The number of countable models

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1 Small theories

Definition 1.1 T is *small* if for all $n < \omega$, $|S_n(\emptyset)| \leq \omega$.

Remark 1.2 *If T is small, then there is a countable $L_0 \subseteq L$ such that for every $\varphi(x) \in L$ there is some $\varphi'(x) \in L_0$ such that in T, $\varphi(x) \equiv \varphi'(x)$. Hence, T is a definitional extension of the countable theory $T_0 = T \restriction L_0$.*

Proof: See Remark 14.25 in [4]. With respect to the second assertion, consider some n-ary relation symbol $R \in L \smallsetminus L_0$. There is some formula $\varphi(x_1, \ldots, x_n) \in L_0$ equivalent to $Rx_1 \ldots x_n$ in T. If we add all the definitions $\forall x_1 \ldots x_n(Rx_1 \ldots x_n \leftrightarrow \varphi(x_1, \ldots, x_n))$ (and similar definitions for constants and function symbols) to T_0 we obtain T. □

Lemma 1.3 *The following are equivalent:*

1. *T is small.*
2. *For all $n < \omega$, for all finite A, $|S_n(A)| \leq \omega$.*
3. *For all finite A, $|S_1(A)| \leq \omega$.*
4. *T has a saturated countable model.*

Proof: See Remark 14.26 in [4]. □

Some topological considerations are helpful for the following discussions. A boolean topological space X can be decomposed using the Cantor-Bendixson derivative as

$$X = (\bigcup_{\alpha \in On} X^{(\alpha)} \smallsetminus X^{(\alpha+1)}) \cup X^{\infty}$$

where $X^{(0)} = X$, $X^{(\alpha+1)}$ is the set of accumulation points of $X^{(\alpha)}$, $X^{(\beta)} = \bigcap_{\alpha<\beta} X_\alpha$ for limit β and $X^{(\infty)} = \bigcap_{\alpha \in On} X_\alpha$. All X_α are closed. The perfect kernel $X^{(\infty)}$ does not contain isolated points (with respect to the induced topology) and hence it is empty or it contains a binary tree $(U_s : s \in 2^{<\omega})$ of nonempty clopen sets U_s with $U_s = U_{s^\frown 0} \dot{\cup} U_{s^\frown 1}$, which gives 2^ω many points in $X^{(\infty)}$. On the other hand if we fix a basis of clopen sets,

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we can assign to every point $a \in \bigcup_{\alpha \in On} X^{(\alpha)} \smallsetminus X^{(\alpha+1)}$ some basic clopen set O_a which isolates a in some $X^{(\alpha)}$. This mapping is one-to-one. In spaces of types $X = S_n(A)$ the size of a basis of clopen sets is $\leq |T(A)|$ and therefore $|\bigcup_{\alpha \in On} X^{(\alpha)} \smallsetminus X^{(\alpha+1)}| \leq |T(A)|$. If T and L are countable, $S_n(A)$ is decomposed in a countable set and a perfect kernel which is either empty or of cardinality 2^ω.

Proposition 1.4 *The theory T is small if and only if the two following conditions hold:*

1. *There is a countable $L_0 \subseteq L$ such that for every $\varphi(x) \in L$ there is some $\varphi'(x) \in L_0$ such that in T, $\varphi(x) \equiv \varphi'(x)$.*
2. *For all $n < \omega$, for all finite A, the space $S_n(A)$ is scattered, that is, every type in $S_n(A)$ has ordinal Cantor-Bendixson rank.*

Proof: See Remark 14.27 in [4] or use the topological description given above. □

Remark 1.5 *1. Countable ω-categorical theories are small.*

2. ω-stable theories are small.

Proof: By Ryll-Nardzewski Theorem, if T is countable, then it is ω-categorical iff $S_n(\emptyset)$ is finite for all n. On the other hand, by definition, T is ω-stable iff $S_1(A)$ is countable for all countable A. □

Lemma 1.6 *Assume $\varphi(x) \in L(A)$ is consistent and there is no isolated $p(x) \in S(A)$ such that $\varphi \in p$. Then, there is a binary tree $(\varphi_s(x) : s \in 2^{<\omega})$ of consistent formulas $\varphi_s(x) \in L(A)$ such that $\varphi = \varphi_\emptyset$ and for each s, $\varphi_s(x) \equiv \varphi_{s^\frown 0}(x) \vee \varphi_{s^\frown 1}(x)$ and $\varphi_{s^\frown 0}(x) \wedge \varphi_{s^\frown 1}(x)$ is inconsistent.*

Proof: If there is no isolated $p(x) \in S_n(A)$ containing φ, then for each $\psi(x) \in L(A)$ consistent with φ there is some $\chi(x) \in L(A)$ such that $\varphi(x) \wedge \psi(x) \wedge \chi(x)$ and $\varphi(x) \wedge \psi(x) \wedge \neg\chi(x)$ are both consistent. This is what one needs to construct the binary tree. In topological terms: the space $[\varphi] = \{p(x) \in S_n(A) : \varphi \in p\}$ is a nonempty perfect set and contains a binary tree of clopen sets as described above. □

Proposition 1.7 *If T is small, then for every finite A there is a prime model over A.*

Proof: By smallness, we may assume L is countable. Then it suffices to find an atomic model $M \supseteq A$. This can be done by the Omitting Types Theorem since by Lemma 1.6 for every $n < \omega$ the set of negations $\neg\varphi(x_1, \ldots, x_n)$ of all atoms $\varphi(x_1, \ldots, x_n) \in L(A)$ is omissible (otherwise we obtain a binary tree which produces 2^ω complete n-types over A). □

Definition 1.8 For every cardinal $\kappa \geq \omega$, let $I(T, \kappa)$ be the number of nonisomorphic models of T of cardinality κ.

Remark 1.9 *If $\kappa \geq |T|$, then $I(T, \kappa) \leq 2^\kappa$.*

Proposition 1.10 *If the countable theory T is not small, then $I(T, \omega) \geq 2^\omega$.*

Proof: Fix some $n < \omega$ for which $S_n(\emptyset)$ is not scattered. The space contains a binary tree of clopen sets, which shows that $|S_n(\emptyset)| \geq 2^\omega$. Since every countable model realizes only countably many n-types and every n-type is realized in some countable model, $I(T, \omega) \geq 2^\omega$. □

Example 1.11 *Let T be the theory of the set ω in a language with an n-ary relation symbol for every relation $R \subseteq \omega^n$. T is ω-categorical but not small.*

2 Scattered theories

This section is based on Morley's article [13].

Definition 2.1 A *regular fragment* of $L_{\omega_1\omega}$ is a countable set of $L_{\omega_1\omega}$-formulas extending the set of first-order formulas that is closed under first-order connectives and quantifiers and under subformulas and substitution of variables by terms. If Φ is such a regular fragment, $S_n(\Phi, T)$ is the set of all complete Φ-types of n-tuples of countable models of T. T is *scattered* if $S_n(\Phi, T)$ is countable for any regular fragment Φ, for every $n < \omega$.

Remark 2.2 *Scattered theories are small.*

Proposition 2.3 *For any regular fragment Φ, either $|S_n(\Phi, T)| \leq \omega$ or $|S_n(\Phi, T)| = 2^\omega$.*

Proof: We identify a set of formulas with its characteristic function. Let Φ_n be the subset of Φ consisting in all formulas with free variables among $x_1, \ldots, x_n$. It is enough to prove that $S_n(\Phi, T)$ is an analytic subset of 2^{Φ_n}, since all analytic subsets of the Cantor space are countable or have cardinality 2^ω. We assume that the variables of Φ are $x_1, x_2, \ldots$. Note that $S_n(\Phi, T)$ is the projection of a subset of 2^Φ, the set $S_\omega(\Phi, t)$ of all Φ-types of ω-sequences enumerating some countable model of T. A subset p of Φ is an element of $S_\omega(\Phi, T)$ iff it satisfies the following conditions:

1. For all $\varphi \in \Phi$, $\neg\varphi \in p$ iff $\varphi \notin p$.
2. For all $\varphi \in \Phi$ of the form $\varphi = \bigwedge \Sigma$, $\varphi \in p$ iff $\psi \in p$ for all $\psi \in \Sigma$.
3. For all $\varphi \in \Phi$ of the form $\varphi = \exists x_n \psi$, $\varphi \in p$ iff $\psi(\frac{x_n}{x_m}) \in p$ for some $m \geq 1$.
4. For all $\varphi \in \Phi$, for all $n \geq 1$, for all terms t, if $\varphi \in p$, and $x_n = t \in p$, then $\varphi(\frac{x_n}{t}) \in p$.
5. $t = t \in p$ for all terms t.
6. $\varphi \in p$ for all $\varphi \in T$.

This shows that $S_\omega(\Phi, T)$ is Borel and hence $S_n(\Phi, T)$ is analytic. □

Corollary 2.4 *If $I(T, \omega) < 2^\omega$, then T is scattered.*

Proof: If T is not scattered then for some regular fragment Φ, for some $n < \omega$, $S_n(\Phi, T)$ is uncountable and by Proposition 2.3 $|S_n(\Phi, T)| = 2^\omega$. If there are $\kappa < 2^\omega$ nonisomorphic countable models of T, since each one of them realizes at most ω different complete Φ-types of n-tuples, it follows that $|S_n(\Phi, T)| \leq \kappa + \omega < 2^\omega$. □

Definition 2.5 Let T be a scattered theory. We define $(\Phi_i : i < \omega_1)$, a chain of regular fragments. Φ_0 is the set of all first-order formulas of L (we may assume L is countable). Φ_{i+1} is the smallest regular fragment containing Φ_i and containing $\bigwedge p$ for every $p \in S_n(\Phi_i, T)$ for every $n < \omega$. For limit δ, Φ_δ is the union of all Φ_i for $i < \delta$.

Proposition 2.6 *If T is scattered, then for every countable model M of T there is some $i < \omega_1$ and some sentence $\varphi \in \Phi_i$ such that for every countable model of T, if $N \models \varphi$ then $M \cong N$.*

Proof: For every two n-tuples a, b in M, either a and b have the same Φ_i-type for all $i < \omega_1$ or there is a smallest ordinal $\alpha_{a,b} < \omega_1$ for which they have different $\Phi_{\alpha_{a,b}}$-type. Let α be the supremum of all $\alpha_{a,b}$ for all n-tuples a, b in M, for all $n < \omega$. Then $\alpha < \omega_1$ and if two finite tuples a, b have the same Φ_α-type, then they have the same Φ_i-type for all $i < \omega_1$. Let φ be the conjunction of the $\Phi_{\alpha+2}$-type without variables. Assume N is a countable model of φ. We claim that the set I of all finite partial isomorphisms between M and N preserving all Φ_α-types is a back-and-forth system, and hence $M \cong N$. Note that $\emptyset \in I$. Assume $f = \{(a_i, b_i) : i = 1, \ldots, n\} \in I$ and a is a new element of M. Let $p(x_1, \ldots, x_n)$ be the Φ_α-type of $a_1, \ldots, a_n$ in M and let $q(x_1, \ldots, x_n, x_{n+1})$ be the Φ_α-type of $a_1, \ldots, a_n, a$ in M. Since $\varphi \vdash \forall x_1 \ldots x_n(\bigwedge p(x_1, \ldots, x_n) \to \exists x_{n+1} \bigwedge q(x_1, \ldots, x_n, x_{n+1}))$ and by assumption $N \models p(b_1, \ldots, b_n)$, there is some $b \in N$ such that $N \models q(b_1, \ldots, b_n, b)$. Therefore $f \cup \{(a, b)\} \in I$. Now assume $b \in N$ is given and let $q(x_1, \ldots, x_n, x_{n+1})$ be the Φ_α-type of $b_1, \ldots, b_n, b$ in N. Then $N \models \exists x \bigwedge q(b_1, \ldots, b_n, x)$. If $p(x_1, \ldots, x_n)$ is again the Φ_α-type of $a_1, \ldots, a_n$ in M and $p'(x_1, \ldots, x_n)$ is its $\Phi_{\alpha+1}$-type, then $\varphi \vdash \forall x_1 \ldots x_n(\bigwedge p(x_1, \ldots, x_n) \to \bigwedge p'(x_1, \ldots, x_n))$ and therefore $N \models p'(b_1, \ldots, b_n)$. Since $p'(x_1, \ldots, x_n)$ is a complete $\Phi_{\alpha+1}$-type, and $\exists x \bigwedge q(x_1, \ldots, x_n, x)$ is $\Phi_{\alpha+1}$-formula consistent with it, $p' \vdash \exists x \bigwedge q(x_1, \ldots, x_n, x)$. Hence $M \models \exists x \bigwedge q(a_1, \ldots, a_n, x)$ and then $M \models q(a_1, \ldots, a_n, a)$ for some $a \in M$. Clearly, $f \cup \{(a, b)\} \in I$. □

