Groups in stable and simple theories*

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1 Preliminaries

T is a complete theory with infinite models. Its language is L and $\mathfrak{C}$ is its monster model. A few results, maybe not well-known, on bounded type-definable relations and canonical bases are needed in the sequel. They are included in the following two lemmas.

Lemma 1.1 *Let E be an equivalence relation type-definable over the set A. If E has only boundedly many equivalence classes (shortly, if E is bounded) then for each model $M \supseteq A$, every E-class is fixed by every $f \in \mathrm{Aut}(\mathfrak{C}/M)$. In particular, for every $p(x) \in S(M)$ there is a unique a (maybe not in M) such that $p(x) \vdash E(x, a)$. It follows that the number of E-classes is at most $2^{|T|+|A|+|\alpha|}$ where α is the length of the sequence of variables x.*

Proof: Let a/E be an equivalence class. Note that the type $E(x, a)$ is finitely satisfiable in M. Otherwise (assuming the type is closed under conjunction) we may fix a formula $\varphi(x, a)$ in the type which is not satisfiable in M. Let λ be a cardinal number greater than the number of E-classes and let $(a_i : i < \lambda)$ a sequence of realizations of $\mathrm{tp}(a/M)$ such that $a = a_0$ and $\mathrm{tp}(a_i/M(a_j : j < i))$ is coheir to $\mathrm{tp}(a_i/M)$. The sequence is M-indiscernible and therefore $\models \neg\varphi(a_i, a_j)$ for $i < j < \lambda$. This contradicts the choice of λ as a bound for the number of equivalence classes. Now let $f \in \mathrm{Aut}(\mathfrak{C}/M)$ and let us check that $\models E(a, f(a))$. Let $\psi(x, y)$ be a formula in the type $E(x, y)$ and choose another formula $\varphi(x, y)$ in the type such that $\models \varphi(x, y) \wedge \varphi(z, y) \rightarrow \psi(x, z)$. Since $\varphi(a, y)$ is satisfiable by some $b \in M$, it follows that $\models \varphi(a, b) \wedge \varphi(f(a), b)$ and hence $\models \psi(a, f(a))$. □

Lemma 1.2 *Let M be a κ-saturated and strongly κ-homogeneous model. If $a \in M$ and A is a subset of M of cardinality $< \kappa$ and for all $f \in \mathrm{Aut}(M/A)$ we have $f(a) = a$, then $a \in \mathrm{dcl}(A)$. Hence, for stable T and $\kappa > |T|$, canonical bases of types over M can be computed in M in the following sense: if $C \subseteq M$ has cardinality $< \kappa$ and $p(x) \in S(M)$, then the following are equivalent*

1. *For all $f \in \mathrm{Aut}(M)$, $f \restriction C = id$ if and only if $p^f = p$.*
2. *$Cb(p) = \mathrm{dcl}^{\mathrm{eq}}(C)$.*

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Proof: For the first assertion observe that for $p(x) = \text{tp}(a/A)$ we get $p(x) \equiv x = a$. To justify this, assume $b \models p$ and find (by κ-saturation) $b' \in M$ such that $b \equiv_{Aa} b'$ and (by strong κ-homogeneity) find $f \in \text{Aut}(M/A)$ which is the identity in A and sends a to b'. By assumption, $b' = f(a) = a$ and this implies $b = a$.

From this it follows that point *2* is equivalent to $\text{Aut}(M/C) = \text{Aut}(M/Cb(p))$. Let $\mathfrak{p} \in S(\mathfrak{C})$ be the nonforking extension of p and let $f \in \text{Aut}(M)$ and $f \subseteq f' \in \text{Aut}(\mathfrak{C})$. Then $p^f = p$ if and only if $\mathfrak{p}^{f'} = \mathfrak{p}$ if and only if $f \in \text{Aut}(M/Cb(p))$. The equivalence of *1* and *2* is an easy consequence of this. □

We will assume that there is a type-definable (without parameters) group G in $\mathfrak{C}$. The group operation $(x, y) \mapsto x \cdot y$ is also type-definable over $\emptyset$. It follows that also the inverse operation $x \mapsto x^{-1}$ is type-definable over $\emptyset$ and that and the identity $1 \in \mathrm{G}$ is 0-definable. We use $\mathrm{G}(x)$ for the partial type defining G over $\emptyset$

We say that a relation $R \subseteq \mathrm{G}^n$ is *relatively definable (over A)* if its the intersection of G with a definable (over A) relation. We say that $R \subseteq \mathfrak{C}^n$ is *φ-definable* (where $\varphi = \varphi(x, y) \in L$) if it is definable by an instance $\varphi(x, a)$ of $\varphi(x, y)$. $R \subseteq \mathrm{G}^n$ is *relatively φ-definable* if it is the intersection of G with a φ-definable relation.

Lemma 1.3 *The group operation is relatively* 0*-definable. The identity is* 0*-definable and the inverse operation of* G *is also relatively* 0*-definable. Moreover we can choose the formula $\varphi(x, y, z)$ for the group operation in such a way that it defines a partial mapping and whenever two of the variables are interpreted in the group the third is in the group too.*

Proof: Let $\Phi(x, y, z)$ be the partial type defining the group operation in G. We may assume that $\Phi(x, y, z) \vdash \mathrm{G}(x) \wedge \mathrm{G}(y) \wedge \mathrm{G}(z)$. Note that

$$\Phi(x, y, u) \wedge \Phi(x, y, v) \vdash u = v$$

Let $\varphi(x, y, u)$ be a finite conjunction of formulas in $\Phi(x, y, u)$ such that $\varphi(x, y, u) \wedge \varphi(x, y, v) \vdash u = v$. It is clear that ϕ defines the group operation in G. The identity 1 is relatively definable by $\varphi(x, x, x)$. Since then 1 is type-definable over $\emptyset$ it is also definable over $\emptyset$. The inverse operation is relatively definable by $\varphi(x, y, 1)$. To obtain the additional properties of the formula defining the group operation one observes that

$$\Phi(x, y, u) \wedge \Phi(x, z, u) \vdash y = z$$

and by compactness we can replace $\Phi(x, z, u)$ by a finite conjunction and add it to $\phi(x, y, z)$. This has to be repeated then for the third and last choice of variables. □

2 Chain conditions

Proposition 2.1 *Assume T does not have the independence property. For each $\varphi = \varphi(x, y) \in L$ there is a natural number $n = n_\varphi$ such that for every $m \geq n$, for each sequence $H_1, \ldots, H_m$ of relatively φ-definable subgroups $H_i \leq \mathrm{G}$, the intersection $H_1 \cap \ldots \cap H_m$ is in fact the intersection of $\leq n$ subgroups H_i.*

Proof: Suppose not. For arbitrarily large m there are $H_1, \ldots, H_m$, relatively φ-definable subgroups whose intersection does not agree with the intersection of any strict subsequence. For each $i = 1, \ldots, m$ there is some $b_i \in \bigcap_{i \neq j} H_j$ such that $b_i \notin H_i$. If $I \subseteq m$, say $I = \{i_0, \ldots, i_k\}$ with $i_0 < \cdots < i_k$, let $b_I = b_{i_0} \cdots b_{i_k}$, and let $b_\emptyset = e$. Since $b_j \in H_i$ whenever $i \neq j$, it follows that $i \notin I \Rightarrow b_I \in H_i$. On the other hand, if $i \in I$, say $i = i_j$, where $I = \{i_0, \ldots, i_k\}$ and $i_0 < \cdots < i_k$, then using the notation $I^- = \{i_0, \ldots, i_{j-1}\}$ and $I^+ = \{i_{j+1}, \ldots, i_k\}$ we see that $b_I = b_{I^-} \cdot b_i \cdot b_{I^+}$, $b_{I^-} \in H_i$, $b_i \notin H_i$ and $b_{I^+} \in H_i$. Therefore $b_I \notin H_i$ in this case. We have shown that

$$i \in I \Leftrightarrow b_I \notin H_i.$$

By hypothesis each H_i is relatively definable by a φ-instance, and hence there is an a_i such that $H_i = \{g \in G : \models \varphi(g, a_i)\}$. We have then $i \in I \Leftrightarrow \models \neg\varphi(b_I, a_i)$ for all $I \subseteq m$ for arbitrarily large m. This implies that $\neg\varphi(x, y)$ has the independence property. □

Proposition 2.2 *If T does not have the strict order property, then for each $\varphi = \varphi(x, y) \in L$ there is a natural number $n = n_\varphi$ such that every chain of φ-definable subgroups $H_i \leq \mathrm{G}$ has at most n elements.*

Proof: Clear. □

Proposition 2.3 *Let T be stable and let $\varphi = \varphi(x, y) \in L$.*

1. *There is a natural number $n = n_\varphi$ such that the intersection of any arbitrary family of relatively φ-definable subgroups of G coincides with the intersection of a subfamily of $\leq n$ members.*
2. *There is a natural number $n = n_\varphi$ such that every chain $(H_i : i \in I)$ of subgroups H_i which are intersection of relatively φ-definable subgroups has $\leq n$ members.*

Proof: *1.* Assume there is no such n, let

$$R = \{(a, b, c) : a \in \mathrm{G},\ b \in \mathrm{G},\ c \in \mathfrak{C} \text{ and } \models \varphi(b^{-1} \cdot a, c)\}$$

and let $\mathfrak{C}'$ be the structure $(\mathfrak{C}, \mathrm{G}, R)$. We will see that the formula $\psi(x; y, z) = R(x, y, z)$ is unstable in $\mathfrak{C}'$ by finding 2^ω complete ψ-types over a countable set. But this will be a contradiction since the order property of ψ in $Th(\mathfrak{C}')$ transfers easily to the order property of $\chi(x; y, z) = \varphi(y^{-1} \cdot x, z)$ in $\mathfrak{C}$. In fact, if ψ has the order property in $Th(\mathfrak{C}')$, then for each $n < \omega$ we can find $(a_i : i < n)$ in G, $(b_i : i < n)$ in G and $(c_i : i < n)$ in $\mathfrak{C}$ such that $\models \psi(a_i, b_j, c_j)$ if and only if $i < j$ and from this it follows that χ has the order property in $\mathfrak{C}$.

By compactness there is a family $(c_i : i < \omega)$ such that for each $i < \omega$, $H_i = \{a \in \mathrm{G} : \models \varphi(a, c_i)\}$ is a subgroup of G and for each $n < \omega$, $\bigcap_{i<n} H_i \not\subseteq H_n$. We now construct a binary tree $(X_s : s \in 2^{<\omega})$ of cosets of intersections of the groups H_i in such a way that $X_s \supseteq X_t$ if $s \subseteq t$ and X_s, X_t are different cosets of $X_{(0,\ldots,0)} = H_0 \cap \ldots \cap H_n$ if $s \neq t$ have length n. We start with $X_\emptyset = H_0$. Assume X_s has been defined for $s \in 2^n$. Let $s = (0, \ldots, 0)$ (of length n) and set $X_{s^\frown 0} = H_0 \cap \ldots \cap H_{n+1} = X_s \cap H_{n+1}$. Since $H_0 \cap \ldots \cap H_{n+1}$ is a proper subgroup of $H_0 \cap \ldots \cap H_n$ we can choose a coset $X_{s^\frown 1} = g \cdot (H_0 \cap \ldots \cap H_{n+1})$ different from the group $X_{s^\frown 0}$. For any other $t \in 2^n$ we first choose g_t such that $X_t = g_t \cdot X_s$ and then we put $X_{t^\frown 0} = g_t \cdot X_{s^\frown 0}$ and $X_{t^\frown 1} = g_t \cdot X_{s^\frown 1}$. Now notice that each X_s is definable in $\mathfrak{C}'$ by a conjunction of ψ-formulas. This gives 2^ω pairwise incompatible ψ-types over a countable set of parameters.

