \} \). \( U \) is a closed subspace of \( S(\bar{a}) \), \( U^i_n \) is a closed subspace of \( U \) and \( U_i \) is a subspace not necessarily closed \( U^i_n \subseteq U_i \subseteq U \). If \( Y \) is \( \mathcal{A} \)-analyzable then, by lemma 2.2, it is also analyzable with respect to \( \{ Y^i_n : n < \omega \} \cup \{ \bigcup_{i \in I} Y^i \} \) and by proposition 2.8 \( U \) is analyzable with respect to \( \{ U^i_n : n < \omega \} \cup \{ \{ \text{tp}(b/a_i) : b \in X \text{ and } \text{Lstp}(b) \neq \text{Lstp}(a_i) \} \} \).

By lemma 2.4 \( U_i \) is analyzable with respect to \( \{ U^i_n : n < \omega \} \). By isomorphism, for any \( b \in X \) the space \( U_b = \{ \text{tp}(c/b) : \text{Lstp}(c) = \text{Lstp}(b) \} \) is analyzable with respect to \( \{ U^i_n : n < \omega \} \) where \( U^i_n = \{ \text{tp}(c/b) : d(c, b) \leq n \} \). On the other hand, by lemma 2.2, if \( Y^i_n = \bigcup_{i \in I} Y^i_n \) then \( Y \) is also analyzable with respect to
\{Y^n : n < \omega \}. Let \((Z_\alpha : \alpha \in \text{On})\) be this last analysis of \(Y\) and let \(\alpha^* + 1\) be its height.

Now we claim that we can find \(b \in X\), ordinals \(\alpha < \beta \leq \alpha^*\), formulas \(\varphi(x, z)\), \(\psi(x, z)\) and natural numbers \(n, m\) such that

1. \(\text{tp}(b/\bar{a}) \in Z_{\beta+1} \setminus Z_\beta\)
2. \(\psi(x, b) \vdash \varphi(x, \bar{a})\)
3. \(\emptyset \not\subseteq U_b \cap [\psi(x, b)] \subseteq U_b^m\)
4. \(Y \cap [\varphi(x, \bar{a})] \subseteq Z_\alpha \cup Y^n\)

We take \(\beta = \alpha^*\) and choose \(b \in X\) arbitrary with \(\text{tp}(b/\bar{a}) \in Z_{\beta+1} \setminus Z_\beta\). Since this is the last level of the analysis, by lemma 2.7 it is covered by just one \(Y^k\). This means that for every \(c \in X\) with \(\text{tp}(c/\bar{a}) \in Z^m_{\beta+1} \setminus Z_\beta\) there is \(i \in I\) such that \(d(c, a_i) \leq k\). Let \((Z^n_\alpha : \alpha \in \text{On})\) be the analysis of \(U_b\) with respect to \(\{U^n_\alpha : n < \omega\}\). If there is a bound \(n\) on \(d(c, b)\) for \(c \in X\) such that \(\text{tp}(c/b) \in Z^n_0\) then, by lemma 2.6, \(Z^n_0 \subseteq U_b^m\) and the analysis stops in one step and \(U_b = Z^n_0\).

But in this case \(\{c \in X : \text{Lstp}(c) = \text{Lstp}(b)\}\) has a diameter bounded by \(n\), contrarily to the initial assumption. Therefore there is no such bound and we can find \(c \in X\) such that \(\text{tp}(c/b) \in Z^n_0\) and \(d(c, b) > 2k\). Choose \(i \in I\) such that \(\text{Lstp}(b) = \text{Lstp}(a_i)\). Since \(\text{tp}(b/\bar{a}) \in Z_{\beta+1} \setminus Z_\beta\), by choice of \(k\), \(d(b, a_i) \leq k\).