Corollary 2.7 *If T is scattered, then $I(T, \omega) \leq \omega_1$.*

Proof: By Proposition 2.6 since there are only countably many sentences in each Φ_i. □

Corollary 2.8 *For any countable T, $I(T, \omega) \in \omega \cup \{\omega, \omega_1, 2^\omega\}$.*

Proof: By corollaries 2.4 and 2.7. □

3 Semi-isolation

Definition 3.1 Let a, b be finite tuples. We say that *a semi-isolates b* if there is some $\varphi(x) \in \text{tp}(b/a)$ such that $\varphi(x) \vdash \text{tp}(b)$. We also say that $\text{tp}(b/a)$ *is semi-isolated.*

Remark 3.2 *1. If $\text{tp}(b)$ is isolated, any tuple semi-isolates b.*

2. If $\text{tp}(b/a)$ is isolated, then a semi-isolates b.

3. Semi-isolation is reflexive and transitive:

- *(a) a semi-isolates a.*
- *(b) If a semi-isolates b and b semi-isolates c, then a semi-isolates c.*

Lemma 3.3 *If $\text{tp}(b/a)$ is isolated and $\text{tp}(a/b)$ is semi-isolated, then $\text{tp}(a/b)$ is isolated.*

Proof: Assume $\varphi(x, a)$ isolates $\text{tp}(b/a)$ and $\psi(b, y) \in \text{tp}(a/b)$ witnesses that this second type is semi-isolated. Then $\varphi(b, y) \wedge \psi(b, y)$ isolates $\text{tp}(a/b)$. □

Lemma 3.4 *Assume $a_0 \equiv a_1$ and $\text{tp}(b_0/a_0a_1)$ does not divide over $\emptyset$. If a_0 is semi-isolates b_0 and b_0 semi-isolates a_1, then a_1 semi-isolates b_0.*

Proof: Choose $\varphi(y,x), \psi(y,x) \in L$ such that $\models \varphi(b_0, a_1) \wedge \psi(b_0, a_0)$, $\varphi(b_0, x) \vdash \mathrm{tp}(a_1)$ and $\psi(y, a_1) \vdash \mathrm{tp}(b_0)$. Since $a_0 \equiv a_1$, we can extend the sequence a_0, b_0, a_1 to the ω-sequence $a_0, b_0, a_1, b_1, a_2, b_2, \ldots,$ in such a way that $a_n b_n a_{n+1} \equiv a_0 b_0 a_1$ for all $n < \omega$. Let $\theta(y; x_0, x_1) = \psi(y, x_0) \wedge \varphi(y, x_1)$. Since $\models \theta(b_0; a_0, a_1)$ and $\mathrm{tp}(b_0/a_0 a_1)$ does not divide over $\emptyset$, for some $n < m < \omega$, $\theta(y; a_{2n}, a_{2n+1}) \wedge \theta(y; a_{2m}, a_{2m+1})$ is consistent. Let b realize this formula. Since a_{2m} semi-isolates b and b semi-isolates a_{2n+1}, by transitivity a_{2m} semi-isolates a_{2n+1}. Since a_{2n+1} semi-isolates b_{2m-1}, by transitivity again a_{2m} semi-isolates b_{2m-1}. Since $b_0 a_1 \equiv b_{2m-1} a_{2m}$, we conclude that a_1 semi-isolates b_0. □

Definition 3.5 Let $p(x), q(y) \in S(\emptyset)$. We say that p is *freely semi-isolated over* q and we write $p <_{si} q$ if there are independent (in the sense of nonforking) $a \models p$ and $b \models q$ such that b semi-isolates a.

Remark 3.6 *Let T be simple.*

1. *If $p <_{si} q$ and $q <_{si} r$, then $p <_{si} r$.*
2. *If p is isolated, then $p <_{si} q$ for any q.*

Proposition 3.7 (Pillay) *In a stable theory, if $p(x) \in S(\emptyset)$ is nonisolated, there is no $q(y) \in S(\emptyset)$ such that $p <_{si} q$.*

Proof: Assume $a \models p$, $b \models q$, $\varphi(x,b) \in tp(a/b)$ semi-isolates the nonisolated type p and $a \mathop{\smile\!\!\!\!\!\!\:|}\, b$. Let $NF_x(b,\emptyset) \subseteq S_x(b)$ be the (closed) set of all types that do not fork over $\emptyset$. The Open Mapping Theorem says that the restriction map $f : NF_x(b,\emptyset) \to S_x(\emptyset)$ is open. Hence the image of the clopen set $[\varphi(x,b)] \cap NF_x(b,\emptyset)$ is open and contains $p(x)$ as its sole element. This image is in fact a clopen set $[\psi(x)]$ and the formula $\psi(x) \in L$ isolates $p(x)$, a contradiction. □

Recall that a formula $\theta(x,y) \in L$ is *thick* if it is symmetric and finite: there is a bound $k < \omega$ for the length of a sequence $(a_i : i < k)$ such that $\models \neg\theta(a_i, a_j)$ for all $i < j < k$.

Lemma 3.8 *If $\theta(x,y)$ is a thick formula and $k < \omega$ is the maximal number for which there is an anticlick $(a_i : i < k)$ (which means $\models \neg\theta(a_i, a_j)$ for all $i < j < k$) then the transitive closure E of θ is definable by*

$$E(x,x') \Leftrightarrow \exists y_0 \ldots y_{2k}(x = y_0 \wedge x' = y_{2k} \wedge \bigwedge_{i<2k} \theta(y_i, y_{i+1}))$$

Proof: We must check transitivity. Assume $E(x,x') \wedge E(x',x'')$, witnessed by the θ-chains $x = y_0, \ldots, y_{2k} = x'$ and $x' = z_0, \ldots, z_{2k} = x''$. Let $u_0, \ldots, u_m$ a minimal subsequence which is still a θ-chain and connects $x = u_0$ with $x'' = u_m$. Note that $m < 2k$, since otherwise $u_0, u_2, \ldots, u_{2k}$ is an anticlick of length $k+1$. Hence $u_0, \ldots, u_m$ witnesses that $E(x,x'')$. □

Proposition 3.9 (Newelski, Tsuboi) *In a simple theory the relation $<_{si}$ on nonisolated types is asymmetric (hence irreflexive).*

Proof: It is enough to check irreflexivity. Assume $p <_{si} p$ and let us prove that p is isolated.. We may find an independent sequence a, b, a' of realizations of p such that b semi-isolates a and a' semi-isolates b. By Lemma 3.4, a semi-isolates b. There is a single formula $\varphi(x,y) \in L$ such that $\models \varphi(a,b)$, $\varphi(x,b) \vdash p(x)$ and $\varphi(a,y) \vdash p(y)$. The first part of the proof consists in showing that we can replace $\varphi(x,y)$ by a thick formula.

Assume $b' \stackrel{\text{Ls}}{\equiv} b$ and $b' \mathop{\smile\!\!\!\!\!\!\,|}\, b$. By the Independence Theorem for Lascar strong types, $\varphi(x,b) \wedge \varphi(x,b')$ is consistent (take a' such that $ab \stackrel{\text{Ls}}{\equiv} a'b'$ and apply the theorem to $\mathrm{tp}(a/b)$ and $\mathrm{tp}(a'/b')$). Equality of Lascar strong type over $\emptyset$ is a bounded type-definable equivalence relation and we can define it by a set of thick formulas. By compactness (and type-definability over b of nonforking over $\emptyset$ for complete types over b) there is some thick formula θ such that $\varphi(x,b) \wedge \varphi(x,b')$ is consistent for every b' such that $\models \theta(b,b')$ and $b \mathop{\smile\!\!\!\!\!\!\,|}\, b'$. Note that consistency of $\varphi(x,b) \wedge \varphi(x,b')$ implies that $b' \models p$ (because we can choose $a'' \models \varphi(x,b) \wedge \varphi(x,b')$ and then $a'' \models p(x)$ and therefore $\varphi(a'',y) \vdash p(y)$). Hence we have: if $\models \theta(b,b')$ and $b \mathop{\smile\!\!\!\!\!\!\,|}\, b'$ then $b' \models p$. We can choose a thick formula θ' (in the type defining equality of Lascar strong type) such that $\theta'(x,y) \wedge \theta'(y,z) \vdash \theta(x,z)$.

We claim now that $\theta'(x,b) \vdash p(x)$. In fact, if $b' \models \theta'(x,b)$ we can choose $b'' \models \theta'(x,b')$ such that $b'' \mathop{\smile\!\!\!\!\!\!\,|}\, bb'$ (because being $\theta'(x,y)$ thick, $\theta'(x,b')$ does not fork over $\emptyset$) and this implies $\models \theta(b'',b)$ and hence $b'' \models p$. Since $\models \theta(b',b'')$ we also get $b' \models p$.

By the claim, $\theta'(x,y) \wedge p(y) \vdash p(x)$. Let E be the transitive closure of $\theta'(x,y)$. By the Lemma 3.8, E is a 0-definable finite equivalence relation and we have proved that $p(\mathfrak{C})$ is a union of E-classes. Hence $p(\mathfrak{C})$ is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Since it is invariant, it is definable over $\emptyset$ by some formula $\psi(x) \in L$. It follows that $\psi(x)$ isolates $p(x)$. □

Definition 3.10 In a small theory T we choose for every $p(x) \in S_n(\emptyset)$ some prime model M_p over some realization of p. Note that the isomorphism type of M_p is independent of the choice of the realization of p.