2 follows from two consecutive applications of *1*. In the first one we fix the number $m < \omega$ of relatively φ-definable subgroups in each intersection. Now every H_i is relatively ψ-definable for $\psi = \bigwedge_{i<m} \varphi(x, y_i)$ and we apply again *1* to bound the length of the chain by some $n < \omega$. □

Corollary 2.4 *For stable T, every intersection of relatively definable subgroups of* G *is the intersection of $\leq |T|$ of these subgroups.*

Proof: Clear. □

Proposition 2.5 *For superstable T, there is no infinite descending chain of definable subgroups $H_0 \supsetneq H_1 \supsetneq \ldots \supsetneq H_i \supsetneq H_{i+1} \ldots$ each one of infinite index in the previous one.*

Proof: Assume there is such a chain $H_i = \varphi_i(G, a_i)$. Let $\alpha_i = R^\infty(H_i)$. It is clear that $\alpha_{i+1} \leq \alpha_i$. We will see that $\alpha_{i+1} < \alpha_i$, which is a contradiction. Write $H_i = \dot{\bigcup}_{j\in J} h_j H_{i+1}$, where $h_j \in H_i$. The mapping $x \mapsto h_j x$ is a definable bijection from H_{i+1} onto $h_j H_{i+1}$. Hence $\alpha_{i+1} = R^\infty(h_j H_{i+1})$. Let $\psi_i(x, y, z) = \exists u(x = y \cdot u \wedge \varphi_{i+1}(u, z))$. Then $\psi_i(x, h_j, a_{i+1})$ defines $h_j H_{i+1}$. There an infinite subset J' of J such that for each $j, k \in J'$, $h_j a_{i+1} \equiv_{a_i} h_k a_{i+1}$. Therefore for $j \in J'$, $\psi_i(x, h_j, a_{i+1})$ forks over a_i and hence $\alpha_{i+1} < \alpha_i$. □

Proposition 2.6 *For totally trascendental T, there is no infinite descending chain of definable subgroup $H_0 \supsetneq H_1 \supsetneq \ldots \supsetneq H_i \supsetneq H_{i+1} \ldots$.*

Proof: Similar to the superstable case but using Morley rank. The existence of the chain H_i implies that (i) $\mathrm{RM}(H_{i+1}) < \mathrm{RM}(H_i)$ or (ii) $\mathrm{RM}(H_{i+1}) = \mathrm{RM}(H_i)$ and $\mathrm{DM}(H_{i+1}) < \mathrm{DM}(H_i)$. □

Corollary 2.7 *Let T be totally trascendental.*

1. *Every intersection of definable subgroups of G is a definable subgroup of G.*
2. *Every definable injective endomorphism from* G *to* G *is onto.*
3. *If* G *is is abelian and torsion-free then it is divisible.*

We will see in a subsequent section that there is also a chain condition in a simple theory: there is no descending chain of length $|T|^+$ made of definable subgroups each of unbounded index in the previous one.

3 Generics

In this section T is a simple theory.

Definition 3.1 *Let $p(x) \in S(A)$ such that $p(x) \vdash \mathrm{G}(x)$. We say that $p(x)$ is* left generic *if for all $a, b \in \mathrm{G}$, if $a \models p$ and $a \perp_A b$, then $b \cdot a \perp Ab$. We say that it is* right generic *if under the same assumption, $a \cdot b \perp Ab$. In both cases the defining property is independent of the choice of the realization $a \models p$. A type is* generic *if it is both left and right generic. Sometimes we consider type-definable subgroups $H \leq G$ and we want to talk about genericity with respect to H. In this case we say that a type $p(x) \vdash H(x)$ or an element $a \in H$ is generic in H as opposed to be generic in* G.

Proposition 3.2 *Si $p(x) \in S(A)$ is generic, then p does not fork over $\emptyset$.*

Proof: Let $a \models p$. Since $a \perp_A e$, we have $a = a \cdot 1 = 1 \cdot a \perp A1$. □

Lemma 3.3 *Let $a, b \in \mathrm{G}$. If $\mathrm{tp}(a/A)$ is left generic and $b \in \mathrm{acl}(A)$, then $\mathrm{tp}(b \cdot a/A)$ is also left generic.*

Proof: Let $c \in \mathrm{G}$ be such that $b{\cdot}a \mathop{\downarrow}_A c$. Then $b{\cdot}a \mathop{\downarrow}_{\mathrm{acl}(A)} c$ and, since $a \in \mathrm{dcl}(b{\cdot}a, \mathrm{acl}(A))$, $a \mathop{\downarrow}_{\mathrm{acl}(A)} c$. Now $c \cdot b \in \mathrm{dcl}(\mathrm{acl}(A), c)$ and therefore $a \mathop{\downarrow}_{\mathrm{acl}(A)} c \cdot b$ and hence $a \mathop{\downarrow}_A c \cdot b$. By genericity $c \cdot b \cdot a \mathop{\downarrow} A, c \cdot b$. Since $c \in \mathrm{acl}(A, c \cdot b)$, we conclude that $c \cdot b \cdot a \mathop{\downarrow} Ac$. □

Lemma 3.4 *Let $p(x) \in S(A)$ be such that $p(x) \vdash x \in \mathrm{G}$ and let $p(x) \subseteq q(x) \in S(B)$ be a nonforking extension. Then p is left generic if and only if q is left generic.*

Proof: Assume p is left generic. Let $a \models q$ and $b \in \mathrm{G}$ be such that $a \mathop{\downarrow}_B b$ and let us see that $b \cdot a \mathop{\downarrow} Bb$. Since $a \mathop{\downarrow}_A B$, we have $a \mathop{\downarrow}_A b$ and by genericity $b \cdot a \mathop{\downarrow} Ab$. On the other hand, $a \mathop{\downarrow}_A Bb$ and thus $a \mathop{\downarrow}_{Ab} B$. Since $b \cdot a \in \mathrm{dcl}(a, Ab)$, it follows that $b \cdot a \mathop{\downarrow}_{Ab} B$. By transitivity of independence, $b \cdot a \mathop{\downarrow} Bb$. Assume now that q is left generic. Let $a, b \in \mathrm{G}$ be such that $a \models p$ and $a \mathop{\downarrow}_A b$. We want to check that $b \cdot a \mathop{\downarrow} Ab$. We can replace a by $a' \models q$ and then b by $b' \mathop{\downarrow}_{Aa'} B$ such that $a'b' \equiv_A ab$. In other words, we can assume that $a \models q$ and $a \mathop{\downarrow}_B b$. Since q is left generic, $b \cdot a \mathop{\downarrow} Bb$ and in particular $b \cdot a \mathop{\downarrow} Ab$. □

Proposition 3.5 *If $p(x) \in S(A)$ is left generic and $B \subseteq A$, then $p \upharpoonright B$ is left generic too.*

Proof: By Proposition 3.2 and Lemma 3.4. □

Definition 3.6 *Let $p(x) \in S(A)$ be such that $p(x) \vdash x \in \mathrm{G}$. We define $p^{-1}(x)$ as $\mathrm{tp}(a^{-1}/A)$ where a is an arbitrary realization of p.*

Lemma 3.7 *If $p(x) \in S(A)$ is left generic, then $p^{-1}(x)$ is left generic too.*

Proof: Let $a \models p$ and choose $b \models p$ such that $a \mathop{\downarrow}_A b$. Since p is left generic, $b \cdot a \mathop{\downarrow} Ab$. From this it follows that $b \mathop{\downarrow}_A a^{-1} \cdot b^{-1}$, and hence, by Lemma 3.3 , $\mathrm{tp}(b/Aa^{-1} \cdot b^{-1})$ is left generic. By Lemma 3.4 $\mathrm{tp}(a^{-1}/Aa^{-1} \cdot b^{-1}) = \mathrm{tp}(a^{-1} \cdot b^{-1} \cdot b/Aa^{-1} \cdot b^{-1})$ is left generic too. By Lemma 3.5, $p^{-1} = \mathrm{tp}(a^{-1}/A)$ is left generic. □

Proposition 3.8 *A type is left generic if and only if it is right generic*

Proof: By Lemma 3.5, since it is clear that p is left generic if and only if p^{-1} is right generic. □

Proposition 3.9 *Let $a, b \in \mathrm{G}$ be such that $a \mathop{\downarrow}_A b$. If $\mathrm{tp}(a/A)$ is generic, then also $\mathrm{tp}(b \cdot a/A)$ is generic.*

Proof: By Lemma 3.4 $\mathrm{tp}(a/Ab)$ is generic and by Lemma 3.3 $\mathrm{tp}(b \cdot a/Ab)$ is generic too. The result follows then from Proposition 3.5. □

4 Stratified local rank

T is simple in this section. Given $\varphi(x)$ and $g \in \mathrm{G}$, by $\varphi(g \cdot x)$ we understand the formula $\exists u(\varphi(u) \wedge u = g \cdot x)$ where the equation $u = g \cdot x$ should be replaced by the formula defining the group operation. Similarly for $\varphi(x \cdot g)$.

Definition 4.1 *Let $\pi(x)$ be a partial type and let $\varphi = \varphi(x, y) \in L$. We inductively define the rank $D^*(\pi(x), \varphi, k)$ as follows:*

1. *$D^*(\pi(x), \varphi, k) \geq 0$ if $\pi(x) \cup \mathrm{G}(x)$ is consistent.*

2. $D^*(\pi(x), \varphi, k) \geq n+1$ *if and only if there are* a_i, $(i < \omega)$ *and* $g_i \in \mathrm{G}$, $(i < \omega)$ *such that* $\{\varphi(g_i \cdot x, a_i) : i < \omega\}$ *is* k*-inconsistent and* $D^*(\pi(x) \cup \{\varphi(g_i \cdot x, a_i)\}, \varphi, k) \geq n$ *for each* $i < \omega$.

We set $D^*(\pi(x), \varphi, k) = \infty$ *if* $D^*(\pi(x), \varphi, k) \geq n$ *for each* $n < \omega$ *and* $D^*(\pi(x), \varphi, k) = \max\{n < \omega : D^*(\pi(x), \varphi, k) \geq n\}$ *otherwise.*

Remark 4.2 $D(\pi, \varphi, k) \leq D^*(\pi, \varphi, k) \leq D(\pi, \varphi^*(x; yz), k)$ *where* $\varphi^*(x; yz) = \exists u(\varphi(u, y) \wedge u = z \cdot x)$*. Hence* $D^*(\pi, \varphi, k) < \omega$ *since* T *is simple. Note also that* $D^*(\pi(x), \varphi, k) = D^*(\mathrm{G}(x) \cup \pi(x), \varphi, k)$.

Proposition 4.3 *1. Given* $\pi(x, z)$ *without parameters,* $\varphi = \varphi(x, y) \in L$*, and* $k, n < \omega$*, there is some partial type* $\Phi(z)$ *over* $\emptyset$ *such that for all* a*,* $\models \Phi(a)$ *if and only if* $D^*(\pi(x, a), \varphi, k) \geq n$.

2. *Given* φ*,* k*, and* $\pi(x)$*, a partial type, there is some conjunction* $\psi(x)$ *of formulas in* π *such that* $D^*(\pi, \varphi, k) = D^*(\psi, \varphi, k)$.

3. $D^*(\pi \cup \{\psi \vee \chi\}, \varphi, k) = \max\{D^*(\pi \cup \{\psi\}, \varphi, k), D^*(\pi \cup \{\chi\}, \varphi, k)\}$.

4. *If* $\pi(x)$ *is over* A*, there is an extension* $p(x) \in S(A)$ *of* π *such that* $D^*(\pi, \varphi, k) = D^*(p, \varphi, k)$.

Proof: In all cases the proofs are the same as for the local D-rank in a simple theory. For *1* we use induction on n. For *2* we use compactness and *1* to justify that if $D^*(\pi, \varphi, k) \ngeq n+1$ then for some conjunction φ of formulas in π, $D^*(\psi, \varphi, k) \ngeq n+1$. In *3* is clear that the maximum is $\leq D^*(\pi \cup \{\psi \vee \chi\}, \varphi, k)$ and it only remains to prove that if $D^*(\pi \cup \{\psi \vee \chi\}, \varphi, k) \geq n$ then also the maximum is $\geq n$. This can be easily done by induction on n. Finally, *4* follows from *3*, since *3* can be used to ensure the consistency of

$$\pi(x) \cup \{\neg\psi(x) \in L(A) : D^*(\pi(x) \cup \{\psi(x)\}, \varphi, k) < D^*(\pi(x), \varphi, k)$$

and any complete extension $p(x) \in S(A)$ of this type over fulfills the requirements. □

Proposition 4.4 *Let* $p(x) \in S(A)$ *be such* $p(x) \vdash \mathrm{G}(x)$ *and let* $q(x) \in S(B)$ *be an extension of* p*. Then* q *is a nonforking extension of* p *if and only if* $D^*(p, \varphi, k) = D^*(q, \varphi, k)$ *for all* k *and* φ.