It follows that \(d(c, a_i) > k\) and therefore \(\text{tp}(c/\bar{a}) \in Z_\beta\). For the same reason, for any other \(c' \models p(x) = \text{tp}(c/b)\) we have \(\text{tp}(c'/\bar{a}) \in Z_\beta\). Now \(Y \setminus Z_\beta\) is a closed subset of \(S(\bar{a})\) and therefore it is the set of types in \(Y\) extending a partial type \(\pi(x, \bar{a})\). We have seen that \(\pi(x, \bar{a}) \cup p(x)\) is inconsistent. Hence there are \(\psi(x, b) \in \text{tp}(c/b)\) and \(\varphi(x, \bar{a}) \in \pi(x, \bar{a})\) such that \(\psi(x, b) \vdash \varphi(x, \bar{a})\) and \(Y \cap [\varphi(x, \bar{a})] \subseteq Z_\beta\). Since \(\text{tp}(c/b) \in Z^n_0\), there is some open set \(W\) in \(S(b)\) and some \(n < \omega\) such that \(\text{tp}(c/b) \in W \cap U_b \subseteq U_b^m\). We may assume that \(W\) is a clopen set defined by \(\psi(x, b)\). Hence \(\emptyset \not\subseteq [\psi(x, b)] \cap U_b \subseteq U_b^m\). Now \(Y \cap [\varphi(x, \bar{a})]\) is compact and it is contained in \(Z_\beta\). If \(\beta\) is limit, clearly it is also contained in \(Z_\alpha\) for some \(\alpha < \beta\). In the case \(\beta = \alpha + 1\) we apply the definition of the analysis and compactness to obtain some \(n < \omega\) such that \(Y \cap [\varphi(x, \bar{a})] \subseteq Z_\alpha \cup Y^n\). Therefore all conditions 1 to 4 are satisfied and the claim is proven.

Let \(\beta\) be minimal for which there are \(\alpha < \beta, b \in X, \psi\) and \(\varphi\) with the properties 1-4. We will show that we can still find a smaller \(\beta\), which is a contradiction and will finish the proof. We start choosing \(\theta(z, \bar{a}) \in \text{tp}(b/\bar{a})\) such that \(\psi(x, z) \land \theta(z, \bar{a}) \vdash \varphi(x, \bar{a})\). For \(\gamma < \beta, Y \cap [\theta(z, \bar{a})]\) is not contained in \(Z_{\gamma+1}\) and therefore, by lemma 2.6, \([\theta(z, \bar{a})] \cap (Z_{\gamma+1} \setminus Z_\gamma)\) can not be covered by just one \(Y_k\). This means that there is no bound on \(d(c, a_i)\) for \(c\) and \(a_i\) such that \(\models \theta(c, \bar{a}), \text{Lstp}(c) = \text{Lstp}(a_i)\) and \(\text{tp}(c/\bar{a}) \in Z_{\gamma+1} \setminus Z_\gamma\). In case \(\beta\) is a successor ordinal we take as \(\beta'\) the predecessor of \(\beta\) and in case \(\beta\) is a limit ordinal we choose \(\beta'\) such that \(\alpha < \beta' < \beta\). Choose now some \(b'\) and \(i \in I\) such that \(\text{tp}(b'/\bar{a}) \in Z_{\beta'+1} \setminus Z_\beta, \models \theta(b', \bar{a}), \text{Lstp}(b') = \text{Lstp}(a_i)\) and \(d(b', a_i) > n + m\).

Since \(b \equiv b'\), we still have that \(\emptyset \not\subseteq U_{b'} \cap [\psi(x, b')] \subseteq U_{b'}^n\). We claim that there is no \(c'\) such that \(\text{Lstp}(c') = \text{Lstp}(b'), \models \varphi(c', \bar{a})\) and \(\text{tp}(c'/\bar{a}) \not\in Z_\alpha\). If there is such a \(c'\), we see that \(\models \varphi(c', \bar{a})\) and hence \(\text{tp}(c'/\bar{a}) \in Y \cap [\varphi(x, \bar{a})] \setminus Z_\alpha\) and
by point 4 $tp(c'/a) \in Y^n$. But this means that $d(c', a_i) \leq n$ which contradicts the facts that $d(b', a_i) > n + m$ and $d(c', b') \leq m$. So, there is no such $c'$.

Take some $c'$ such that $Lstp(c') = Lstp(b')$ and $\models \psi(c', b')$, and let $p'(x) = tp(c'/b')$. If $\pi'(x, a)$ is a partial type characterizing the closed set $Y \triangle Z_\alpha$, then $p'(x) \cup \pi'(x, a)$ is inconsistent. As above, we find $\psi'(x, b') \in p'(x)$ and $\phi'(x, a)$ such that $\psi(x, b') \vdash \phi'(x, a)$ and $Y \cap [\phi'(x, a)] \subseteq Z_\alpha$. As in the initial situation from this follows that $Y \cap [\phi(x, a)] \subseteq Z_{\alpha'} \cup Y^{n'}$ for some $a' < \alpha$ and some $n' < \omega$. We may assume that $\psi'(x, b') \vdash \phi(x, b')$ and therefore $\emptyset \not\in U_{b'} \cap [\phi'(x, b')] \subseteq U_{b'}^m$ is still true.