Lemma 3.11 *Let T be small and simple. If $p(x) \in S_n(\emptyset)$ is nonisolated and has finite weight $w(p) = k < \omega$, then M_p does not contain an independent sequence $a_0, \ldots, a_k$ of realizations a_i of p.*

Proof: Let $a \models p$ and let M_p be prime over a and assume all a_i are in M_p. By definition of weight, there must be some $i < k$ such that $a \mathop{\smile\!\!\!\!\!\!\,|}\, a_i$. Since a_i is isolated over a, a semi-isolates a_i. Hence $p <_{si} p$, in contradiction with Proposition 3.9. □

Theorem 3.12 (Kim) *If T is a countable supersimple theory, then either $I(T,\omega) = 1$ or $I(T,\omega) \geq \omega$.*

Proof: If T is countable but not ω-categorical, then some $p(x) \in S(\emptyset)$ is nonisolated. We can assume T is small, since otherwise $I(T,\omega) \geq 2^\omega$. Choose $(a_i : i < \omega)$, a Morley sequence of realizations of p and let M_n be a prime model over $a_{<n}$. Since T is supersimple all types have finite weight. Let $k = w(\mathrm{tp}(a_{<n}))$. We claim that $M_m \ncong M_n$ for all $m > n(k+1)$ (and this will guarantee $I(T,\omega) \geq \omega$). Assume $M_n \cong M_m$. Then M_n contains an independent $k+1$-sequence of realizations of the nonisolated type $\mathrm{tp}(a_{<n})$, in contradiction with Lemma 3.11. □

Remark 3.13 *In fact the proof of Proposition 3.12 shows that the result holds more generally for every countable simple theory where all types $p \in S_n(\emptyset)$ have finite weight.*

Simple small one-based theories are contained in the wider class of simple small theories with finite coding (or finitely based). By definition, a simple theory T is *one-based* if for every Lascar strong type $\mathrm{Lstp}(a/A)$, the canonical base $\mathrm{Cb}(a/A)$ is contained in the bounded closure $\mathrm{bdd}(a)$. It is *finitely based* if for every finite tuple a, $\mathrm{Cb}(a/A)$ is contained in $\mathrm{bdd}(B)$ for some finite set B. Clearly one-based simple theories are finitely based. Supersimple theories are also finitely based. T is said to have no *dense forking chains* if there is no chain of types $(p_q(x) : q \in \mathbb{Q})$ such that $p_{q'}$ is a forking extension of p_q whenever $q' > q$. It is known that finitely based small simple theories have no dense forking chains and that in simple theories with no dense forking chains all finitary types have finite weight. Therefore Theorem 3.12 can be generalized to countable simple theories without dense forking chains and, in particular, to countable one-based simple theories. Details of all this can be found in section 6.1.3 of [24]. The case of one-based stable theories was proved by Pillay. It was generalized to stable finitely based theories by Hrushovski (see [9]) and to stable theories without dense forking chains by Herwig, Loveys, Pillay, Tanović, and Wagner (see [6]).

The notion of *weight* is not as plain as one might think. There are some issues concerning how finite or infinite weight should be defined. Finite weight of $p(x) \in S(A)$ means that there is a fixed $k < \omega$ such that for every nonforking extension $\mathrm{tp}(a/B)$ of $p(x)$ over a larger set $B \supseteq A$, there is no B-independent $k+1$-sequence of tuples $(a_i : i < k)$ such that $a \not\perp_B a_i$ for all $i < k$. Finite preweight of $p(x) \in S(A)$ means that there is a fixed $k < \omega$ such that there is no $a \models p$ and a A-independent $k+1$-sequence of tuples $(a_i : i < k)$ such that $a \not\perp_A a_i$ for all $i < k$. In fact we only need that all finitary types over $\emptyset$ have finite preweight. See [17], [24] and [1] for more information.

4 Ehrenfeucht theories and powerful types

Definition 4.1 A complete theory T is an *Ehrenfeucht theory* if it is countable and $1 < I(T, \omega) < \omega$. *Lachlan's problem* is to find some stable Ehrenfeucht theory or to show that such theories do not exist. Note that Ehrenfeucht theories are small.

Example 4.2 (A theory with $I(T, \omega) = n$ where $2 < n < \omega$) *Consider first the case $n = 3$. The language is $L = \{<\} \cup \{c_n : n < \omega\}$. A prime model of T is $(\mathbb{Q}, <, n)_{n<\omega}$. If we add a copy of $\mathbb{Q}$ at the end we get the countable saturated model. A third model is obtained from the countable saturated model adding a point between $\mathbb{Q}$ and its copy; this point is the supremum of the natural numbers (named by the constants). The third model is universal but not saturated. For the case $n > 3$ we add unary predicates $\{P_i : i = 1, \dots n-2\}$ which define a partition of the model into dense subsets. One should specify in which sets are the constants. The third model gives now rise to $n-2$ models according to in which P_i lies the added point.*

Remark 4.3 (Vaught) $I(T, \omega) \neq 2$.

Definition 4.4 A type $p(x) \in S_n(\emptyset)$ is *powerful* if every model that realizes p realizes any other $q(y) \in S(\emptyset)$ (in other words, it is weakly saturated). If M_p is well-defined (for instance, in a small theory) this only means that M_p is weakly saturated.

Remark 4.5 *If T is countable and has an isolated powerful type, then T is ω-categorical.*

Proposition 4.6 (Benda) *Ehrenfeucht theories have powerful types.*

Proof: Let T be an Ehrenfeucht theory without powerful types. We construct a sequence $(a_n : n < \omega)$ of finite tuples a_n such that if $p_n = \mathrm{tp}(a_0, \dots, a_n)$ then p_{n+1} is omitted in M_{p_n}, which implies $M_{p_n} \not\cong M_{p_m}$ for $m > n$. We start with a_0 arbitrary. To obtain a_{n+1} we observe that p_n is not powerful and therefore there is a type $q \in S(\emptyset)$ omitted in M_{p_n}. Then we take as a_{n+1} some realization of q. □

Definition 4.7 For a finitary type $p(x) \in S(\emptyset)$ we define

$$\mathrm{SI}_p = \{(a, b) : a, b \models p \text{ and } a \text{ semi-isolates } b\}.$$

Lemma 4.8 *If $p(x) \in S_n(\emptyset)$ is nonisolated, then for every (some) $a \models p$ there is some $b \models p$ such that $(a, b) \notin \mathrm{SI}_p$.*

Proof: Let $\Phi(x, y)$ be the set of all $\varphi(x, y) \in L$ such that $p(x) \wedge \varphi(x, y) \vdash p(y)$. Note that Φ is closed under disjunctions and that for $a, b \models p$, $(a, b) \notin \mathrm{SI}_p$ iff $\models \neg\varphi(a, b)$ for all $\varphi \in \Phi$. We need to check the consistency of $p(x) \cup p(y) \cup \{\neg\varphi(x, y) : \varphi(x, y) \in \Phi\}$. If it is inconsistent, then for some $\theta(x) \in p(x)$, for some $\varphi(x, y) \in \Phi$, $\theta(x) \wedge \theta(y) \vdash \varphi(x, y)$. Let $a \models p$. Since p is nonprincipal, there is some b such that $\models \theta(b)$ but $b \not\models p$. Since $\models \varphi(a, b)$ and $a \models p$ we get that $b \models p$, a contradiction. □

Proposition 4.9 (Pillay) *In a small theory, if $p(x) \in S(\emptyset)$ is a nonisolated powerful type, then SI_p is not symmetric.*

Proof: Let $p(x)$ be a nonprincipal powerful type. If SI_p is symmetric, it is an equivalence relation and all realizations of p in M_p are SI_p-equivalent. By Lemma 4.8 there are $a, b \models p$ in M_p such that $(a, b) \notin \mathrm{SI}_p$, a contradiction. □

Definition 4.10 Let $p(x) \in S(\emptyset)$ be nonisolated. Following Tsuboi, we say that a complete type $r(x, y) \in S(\emptyset)$ extending $p(x) \cup p(y)$ is an *order expression* if $(a, b) \in \mathrm{SI}_p$ and $(b, a) \notin \mathrm{SI}_p$ for all $\models r(a, b)$. Lemma 4.9 implies that there is an order expression extending $p(x) \cup p(y)$ if T is small and $p(x)$ is powerful and nonisolated.

Proposition 4.11 (Tsuboi) *If $(T_i : i < \omega)$ is a chain of countable ω-categorical theories, and $T = \bigcup_{i<\omega} T_i$ is an Ehrenfeucht theory, then T is unstable.*

Proof: T is small and has a nonisolated powerful type $p(x) \in S(\emptyset)$. Choose an order expression $r(x, y)$ extending $p(x) \cup p(y)$. It is easy to construct a sequence $(a_i : i < \omega)$ of realizations a_i of p such that $\models r(a_i, a_{i+1})$ for all $i < \omega$. By transitivity of SI_p, if $i < j$ then $\mathrm{tp}(a_i, a_j)$ is an order expression. It follows that $i < j$ iff $\mathrm{tp}(a_i, a_j)$ is an order expression. Since p is powerful, $r(x, y)$ realized in M_p and there is some $\varphi(x, y) \in r(x, y)$ such that $p(x) \wedge \varphi(x, y) \vdash r(x, y)$. Let $m < \omega$ be such $\varphi(x, y) \in L_m$ (the language of T_m) and let

$$\varphi_i(x, y) = \exists x_0 \ldots x_i(\varphi(x, x_0) \wedge \varphi(x_0, x_1) \wedge \ldots \wedge \varphi(x_i, y))$$

By ω-categoricity of T_m, the infinite disjunction $\bigvee_{i<\omega} \varphi_i(x, y)$ is in fact equivalent to a finite disjunction $\psi(x, y) = \bigvee_{i<n} \varphi_i(x, y)$. It is easy to see that $\models \psi(a_i, a_j)$ iff $i < j$. Hence T is unstable. □

5 Rudin-Keisler order and limit models

This section is based on the results of Section 1.1 of Sudoplatov's book [19].

Lemma 5.1 *Let T be small. If $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n$ is a chain of finite sets contained in a model M and $\varphi(x) \in L(A_0)$ is consistent, then some realization of $\varphi(x)$ in M is isolated over every A_i.*

Proof: Choose a prime model $M_n \preceq M$ over A_n as starting point of an elementary chain $M_0 \preceq M_1 \preceq \ldots \preceq M_n$ such that each M_i is prime over A_i. A realization of $\varphi(x)$ in M_0 satisfies the requirements. □

Proposition 5.2 *If M is a countable model of a small theory, then M is the union of an elementary chain $(M_i : i < \omega)$ such that every M_i is prime over a finite set $A_i \subseteq M_i$.*

Proof: It is an inductive construction. Let $M = \{a_i : i < \omega\}$ and let $(\varphi_i(x) : i < \omega)$ be an enumeration of all $L(M)$-formulas in the single variable x. In order to get an easier notation we may assume that each formula appears infinitely many times in the enumeration. We construct a family $(A_i^j : i, j < \omega)$ of finite subsets of M such that $a_i \in A_i^i$, $A_i^j \subseteq A_{i+1}^j$, $A_i^j \subseteq A_i^{j+1}$, each A_i^j is atomic over A_k^0 for all $k \geq i$, and whenever $\varphi_i(x) \in L(A_j^i)$ (with $j \leq i$) is consistent then it is realized in A_i^{j+1}. We start with $A_0^0 = \{a_0\}$ and we want to extend $(A_i^j : i, j \leq n)$ to $(A_i^j : i, j \leq n+1)$. Put $A_{i+1}^j = A_i^i \cup \{a_{i+1}\}$ for $j \leq i$. If $\varphi_i(x) \in L(A_k^i)$ is consistent and $k \leq i+1$ is minimal with this property, choose with Lemma 5.1 some realization a of $\varphi_i(x)$ which is isolated over every set A_l^i for all $k \leq l \leq i+1$ and put $A_{i+1}^j = A_i^j$ for $j < k$ and $A_{i+1}^j = A_i^j \cup \{a\}$ for $k \leq j \leq i+1$. If $\varphi_i(x)$ is inconsistent or it is not over A_{i+1}^i, we put $A_{i+1}^j = A_i^j$ and $A_{i+1}^{i+1} = A_{i+1}^i$. Then $M_i = \bigcup_{j<\omega} A_i^j$ is a model prime over A_i^0, $(M_i : i < \omega)$ is an elementary chain and $M = \bigcup_{i<\omega} M_i$. □

Corollary 5.3 *If T is an Ehrenfeucht theory, then for every countable model M of T there is some finitary type $p(x) \in S(\emptyset)$ such that M is the union of an elementary chain $(M_i : i < \omega)$ with $M_i \cong M_p$ for all $i < \omega$.*

Proof: Use Proposition 5.2 and the fact that there are only finitely many nonisomorphic countable models. □

Definition 5.4 A countable model M is a *limit model* if it is not prime over a finite set and it is the union $M = \bigcup_{i<\omega} M_i$ of an elementary chain $(M_i : i < \omega)$ where every M_i is prime over a finite set. It is *p-limit* or *limit over p* (where $p(x) \in S_n(\emptyset)$) if $M_i \cong M_p$ for all $i < \omega$ (that is, every M_i is prime over a realization of p). Notice that, according to Proposition 5.2, in a small theory a countable model is either prime over a finite set or limit.