Proof: Assume q forks over A. Then for some $\varphi(x, y) \in L$ there is $b \in B$ such that $\varphi(x, b) \in q$ and $\varphi(x, b)$ k-divides over A for some k. This means that there are b_i, $(i < \omega)$ such that $b_i \equiv_A b$ for all $i < \omega$ and $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent. If we put $g_i = 1$ we see that the g_i, b_i witness that

$$D^*(p, \varphi, k) > D^*(p \cup \{\varphi(x, b)\}, \varphi, k) \geq D^*(q, \varphi, k).$$

For the other direction, assume $a \models q$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_A B$. We show by induction on m that $D^*(p, \varphi, k) \geq m$ implies $D^*(q, \varphi, k) \geq m$. This is clear for $m = 0$. Let $D^*(p, \varphi, k) \geq m+1$. There are b_i $(i < \omega)$ and $g_i \in \mathrm{G}$ $(i < \omega)$ such that $D^*(p \cup \{\varphi(g_i \cdot x, b_i)\}, \varphi, k) \geq m$ for all $i < \omega$ and $\{\varphi(g_i \cdot x, b_i) : i < \omega\}$ is k-inconsistent. By Ramsey's Theorem we may assume that $(b_i, g_i : i < \omega)$ is A-indiscernible. Choose an extension $q' \in S(Ag_0b_0)$ of $p \cup \{\varphi(g_0 \cdot x, b_0)\}$ with $D^*(q', \varphi, k) \geq m$ and let $a' \models q'$. We may then choose $(g'_i, b'_i : i < \omega)$ such that $(g'_i, b'_i : i < \omega)a \equiv_A (g_i, b_i : i < \omega)a'$ and also choose $B' \mathop{\smile\!\!\!\!\!\!\,|}_{Aa} (g'_i, b'_i : i < \omega)$ such that $B \equiv_{Aa} B'$. In other words, we may assume that $a \models q'$ and $B \mathop{\smile\!\!\!\!\!\!\,|}_{Aa} (g_i, b_i : i < \omega)$.

Then $B \perp_A (g_i, b_i : i < \omega)$ and (changing again $(g_i, b_i : i < \omega)$ by an A-isomorphic copy if necessary) we may assume that $(g_i, b_i : i < \omega)$ is also B-indiscernible. Since $a \perp_{Ag_0b_0} B$, we may apply the inductive hypothesis to obtain that $D^*(\mathrm{tp}(a/Bg_0b_0), \varphi, k) \geq m$ and thus $D^*(q \cup \{\varphi(g_i \cdot x, b_i)\}, \varphi, k) \geq m$ for all $i < \omega$. From this we may conclude that $D^*(q, \varphi, k) \geq m + 1$. □

Definition 4.5 *For a formula $\varphi(x)$ and an element of the group $g \in \mathrm{G}$, we define*

$$g \cdot \varphi(x) = \varphi(g^{-1} \cdot x)$$

considered as a formula over g (and the rest of parameters of $\varphi(x)$), namely, as $\exists u(\varphi(u) \wedge g \cdot u = x)$. For a partial type $\pi(x)$ over A, $g \cdot \pi(x)$ is $\{g \cdot \varphi(x) : \varphi \in \pi\}$, a partial type over Ag. Note that

$$h \models \pi \iff g \cdot h \models g \cdot \pi$$

Proposition 4.6 *$D^*(\pi, \varphi, k) = D^*(g \cdot \pi, \varphi, k)$.*

Proof: Since $\pi \equiv g^{-1} \cdot g \cdot \pi$, it is enough to prove that $D^*(\pi, \varphi, k) \geq D^*(g \cdot \pi, \varphi, k)$. For this we check by induction on n that $D^*(\pi, \varphi, k) \geq n$ implies $D^*(g \cdot \pi, \varphi, k) \geq n$. The case $n = 0$ is clear. Assume $D^*(\pi, \varphi, k) \geq n + 1$. There are a_i, $(i < \omega)$ and $g_i \in \mathrm{G}$, $(i < \omega)$ such that $\{\varphi(g_i \cdot x, a_i) : i < \omega\}$ is k-inconsistent and for each $i < \omega$, $D^*(\pi(x) \cup \{\varphi(g_i \cdot x, a_i)\}, \varphi, k) \geq n$. Now $\{\varphi(g_i \cdot g^{-1} \cdot x, a_i) : i < \omega\}$ is also k-inconsistent and by the inductive hypothesis for each $i < \omega$, $D^*(g \cdot \pi(x) \cup \{g \cdot \varphi(g_i \cdot x, a_i)\}, \varphi, k) \geq n$. But $g \cdot \varphi(g_i \cdot x, a_i) = \varphi(g_i \cdot g^{-1} \cdot x, a_i)$ and therefore the sequences $(a_i : i < \omega)$ and $(g_i \cdot g^{-1} : i < \omega)$ witness that $D^*(g \cdot \pi, \varphi, k) \geq n + 1$. □

Corollary 4.7 *If $g, h \in \mathrm{G}$ and $h \in \mathrm{acl}(A)$, then $D^*(\mathrm{tp}(g/A), \varphi, k) = D^*(\mathrm{tp}(h \cdot g/A), \varphi, k)$*

Proof: Let $p(x) = \mathrm{tp}(g/A)$. By Proposition 4.6 we know that $D^*(p, \varphi, k) = D^*(h \cdot p, \varphi, k)$. Now, $h \cdot p$ is a partial type over Ah and $(h \cdot p) \restriction A = \mathrm{tp}(h \cdot g/A)$. Since $h \in \mathrm{acl}(A)$, $h \cdot g \perp_A h$ and by Proposition 4.4, $D^*(\mathrm{tp}(h \cdot g/A), \varphi, k) = D^*(\mathrm{tp}(h \cdot g/Ah), \varphi, k) \leq D^*(h \cdot p, \varphi, k)$. Thus $D^*(\mathrm{tp}(h \cdot g/A), \varphi, k) \leq D^*(\mathrm{tp}(g/A), \varphi, k)$. Using now h^{-1} instead of h we get the equality. □

Theorem 4.8 *For any A, there is a generic type $p(x) \in S(A)$.*

Proof: Since a nonforking extension of a generic type is generic, it is enough to check it for the case $A = \emptyset$. Fix an enumeration $(\varphi_i, k_i : i < \kappa)$ of all pairs (φ, k) where $\varphi = \varphi(x, y) \in L$ and $2 \leq k < \omega$. We define inductively a partial type $\pi_i(x)$ over $\emptyset$ and a natural number n_i as follows:

1. $\pi_0(x) = \mathrm{G}(x)$
2. $n_i = D^*(\pi_i(x), \varphi_i, k_i)$
3. $\pi_{i+1} = \pi_i(x) \cup \{\neg\psi(x) : \psi(x) \in L \text{ and } D^*(\pi_i(x) \cup \{\psi(x)\}, \varphi_i, k_i) < n_i\}$
4. $\pi_\alpha = \bigcup_{i<\alpha} \pi_i$ for any limit ordinal $\alpha \leq \kappa$.

Let $p(x) \in S(\emptyset)$ be an extension of $\pi_\kappa(x)$. We will show that p is generic. Let $g \models p$ and let $h \in \mathrm{G}$ be such that $g \perp h$. We have to prove that $h \cdot g \perp h$. For this we use Proposition 4.4. We have to check that for each $i < \kappa$, $D^*(\mathrm{tp}(h \cdot g), \varphi_i, k_i) = D^*(\mathrm{tp}(h \cdot g/h), \varphi_i, k_i)$. Note that any complete type q over $\emptyset$ extending π_{i+1} verifies $D^*(q,\varphi_i,k_i) = n_i$. Hence $n_i = D^*(p,\varphi_i,k_i)$. Observe that by Propositions 4.4 and Corollary 4.7

$$D^*(p,\varphi_i,k_i) = D^*(\mathrm{tp}(g/h),\varphi_i,k_i) = D^*(\mathrm{tp}(h\cdot g/h),\varphi_i,k_i) \le D^*(\mathrm{tp}(h\cdot g),\varphi_i,k_i)$$

Hence $D^*(\mathrm{tp}(h\cdot g),\varphi_i,k_i)$ for all i and this can be used to see that $h\cdot g \models \pi_i$ for all i. Therefore $D^*(\mathrm{tp}(h\cdot g),\varphi_i,k_i) \le D^*(\pi_i,\varphi_i,k_i) = n_i$. Thus n_i is a lower and upper bound of the ranks and all them are equal. In particular $D^*(\mathrm{tp}(h\cdot g/h),\varphi_i,k_i) = D^*(\mathrm{tp}(h\cdot g),\varphi_i,k_i)$. □

Definition 4.9 *A partial type $\pi(x)$ over A is* generic *if it can be extended to a complete generic type over A. In particular, a formula is generic if it belongs to a complete generic type. An element g of the group* G *is* generic over A *if* $\mathrm{tp}(g/A)$ *is generic. A global type $\mathfrak{p} \in S(\mathfrak{C})$ is generic if for every set A, $\mathfrak{p} \restriction A$ is generic.*

Remark 4.10 *Notice that if $\pi(x)$ is a partial type over A and it is also a partial type over B, then it is generic with respect to A if and only if it is generic with respect to B (because a nonforking extension of a generic type is generic). Hence there is no harm in the terminology.*

Proposition 4.11 *The following conditions are equivalent for any partial type $\pi(x)$ over A.*

1. *π is generic.*
2. *$D^*(\pi,\varphi,k) = D^*(\mathrm{G}(x),\varphi,k)$ for all φ,k.*
3. *For all $g \in \mathrm{G}$, $g\cdot\pi$ does not fork over $\emptyset$.*
4. *For all $g \in \mathrm{G}$, $g\cdot\pi$ does not fork over A.*

Proof: *1* ⇒ *2*. Since π is generic, there is some $g \models \pi$ which is generic over A. Let φ,k be given and choose $h \in \mathrm{G}$ with $D^*(\mathrm{tp}(h/A),\varphi,k) = D^*(\mathrm{G}(x),\varphi,k)$. We may choose it such that additionally $h \mathop{\smile\!\!\!\!\!\!\;\mid}_A g$. Then $g \mathop{\smile\!\!\!\!\!\!\;\mid}_A h^{-1}$ and, since g is generic over A, $g\cdot h^{-1} \mathop{\smile\!\!\!\!\!\!\;\mid} Ah^{-1}$. Hence $h \mathop{\smile\!\!\!\!\!\!\;\mid}_A g\cdot h^{-1}$. By Proposition 4.4 and Corollary 4.7

$$D^*(\mathrm{tp}(h/A),\varphi,k) = D^*(\mathrm{tp}(h/A,g\cdot h^{-1}),\varphi,k) = D^*(\mathrm{tp}(g\cdot h^{-1}\cdot h/A,g\cdot h^{-1}) =$$
$$= D^*(\mathrm{tp}(g/A,g\cdot h^{-1},\varphi,k) \le D^*(\mathrm{tp}(g/A),\varphi,k) \le D^*(\pi,\varphi,k).$$

2 ⇒ *3* Assume $g\cdot\pi$ forks over $\emptyset$. Then, for some $\varphi(x,y) \in L$, some $a \in A$, and some $k < \omega$, $\pi(x) \vdash \varphi(x,a)$ and $\varphi(g^{-1}\cdot x,a)$ k-divides over $\emptyset$. This means that there are a_i, $(i<\omega)$ and $g_i \in \mathrm{G}$, $(i<\omega)$ such that $g^{-1},a \equiv g_i,a_i$ for all $i<\omega$ and $\{\varphi(g_i\cdot x,a_i) : i<\omega\}$ is k-inconsistent. Note that by isomorphism and by Proposition 4.6,

$$D^*(\mathrm{G}(x)\cup\{\varphi(g_i\cdot x,a_i)\},\varphi,k) = D^*(\varphi(g^{-1}\cdot x,a),\varphi,k) = D^*(\varphi(x,a),\varphi,k)$$

and therefore

$$D^*(\mathrm{G}(x),\varphi,k) \ge D^*(\varphi(x,a),\varphi,k)+1 > D^*(\pi(x),\varphi,k).$$

It is clear that *3* implies *4*, so we finish by proving *4* ⇒ *1*. Let $g \in \mathrm{G}$ be generic over A. Since $g\cdot\pi$ does not fork over A, there is some $h \models \pi$ such that $g\cdot h \mathop{\smile\!\!\!\!\!\!\;\mid}_A g$. By Lemma 3.4 g is generic over $A, g\cdot h$ and by Lemma 3.3, $(g\cdot h)^{-1}\cdot g = h^{-1}$ is generic over $A, g\cdot h$. By Proposition 3.5 h^{-1} is generic over A and by Lemma 3.7 h is also generic over A. Then π has a realization which is generic over A and consequently it is generic. □

Corollary 4.12 *1. A type (closed under conjunction) is generic if and only if all its formulas are generic.*

2. A partial type $\pi(x)$ is generic if and only if $g \cdot \pi(x)$ is generic.

Proof: Clear by point *2* of Proposition 4.11, point *2* of Proposition 4.3 and Proposition 4.6. □

5 The connected component

T is simple in this section too.