2. We know that $Y$ is not $\mathcal{A}$-analyzable and therefore the core $C$ is a non-empty closed subset of $S(a)$. Hence the set $X' = \{ b \in X : tp(b/a) \in C \}$ is type-definable over $a$. Assume $\varphi(x)$ is a formula over $a$ which is realized in $X'$ but all whose realizations in $X'$ have the same Lascar strong type over the empty set, say the same Lascar strong type as $a_i$. Then $\emptyset \notin C \cap [\varphi(x)] \subseteq Y = \bigcup_{n<\omega} Y_i^n$. Now we use the Baire category theorem in $C$. Since $C \cap [\varphi(x)]$ is a non-empty open set in $C$, it is not meager, i.e., it is not a countable union of nowhere dense sets. Hence for some $n, C \cap Y_i^n$ has non-empty interior in $C$, that is, there is some non-empty open set $W$ in $S(a)$ such that $W \cap C \subseteq Y_i^n$. Then $W \cap Y \subseteq Y_i^n \cup (Y \setminus C)$, which means that the analysis continues beyond $Y \setminus C$, a contradiction.

**Theorem 3.2** Let $X$ be a union of Lascar strong types of infinite diameter. Assume that all elements of $X$ have the same type over the empty set and that $X$ is type-definable over some parameters. Then there are at least $2^{\omega_2}$ Lascar strong types realized in $X$.

**Proof.** Let $X'$ as in point 2 of theorem 3.1. We first observe that whenever we have $a, b \in X'$ with different Lascar strong type, then for every $n < \omega$ we can find formulas $\varphi(x) \in tp(a/b)$ and $\psi(x) \in tp(b/a)$ such that $d(a', b') > n$ for all $a', b' \in X'$ such that $\models \varphi(a')$ and $\models \psi(b')$. This follows from the fact that if $p(x) = tp(a/b)$ and $q(x) = tp(b/a)$ then $nc_n(x, y) \cup p(x) \cup q(y)$ is inconsistent. Now this allows us to construct a tree of formulas $(\varphi_s(x) : s \in \omega_2^2)$ such that

1. $\varphi_s(x)$ is a formula over $a$ and it is realized in $X'$.

2. $\varphi_s(x) \vdash \varphi_t(x)$ if $t \subseteq s$.

3. If $s, t \in \omega_2^2$ are different then $d(a, b) \geq n$ for all $a, b \in X'$ such that $\models \varphi_s(a)$ and $\models \varphi_t(b)$.

Let $\Phi(x, a)$ a type over $a$ defining $X'$. For all $\eta \in \omega_2$, we have a type $p_\eta(x) = \Phi(x, a) \cup \{ \varphi_{s(a)}(x) : s < \omega \}$ and for different $\eta, \eta'$ if $a \models p_\eta$ and $a' \models p_\eta'$, then $Lstp(a) \neq Lstp(a')$.

**Corollary 3.3** 1. If a Lascar strong type is type-definable over some parameters, then it has finite diameter.
2. If a $E_{KP}$-class is not a Lascar strong type, then it splits into at least $2^\omega$ Lascar strong types.

3. If for each $n < \omega$ there is a Lascar strong type of diameter at least $n$, then there is a Lascar strong type which is not type-definable, even with parameters, and therefore $E_L \neq E_{KP}$.

4. If $E_L$ is type-definable over some parameters, then for some $n$ it is defined by $nc^n(x, y)$.

**Proof.** 1 follows directly from theorem 3.2.

2. Consider a $E_{KP}$-class $a/E_{KP}$ which is not a Lascar strong type. It is a union of Lascar strong types and it is type-definable over $a$. To apply theorem 3.2 we have to show that all its Lascar strong types have infinite diameter. Assume not. Let $p(x) = tp(a)$ and let $X$ be the set of all realizations of $p(x)$. Since $X$ includes $a/E_{KP}$ and all Lascar strong types contained in $X$ are isomorphic, all have diameter bounded by $n$ for some fixed $n$. Therefore in $X$ the relation $E_L$ of equality of Lascar strong types is type-definable by $nc^n(x, y)$. Since $E_L$ restricted to $X$ is a bounded type-definable relation, it is refined by $E_{KP}$ restricted to $X$. But this means that on $X$ they coincide and therefore that $a/E_{KP}$ is a Lascar strong type, which is a contradiction.

3. For each $n < \omega$ fix a sequence $a_n$ whose Lascar strong type $Lstp(a_n) = a_n/E_L$ has diameter at least $n$ and consider $a = (a_n : n < \omega)$. It is easy to check that $Lstp(a)$ has infinite diameter. By point 1 it is not type-definable.

4. Assume $E_L$ is type-definable. By 3 there is a bound $n < \omega$ for the diameter of any Lascar strong type. Therefore $nc^n(x, y)$ defines $E_L$. 

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