Remark 5.5 *If M is not limit and $M = \bigcup_{i<\omega} M_i$ for an elementary chain $(M_i : i < \omega)$ with $M_i \cong M_p$, then $M \cong M_p$.*

Proof: Assume M is prime over the finite set $A \subseteq M_i$ and M_i is prime over $a_i \models p$. Since a_i is a finite tuple, M is atomic over Aa_i. Since A is atomic over a_i, it follows that M is atomic (and prime) over a_i. □

Remark 5.6 *If T is small and it is not ω-categorical, then the countable saturated model of T is not prime over a finite set, and therefore it is a limit model.*

Proposition 5.7 *There is a p-limit model iff there are $a, b \in M_p$ such that $a \models p$ and $\mathrm{tp}(b/a)$ is nonisolated.*

Proof: $\Rightarrow$. Assume M is a p-limit, say $M = \bigcup_{i<\omega} M_i$ with M_i prime over $a_i \models p$. If the right hand side fails, then every finite tuple $b \in M_i$ is isolated over any $a \models p$ in M_i, in particular over a_0. It follows that every finite tuple $b \in M$ is isolated over a_0 and hence $M \cong M_p$.

$\Leftarrow$. Assume M_1 is prime over $a_1 \models p$ and contains a, b such that $a \models p$ and $\mathrm{tp}(b/a)$ is nonisolated. Let $a_0 = a$ and extend the sequence a_0a_1 to $(a_n : n < \omega)$ with $a_0a_1 \equiv a_na_{n+1}$. We choose $M_0 \preceq M_1$ prime over a_0 and M_n, M_{n+1} such that $a_0a_1M_0M_1 \equiv a_na_{n+1}M_nM_{n+1}$. Then $(M_n : n < \omega)$ is an elementary chain and each M_n is prime over $a_n \models p$. Let $M = \bigcup_{n<\omega} M_n$. We claim that $M \not\cong M_p$ and therefore it is a p-limit. Assume $M \cong M_p$, say M prime over some $d \models p$. Then $d \in M_n$ for some $n < \omega$. Let $q(x, y) = \mathrm{tp}(ba)$. Then $q(x, a) = q(x, a_0)$ is omitted in M_0 and realized in M_1 and hence $q(x, a_n)$ is omitted in M_n and realized by some $c \in M_{n+1}$. Since c is isolated over d, it is also isolated over da_n. But d is isolated over a_n and hence c is isolated over a_n, which implies that $q(x, a_n)$ must be realized in M_n, a contradiction. □

Corollary 5.8 *If* SI_p *is not symmetric in* M_p*, then there is a* p*-limit model.*

Proof: By Proposition 5.7. □

Remark 5.9 *If* $p(x)$ *is a powerful nonisolated type in a small theory, then there is a* p*-limit model.*

Proof: By and Proposition 4.9 and Corollary 5.8. □

Definition 5.10 Let $p(x), q(y) \in S(\emptyset)$. We say that p *is weaker than* q *in the Rudin-Keisler order* and we write $p \leq_{\mathrm{RK}} q$ if any model that contains a realization of q contains also a realization of p. If $p \leq_{\mathrm{RK}} q$ and $q \leq_{\mathrm{RK}} p$, we say that p *and* q *are Rudin-Keisler equivalent* and we write $p \sim_{\mathrm{RK}} q$. Note that if p is isolated then $p \leq_{\mathrm{RK}} q$ for any q. Note also that if q is powerful, then $p \leq_{\mathrm{RK}} q$ for any p. We say that $p(x)$ and $q(y)$ are *strongly Rudin-Keisler equivalent* and we write $p \equiv_{\mathrm{RK}} q$ if there are $a \models p$ and $b \models q$ such that $\mathrm{tp}(a/b)$ and $\mathrm{tp}(b/a)$ are isolated. Clearly $p \equiv_{\mathrm{RK}} q$ implies $p \sim_{\mathrm{RK}} q$.

Remark 5.11 *If* M_q *exists (in particular, if* T *is small), then the following are equivalent for any* p*:*

1. $p \leq_{\mathrm{RK}} q$
2. p *is realized in* M_q.
3. *There are* $a \models p$ *and* $b \models q$ *such that* $\mathrm{tp}(a/b)$ *is isolated.*
4. *There is some* $\varphi(x, y) \in L$ *such that* $q(y) \vdash \exists x\varphi(x, y)$ *and* $q(y) \cup \{\varphi(x, y)\} \vdash p(x)$.

Definition 5.12 Assume $p(x), q(y) \in S(\emptyset)$. We say that $\varphi(x, y) \in L$ is (q, p)*-principal* if $q(y) \vdash \exists x\varphi(x, y)$, $q(y) \cup \{\varphi(x, y)\} \vdash p(x)$ and $q(y) \cup \{\varphi(x, y)\}$ is complete (it has a unique extension to a type $r(x, y) \in S(\emptyset)$).

Remark 5.13 *If* T *is small and* $p \leq_{\mathrm{RK}} q$*, then there is some* (q, p)*-principal* $\varphi(x, y) \in L$.

Proof: Let $b \models q$ and let $a \models p$ be such that $\mathrm{tp}(a/b)$ is isolated. Let $r(x, y) = \mathrm{tp}(ab)$. Then $\mathrm{tp}(a/b) = r(x, b)$ and for some $\varphi(x, y) \in L$, $\varphi(x, b)$ isolates $\mathrm{tp}(a/b)$. Hence $\models \varphi(a, b)$ and $q(y) \cup \{\varphi(x, y)\} \vdash r(x, y)$. Clearly, $\varphi(x, y)$ satisfies the requirements. □

Proposition 5.14 *If* T *is small, the following are equivalent for any* $p(x), q(y) \in S(\emptyset)$*:*

1. $M_p \cong M_q$

2. $p \equiv_{\mathrm{RK}} q$

3. There is some (q,p)*-principal* $\varphi(x,y) \in L$ *and some* (p,q)*-principal* $\psi(y,x) \in L$ *such that* $p(x) \cup q(y) \cup \{\varphi(x,y), \psi(y,x)\}$ *is consistent.*

4. For some (q,p)*-principal* $\varphi(x,y) \in L$*,* $\varphi^{-1}(y,x) = \varphi(x,y)$ *is* (p,q)*-principal and* $p(x) \cup q(y) \cup \{\varphi(x,y)\}$ *is consistent.*

Proof: *1* ⇒ *2*. Let M_p be prime over $a \models p$. Since $M_q \cong M_p$, M_p is also prime over some $b \models q$. Clearly, $\mathrm{tp}(a/b)$ and $\mathrm{tp}(b/a)$ are isolated.

2 ⇒ *3*. Assume $a \models p$, $b \models q$ and the types $\mathrm{tp}(a/b)$, $\mathrm{tp}(b/a)$ are isolated. Let $r(x,y) = \mathrm{tp}(a,b)$. Since $\mathrm{tp}(a/b) = r(x,b)$ and $\mathrm{tp}(b/a) = r(a,y)$, we can find $\varphi(x,y), \psi(y,x) \in L$ such that $\models \varphi(a,b) \wedge \psi(b,a)$, $\varphi(x,b) \vdash r(x,b)$ and $\psi(y,a) \vdash r(a,y)$. Then $q(y) \cup \{\varphi(x,y)\} \vdash r(x,y)$ and $p(x) \cup \{\varphi(y,x)\} \vdash r(x,y)$, and $p(x) \cup q(y) \cup \{\varphi(x,y), \psi(y,x)\}$ is consistent.

It is clear that *3* implies *4* (take the conjunction of $\varphi(x,y)$ and $\psi(y,x)$) and that *4* implies *3*. We prove now *4* ⇒ *1*. We can find $a \models p$ and $b \models q$ such that M_p is prime over a, M_q is prime over b and $\models \varphi(a,b)$. There are $a' \in M_q$ and $b' \in M_p$ such that $\models \varphi(a',b)$ and $\models \varphi(b',a)$. Then $ab' \equiv ab \equiv a'b$ and hence $\mathrm{tp}(a/b')$ is isolated. It follows that M_p is prime over $b' \models q$ and hence $M_p \cong M_q$. □

Corollary 5.15 *Let* T *be small. If* $p \sim_{\mathrm{RK}} q$ *and* $M_p \not\cong M_q$*, then some model is a* p*-limit and a* q*-limit.*

Proof: It is easy to construct an elementary chain $(M_n : n < \omega)$ with $M_{2n} \cong M_p$ and $M_{2n+1} \cong M_q$. Let $M = \bigcup_{n<\omega} M_n$. If M is not limit, then by Proposition 5.5 $M_p \cong M \cong M_q$. □

Remark 5.16 *It is clear that the previous result obviously hold also for finitely many Rudin-Keisler equivalent types.*

Corollary 5.17 *Let* T *be small. If* $p \sim_{\mathrm{RK}} q$ *and there is a* p*-limit model, then some model is a* p*-limit and a* q*-limit.*

Proof: If $M_p \cong M_q$, any p-limit is a q-limit. If $M_p \not\cong M_q$, then apply Corollary 5.15. □

Remark 5.18 *If some model* M *is a* p*-limit and a* q*-limit, then* $p \sim_{\mathrm{RK}} q$*.*

Proof: Let M be the union of the elementary chain $(M_i : i < \omega)$ with $M_i \cong M_p$ and of the elementary chain $(N_i : i < \omega)$ with $N_i \cong M_q$. Since q is realized in M_q, it is realized in M. The realization appears in some M_i and hence q is realized in M_p. This shows $q \leq_{\mathrm{RK}} p$. Similarly, $p \leq_{\mathrm{RK}} q$. □

Definition 5.19 Let $\mathrm{RK}(T)$ be the set of isomorphism types of countable models of T that are prime over a finite set. By Proposition 5.14, in a small theory we can identify $\mathrm{RK}(T)$ with $S(\emptyset)/\equiv_{\mathrm{RK}}$. The Rudin-Keisler order induces a preordering $\leq_{\mathrm{RK}}$ of $\mathrm{RK}(T)$ with equivalence relation $\sim_{\mathrm{RK}}$. For each class $X \in \mathrm{RK}(T)/\sim_{\mathrm{RK}}$ let $\mathrm{IL}(X)$ the cardinality of the set of isomorphism types of countable models that are p-limit for some p with $M_p \in X$.