Definition 5.1 *The A-*connected component of G *is the intersection G_A^{00} of all subgroups $H \leq G$ which are type-definable over A and have bounded index $|\mathrm{G} : H|$. We say that* G *is* connected over A *if $\mathrm{G} = \mathrm{G}_A^{00}$, that is, if* G *does not have proper subgroups of bounded index which are type-definable over A.*

Lemma 5.2 G_A^{00} *is a normal subgroup of* G *and* $|\mathrm{G} : \mathrm{G}_A^{00}| \leq 2^{|T|+|A|}$.

Proof: The equivalence relation on G having the cosets $h \cdot \mathrm{G}_A^{00}$ as equivalence clases is type-definable over A and it is intersection of bounded type-definable over A equivalence relations. Hence it is also bounded and has $\leq 2^{|T|+|A|}$ classes. This means that $|\mathrm{G} : \mathrm{G}_A^{00}| \leq 2^{|T|+|A|}$. We prove now that G_A^{00} is a normal subgroup of G. Let λ be the group homomorphism $g \mapsto \lambda_g$ from G into $Sym(\mathrm{G}/\mathrm{G}_A^{00})$ defined by $\lambda_g(h \cdot \mathrm{G}_A^{00}) = g \cdot h \cdot \mathrm{G}_A^{00}$. We show that G_A^{00} is the kernel of λ. It is obvious that $ker(\lambda) \leq \mathrm{G}_A^{00}$. It is clear that the kernel $ker(\lambda)$ is bounded since $Sym(\mathrm{G}/\mathrm{G}_A^{00})$ has bounded size and $\mathrm{G}/ker(\lambda) \cong imag(\lambda) \leq Sym(\mathrm{G}/\mathrm{G}_A^{00})$. It is also type-definable, since if $(h_i : i \in I)$ are representatives of all the cosets $h \cdot \mathrm{G}_A^{00}$ then $ker(\lambda)$ can be defined by the type $\pi(x)$ over $A \cup \{h_i : i \in I\}$ expressing that for each $i \in I$, $h_i^{-1} \cdot x \cdot h_i \in \mathrm{G}_A^{00}$. But $ker(\lambda)$ is A-invariant and therefore it is also type-definable over A. Then $\mathrm{G}_A^{00} \leq ker(\lambda)$. □

Definition 5.3 *We say that a subgroup H of* G *is* generic over A *if H is type-definable over A by a generic type.*

Proposition 5.4 *The following are equivalent for a subgroup $H \leq$ G type-definable over A:*

1. *H is generic over A.*
2. $\mathrm{G}_A^{00} \leq H$.
3. *H has bounded index in* G.
4. *$D^*(H, \varphi, k) = D^*(\mathrm{G}, \varphi, k)$ for all φ, k.*

Proof: *1* ⇔ *4* follows directly from Proposition 4.11. On the other hand it is obvious that *2* ⇔ *3*. We prove now the equivalence of *3* with point *4* of Proposition 4.11 and therefore also with point *1* of the present proposition. Assume first $|\mathrm{G} : H|$ is unbounded. In this case there is an A-indiscernible sequence $(g_i : i < \omega)$ of elements $g_i \in \mathrm{G}$ such that $g_i \cdot H \neq g_j \cdot H$ for all $i \neq j$. The A-indiscernibility of the sequence can be assured using the indiscernibility lemma based on Erdös-Rado's Lemma. Then $g_i \cdot H \cap g_j \cdot H = \emptyset$ for all $i \neq j$ and hence $g_0 \cdot H$ forks over A. For the other direction, assume $g \in \mathrm{G}$ and $g \cdot H$ forks over A. Then, there is an A-indiscernible sequence $(g_i : i < \omega)$ such that $g = g_0$ and $\bigcap_{i<\omega} g_i \cdot H = \emptyset$. It follows that all $g_i \cdot H$ are disjoint. But we can extend the indiscernible sequence as long as we want, which contradicts boundedness of the index. □

Proposition 5.5 *If $(H_i : i < \alpha)$ is a descending chain of type-definable subgroups $H_i \leq \mathrm{G}$ such that $|H_i : H_{i+1}|$ is unbounded for all $i < \alpha$, then $\alpha < |T|^+$.*

Proof: We prove first that the unboundedness of $|H_i : H_{i+1}|$ implies that for some φ_i and some $k_i < \omega$, $D^*(H_i, \varphi_i, k_i) > D^*(H_{i+1}, \varphi_i, k_i)$. Assume H_i, H_{i+1} are type-definable over A. By the Proposition 5.4 H_{i+1} is not generic over A (as a subgroup of H_i) and by Proposition 4.11 for some $h \in H_i$, $h \cdot H_{i+1}$ forks over A. Therefore there is a formula $\varphi_i(x, y) \in L$, there is some $a \in A$ such that $\varphi_i(x, a)$ belongs to the partial type $\pi_{i+1}(x)$ defining H_{i+1}, and there is an A-indiscernible sequence $(g_j : j < \omega)$ where $g_j \in \mathrm{H}_i$ and some $k_i < \omega$ such that $h^{-1} = g_0$, and $\{\varphi_i(g_j \cdot x, a) : j < \omega\}$ is k_i-inconsistent. Then if the partial type π_i defines H_i over A, $D^*(\pi_i(x) \cup \{\varphi_i(g_j \cdot x, a)\}, \varphi_i, k_i) \geq D^*(g_j^{-1} \cdot \pi_{i+1}(x) \cup \{\varphi_i(g_j \cdot x, a)\}, \varphi_i, k_i) = D^*(g_j^{-1} \cdot (\pi_{i+1}(x) \cup \{\varphi_i(x, a)\}), \varphi_i, k_i) = D^*(\pi_{i+1}(x) \cup \{\varphi_i(x, a)\}, \varphi_i, k_i) = D^*(\pi_{i+1}(x), \varphi, k)$ and hence $D^*(H_i, \varphi_i, k_i) \geq D^*(H_{i+1}, \varphi_i, k_i) + 1$.

Now assume $\alpha \geq |T|^+$. For some infinite $I \subseteq \alpha$ there is a fixed $\varphi(x, y) \in L$ and $k < \omega$ such that $\varphi = \varphi_i$ and $k = k_i$ for all $i \in I$. Then $D^*(H_i, \varphi, k) > D^*(H_j, \varphi, k)$ whenever $i, j \in I$ and $i < j$. This contradicts the finiteness of the rank. □

6 Stabilizers

Definition 6.1 *For $p(x) \in S(A)$ such that $p(x) \vdash \mathrm{G}(x)$, we define*

$$S(p) = \{g \in \mathrm{G} : g \cdot p \cup p \text{ does not fork over } A\}$$

and we define the stabilizer of p *as* $\mathrm{Stab}(p) = S(p) \cdot S(p)$.

Lemma 6.2 *Let $p(x) \in S(A)$. If $g \in S(p)$, then there is some $h \models p$ such that $g \cdot h \models p$, $g \cdot h \mathop{\smile\!\!\!\!\!\!|}_A g$. It follows from this that $h \mathop{\smile\!\!\!\!\!\!|}_A g$.*

Proof: Let $g \in S(p)$. Then there is some $a \models g \cdot p \cup p$ such that $a \mathop{\smile\!\!\!\!\!\!|}_A g$. Clearly, $a \models p$ and there is some $h \models p$ such that $a = g \cdot h$. Then for all φ, k

$$D^*(\mathrm{tp}(h/A), \varphi, k) = D^*(p, \varphi, k) = D^*(\mathrm{tp}(g \cdot h/A), \varphi, k) = D^*(\mathrm{tp}(g \cdot h/Ag), \varphi, k) =$$

$$= D^*(\mathrm{tp}(g^{-1} \cdot g \cdot h/Ag), \varphi, k) = D^*(\mathrm{tp}(h/Ag), \varphi, k)$$

Consequently, $h \mathop{\smile\!\!\!\!\!\!|}_A g$. □

Proposition 6.3 *Let $p(x) \in S(A)$ such that $p(x) \vdash \mathrm{G}(x)$. The class $S(p)$ is type-definable over A and it is closed under inverses.*

Proof: Let $g \in S(p)$ and find h for g as in Lemma 6.2. Then $g \cdot h \models g^{-1} \cdot p \cup p$ and $g \cdot h \mathop{\smile\!\!\!\!\!\!|}_A g^{-1}$. Hence $g^{-1} \in S(p)$.

With respect to the type-definability, we need to use the fact that in a simple theory, for any $p(x) \in S(A)$ there is a partial type $\pi(x, y)$ over A such that for any $a \models p$ for any b, $a \mathop{\smile\!\!\!\!\!\!|}_A b$ if and only if $\models \pi(a, b)$. This type π only expresses that for all φ, k, the local rank $D(\mathrm{tp}(a/Ab), \varphi, k)$ is at least $D(p, \varphi, k)$. Using the partial type π we characterize $g \in S(p)$ as $\exists x(\pi(x, g) \wedge g \cdot p(x) \wedge p(x))$. □

Proposition 6.4 *Let $p(x) \in S(M)$ such that $p(x) \vdash \mathrm{G}(x)$. If $g, g' \in S(p)$ and $g \mathop{\smile\!\!\!\!\!\mid}_M g'$, then $g \cdot g' \in S(p)$.*

Proof: By Lemma 6.2 we may choose h, h' such that $h \models p$, $h' \models p$, $g \cdot h \models p$, $g' \cdot h' \models p$, $g \cdot h \mathop{\smile\!\!\!\!\!\mid}_M g$, $h \mathop{\smile\!\!\!\!\!\mid}_M g$, $g' \cdot h' \mathop{\smile\!\!\!\!\!\mid}_M g'$, and $h' \mathop{\smile\!\!\!\!\!\mid}_M g'$. We can apply the Independence Theorem over M to the types $p(x)$, $p_1(x) = \mathrm{tp}(h/Mg)$ and $p_2(x) = \mathrm{tp}(g' \cdot h'/Mg')$ obtaining this way a realization a of $p_1(x) \cup p_2(x)$ such that $a \mathop{\smile\!\!\!\!\!\mid}_M g, g'$. We can then interchange a and h, that is, we can assume that $h \equiv_{Mg'} g' \cdot h'$ and $h \mathop{\smile\!\!\!\!\!\mid}_M g, g'$. Then $h = g' \cdot f$ for some f such that $f \equiv_{Mg'} h'$. Again, we can interchange f and h', that is, we can assume that $h = g' \cdot h'$. To ensure that $g \cdot g' \in S(p)$, we need to find some e such that $e \mathop{\smile\!\!\!\!\!\mid}_M g \cdot g'$, $e \models p$ and $e \models g \cdot g' \cdot p$. The solution is $e = g \cdot g' \cdot h' = g \cdot h$. Only the independence needs some checking. We know that $g \cdot h \mathop{\smile\!\!\!\!\!\mid}_M g$ and $g \cdot h \mathop{\smile\!\!\!\!\!\mid}_{Mg} g'$. By transitivity, $g \cdot h \mathop{\smile\!\!\!\!\!\mid}_M g, g'$. Hence $g \cdot h \mathop{\smile\!\!\!\!\!\mid}_M g \cdot g'$. □

Lemma 6.5 *Let $X \subseteq \mathrm{G}$ be nonempty and type-definable over A and assume X is closed under inverses and $g \cdot h \in X$ whenever $g, h \in X$ and $g \mathop{\smile\!\!\!\!\!\mid}_A h$. Then $Y = X \cdot X$ is a subgroup of G, is type-definable over A and every $g \in Y$ generic over A belongs to X.*

Proof: It is clear that Y is closed under inverses and it is type-definable over A. We will show first that $g \cdot g' \cdot g'' \in Y$ if $g, g', g'' \in X$. From this it follows easily that Y is closed under product. But first we choose a particular type $p(x) \in S(A)$. We fix an enumeration (φ_i, k_i) $(i < \mu)$ of all pairs consisting in a formula $\varphi_i(x, y_i) \in L$ and a natural number k_i. As in the proof of Theorem 4.8, we define partial types $\pi_i(x)$ over A and corresponding natural numbers $n_i = D^*(\pi_i, \varphi_i, k_i)$:

1. $\pi_0(x) = X(x)$ (the type over A defining X)
2. $\pi_{i+1}(x) = \pi_i(x) \cup \{\neg\varphi(x) : \varphi(x) \in L(A) \text{ and } D^*(\pi_i(x) \cup \{\varphi(x)\}, \varphi_i, k_i) < n_i\}$
3. $\pi_\beta(x) = \bigcup_{i<\beta} \pi_i(x)$ for limit β.