Remark 5.20 *Let* T *be small. There is a smallest element in* $\mathrm{RK}(T)$ *(the prime model) and every two elements have a supremum: if* $a \models p$ *and* $b \models q$*, and* $r = \mathrm{tp}(ab)$*, then* M_r *is the supremum of* M_p *and* M_q*. If* T *has only finitely many models, then the order is finite and has a greatest element:* M_p *for* p *a powerful type.*

Remark 5.21 *Let T be small.*

1. *T is ω-categorical iff $|\mathrm{RK}(T)| = 1$.*
2. *If $|\mathrm{RK}(T)| = 2$, then every nonisolated type is powerful.*

Proof: *1.* If T is not ω-categorical, then it has a nonisolated type p. Then M_p is not isomorphic to the prime model, which shows $|\mathrm{RK}(T)| \geq 2$.

2. If p, q are nonisolated then M_p, M_q are not prime and therefore $|\mathrm{RK}(T)| = 2$ implies $M_p \cong M_q$. □

Proposition 5.22 *For any countable T, the following are equivalent:*

1. *$I(T, \omega) < \omega$.*
2. *T is small, $|\mathrm{RK}(T)| < \omega$, and $\mathrm{IL}(X) < \omega$ for every class $X \in \mathrm{RK}(T)/\sim_{\mathrm{RK}}$.*

In fact, if these conditions hold, then

$$I(T, \omega) = |\mathrm{RK}(T)| + \sum_{i=0}^{m} \mathrm{IL}(X_i)$$

(where $X_0, \ldots, X_m$ are the different elements of $\mathrm{RK}(T)/\sim_{\mathrm{RK}}$) and

a. *If X is the class of the prime model, $\mathrm{IL}(X) = 0$.*

b. *If X is the class of M_p with p powerful, then $|\mathrm{RK}(T)| > 1$ implies $\mathrm{IL}(X) \geq 1$.*

c. *If $|X| > 1$, then $\mathrm{IL}(X) \geq 1$.*

Proof: *a.* The union of an elementary chain $(M_i : i < \omega)$ is prime if every M_i is prime.

b. The assumption $|\mathrm{RK}(T)| > 1$ means that T is not ω-categorical and hence a powerful type p is nonisolated. Then apply Remark 5.9.

c. By Corollary 5.15. □

Corollary 5.23 *For any countable T, the following are equivalent:*

1. *$I(T, \omega) = 3$.*
2. *T is small, $|\mathrm{RK}(T)| = 2$, and any two p-limit models are isomorphic for any p.*

Proof: It is clear that *2* implies *1*: the models are the prime model, the model M_p with p powerful and the unique p-limit, which is saturated. *1* ⇒ *2* follows from Proposition 5.22. □

6 Smooth classes

This section is based on a previous script partially done jointly with Hans Adler and Silvia Barbina. Some topics have been discussed some time ago with Daniel Palacín.

We assume L is a relational language.

Definition 6.1 Let K be a class of finite structures and assume $\leq$ is a binary relation on K. We say that $(K, \leq)$ is *smooth* if K is closed under isomorphism and $\leq$ is a partial order on K refining $\subseteq$ and satisfying the following conditions:

1. If $A \leq B$ and $f : B \cong B'$ is an isomorphism such that $f(A) = A'$, then $A' \leq B'$.
2. If $A \subseteq B \subseteq C$, $B \in K$ and $A \leq C$, then $A \leq B$.

If M is an L-structure, we extend the relation $\leq$ for $A \in K$ by

$$A \leq M \text{ iff } A \leq B \text{ for all } B \in K \text{ such that } A \subseteq B \subseteq M$$

Clearly, if f is an automorphism of M and $A \leq M$, then $f(A) \leq M$.

Definition 6.2 Let $(K, \leq)$ be a smooth class. The L-structure M is a $(K, \leq)$-*union* if $M = \bigcup_{n<\omega} C_n$ for a $\leq$-chain $(C_n : n < \omega)$ of structures $C_n \in K$.

An embedding $f : A \to B$ is *strong* if $f(A) \leq B$. In this case we say that A is *strongly embeddable* in B. Similarly for an embedding $f : A \to M$.

We say that M is $(K, \leq)$-*rich* if for all $A \leq B$ such that $A \leq M$, there is some strong embedding $f : B \to M$ over A, that is, such that $f \restriction A$ is the identity.

We say that M is $(K, \leq)$-*generic* if

1. M is a $(K, \leq)$-union
2. All $A \in K$ are strongly embeddable in M.
3. M is $(K, \leq)$-rich.

Remark 6.3 *Let $(K, \leq)$ be a smooth class, let M be a $(K, \leq)$-union, say $M = \bigcup_{n<\omega} C_n$ where $C_n \leq C_{n+1}$. Then, $A \leq M$ iff $A \leq C_n$ for some $n < \omega$ iff there is some $n < \omega$ such that $A \leq C_m$ for all $m \geq n$. Hence, if $A \leq B \leq M$, then $A \leq M$.*

Definition 6.4 Let $(K, \leq)$ be smooth.

1. $(K, \leq)$ has the *joint embedding property* (JEP) if any $A, B \in K$ are strongly embeddable in some $C \in K$.
2. $(K, \leq)$ has the *amalgamation property* (AP) if for any $A, B_1, B_2 \in K$ if $f_1 : A \to B_1$ and $f_2 : A \to B_2$ are strong embeddings, then there is some $C \in K$ and strong embeddings $g_1 : B_1 \to C$, $g_2 : B_2 \to C$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Note that if we admit $\emptyset \in K$ and $\emptyset \leq A$ for every $A \in K$, then AP implies JEP.

Theorem 6.5 *Let $(K, \leq)$ be smooth.*

1. *There is a $(K, \leq)$-generic model M if and only if $K/\cong$ is countable, and $(K, \leq)$ satisfies AP and JEP.*
2. *If M, N are $(K, \leq)$-generic, then $M \cong N$.*

Proof: *2.* The set of isomorphisms between strong substructures $A \leq M$ and $B \leq N$ is a nonempty system of partial isomorphisms with the back-and-forth properties. □

Definition 6.6 A class K of finite structures is *oligomorphic* if for every $n < \omega$ there are only finitely many nonisomorphic structures of cardinality n in K.

Remark 6.7 *If $(K, \leq)$ is smooth and M is generic, then M is homogeneous in the following sense: if $A \leq M$, $B \leq M$ and $f : A \cong B$ is an isomorphism, then f can be extended to an automorphism of M. Moreover, this homogeneity property can replace richness in the definition of a generic structure.*

Corollary 6.8 *Let $(K, \leq)$ be an oligomorphic smooth class. If M is an L-structure, $A \leq M$, and $M \preceq N$, then $A \leq N$.*

Proof: Let $|A| = n$ and fix some $m > 0$. We want to express in N that if $A \subseteq B \subseteq N$ with $B \in K$ and $|B| = n + m$, then $A \leq B$. Since K is oligomorphic the number of isomorphism types over A of structures in $K_0 = \{B \in K : A \subseteq B \text{ and } |B| = n + m\}$ is finite. Let $B_1, \dots, B_k$ be a list of representatives and let $\sigma_1(x_1, \dots, x_m), \dots, \sigma_k(x_1, \dots, x_m)$ be formulas $\sigma_i(x_1, \dots, x_m) \in L(A)$ such that for any model N, for all $b_1, \dots, b_m \in N$ with $A \cup \{b_1, \dots, b_m\} \in K_0$:

$$N \models \sigma_i(b_1, \dots, b_m) \Leftrightarrow A \cup \{b_1, \dots, b_m\} \cong_A B_i.$$

(where $A \cup \{b_1, \dots, b_m\}$ is the substructure of N with this universe). If $K_1 = \{B \in K_0 : A \leq B\}$ we may assume that $B_1, \dots, B_j$ (with $j \leq k$) is a list of representatives of isomorphism types over A of structures in K_1. Then the condition can be expressed by $\forall x_1 \dots x_m \bigwedge_{j<i\leq k} \neg\sigma_i(x_1, \dots, x_m)$. □

Corollary 6.9 *Let $(K, \leq)$ be an oligomorphic smooth class, M an L-structure, and $\mathfrak{C}$ the monster model of* $\mathrm{Th}(M)$. *Assume $\overline{a}$ enumerates some $A \leq M$ and $\overline{b}$ enumerates some finite substructure $B \subseteq \mathfrak{C}$. If $\overline{a} \equiv \overline{b}$, then $B \leq \mathfrak{C}$.*

Proof: Let f be an automorphism of $\mathfrak{C}$ such that $f(\overline{a}) = \overline{b}$. Then $B \leq f(M)$ and by Corollary 6.8 $B \leq \mathfrak{C}$. □

7 Closures

L is a relational language.

Definition 7.1 The class K is *cofinal* in an L-structure M if for each finite $A \subseteq M$ there is some $B \in K$ such that $A \subseteq B \subseteq M$. The ordered class $(K, \leq)$ is *cofinal* in M if for each finite $A \subseteq M$ there is some $B \in K$ such that $A \subseteq B \leq M$. In [2] this last definition is rephrased as: *M has finite closures.*

Remark 7.2 *Let $(K, \leq)$ be smooth.*

1. *If M is countable, then (i) K is cofinal in M iff M is a union of an ascending chain of structures of K, and (ii) $(K, \leq)$ is cofinal in M iff M is a $(K, \leq)$-union.*
2. *In general, (i) K is cofinal in M iff M is a union of a $\subseteq$-directed system of structures in K, and (ii) $(K, \leq)$ is cofinal in M iff M is a union of a $\leq$-directed system of structures in K.*
3. *If $(K, \leq)$ is cofinal in M and $A \leq B \leq M$, then $A \leq M$.*

Proof: Clear. □

Remark 7.3 *If L is finite and $(K, \leq)$ is smooth and it is closed under substructures, with generic M. Then K is cofinal in every $N \equiv M$.*

Proof: Let $n < \omega$. There are only finitely many L-structures of cardinality n up to isomorphism and the isomorphism type of one can be given by a formula $\varphi(x_1, \dots, x_n)$. Let $\varphi_1(x_1, \dots, x_n), \dots, \varphi_k(x_1, \dots, x_n)$ be formulas characterizing the isomorphism types of all structures in K of size n. Then $M \models \forall x_1 \dots x_n (\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \to \bigvee_{1 \leq i \leq k} \varphi_i(x_1, \dots, x_n)$ and this sentence hold also in N, which implies that every substructure of N of cardinality n belongs to K. □

Definition 7.4 Let $(K, \leq)$ be a smooth class and let M be an L-structure. Assume $A \subseteq B \subseteq M$ and $B \in K$. We say that B is a *$\leq$-closure of A in M* if B is a minimal extension of A such that $B \leq M$. If $(K, \leq)$ is cofinal in M, then any $A \subseteq M$ has at least one $\leq$-closure.

Remark 7.5 *Let $(K, \leq)$ be a smooth class cofinal in M. If $B \in K$ is a $\leq$-closure of $A \subseteq M$ in M and $A \subseteq C \subseteq B$ is such that $C \leq B$, then $C = B$.*

Proof: Since $C \leq B \leq M$, we get $C \leq M$. Minimality of B gives $B = C$. □

Definition 7.6 We say that a smooth class $(K, \leq)$ satisfies the *diamond principle* $\diamondsuit$ if for all $A_1, A_2, B \in K$, if $A_1 \subseteq B$ and $A_2 \leq B$ then $A_1 \cap A_2 \leq A_1$. Similarly, we say that $(K, \leq)$ satisfies the *weak diamond principle* $\diamondsuit_w$ if the same happens with the additional hypothesis that $A_1 \leq B$. In both cases this implies $A_1 \cap A_2 \in K$.

Proposition 7.7 *Let $(K, \leq)$ be a smooth class cofinal in M. If $(K, \leq)$ satisfies the $\diamondsuit_w$ principle, then any $A \subseteq M$ has at most one $\leq$-closure.*

Proof: Assume $A_1, A_2 \in K$ are $\leq$-closures in M of $A \subseteq M$. Since $A_1 \leq M$ and $A_2 \leq M$, there is some $B \in K$ such that $A_1 \leq B$ and $A_2 \leq B$. By $\diamondsuit_w$, $A_1 \cap A_2 \leq A_1$. By Remark 7.5, $A_1 \cap A_2 = A_1$. Similarly, $A_1 \cap A_2 = A_2$. □

Definition 7.8 Let $(K, \leq)$ be a smooth class cofinal in M, and assume $(K, \leq)$ satisfies the $\diamondsuit_w$ principle. In M there is a unique $\leq$-closure of $A \subseteq M$ and we denote it by $\mathrm{cl}_M(A)$. If M is understood we just write $\mathrm{cl}(A)$. A finite substructure $A \subseteq M$ is called *closed* if $\mathrm{cl}(A) = A$. Note that A is closed iff $A \leq M$.

Remark 7.9 *Let $(K, \leq)$ be a smooth class cofinal in M, and assume $(K, \leq)$ satisfies the $\diamondsuit_w$ principle, and A, B are finite substructures of M.*

1. $A \subseteq \mathrm{cl}(A)$
2. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
3. *If* $A \subseteq B$, *then* $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
4. $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B) = \mathrm{cl}(\mathrm{cl}(A) \cap \mathrm{cl}(B))$.

Proof: *2.* By Remark 7.5 since $\mathrm{cl}(A) \leq \mathrm{cl}(\mathrm{cl}(A))$.

3. Since $A \subseteq \mathrm{cl}(B)$ and $\mathrm{cl}(B)$ is closed, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

4. This follows directly from *1*, *2*, and *3*. □

Definition 7.10 Let $(K, \leq)$ be a smooth class cofinal in M, and assume $(K, \leq)$ satisfies the $\diamondsuit_w$ principle. For an arbitrary substructure $A \subseteq M$ we define $\mathrm{cl}_M(A) = \bigcup\{\mathrm{cl}_M(A_0) : A_0 \subseteq A \text{ finite}\}$. This clearly agrees with the previous use. Again, we omit the subscript M when possible and we call an arbitrary substructure $A \subseteq M$ *closed* if $\mathrm{cl}(A) = A$. It follows from Remark 7.9 that cl is a finitary closure operator.

Remark 7.11 *Assume $(K, \leq)$ is smooth and oligomorphic, with $\diamondsuit_w$ and cofinal in M. If $A \subseteq M$ is finite and f is an automorphism of the monster model such that $f(A) \subseteq M$, then $f(\mathrm{cl}(A)) = \mathrm{cl}(f(A))$.*

Proof: By Corollary 6.9, $f(\mathrm{cl}(A))$ and $\mathrm{cl}(f(A))$ are closures of $f(A)$ in M. □

Definition 7.12 Let $(K, \leq)$ be smooth. A *minimal pair* is a pair of elements (A, B) of K such that $A \subsetneq B$, $A \not\leq B$ and $A \leq C$ for all $C \in K$ such that $A \subseteq C \subsetneq B$. A *chain of minimal pairs* is an ascending chain $(A_i : i < \omega)$ of structures $A_i \in K$ such that each (A_i, A_{i+1}) is a minimal pair. The statement that there are no chains of minimal pairs is condition C2 in [23].

Proposition 7.13 *Let $(K, \leq)$ be a smooth class and let M be generic.*

1. *If $N \equiv M$ is an ω-saturated model and $(K, \leq)$ is cofinal in N and satisfies the $\diamondsuit$ principle, then there are no chains of minimal pairs.*
2. *If there are no chains of minimal pairs and K is cofinal in the L-structure N, then $(K, \leq)$ is cofinal in N.*

Proof: *1.* If there is a chain of minimal pairs, since N is ω-saturated, there is such a chain contained in N. Take $C \leq N$ such that $A_0 \subseteq C$ and choose $i < \omega$ such that $A_i \subseteq C$ but $A_{i+1} \not\subseteq C$. Then $A_i \subseteq A_{i+1} \cap C \subsetneq A_{i+1}$. By $\diamondsuit$, $A_{i+1} \cap C \leq A_{i+1}$. Since (A_i, A_{i+1}) is a minimal pair, $A_i \leq A_{i+1} \cap C$. Hence $A_i \leq A_{i+1}$, a contradiction.

2. We show that any finite $A \subseteq N$, $A \in K$, has a $\leq$-closure in N. If not, there is some $A_0 \in K$, such that $A \subseteq A_0 \subseteq N$ and $A \not\leq A_0$. We take it minimal with this property. By iteration we construct an ascending $\subseteq$-chain $(A_i : i < \omega)$ of minimal pairs (A_i, A_{i+1}). □

Proposition 7.14 *Let $(K, \leq)$ be a smooth oligomorphic class that satisfies the $\diamondsuit_w$ principle and let M be generic.*

1. *M is homogeneous.*
2. *For any finite substructure $A \subseteq M$, $\mathrm{cl}(A) \subseteq \mathrm{acl}(A)$.*

Proof: *1.* Let A, B finite subsets of M such that $f(A) = B$ for some automorphism f of the monster model. By Remark 7.11, $f(\mathrm{cl}(A)) = \mathrm{cl}(B)$ and hence $f \restriction \mathrm{cl}(A)$ can be extended to an automorphism of M.

2. Let $\mathfrak{C}$ be the monster model of $\mathrm{Th}(M)$. By Remark 7.11 every automorphism f of $\mathfrak{C}$ fixing pointwise A fixes setwise $\mathrm{cl}(A)$. Since $\mathrm{cl}(A)$ is finite, $\mathrm{cl}(A) \subseteq \mathrm{acl}(A)$. □

Definition 7.15 Let $(K, \leq)$ be a smooth class. We say that $(K, \leq)$ has *disjoint amalgamations* if for any $A, B_1, B_2 \in K$ if $f_1 : A \to B_1$ and $f_2 : A \to B_2$ are strong embeddings, then there is some $C \in K$ and strong embeddings $g_1 : B_1 \to C$, $g_2 : B_2 \to C$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and $g_1(B_1) \cap g_2(B_2) = g_1(f_1(A))$. It is enough to check this for the case when $f_1 = f_2$ is the identity on A.

Proposition 7.16 *Let $(K, \leq)$ be an oligomorphic smooth class, let M be generic of $(K, \leq)$, and assume $(K, \leq)$ satisfies the $\diamondsuit_w$ principle. The following are equivalent:*

1. *$(K, \leq)$ has disjoint amalgamations.*
2. *For any finite $A \subseteq M$, $\mathrm{cl}(A) = \mathrm{acl}(A)$.*

Proof: *1 ⇒ 2.* By Proposition 7.14, $\mathrm{cl}(A) \subseteq \mathrm{acl}(A)$. Assume $b \in \mathrm{acl}(A)$ and choose $B \leq M$ such that for all $b' \equiv_A b$, $b' \in B$. By disjoint amalgamation, there is some $C \in K$ and strong embeddings $g_1 : B \to C$, $g_2 : B \to C$ such that $g_1 \restriction \mathrm{cl}(A) = g_2 \restriction \mathrm{cl}(A)$ and $g_1(B) \cap g_2(B) = g_1(\mathrm{cl}(A))$. We may assume $C \leq M$ and g_1 is the identity. Since g_2 can be extended to an automorphism of M (which fixes setwise the orbit of b under $\mathrm{Aut}(M/A)$) it follows that $b \in g_1(B) \cap g_2(B) = \mathrm{cl}(A)$.

2 ⇒ 1. Without loss of generality $f_1 = f_2$ is the identity on A and $A \leq M$. Then $A = \mathrm{cl}(A) = \mathrm{acl}(A)$. By Neumann's Lemma, whenever $\mathrm{acl}(A) \subseteq B \cap C \subseteq M$ and B is finite, there is some elementary mapping $f : B \to M$ fixing pointwise $\mathrm{acl}(A)$ and such that $(f(B) \smallsetminus \mathrm{acl}(A)) \cap C = \emptyset$. □

Proposition 7.17 *Let $(K, \leq)$ be a smooth oligomorphic class. If a generic model M of $(K, \leq)$ is weakly saturated, then it is saturated.*

Proof: Since $\mathrm{Th}(M)$ has a countable weakly saturated model, it is small and has a countable saturated model $N \succeq M$. We will show that N is generic, which implies $M \cong N$ and hence that M is saturated. By Corollary 6.8, every $A \in K$ is strongly embeddable in N. If $A \subseteq N$ is finite, by weakly saturation there is an elementary mapping $f : A \to M$ and hence there is some $B \leq M$ such that $f(A) \subseteq B$. By ω-saturation there exists also some finite B' such that $A \subseteq B' \subseteq N$ and f extends to some elementary bijection $g : B' \to B$. By Corollary 6.9, $B' \leq N$. It follows that N is a $(K, \leq)$-union. Finally, consider some $A \subseteq N$ and some $B \in K$ such that $A \leq B$. By weakly saturation of M there is some elementary mapping $f : A \to M$. By ω-saturation of N, there is some $M_1 \preceq N$ and some automorphism f_1 of the monster model such that $f \subseteq f_1$ and $f_1(M_1) = M$. By genericity of M_1 there is some strong embedding $g : B \to M_1$ fixing pointwise A. By Corollary 6.8 it is also an strong embedding in N. □

8 Predimension and dimension

In this and in the next sections L is a countable relational language and $\emptyset$ is accepted as L-structure.

Definition 8.1 Let K be a class of finite L-structures closed under isomorphism and substructures. A mapping $\delta : K \to \mathbb{R}^{\geq 0}$ is a *predimension* if

1. $\delta(\emptyset) = 0$
2. If $A \cong B$, then $\delta(A) = \delta(B)$.
3. $\delta(AB) + \delta(A \cap B) \leq \delta(A) + \delta(B)$.

In 3, AB is some structure in K whose universe is $A \cup B$ and extends the structures $A, B \in K$. In this context $A \cap B$ is well-defined.

In [23] the definition of predimension has an additional condition that we will only use occasionally:

4. There is no sequence $(A_i : i < \omega)$ such that $A_i \subseteq A_{i+1}$ and $\delta(A_i) > \delta(A_{i+1})$ for all $i < \omega$.

Remark 8.2 *If K is closed under isomorphisms and substructures and* $\mathrm{Age}(M)$ *is the set of all finite structures embeddable in M, then: K is cofinal in M iff* $\mathrm{Age}(M) \subseteq K$. *Note that in the case of a finite language L, $M \equiv N$ implies* $\mathrm{Age}(M) = \mathrm{Age}(N)$.

Definition 8.3 Let δ be a predimension in K. We define two relations $\leq_\delta$ and $\leq'_\delta$ on K by

$$A \leq_\delta B \Leftrightarrow A \subseteq B \text{ and } \delta(X) \geq \delta(A) \text{ for all } X \text{ such that } A \subseteq X \subseteq B$$

and

$$A \leq'_\delta B \Leftrightarrow A \subseteq B \text{ and } \delta(X) > \delta(A) \text{ for all } X \text{ such that } A \subsetneq X \subseteq B$$

If the context allows it, we write $\leq$ and $\leq'$ instead of $\leq_\delta$ and $\leq'_\delta$.