Let now $p(x) \in S(A)$ be any type extending $\bigcup_{i<\mu} \pi_i(x)$. Observe that $D^*(p(x), \varphi_i, k_i) = n_i$. Moreover, if $q(x) \in S(A)$ and $D^*(q(x), \varphi_i, k_i) \geq n_i$ for all $i < \mu$, then $D^*(q(x), \varphi_i, k_i) = n_i$ for all $i < \mu$.

Let $g, g', g'' \in X$ and choose $h \models p$ such that $h \mathop{\smile\!\!\!\!\!\mid}_A g, g', g''$. Then $h \in X$ and, by independence, $g' \cdot h \in X$. Clearly, $g' \cdot h \mathop{\smile\!\!\!\!\!\mid}_{Ag'} g, g''$. For all $i < \mu$ we have

$$n_i = D^*(\mathrm{tp}(h/A), \varphi_i, k_i) = D^*(\mathrm{tp}(h/Ag'), \varphi_i, k_i) = D^*(\mathrm{tp}(g' \cdot h/Ag'), \varphi_i, k_i) \leq D^*(\mathrm{tp}(g' \cdot h/A), \varphi_i, k_i)$$

and hence for all $i < \mu$,

$$D^*(\mathrm{tp}(g' \cdot h/A), \varphi_i, k_i) = n_i = D^*(\mathrm{tp}(g' \cdot h/Ag'), \varphi_i, k_i).$$

As a consequence, $g' \cdot h \mathop{\smile\!\!\!\!\!\mid}_A g'$. By transitivity of independence, $g' \cdot h \mathop{\smile\!\!\!\!\!\mid}_A g, g', g''$. Therefore $g \cdot g' \cdot h \in X$. On the other hand $h^{-1} \in X$ and $h \mathop{\smile\!\!\!\!\!\mid}_A g''$ and thus $h^{-1} \cdot g'' \in X$. We conclude that

$$g \cdot g' \cdot g'' = (g \cdot g' \cdot h) \cdot (h^{-1} \cdot g'') \in X \cdot X = Y.$$

Now let $g \in Y$ be generic over A. We show that $g \in X$. Choose $g', g'' \in X$ such that $g = g' \cdot g''$ and choose $h \models p$ such that $h \mathop{\smile\!\!\!\!\!\mid}_A g', g''$. Then $g'' \cdot h \in X$. As above, for all $i < \mu$,

$$n_i = D^*(\mathrm{tp}(h/A), \varphi_i, k_i) = D^*(\mathrm{tp}(h/Ag''), \varphi_i, k_i) = D^*(\mathrm{tp}(g'' \cdot h/Ag''), \varphi_i, k_i)$$

$$\leq D^*(\mathrm{tp}(g'' \cdot h/A), \varphi_i, k_i)$$

and hence for all $i < \mu$,

$$D^*(\mathrm{tp}(g'' \cdot h/A), \varphi_i, k_i) = n_i = D^*(\mathrm{tp}(g'' \cdot h/Ag'), \varphi_i, k_i)$$

and consequently $g'' \cdot h \downarrow_A g''$. But $g'' \cdot h \downarrow_{Ag''} g'$ and then, by transitivity, $g'' \cdot h \downarrow_A g', g''$. By the assumption on X, $g \cdot h = g' \cdot g'' \cdot h \in X$. Since g is generic over A and $g \downarrow_A h^{-1}$, we have $g \cdot h \downarrow_A h^{-1}$ and therefore $g = (g \cdot h) \cdot h^{-1} \in X$. □

Proposition 6.6 *Let* $p(x) \in S(M)$ *such that* $p(x) \vdash \mathrm{G}(x)$*. Then* $\mathrm{Stab}(p) \leq \mathrm{G}^{00}_M$*. Moreover,* p *is generic if and only if* $\mathrm{Stab}(p) = \mathrm{G}^{00}_M$*.*

Proof: By Proposition 6.4 and Lemma 6.5 we know that $\mathrm{Stab}(p)$ is a subgroup of G. To check that is is a subgroup of G^{00}_M it is enough to prove that $S(p) \subseteq \mathrm{G}^{00}_M$. Let $g \in S(p)$. Then $p \cup (g \cdot p)$ is consistent. Let E be the equivalence relation determined by G^{00}_M, as $E(x, y) \Leftrightarrow x \cdot y^{-1} \in \mathrm{G}^{00}_M$. It is type-definable over M and has only boundedly many classes. Since M is a model, each equivalence class is fixed by each $f \in \mathrm{Aut}(\mathfrak{C}/M)$ (see Lemma 1.1). Then there is some $h \in \mathrm{G}$ (perhaps not in M) such that $p(x) \vdash E(x, h)$. Then $g \cdot p \vdash E(x, g \cdot h)$. It follows that $\models E(h, g \cdot h)$ and therefore that $g = g \cdot h \cdot h^{-1} \in \mathrm{G}^{00}_M$.

Assume now that p is generic. Choose $g \models p$ and $g' \models p$ such that $g \downarrow_M g'$. Then $g' \cdot g^{-1}$ is generic over M and $g' \cdot g^{-1} \downarrow_M g$ and $g' \cdot g^{-1} \downarrow_M g'$. Note that $(g' \cdot g^{-1}) \cdot g = g'$ and thus $g' \models g' \cdot g^{-1} \cdot p$. Hence $g' \cdot g^{-1} \in S(p) \subseteq \mathrm{Stab}(p)$. The group $\mathrm{Stab}(p)$ contains an element generic over M and then it is itself generic over M. By Proposition 5.4 $\mathrm{G}^{00}_M \leq \mathrm{Stab}(p)$.

Assume, to finish the proof, that $\mathrm{Stab}(p) = \mathrm{G}^{00}_M$. By Proposition 5.4, $\mathrm{Stab}(p)$ is generic over M and therefore some $g \in \mathrm{Stab}(p)$ is generic over M. By Lemma 6.5, $g \in S(p)$. This means that for some $g' \models p$, $g' \downarrow_M g$, we have $g \cdot g' \models p$ and $g \cdot g' \downarrow_M g$. Then g is also generic over Mg'. It follows that $g \cdot g'$ is generic over Mg' and also over M. Thus, $p = \mathrm{tp}(g \cdot g'/M)$ is generic. □

Proposition 6.7 *Let* $p(x) \in S(A)$ *be a stationary type such that* $p(x) \vdash \mathrm{G}(x)$ *and let* $q(x) \in S(B)$ *be a nonforking extension of* p *over* $B \supseteq A$*. Then* $\mathrm{Stab}(p) = \mathrm{Stab}(q) = S(p) = S(q)$*.*

Proof: First notice that even when p is not stationary, $S(q) \subseteq S(p)$. To see this, assume $g \in S(q)$. Then for some $h \models q$, $g \cdot h \models q$ and $g \cdot h \downarrow_B g$. Since $g \cdot h \downarrow_A B$ it follows that $g \cdot h \downarrow_A g$. Hence $g \in S(p)$.

Now we show that $S(p) \subseteq S(q)$ using stationarity of p. Let $g \in S(p)$. For some $h \models p$, $g \cdot h \models p$ and $g \cdot h \downarrow_A g$. It follows that $g \downarrow_A h$ and we may assume that $h \downarrow_{Ag} B$. Then $g \cdot h \downarrow_{Ag} B$ and by transitivity $g \cdot h \downarrow_A Bg$. In particular $g \cdot h \downarrow_B g$ and $g \cdot h \downarrow_A B$. Since p is stationary, $g \cdot h \models q$. Similarly, since $h \downarrow_A B$, by stationarity $h \models q$. Now $h \models q$, $g \cdot h \models q$ and $g \cdot h \downarrow_B g$ and this implies $g \in S(q)$.

The next step is to show that $S(p)$ is closed under product. This will imply straightforwardly that $S(p) = \mathrm{Stab}(p)$ (and $S(q) = \mathrm{Stab}(q)$). Since $S(p)$ is closed under inverses, it is enough to check that $g' \cdot g^{-1} \in S(p)$ whenever $g, g' \in S(p)$. Choose $h, h' \in \mathrm{G}$ such that $h \models p$, $h' \models p$, $g \cdot h \models p$, $g' \cdot h' \models p$, $g \cdot h \downarrow_A g$ and $g' \cdot h' \downarrow_A g'$. As usual, it follows that $h \downarrow_A g$ and $h' \downarrow_A g'$. In fact we can choose h, h' such that additionally $h \downarrow_{Ag} g'$ and $h' \downarrow_{Ag'} g$. Hence $h \downarrow_A g, g'$ and $h' \downarrow_A g, g'$. Since p is stationary, $h \equiv_{Agg'} h'$. This means that we can substitute h for h'. In other words, we can assume that $h = h'$. Notice now that $g \cdot h \models p$, $(g' \cdot g^{-1}) \cdot (g \cdot h) = g' \cdot h \models p$ and $g' \cdot h \downarrow_A g' \cdot g^{-1}$ and therefore $g' \cdot g^{-1} \in S(p)$. □

7 Stable groups

Proposition 7.1 *If T is stable, then $\mathrm{G}^{00}_A = \mathrm{G}^{00}_\emptyset$*

Proof: Let $M \preceq N$ and choose $p(x) \in S(M)$ generic and $q(x) \in S(N)$ a nonforking extension of p. Then also q is generic and by Proposition 6.6, $\mathrm{G}^{00}_M = \mathrm{Stab}(p)$ and $\mathrm{G}^{00}_N = \mathrm{Stab}(q)$. Since types over models are stationary, by Proposition 6.7, $\mathrm{G}^{00}_M = \mathrm{Stab}(p) = \mathrm{Stab}(q) = \mathrm{G}^{00}_N$. From this it follows that for arbitrary models M, N, $\mathrm{G}^{00}_M = \mathrm{G}^{00}_N$. And this implies that G^{00}_M is invariant. Since it is type-definable over M and invariant, it must be type-definable over $\emptyset$. Hence $\mathrm{G}^{00}_M = \mathrm{G}^{00}_\emptyset$. Finally, if A is an arbitrary set and we choose a model $M \supseteq A$, we see that

$$\mathrm{G}^{00}_\emptyset = \mathrm{G}^{00}_M \leq \mathrm{G}^{00}_A \leq \mathrm{G}^{00}_\emptyset.$$

□

Definition 7.2 *When $\mathrm{G}^{00}_A = \mathrm{G}^{00}_\emptyset$ for all A, we use also the notation G^{00} for it. In this case it is sometimes called the* absolute *connected component.*

Proposition 7.3 *Let T be stable. A formula $\varphi(x) \in L(A)$ is generic if and only if for some $n < \omega$ there are $g_1, \ldots, g_n \in \mathrm{G}$ such that $\mathrm{G} \subseteq \bigcup_{i=1}^n g_i \cdot \varphi(\mathfrak{C})$. The same is true for right translates.*

Proof: From right to left it is true also in simple theories. Assume that there are $g_1, \ldots, g_n \in \mathrm{G}$ such that $\mathrm{G} \subseteq \bigcup_{i=1}^n g_i \cdot \varphi(\mathfrak{C})$. Choose $B \supseteq A$ with $g_1, \ldots, g_n \in B$ and choose $p(x) \in S(B)$ generic. Then $p(x) \vdash \bigvee_{i=1}^n g_i \cdot \varphi(x)$ and hence $p(x) \vdash g_i \cdot \varphi(x)$ for some i. Hence $g_i \cdot \varphi(x)$ is generic. It follows that also $\varphi(x)$ is generic.

From left to right. Choose an ω-saturated model $M \supseteq A$ and a generic type $p(x) \in S(M)$ such that $\varphi(x) \in p(x)$. Let $g \in \mathrm{G}$. Since $g \cdot p$ does not fork over M, it is finitely satisfiable in M. Observe that $\mathrm{G}(x) \cup \{g \cdot \varphi(x)\} \subseteq g \cdot p(x)$. By definability of types, there is some $\psi(x) \in L(M)$ such that $\psi(M) = M \cap g \cdot \varphi(\mathfrak{C})$. Hence $\mathrm{G}(x) \cup \{\psi(x)\}$ is finitely satisfiable in M. By ω-saturation, there is some $h \in M \cap \mathrm{G}$ such that $h \models g \cdot \varphi(x)$. This means that $\models \varphi(g^{-1} \cdot h)$, that is $g^{-1} \models \varphi(x) \cdot h^{-1}$. We have then shown that for every $g \in \mathrm{G}$ there is some $h \in M \cap \mathrm{G}$ such that $g \models \varphi(x) \cdot h$, and hence

$$\mathrm{G}(x) \vdash \bigvee_{h \in M \cap \mathrm{G}} \varphi(x) \cdot h$$

By compactness, for some $n < \omega$ there are $h_1 \ldots, h_n \in M \cap \mathrm{G}$ such that $\mathrm{G}(x) \vdash \varphi(x) \cdot h_1 \vee \ldots \vee \varphi(x) \cdot h_n$, that is $\mathrm{G} \subseteq \bigcup_{i=1}^n \varphi(\mathfrak{C}) \cdot h_i$. Since left generics are right generics, the same is true for some left translates of $\varphi(x)$. □

Definition 7.4 *Let $p(x) \in S(A)$ be a stationary type such that $p(x) \vdash \mathrm{G}(x)$. If $g \in \mathrm{G}$, by $g * p$ we denote the complete type over A of $g \cdot a$ where $a \models p$ is such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A g$. By stationarity of p, this is independent of the choice of a and it is therefore well defined. Observe that $g * p \vdash g \cdot p$. Moreover $g * p = g \cdot p$ if $g \in A \cap \mathrm{G}$.*

Proposition 7.5 *Let T be stable and $A = \mathrm{acl}^{\mathrm{eq}} A$. Then the mapping $(g, p) \mapsto g * p$ defines an action of* G *on the generic types $p(x) \in S(A)$. Moreover $\mathrm{Stab}(p) = \{g \in \mathrm{G} : g * p = p\}$ for any $p(x) \in S(A)$ such that $p(x) \vdash \mathrm{G}(x)$ (generic or not), and therefore in particular it is the stabilizer of p in this action when p is generic.*

Proof: It is clear that if $p(x) \in S(A)$ is generic, then also $g * p$ is generic. We check that $(g \cdot h) * p = g * (h * p)$. It is enough to show that these two types have a common realization. Choose $a \models p$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A g, h$. Then $a \mathop{\smile\!\!\!\!\!\!\mid}_A g \cdot h$ and therefore $g \cdot h \cdot a \models (g \cdot h) * p$. Now note that $a \mathop{\smile\!\!\!\!\!\!\mid}_{Ah} g$ and hence $h \cdot a \mathop{\smile\!\!\!\!\!\!\mid}_{Ah} g$. By genericity of a over A and the fact that $a \mathop{\smile\!\!\!\!\!\!\mid}_A h$ we also have $h \cdot a \mathop{\smile\!\!\!\!\!\!\mid} h, A$. By transitivity $h \cdot a \mathop{\smile\!\!\!\!\!\!\mid} hAg$ and in particular $h \cdot a \mathop{\smile\!\!\!\!\!\!\mid}_A g$. Hence $g \cdot h \cdot a \models g * (h * p)$.

Genericity of p is not used in the rest. By Proposition 6.2 we know that $\mathrm{Stab}(p) = S(p)$. We show that $g \in S(p)$ if and only if $g * p = p$. Assume first $g \in S(p)$. This means that there is some $h \models p$ such that $g \cdot h \models p$ and $g \cdot h \mathop{\smile\!\!\!\!\!\!\mid}_A g$. It follows that $g \mathop{\smile\!\!\!\!\!\!\mid}_A h$ and therefore $g \cdot h \models g * p$. Hence $p = g * p$ because they have a common realization. For the other direction, assume now $g * p = p$ and take $a \models p$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A g$. Then $a \models g * p$ and therefore $a = g \cdot h$ for some $h \models p$ such that $h \mathop{\smile\!\!\!\!\!\!\mid}_A g$. We have then $h \models p$, $g \cdot h \mathop{\smile\!\!\!\!\!\!\mid}_A g$ and $g \cdot h \models p$. This clearly means that $g \in S(p)$. □

Proposition 7.6 *Let T be stable and $A = \mathrm{acl}^{\mathrm{eq}}(A)$. There is only one generic type $p(x) \in S(A)$ such that $p(x) \vdash \mathrm{G}^{00}(x)$.*

Proof: Existence of at least one such generic type follows from the fact that $\mathrm{G}^{00}(x)$ is a partial generic type. For the uniqueness, let p, q be generic types over A such that $p(x) \vdash \mathrm{G}^{00}(x)$ and $q(x) \vdash \mathrm{G}^{00}(x)$. Let $a \models p$ and $b \models q$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$ and let $c = b \cdot a^{-1}$. Then $c \in \mathrm{G}^{00}$ and $c \mathop{\smile\!\!\!\!\!\!\mid}_A a$. By Proposition 6.6 (and propositions 6.7 and 7.1 to be able to apply it to A) $\mathrm{Stab}(p) = \mathrm{G}^{00}$. Hence $c * p = p$ and therefore $b = c \cdot a \models c * p = p$. Since p, q have b as a common realization, $p = q$. □

Corollary 7.7 *Let T be stable and $A = \mathrm{acl}^{\mathrm{eq}}(A)$.*

1. *The action $(g, p) \mapsto g * p$ of G on the generic types $p(x) \in S(A)$ is transitive.*
2. *For any $g \in \mathrm{G}$ there is only one generic type $p(x) \in S(A)$ such that $p(x) \vdash g \cdot \mathrm{G}^{00}(x)$. Hence the generics over A are in a one-to-one correspondence with the quotient group $\mathrm{G}/\mathrm{G}^{00}$.*

Proof: *1.* Observe that if p', q' are nonforking extensions respectively of p, q over a bigger set B and $p' = g * q'$, then also $p = g * q$. Hence we may assume that A is in fact a $(2^{|T|})^+$-saturated model and hence every coset of G^{00} meets A. Let p, q be generic types over A. By Lemma 1.1 there are $a, b \in \mathrm{G}$ such that $p(x) \vdash a \cdot \mathrm{G}^{00}(x)$ and $q(x) \vdash b \cdot \mathrm{G}^{00}(x)$. By the saturation of A, we may assume that $a, b \in A$. Let $g \models p$ and $h \models q$. Then $a^{-1} \cdot g \in \mathrm{G}^{00}$ and it realizes the generic type $a^{-1} * p$. Similarly, $b^{-1} \cdot h \in \mathrm{G}^{00}$ and realizes the generic type $b^{-1} * q$. By Proposition 7.6, $a^{-1} * p = b^{-1} * q$ and hence $p = a \cdot b^{-1} * q$. For point *2* observe that in case $a = b$ we get $p = q$. □

Definition 7.8 *For any set A, G^0_A is the intersection of all subgroups of G of finite index which are relatively A-definable. In case $\mathrm{G}^0_A = \mathrm{G}^0_\emptyset$ for any A, we use the notation G^0. Clearly, $\mathrm{G}^{00}_A \leq \mathrm{G}^0_A$.*

Proposition 7.9 *If T is stable then $\mathrm{G}^0_A = \mathrm{G}^0$.*

Proof: For any $\varphi = \varphi(x, y) \in L$, let G_φ be the intersection of all subgroups of finite index which are relatively φ-definable (over $\mathfrak{C}$). By Proposition 2.3 G_φ is the intersection of finitely many of these groups. Therefore G_φ is relatively definable and hence has finite index. Since it is invariant, it is in fact relatively definable over $\emptyset$. Hence $\mathrm{G}^0_\emptyset \leq \bigcap_{\varphi \in L} G_\varphi \leq \mathrm{G}^0_A \leq G^0_\emptyset$. □

Remark 7.10 *The group* G *acts on* $S(\mathfrak{C})^{\mathrm{G}} = \{\mathfrak{p} \in S(\mathfrak{C})) : \mathfrak{p} \vdash \mathrm{G}(x)\}$ *by* $(g, \mathfrak{p}) \mapsto g \cdot \mathfrak{p}$. *Let* T *be stable and* $A = \mathrm{acl}^{\mathrm{eq}}(A)$. *The action of* G *on the generics over* A *defined in Proposition 7.5 can now be explained in terms of this new action. Let* $p(x) \in S(A)$ *and let* $\mathfrak{p} \in S(\mathfrak{C})^{\mathrm{G}}$ *be its corresponding nonforking extension over* $\mathfrak{C}$. *Then*

1. $g * p = (g \cdot \mathfrak{p}) \upharpoonright A$
2. $\mathrm{Stab}(p) = \{g \in G : g \cdot \mathfrak{p} = \mathfrak{p}\}$ $(= \mathrm{Stab}(\mathfrak{p}))$
3. *If* $q(x) \in S(A)$ *is generic and* $\mathfrak{q} \in S(\mathfrak{C})^{\mathrm{G}}$ *is its nonforking extension, then* $g * p = q$ *if and only if* $g \cdot \mathfrak{p} = \mathfrak{q}$

Proof: *1.* We need to find a common realization of these two complete types over A. Let $a \models p$ be such that $a \mathop{\smile\!\!\!\!\!\mid}_A g$. Then $g \cdot a \models g * p$. On the other hand $a \models \mathfrak{p} \upharpoonright Ag$ and therefore $g \cdot a \models g \cdot (\mathfrak{p} \upharpoonright Ag)$. But $g \cdot (\mathfrak{p} \upharpoonright Ag) \subseteq g \cdot \mathfrak{p}$ and therefore $g \cdot (\mathfrak{p} \upharpoonright Ag) = (g \cdot \mathfrak{p}) \upharpoonright Ag$. Thus $g \cdot a \models (g \cdot \mathfrak{p}) \upharpoonright A$.

2. If $g \cdot \mathfrak{p} = \mathfrak{p}$ then $(g \cdot \mathfrak{p}) \upharpoonright A = p$ and by *1*, $g * p = p$. For the other direction, assume $g * p = p$, that is $(g \cdot \mathfrak{p}) \upharpoonright A = \mathfrak{p} \upharpoonright A = p$. By stationarity of p is it enough now to prove that $g \cdot \mathfrak{p}$ is a nonforking extension of p. But this follows from Proposition 4.4 and Proposition 4.6 since for each φ and k, for each $B \supseteq gA$,

$$D^*(g \cdot \mathfrak{p} \upharpoonright B, \varphi, k) = D^*(\mathfrak{p} \upharpoonright B, \varphi, k) = D^*(p, \varphi, k).$$

3. From right to left is like *2*. For the other direction, assume $g * p = q$. Then $(g \cdot \mathfrak{p}) \upharpoonright A = \mathfrak{q} \upharpoonright A = q$. By genericity $g \cdot \mathfrak{p}$ does not fork over A and then $g \cdot \mathfrak{p} = \mathfrak{q}$ because the types are stationary. □

Remark 7.11 *Let* T *be stable and* $H \leq \mathrm{G}$ *a connected subgroup type-definable over* $A = \mathrm{acl}^{\mathrm{eq}}(A)$. *If* $p(x) \in S(A)$ *is the generic of* H *then* $H = \mathrm{Stab}(p)$.

Proof: Let $\mathrm{Stab}_H(p)$ be the stabilizer of p in H. Clearly, $\mathrm{Stab}_H(p) = H \cap \mathrm{Stab}(p)$. By connectedness and genericity $H = H_0 = \mathrm{Stab}_H(p)$. It remains to show that $\mathrm{Stab}(p) \subseteq H$. Let $g \in \mathrm{Stab}(p)$. For some $a \models p$, $a \mathop{\smile\!\!\!\!\!\mid}_A g$ and $g \cdot a \equiv_A a$. Then $g \cdot a \models p$. Since $p(x) \vdash H(x)$, $g \cdot a \in H$. Since $a \in H$ too, we conclude $g \in H$. □

Remark 7.12 *Let* $\varphi(x, y) \in L$ *and let us define* $\varphi'(x; y, z) = \varphi(x \cdot z, y)$. *Note that*

$$g \cdot \varphi'(x; a, b) = \varphi'(x \cdot g^{-1}; a, b) = \varphi(x \cdot g^{-1} \cdot b, a) = \varphi'(x; a, g^{-1} \cdot b)$$

and therefore the product by elements of G *is also an action on* $S_{\varphi'}(\mathfrak{C})$. *To be precise we must point out that by* $\neg\varphi'(x; a, b)$ *we understand the negation of* $\varphi'(x; a, b)$ *and not the corresponding substitution in* $\neg\varphi(x, a)$. *Thus, for* $\mathfrak{p} \in S_{\varphi'}(\mathfrak{C})$,

$$g \cdot \mathfrak{p} = \{g \cdot \varphi'(x; a) : a \in \mathfrak{C} \text{ and } \varphi'(x; a) \in \mathfrak{p}\} \cup \{\neg g \cdot \varphi'(x; a) : a \in \mathfrak{C} \text{ and } \varphi'(x; a) \notin \mathfrak{p}\}$$

and the mapping $(g, \mathfrak{p}) \mapsto g \cdot \mathfrak{p}$ *is an action of* G *on* $S_{\varphi'}(\mathfrak{C})$. *For* $\mathfrak{p} \in S_{\varphi'}(\mathfrak{C})$, *let* $\mathrm{Stab}_{\varphi'}(\mathfrak{p}) = \{g \in \mathrm{G} : g \cdot \mathfrak{p} = \mathfrak{p}\}$ *be the stabilizer of the action. Clearly it is a subgroup of* G. *Moreover the action is transitive when restricted to generics and for any* $\mathfrak{p} \in S(\mathfrak{C})^{\mathrm{G}}$,

$$\mathrm{Stab}(\mathfrak{p}) = \bigcap_{\varphi \in L} \mathrm{Stab}_{\varphi'}(\mathfrak{p} \upharpoonright \varphi').$$