Remark 8.4 *$A \leq' B$ implies $A \leq B$.*

Proposition 8.5 *Let δ be a predimension in K. Then $(K, \leq_\delta)$ is a smooth class with $\diamondsuit$. Condition 4 implies additionally that there are no chains of minimal pairs. Hence, if $(K, \leq)$ is cofinal in M, every finite $A \subseteq M$ has a unique closure $\mathrm{cl}_M(A) \in K$, a minimal extension of A which is strong in M. Assuming condition 4, then $\mathrm{Age}(M) \subseteq K$ implies that $(K, \leq)$ is cofinal in M and hence $\mathrm{cl}_M(A)$ is well-defined for every $A \subseteq M$.*

Proof: Only transitivity and $\diamondsuit$ need some checking. We begin with transitivity. Assume $A \leq B$, $B \leq C$ and $A \subseteq X \subseteq C$. We show that $\delta(X) \geq \delta(A)$. Note that $\delta(XB) \geq \delta(B)$ and $\delta(X \cap A) \geq \delta(A)$. Hence $\delta(X) + \delta(B) \geq \delta(XB) + \delta(X \cap B) \geq \delta(B) + \delta(A)$ and therefore $\delta(X) \geq \delta(A)$.

For $\diamondsuit$, assume $A \leq AB$ and let us check that $A \cap B \leq B$. Suppose $A \cap B \subseteq X \subseteq B$. Then $A \subseteq AX \subseteq AB$ and therefore $\delta(AX) \geq \delta(A)$. Then $\delta(X) + \delta(A) \geq \delta(XA) + \delta(X \cap A) \geq \delta(A) + \delta(X \cap A)$. Since $X \cap A = B \cap A$, we conclude $\delta(X) \geq \delta(B \cap A)$.

If (A, B) is a minimal pair, then $\delta(A) > \delta(B)$. Therefore, inexistence of chains of minimal pairs follows from condition 4 in the definition of predimension. By Remark 8.2 and propositions 7.13 and 7.7 we obtain existence and uniqueness of closures. □

There is a corresponding proposition for $\leq'$:

Proposition 8.6 *Let δ be a predimension in K. Then $(K, \leq'_\delta)$ is a smooth class with $\diamondsuit$. Condition 4 implies additionally that there are no chains of minimal pairs. Hence, if $(K, \leq')$ is cofinal in M, every finite $A \subseteq M$ has a unique closure $\mathrm{cl}'_M(A) \in K$, a minimal extension of A which is strong in M. Assuming condition 4, then $\mathrm{Age}(M) \subseteq K$ implies that $(K, \leq')$ is cofinal in M and hence $\mathrm{cl}'_M(A)$ is well-defined for every $A \subseteq M$.*

Proof: Similar to the proof of 8.5. For transitivity, we assume $A \subsetneq X \subseteq C$ and we distinguish two cases, according to whether $B = BX$ or not. In the first case, $A \subsetneq X \subseteq B$ and hence $\delta(X) > \delta(A)$. In the second case, $B \subsetneq BX \subseteq C$ and hence $\delta(BX) > \delta(B)$. On the other hand, $\delta(B \cap X) \geq \delta(A)$ and therefore $\delta(X) + \delta(B) \geq \delta(XB) + \delta(X \cap B) > \delta(B) + \delta(A)$ and hence $\delta(X) > \delta(A)$. For $\diamondsuit$, it is essentially the same proof. □

Lemma 8.7 *Let δ be a predimension in K and assume $(K, \leq)$ is cofinal in M and $A \subseteq B \subseteq M$.*

1. $\delta(B) \geq \delta(\mathrm{cl}(A))$.

2. If $\delta(B) = \delta(\mathrm{cl}(A))$*, then* $\mathrm{cl}(A) \subseteq B$.

If moreover $(K, \leq')$ *is cofinal in* M*, then*

3. $\delta(B) \geq \delta(\mathrm{cl}'(A))$.

4. If $\delta(B) = \delta(\mathrm{cl}'(A))$*, then* $B \subseteq \mathrm{cl}'(A)$.

5. $\mathrm{cl}(A) \subseteq \mathrm{cl}'(A)$.

Proof: We first claim that for every X such that $A \subseteq X \subsetneq \mathrm{cl}(A)$ we have $\delta(X) > \delta(\mathrm{cl}(A))$. Assume not and let X be a maximal counterexample. Then for every Y such that $X \subseteq Y \subseteq \mathrm{cl}(A)$ we have $\delta(Y) \geq \delta(X)$ (by maximality) and therefore $X \leq \mathrm{cl}(A)$. It follows $X \leq M$ and hence $X = \mathrm{cl}(A)$, a contradiction.

1. We apply $\diamondsuit$ to B and $\mathrm{cl}(A)$: since $\mathrm{cl}(A) \leq \mathrm{cl}(A)B$, we get $\mathrm{cl}(A) \cap B \leq B$. Hence $\delta(B) \geq \delta(\mathrm{cl}(A) \cap B)$ and by the claim $\delta(\mathrm{cl}(A) \cap B) \geq \delta(\mathrm{cl}(A))$. Thus, $\delta(B) \geq \delta(\mathrm{cl}(A))$.

2. If $\mathrm{cl}(A) \not\subseteq B$, then $A \subseteq B \cap \mathrm{cl}(A) \subsetneq \mathrm{cl}(A)$ and by the claim $\delta(B \cap \mathrm{cl}(A)) > \delta(\mathrm{cl}(A))$. By $\diamondsuit$, $\delta(B) \geq \delta(B \cap \mathrm{cl}(A))$ and therefore $\delta(B) > \delta(\mathrm{cl}(A))$.

Now we assume $(K, \leq')$ is cofinal in M and we claim that for every X such that $A \subseteq X \subsetneq \mathrm{cl}'(A)$ we have $\delta(X) \geq \delta(\mathrm{cl}'(A))$. If not and X is a maximal counterexample, then $X \leq' \mathrm{cl}'(A)$ and therefore $X = \mathrm{cl}'(A)$.

3. Assume $\delta(B) < \delta(\mathrm{cl}'(A))$. By $\diamondsuit$, $\delta(B) \geq \delta(B \cap \mathrm{cl}'(A))$. If $B \cap \mathrm{cl}'(A) = \mathrm{cl}'(A)$ we get a contradiction. Otherwise $B \cap \mathrm{cl}'(A) \subsetneq \mathrm{cl}'(A)$ and by the claim $\delta(B \cap \mathrm{cl}'(A)) \geq \delta(\mathrm{cl}'(A)$.

4. Assume $B \not\subseteq \mathrm{cl}'(A)$. Then $A \subseteq B \cap \mathrm{cl}'(A) \subsetneq B$ and, by $\diamondsuit$, $\delta(B) > \delta(B \cap \mathrm{cl}'(A))$. The case $B \cap \mathrm{cl}'(A) = \mathrm{cl}'(A)$ is impossible by assumption. Then $B \cap \mathrm{cl}'(A) \subsetneq \mathrm{cl}'(A)$ and by the claim $\delta(B \cap \mathrm{cl}'(A)) \geq \delta(\mathrm{cl}'(A))$. Hence $\delta(B) > \delta(\mathrm{cl}'(A))$, a contradiction.

5. This follows from all the previous points. $\square$

Definition 8.8 Assuming δ is a predimension in K and assuming $(K, \leq)$ is cofinal in M we define a dimension in M. For every finite $A \subseteq M$ the *dimension of A in M* is

$$\mathrm{d}_M(A) = \min\{\delta(B) : A \subseteq B \subseteq M\}$$

and we write $\mathrm{d}(A)$ if M is understood. The existence of the minimum follows from Lemma 8.7:

$$\mathrm{d}_M(A) = \delta(\mathrm{cl}_M(A)).$$

Assuming $(K, \leq')$ is cofinal in M we get the same dimension as $\mathrm{d}_M(A) = \delta(\mathrm{cl}'(A))$.

Remark 8.9 *Let δ be a predimension in K and assume $(K, \leq)$ is cofinal in M and $A \subseteq B \subseteq M$. Then*

$$\mathrm{cl}(A) = \bigcap\{B : A \subseteq B \subseteq M \textit{ and } \delta(B) = \mathrm{d}(A)\}.$$

If $(K, \leq')$ is cofinal in M, then

$$\mathrm{cl}'(A) = \bigcup\{B : A \subseteq B \subseteq M \textit{ and } \delta(B) = \mathrm{d}(A)\}.$$

Proof: By Lemma 8.7 and by the fact that $\mathrm{d}(A) = \delta(\mathrm{cl}(A)) = \delta(\mathrm{cl}'(A))$. $\square$

Proposition 8.10 *Let δ be a predimension in K and assume $(K, \leq)$ is cofinal in M. Then:*

1. $\mathrm{d}(\emptyset) = 0$
2. $\mathrm{d}(A) = \mathrm{d}(f(A))$ *for every* $f \in \mathrm{Aut}(M)$.
3. *If* $A \subseteq B$, *then* $\mathrm{d}(A) \leq \mathrm{d}(B)$.
4. $\mathrm{d}(AB) + \mathrm{d}(A \cap B) \leq \mathrm{d}(A) + \mathrm{d}(B)$

If moreover $d(A) \in \omega$ for every A, then

5. *the operator*
$$A \mapsto \mathrm{cl_d}(A) = \{a \in M : \mathrm{d}(A \cup \{a\}) = \mathrm{d}(A)\}$$
determines a pregeometry on subsets of M (if $A \subseteq M$ is infinite we take $\mathrm{cl_d}(A) = \bigcup\{\mathrm{cl_d}(B) : B \subseteq A$ *is finite* $\}$*). For finite A:* $\mathrm{cl}_M(A) \subseteq \mathrm{cl_d}(A)$ *and* $\mathrm{d}(A)$ *is the dimension of A in the pregeometry.*

Proof: We check *4*. Choose $A' \supseteq A$ and $B' \supseteq B$ with $\mathrm{d}(A) = \delta(A')$ and $\mathrm{d}(B) = \delta(B')$. Then $\mathrm{d}(AB) + \mathrm{d}(A \cap B) \leq \mathrm{d}(A'B') + \mathrm{d}(A' \cap B') \leq \delta(A'B') + \delta(A' \cap B') \leq \delta(A') + \delta(B') = \mathrm{d}(A) + \mathrm{d}(B)$. □

9 Some predimensions

Definition 9.1 Let L consist only of an n-ary relational symbol R and let K be the class of all finite L-structures A such that R^A is symmetric and irreflexive, that is:

1. $R^A(a_1, \ldots, a_n)$ implies $R^A(a_{\pi(1)}, \ldots, a_{\pi(n)})$ for any permutation π of $\{1, \ldots, n\}$.
2. $R^A(a_1, \ldots, a_n)$ implies $a_i \neq a_j$ for all $i \neq j$.

The case $A = \emptyset$ is allowed. Clearly, R^A can always be identified with a subset of $[A]^n$, that is, with a collection of n-element subsets of A. The number of edges of R^A is, by definition, the number of tuples in R^A up to permutation. We will denote it by $|R^A|$ in the hope that this won't be understood literally as the plain cardinality of R^A.

Given a real number $\alpha \geq 0$, we define $\delta_\alpha(A) = |A| - \alpha \cdot |R^A|$ for $A \in K$. We define

1. $A \leq B$ iff $\delta_\alpha(X) \geq \delta_\alpha(A)$ for all X such that $A \subseteq X \subseteq A$.
2. $A \leq' B$ iff $\delta_\alpha(X) > \delta_\alpha(A)$ for all X such that $A \subsetneq X \subseteq A$.
3. $K_0 = \{A \in K : \emptyset \leq A\}$
4. $K_0' = \{A \in K : \emptyset \leq' A\}$

Proposition 9.2 *K_0 and K_0' are closed under isomorphisms and substructures, δ_α is a predimension in K_0, in K_0', and more generally in every subclass of K_0 closed under isomorphisms and substructures. Hence, in all these classes $\leq$ and $\leq'$ define smooth classes with $\diamondsuit$.*

Proof: We check that $\delta(AB) + \delta(A \cap B) \leq \delta(A) + \delta(B)$. Let X be the set of edges of R^{AB} with some vertex in $A \smallsetminus B$ and some vertex in $B \smallsetminus A$, let Y be the set of edges of R^{AB} with some vertex in $A \smallsetminus B$ and no vertex in $B \smallsetminus A$, and let Z be the set of edges of R^{AB} with some vertex in $B \smallsetminus A$ and no vertex in $A \smallsetminus B$. Then $R^A = Y \cup R^{A\cap B}$, $R^B = Z \cup R^{A\cap B}$ and $R^{AB} = X \cup Y \cup Z \cup R^{A\cap B}$. Now observe that $\delta(AB) + \delta(A \cap B) = |AB| + |A \cap B| - \alpha \cdot (|R^{AB}| + |R^{A\cap B}|) = |A| + |B| - \alpha \cdot (|X \cup Y \cup Z \cup R^{A\cap B}| + |R^{A\cap B}|) = |A| + |B| - \alpha \cdot (|R^A| + |R^B| + |X|) = \delta(A) + \delta(B) - \alpha \cdot |X| \leq \delta(A) + \delta(B)$. □

Remark 9.3 *If α is an irrational number, then $\leq = \leq'$.*

Proof: Assume $A \leq B$ but $A \not\leq' B$. Then $\delta(X) = |X| - \alpha \cdot |R^X| = \delta(A) = |A| - \alpha \cdot |R^A|$ for some X such that $A \subsetneq X \subseteq A$. Then $0 \neq |X| - |A| = \alpha \cdot (|R^X| - |R^A|)$ and therefore $\alpha \in \mathbb{Q}$. □

Definition 9.4 Let A, B be arbitrary L-structures and assume $A \cap B$ is a common substructure of them. The *free amalgam* of the structures A, B is the structure of universe AB with $R^{AB} = R^A \cup R^B$ for every relation symbol $R \in L$. In other words, there are no tuples in R^{AB} with some point in $A \smallsetminus B$ and some point in $B \smallsetminus A$. We denote it by $A \sqcup B$. We say that a class K has *free amalgamation* if $A \sqcup B \in K$ whenever $A, B \in K$

Remark 9.5 *If K is closed under isomorphism and has free amalgamation, then it has disjoint amalgamation.*

Proof: In order to amalgamate A and B over $C \subseteq A, B$ one can always assume that $A \cap B = C$ just by taking some $B' \cong_C B$ such that $B' \cap A = C$. Then $A \sqcup B$ works. □

Remark 9.6 *Let $A, B \in K$ and assume AB is a common extension with universe $A \cup B$. Then $AB = A \sqcup B$ iff $\delta(AB) + \delta(A \cap B) = \delta(A) + \delta(B)$.*

Remark 9.7 *Assume $A, B \in K$ and $A \cap B$ is a common substructure. For any $X \subseteq A$ and $Y \subseteq B$, $X \sqcup Y$ is the substructure of $A \sqcup B$ with universe $X \cup Y$.*

Proposition 9.8 *Assume $A, B \in K_0$ and $A \cap B$ is a common substructure. If $A \cap B \leq A$ then $B \leq A \sqcup B$ and $A \sqcup B \in K_0$. Similarly, in case $A, B \in K_0'$ and $A \cap B \leq' A$, we get $A \sqcup B \in K_0'$ and $B \leq' A \sqcup B$.*

Proof: Let $X \subseteq A \sqcup B$. Then $X = (A \cap X) \sqcup (B \cap X)$ and hence $\delta(X) + \delta(X \cap A \cap B) = \delta(A \cap X) + \delta(B \cap X)$. By $\diamondsuit$, $A \cap B \cap X \leq A \cap X$ and thus $\delta(A \cap X) \geq \delta(A \cap B \cap X)$. It follows that $\delta(X) \geq \delta(B \cap X) + (\delta(A \cap X) - \delta(A \cap B \cap X)) \geq 0$. Hence $A \sqcup B \in K_0$.

Let $B \subseteq X \subseteq A \sqcup B$. Then $A \cap B \subseteq A \cap X \subseteq A$ and therefore $\delta(A \cap X) \geq \delta(A \cap B)$. Notice that $X = B \sqcup (A \cap X)$ and $\delta(B \sqcup (A \cap X)) + \delta(A \cap B) = \delta(B) + \delta(A \cap X)$. Hence $\delta(X) \geq \delta(B) + (\delta(A \cap X) - \delta(A \cap B)) \geq \delta(B)$. This shows that $B \leq A \sqcup B$.

The case of K_0' is similar. □

Proposition 9.9 $(K_0, \leq)$ *and* $(K_0', \leq')$ *have free amalgamation and therefore they have the amalgamation property (AP) and the joint embedding (JEP) properties and in the generic model* $\mathrm{cl}(A) = \mathrm{acl}(A) = \mathrm{cl}'(A)$.

Proof: By Proposition 9.8, Remark 9.5 and Proposition 7.16. □

10 Predimension and ω-categoricity

Definition 10.1 Given some strictly increasing $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with $\lim_{x\to\infty} f(x) = \infty$, let $K_f = \{A \in K_0 : \delta(B) \geq f(|B|)$ for every $B \subseteq A\}$ and let $K'_f = \{A \in K'_0 : \delta(B) \geq f(|B|)$ for every $B \subseteq A\}$

Proposition 10.2 *If $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is strictly increasing and $\lim_{x\to\infty} f(x) = \infty$, then K_f and K'_f are smooth classes with $\diamondsuit$ closed under substructures. If some of these classes has the amalgamation property, then the generic is ω-categorical.*

Proof: We check the ω-categoricity of the generic M. We claim that for every $n < \omega$ there is some $m < \omega$ such that for every finite substructure $A \subseteq M$, with $|A| \leq n$ we get $|\mathrm{cl}(A)| \leq m$. The reason is that $f(|\mathrm{cl}(A)|) \leq \delta(\mathrm{cl}(A)) \leq \delta(A) \leq |A|$ and there is some real r in the range of f with $r \geq |A|$ and hence if we take $m \geq f^{-1}(r)$ we obtain $|\mathrm{cl}(A)| \leq m$. Now, we count orbits of of n-tuples of M under $\mathrm{Aut}(M)$. There are only finitely many atomic types of m-tuples in M and in particular of m-tuples enumerating closed sets. The atomic type of a closed set determines its elementary type and its orbit under $\mathrm{Aut}(M)$. We have proved that the elementary type of an n-tuple can be extended to the elementary type of a closed m-tuple. Therefore, the mapping sending a type of a closed m-tuple to its restriction to the first n variables is surjective and we conclude that there are only finitely many orbits of n-tuples. □

11 Dimension and stability

Definition 11.1 A *dimension function in M* is a mapping d assigning a real number $\mathrm{d}(A) \in \mathbb{R}^{\geq 0}$ to each finite subset $A \subseteq M$ and such that

1. $\mathrm{d}(\emptyset) = 0$
2. $\mathrm{d}(A) = \mathrm{d}(f(A))$ for every $f \in \mathrm{Aut}(M)$.
3. If $A \subseteq B$, then $\mathrm{d}(A) \leq \mathrm{d}(B)$.
4. $\mathrm{d}(AB) + \mathrm{d}(A \cap B) \leq \mathrm{d}(A) + \mathrm{d}(B)$.

Let cl be a closure operator defined also for all finite subsets of M and such that

1. $\mathrm{cl}(A)$ is finite for every finite $A \subseteq M$.
2. $f(\mathrm{cl}(A)) = \mathrm{cl}(f(A))$ for every finite $A \subseteq M$.
3. If $A, B \subseteq M$ are closed and $A \cong B$, then $f(A) = B$ for some automorphism $f \in \mathrm{Aut}(M)$.

We say that cl and d are *compatible* if $\mathrm{d}(A) = \mathrm{d}(\mathrm{cl}(A))$ for every A.

For $A \subseteq M$ of arbitrary cardinality, the closure is defined as $\mathrm{cl}(A) = \bigcup\{X : X \subseteq A$ is finite $\}$. For finite $A, B \subseteq M$ the *relative dimension* $\mathrm{d}(A/B)$ of A over B is defined as

$$\mathrm{d}(A/B) = \mathrm{d}(AB) - \mathrm{d}(B)$$

and for finite $A \subseteq M$ and arbitrary $B \subseteq M$ this notion is extended as

$$\mathrm{d}(A/B) = \inf\{\mathrm{d}(A/X) : X \subseteq B \text{ is finite }\}$$

In this context we define the ternary relation $A \mathop{\smile\!\!\!\!\!\!\;|}^{\rm d}_C B$ for arbitrary subsets A, B, C of M by the condition:

$$\mathrm{cl}(AC) \cap \mathrm{cl}(BC) = \mathrm{cl}(C) \text{ and } \mathrm{d}(A'/CB') = \mathrm{d}(A'/C) \text{ for all finite } A' \subseteq A \text{ and } B' \subseteq B.$$

Remark 11.2 *Let* d *be a dimension function in M compatible with the closure operator* cl. *If $A \subseteq M$ is finite and $B \subseteq M$ is arbitrary, there is a countable subset $C \subseteq B$ such that $A \mathop{\smile\!\!\!\!\!\!\;|}^{\rm d}_C B$.*

Proposition 11.3 *Assume* d *is a dimension function in the monster model $\mathfrak{C}$ of T and* cl *is a closure operator compatible with* d. *Assume additionally that for all closed sets A, B, C such that $A \mathop{\smile\!\!\!\!\!\!\;|}^{\rm d}_C B$, we have $AB = A \sqcup B$ and it is closed. Then T is stable. If moreover there are only finitely many nonisomorphic substructures of $\mathfrak{C}$ of the same finite cardinality n, and for each finite A, for each B there is some finite $C \subseteq B$ such that $\mathrm{d}(A/B) = \mathrm{d}(A/C)$, then T is ω-stable.*

Proof: It is enough to prove that for any closed X, $|S_n(X)| \leq 2^\omega + |X|^\omega$. Given some finite tuple a, choose some countable $X_0 \subseteq X$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_{X_0} X$. Let $A = \mathrm{cl}(aX_0)$ and let $X_1 = A \cap X$. Notice that X_1 is countable and closed. Let a' be another tuple of the same length and choose a corresponding X_0' obtaining $A' = \mathrm{cl}(a'X_0')$ and $X_1' = A' \cap X$. We claim that if $X_1 = X_1'$ and $A \cong_{X_1} A'$, then $a \equiv_X a'$. To check this, notice that $A \cong_{X_1} A'$ implies $A \sqcup X \cong_X A' \sqcup X$. By choice of X_1, $A \mathop{\smile\!\!\!\!\!\!\;|}^{\rm d}_{X_1} X$ and hence $A \sqcup X = AX$ is closed. Similarly, $A' \sqcup X = A'X$ is closed. Hence, $f(AX) = A'X$ for some automorphism f fixing X. It follows that $a \equiv_X a'$. Now observe that the number of choices for X_1 is $|X|^\omega$ and that for each given X_1 there are at most 2^ω types of tuples of countable length over X_1. This gives the upper bound on the number of types.

For ω-stability, with the additional assumption we can get X_0 and X_1 finite and the assumption on the number of nonisomorphic structures of a given finite cardinality implies that there are only finitely many atomic types over X_1. Altogether this gives a bound of $|X| + \omega$ for the number of types over X. □

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