Proof: Let $\mathfrak{p} \in S(\mathfrak{C})^{\mathrm{G}}$. It is easy to check that $\mathrm{Stab}_{\varphi}(\mathfrak{p} \restriction \varphi')$ is a subgroup that contains $\mathrm{Stab}(\mathfrak{p})$. Assume now $g \in \bigcap_{\varphi \in L} \mathrm{Stab}_{\varphi}(\mathfrak{p} \restriction \varphi')$ and let us prove that for any $\varphi(x, y) \in L$, for any $a \in \mathfrak{C}$: $\varphi(x, a) \in \mathfrak{p}$ if and only if $g \cdot \varphi(x, a) \in \mathfrak{p}$ (which proves $g \in \mathrm{Stab}(\mathfrak{p})$). It is enough to notice that $\varphi(x, a) \in \mathfrak{p}$ iff $\varphi'(x; a, 1) \in \mathfrak{p} \restriction \varphi'$ iff $g \cdot \varphi'(x; a, 1) \in \mathfrak{p} \restriction \varphi'$ iff $g \cdot \varphi(x, a) \in \mathfrak{p}$. It only remains to prove that the action is transitive on the generic types. Let $\mathfrak{p}, \mathfrak{q} \in S(\mathfrak{C})^{\mathrm{G}}$ be generic types, fix some set $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and let $p = \mathfrak{p} \restriction A$ and $q = \mathfrak{q} \restriction A$. By genericity, $\mathfrak{p}$ and $\mathfrak{q}$ do not fork over A. By Proposition 7.7 there is some $g \in \mathrm{G}$ such that $g * p = q$ and by Remark 7.10 $g \cdot \mathfrak{p} = \mathfrak{q}$. In particular $g \cdot \mathfrak{p} \restriction \varphi' = \mathfrak{q} \restriction \varphi'$ □

Proposition 7.13 *If T is stable then $\mathrm{G}^{00} = \mathrm{G}^{0}$ and for any $\varphi(x, y) \in L$ the set $\{\mathfrak{p} \restriction \varphi :$ $\mathfrak{p}$ is generic $\}$ is finite.*

Proof: Let $p(x) \in S(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ be generic and let $\mathfrak{p}$ be its nonforking extension over $\mathfrak{C}$. By Proposition 6.6, $\mathrm{Stab}(p) = \mathrm{G}^{00}$. With the notation of Remark 7.12, let $H_{\varphi} = \mathrm{Stab}_{\varphi'}(\mathfrak{p} \restriction \varphi')$. We know that $\mathrm{Stab}(p) = \mathrm{Stab}(\mathfrak{p}) = \bigcap_{\varphi \in L} H_{\varphi}$ and each H_{φ} is a subgroup of G. By stability, $\mathfrak{p} \restriction \varphi'$ is definable by a formula $d_{\mathfrak{p}} x \varphi'(x; y, z)$. Then for $g \in \mathrm{G}$

$$g \in H_{\varphi} \text{ if and only if } \models \forall yz(d_{\mathfrak{p}} x \varphi'(x; y, z) \leftrightarrow d_{\mathfrak{p}} x \varphi'(x; y, g^{-1} \cdot z))$$

and hence H_{φ} is relatively definable (over the canonical base of $\mathfrak{p}$, which is contained in any model M). Since $\mathrm{G}^{00} \leq H_{\varphi}$ and G^{00} is of bounded index, H_{φ} is of finite index. Now if M is a model

$$\mathrm{G}^{0} = \mathrm{G}^{0}_{M} \leq \bigcap_{\varphi \in L} H_{\varphi} = \mathrm{G}^{00}$$

and hence $\mathrm{G}^{0} = \mathrm{G}^{00}$.

By Remark 7.12 the action of G on $\{\mathfrak{p} \restriction \varphi' : \mathfrak{p} \in S(\mathfrak{C})$ is generic $\}$ is transitive and therefore the size of the orbit of any generic $\mathfrak{p} \restriction \varphi'$ is the finite index $[\mathrm{G} : \mathrm{Stab}_{\varphi'}(\mathfrak{p} \restriction \varphi')]$. Hence $\{\mathfrak{p} \restriction \varphi' : \mathfrak{p}$ is generic $\}$ is finite. Since $\mathfrak{p} \restriction \varphi' \vdash \mathfrak{p} \restriction \varphi$ (because $\varphi(x, a) \equiv \varphi'(x; a, 1)$) also $\{\mathfrak{p} \restriction \varphi : \mathfrak{p}$ is generic $\}$ is finite. □

Theorem 7.14 *If T is stable, G is a subgroup of a 0-definable group H and there is a family $(H_i : i \in I)$ of 0-definable subgroups of H such that $\mathrm{G} = \bigcap_{i \in I} H_i$.*

Proof: We will assume that the type $\mathrm{G}(x)$ defining the group G is closed under conjunction. By compactness we can find a formula $\varphi_0(x) \in \mathrm{G}(x)$ such that if $\models \varphi_0(a) \wedge \varphi_0(b) \wedge \varphi_0(c)$ then $a \cdot (b \cdot c)$ and $(a \cdot b) \cdot c$ are defined, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, and $a \cdot 1 = 1 \cdot a = a$. For any $\varphi(x) \in L$, let

$$\hat{\varphi}(x, y) = \varphi_0(x) \wedge \varphi_0(y) \wedge \varphi(y \cdot x).$$

By Proposition 7.13 there are only finitely many $\hat{\varphi}$-restrictions of generic types. For each generic $\mathfrak{p}$, the restriction $\mathfrak{p} \restriction \hat{\varphi}$ is definable by some formula $d_{\mathfrak{p}} x \hat{\varphi}(x, y)$. Let

$$\delta_{\varphi}(y) = \bigwedge_{\mathfrak{p} \text{ generic}} d_{\mathfrak{p}} x \hat{\varphi}(x, y).$$

Note that $\delta_{\varphi}(y)$ is invariant and therefore we may assume it is over $\emptyset$. Notice that if $\varphi(x) \vdash \psi(x)$, then $\hat{\varphi}(x, y) \vdash \hat{\psi}(x, y)$ and therefore $\delta_{\varphi}(y) \vdash \delta_{\psi}(y)$. Note also that for all b

$$\begin{array}{lll} \models \delta_{\varphi}(b) & \text{if and only if} & \hat{\varphi}(x, b) \in \mathfrak{p} \text{ for all } \mathfrak{p} \text{ generic} \\ & \text{if and only if} & \models \hat{\varphi}(a, b) \text{ for all } a \in \mathrm{G} \text{ generic over } b \\ & \text{if and only if} & \models \varphi_0(b) \text{ and } \models \varphi(b \cdot a) \text{ for all } a \in \mathrm{G} \text{ generic over } b. \end{array}$$

The group G can be also defined by the set of formulas $\{\varphi_0(x)\}\cup\{\delta_\varphi(x) : \varphi(x) \in \mathrm{G}(x)\}$. On the one hand is obvious that all elements in G satisfy these formulas. On the other hand, if b realizes them and we choose $a \in \mathrm{G}$ generic over b, then $\models \varphi(b \cdot a)$ for all $\varphi(x) \in \mathrm{G}(x)$ and hence $b \cdot a \in \mathrm{G}$. It follows that $b \in \mathrm{G}$.

By compactness there is some $\varphi(x) \in \mathrm{G}(x)$ such that if H_0 is defined by $\varphi_0(x) \wedge \delta_\varphi(x)$, then $\models \varphi_0(a \cdot b)$ for all $a, b \in H_0$. Clearly $\mathrm{G} \subseteq H_0$. We now claim that

$$a \in \mathrm{G}, b \in H_0 \Rightarrow b \cdot a \in H_0 \tag{1}$$

In order to prove it, assume $a \in \mathrm{G}$ and $b \in H_0$. By definition of H_0 and choice of φ is obvious that $\models \varphi_0(b \cdot a)$ and hence only remains to check that $\models \delta_\varphi(b \cdot a)$, that is, $\models \varphi((b \cdot a) \cdot c)$ for all $c \in \mathrm{G}$ generic over $b \cdot a$. Let $c \in \mathrm{G}$ be generic over $b \cdot a$. To prove that $\models \varphi((b \cdot a) \cdot c)$ we can clearly assume that $c \downarrow_{b \cdot a} b, a$. Hence c is also generic over a, b and $a \cdot c$ is generic over a, b and in particular over b. Since $\models \delta_\varphi(b)$ it follows that $\models \varphi(b \cdot (a \cdot c))$. By choice of φ_0, $(b \cdot a) \cdot c = b \cdot (a \cdot c)$ and therefore $\models \varphi((b \cdot a) \cdot c)$.

Now let $H_1 = \{a \in H_0 : b \cdot a \in H_0 \text{ for all } b \in H_0\}$. It is definable over $\emptyset$ and by claim 1 $\mathrm{G} \subseteq H_1 \subseteq H_0$. It is easy to check that H_1 is closed under the group operation. Finally let us define the set of invertible elements of H_1:

$$H = \{a \in H_1 : a \cdot b = b \cdot a = 1 \text{ for some } b \in H_1\}$$

H is also closed under product and it is therefore a group definable over $\emptyset$ and such that $\mathrm{G} \leq H$.

To find the family $(H_i : i \in I)$ of subgroups $H_i \leq H$ it is enough to notice that for each $\psi(x) \in \mathrm{G}(x)$, if we repeat the construction of H starting with $\psi(x) \wedge \varphi_0(x)$ in place of $\varphi_0(x)$ we obtain a group $H_\psi \leq H$ such that $\mathrm{G} \leq H_\psi \subseteq \psi(\mathfrak{C})$. □

Corollary 7.15 *Let T stable and assume $H \leq \mathrm{G}$ is type-definable over A and $a \in \mathrm{G}$. There is a sequence $(c_i : i \in I)$ of imaginaries such that for every automorphism $f \in \mathrm{Aut}(\mathfrak{C})$, $f(H \cdot a) = H \cdot a$ if and only if $f(c_i) = c_i$ for all $i \in I$.*

Proof: By Theorem 7.14 we may assume (changing if necessary G by a larger group) that G is definable over $\emptyset$. We apply Theorem 7.14 now to H (in $T(A)$) Theorem 7.14 obtaining a family $(H_i : i \in I)$ of A-definable subgroups $H_i \leq \mathrm{G}$ such that $H = \bigcap_{i\in I} H_i$. Fix $i \in I$ and consider all the groups $f(H_i)$ where $f \in \mathrm{Aut}(\mathfrak{C})$ is such that $f(H \cdot a) = H \cdot a$. Note that $f(H) = H$ for any such f because $f(H) \cdot f(a) = H \cdot a$ and hence $f(H)$ is a group and a coset of H. By the chain condition in Proposition 2.3 the intersection G_i of all the groups $f(H_i)$ for all f as above is in fact the intersection of a finite number of them and it is therefore definable. Note that $H \leq G_i \leq H_i$. Clearly, $f(G_i) = G_i$ if $f(H \cdot a) = H \cdot a$. Moreover $f(G_i \cdot a) = G_i \cdot f(a)$ and hence $H \cdot a$ is a common subset of $G_i \cdot f(a)$ and $G_i \cdot a$, which implies $f(G_i \cdot a) = G_i \cdot a$. Let c_i be the canonical parameter of $G_i \cdot a$. An automorphism $f \in \mathrm{Aut}(\mathfrak{C})$ fixes c_i if and only if f fixes setwise $G_i \cdot a$. Notice that $H \cdot a = (\bigcap_{i\in I} G_i) \cdot a = \bigcap_{i\in I} G_i \cdot a$. Hence $f(c_i) = c_i$ for all $i \in I$ if and only if $f(H \cdot a) = H \cdot a$. □

8 Transitive group actions

We assume T is a simple theory in this section.

Let S be type-definable over the empty set and assume there is a type-definable (over $\emptyset$) transitive action of G on S. We use $\mathrm{S}(x)$ for the type defining S. With the same arguments of Lemma 1.3 one sees easily that the action $\mathrm{G} \times \mathrm{S} \to \mathrm{S}$ is in fact relatively definable over $\emptyset$ and that the formula $\psi(x, y, z)$ which defines the action can be assumed to be generally a partial mapping on the variables x, y an also in the variables x, z. We will use the notation $g \cdot a$ for the group action.

Definition 8.1 *Let $p(x) \in S(A)$ be a type such that $p(x) \vdash \mathrm{S}(x)$. We say that p is* generic *if for any $a \models p$, for any $g \in \mathrm{G}$ such that $g \perp_A a$, we have $g \cdot a \perp A, g$. An element $a \in \mathrm{S}$ is called* generic over A *if $\mathrm{tp}(a/A)$ is generic. A partial type over A is* generic *if it can be extended to a complete type $p(x) \vdash \mathrm{S}(x)$ generic over A. To distinguish between genericity with respect to* G *and genericity with respect to* S *we sometimes say that an element or a type is* generic in G *or that it is* generic in S. *As in the group case, genericity of a partial type $\pi(x)$ is independent of the choice of the set A as long as it contains the parameters of $\pi(x)$.*

Proposition 8.2 *Let $a \in \mathrm{S}$*

1. *If a is generic over A then $a \perp A$.*
2. *If a is generic over A and $g \in \mathrm{G} \cap \mathrm{acl}(A)$ then $g \cdot a$ is generic over A.*
3. *If $A \subseteq B$ and $a \perp_A B$, then a is generic over A if and only if it is generic over B.*
4. *If a is generic over A and $g \perp_A a$ then $g \cdot a$ is generic over A,g.*
5. *If $A \subseteq B$ and a is generic over B, then a is generic over A.*

Proof: It is a copy of the proof for the group case. □

Proposition 8.3 *If $g \in \mathrm{G}$ is generic over Aa and $a \in S$ is such that $a \perp A$, then $g \cdot a$ is generic over A. In particular (case $A = \emptyset$) if $g \in \mathrm{G}$ is generic over $a \in \mathrm{S}$, then $g \cdot a$ is generic.*

Proof: Let $h \in \mathrm{G}$, $h \perp_A g \cdot a$. We want to show that $h \cdot g \cdot a \perp h, A$. Without loss of generality $h \perp_{A, g \cdot a} g, a$. Then $h \perp_A g, a$. Since $g \perp A, a$ we conclude then that $g \perp A, a, h$. Therefore g is also generic over Aah and $h \cdot g$ is generic over Aah. This implies that $h \cdot g \perp A, a, h$ and hence $h \cdot g \perp_a h, A$. By definability, $h \cdot g \cdot a \perp_a h, A$. Now we have $a \perp A$ and $a \perp_A h$ and hence $A, h \perp a$. By transitivity $A, h \perp h \cdot g \cdot a$. □

Proposition 8.4 *For any A there is some $a \in \mathrm{S}$ generic over A.*

Proof: Let $a \in \mathrm{S}$ $a \perp A$ and let $g \in \mathrm{G}$ generic over Aa. By Proposition 8.3, $g \cdot a$ is generic over A. □

Proposition 8.5 *Let $b \in \mathrm{S}$ be generic over A and let $a \in \mathrm{S}$, $a \perp_A b$. Then $g \cdot a = b$ for some $g \in \mathrm{G}$ generic over Aa.*

Proof: Let $g \in \mathrm{G}$ be generic over Aba. Since the action is transitive, there is some $h \in \mathrm{G}$ such that $h \cdot a = g \cdot b$. We can additionally require that $h \perp_{a, g \cdot b} Ag$. Since b is generic over Aa and $g \perp_{Aa} b$ we see that $g \cdot b \perp Aag$. Now $g \perp_{Aa} g \cdot b$ and $g \perp_{Aa, g \cdot b} h$ and hence $g \perp_{Aa} h$. Therefore $g^{-1} \cdot h$ is generic over Aa. But $g^{-1} \cdot h \cdot a = b$. □

Remark 8.6 *A version of* $D^*(\pi,\varphi,k)$ *rank for this context is easily available, changing only* $\mathrm{G}(x)$ *by* $\mathrm{S}(x)$*. The product* $g\cdot\varphi(x)$ *for* $\varphi(x)$ *consistent with* $\mathrm{S}(x)$ *is defined similarly as in the group case. It is routine to check that the analogues of Proposition 4.3, Proposition 4.6 and Corollary 4.7 hold. This will be used in the sequel.*

Proposition 8.7 *The following conditions are equivalent for any partial type* $\pi(x)$ *over* A *consistent with* $\mathrm{S}(x)$*.*

1. π *is generic in* S.
2. $D^*(\pi,\varphi,k) = D^*(\mathrm{S}(x),\varphi,k)$ *for all* φ, k.
3. *For all* $g\in\mathrm{G}$, $g\cdot\pi$ *does not fork over* $\emptyset$.

Proof: *1* $\Rightarrow$ *2* Let $b\models\pi$ be generic over A and choose $a\in\mathrm{S}$, $a\mathop{\smile\!\!\!\!\!\!\mid}_A b$ and such that $D^*(\mathrm{tp}(a/A),\varphi,k) = D^*(\mathrm{S}(x),\varphi,k)$. By Proposition 8.5 there is some $g\in\mathrm{G}$ generic over Aa and such that $g\cdot a = b$. Then $a\mathop{\smile\!\!\!\!\!\!\mid}_A g$ and

$$D^*(\mathrm{tp}(a/A),\varphi,k) = D^*(\mathrm{tp}(a/Ag),\varphi,k) = D^*(\mathrm{tp}(g\cdot a/Ag),\varphi,k) \le D^*(\pi,\varphi,k)$$

2 $\Rightarrow$ *3*. Like the corresponding implication in the proof of Proposition 4.11.

3 $\Rightarrow$ *1*. Let $g\in\mathrm{G}$ be generic over A. Since $g\cdot\pi$ does not fork over $\emptyset$, for some $a\models\pi$, $g\cdot a\mathop{\smile\!\!\!\!\!\!\mid} Ag$. Then g^{-1} is generic over $A, g\cdot a$ and $g\cdot a\mathop{\smile\!\!\!\!\!\!\mid} A$. By Proposition 8.3, $a = g^{-1}\cdot g\cdot a$ is generic over A. Then π is generic. □

Corollary 8.8 *A type (closed under conjunction) is generic in* S *if and only if all its formulas are generic in* S.

Proof: Clear by point *2* of Proposition 8.7. □

Proposition 8.9 *Let* T *be stable. A formula* $\varphi(x)\in L(A)$ *is generic in* S *if and only if for some* $n<\omega$ *there are* $g_1,\dots,g_n\in\mathrm{G}$ *such that* $\mathrm{S}\subseteq\bigcup_{i=1}^n g_i\cdot\varphi(\mathfrak{C})$.

Proof: It is a modification of the proof of Proposition 7.3. From right to left is the same argument. For the other direction, we fix an ω-saturated model $M\supseteq A$ and a generic type $p(x)\in S(M)$ in S containing the formula $\varphi(x)$. Like in the proof of Proposition 7.3, it will be enough to prove that

$$\mathrm{S}(x)\vdash\bigvee_{h\in M\cap\mathrm{G}} h\cdot\varphi(x)$$

Choose some $a_0\in M\cap\mathrm{S}$. Since the action is transitive, it suffices to show that for each $g\in\mathrm{G}$ there is some $h\in M\cap\mathrm{G}$ such that $g\cdot a_0\in h\cdot\varphi(\mathfrak{C})$, that is $g\cdot a_0 = h\cdot b$ for some $b\in\mathrm{S}$ such that $\models\varphi(b)$. In other words, we have to show that $\varphi(y^{-1}\cdot g\cdot a_0)$ is satisfiable in M. Since $g\cdot p$ does not fork over M, there is some $a\models p$ such that $g\cdot a\mathop{\smile\!\!\!\!\!\!\mid}_M g$. Choose h such that $g\cdot a = h\cdot a_0$ and $h\mathop{\smile\!\!\!\!\!\!\mid}_{g\cdot a,a_0} Mag$. Then $g\mathop{\smile\!\!\!\!\!\!\mid}_M h$, that is $\mathrm{tp}(h/Mg)$ does not fork over M and it is therefore finitely satisfiable in M. Using the same trick as in the proof of Proposition 7.3 one sees that the type $\mathrm{G}(x)\cup\{\varphi(y^{-1}\cdot g\cdot a_0)\}$ (which is contained in $\mathrm{tp}(h/Mg)$) is satisfiable in M and we get this way some $h'\in M\cap\mathrm{G}$ that satisfies $\varphi(y^{-1}\cdot g\cdot a_0)$. □

Corollary 8.10 *Let* T *be stable.*

1. *A partial type $\pi(x)$ over A is generic in* S *if and only if for all $g \in$ G, $g \cdot \pi$ does not fork over A.*

2. *$a \in$ S is generic over A if and only if for all $g \in$ G, if $a \mathop{\smile\!\!\!\!\!\!\mid}_A g$, then $g \cdot a \mathop{\smile\!\!\!\!\!\!\mid}_A g$.*

3. *Let $a \in$ S. If $g \in$ G is generic over Aa, then $g \cdot a$ is generic over A.*

Proof: *1.* By Proposition 8.7, the right hand expresses genericity of π in the theory $T(A)$. Now π is generic in $T(A)$ if and only if all finite conjunctions of formulas in π are generic in $T(A)$. By Proposition 8.9, a formula $\varphi(x) \in L(A)$ is generic over A in the theory T if and only if it is generic over A in the theory $T(A)$.

2 follows from *1.*

3 follows from *2* along the lines of the proof of Proposition 8.3. □

Definition 8.11 *For stationary $p(x) \in S(A)$ such that $p(x) \vdash$ S(x) and $g \in$ G we define the product $g * p$ as the type* tp$(g \cdot a/A)$ *where $a \models p$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_A g$. Moreover we define the stabilizer of p:*

$$\mathrm{Stab}(p) = \{g \in \mathrm{G} : g * p = p\}$$

Remark 8.12 *If $p(x) \in S(A)$, then* Stab(p) *is type-definable over A.*

Proof: As in the proof of Proposition 6.3. □

Proposition 8.13 *Let T be stable and $A = \mathrm{acl}^{\mathrm{eq}}(A)$. The mapping $(g, p) \mapsto g * p$ is a transitive action of* G *on the generic types $p(x) \in S(A)$ such that $p(x) \vdash$ S(x). A type $p(x) \in S(A)$ is generic in* S *if and only if its stabilizer* Stab(p) *contains the connected component* G^0.

Proof: We prove that $g*p$ is generic. Let $a \models p$ and let $g \in$ G be such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A g$. Then $g \cdot a \models g * p$. Let $h \in$ G be such that $h \mathop{\smile\!\!\!\!\!\!\mid}_A g \cdot a$. We have to show that $h \cdot g \cdot a \mathop{\smile\!\!\!\!\!\!\mid} Ah$. We can assume $h \mathop{\smile\!\!\!\!\!\!\mid}_{A, g \cdot a} ag$. Then $h \mathop{\smile\!\!\!\!\!\!\mid}_A ag$. From this we see that $a \mathop{\smile\!\!\!\!\!\!\mid}_{Ag} h$ and hence that $a \mathop{\smile\!\!\!\!\!\!\mid}_{Ag} h \cdot g$. Since a is generic over Ag, it follows that $h \cdot g \cdot a \mathop{\smile\!\!\!\!\!\!\mid} Ag, h \cdot g$ and therefore $h \cdot g \cdot a \mathop{\smile\!\!\!\!\!\!\mid} Ah$.

Checking that $(g \cdot h) * p = g * (h * p)$ is like in the proof of Proposition 7.5, so we omit it. To see that the action is transitive, assume $p(x), q(x) \in S(A)$ are generic types extending S(x). Let $a \models p$, $b \models q$ be such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$. By Proposition 8.4 $g \cdot a = b$ for some $g \in$ G generic over Aa. Then $g \mathop{\smile\!\!\!\!\!\!\mid}_A a$ and therefore $g \cdot a \models g * p$. Hence $g * p = q$.

Assume $p(x) \in S(A)$ extends S(x) and is generic. Let $a \models p$, $b \models p$ be such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$. By Proposition 8.4 there is some $g \in$ G generic over Aa and such that $g \cdot a = b$. Then $g \cdot a \models g * p$ and hence $g \in$ Stab(p). Since g is generic over A, Stab(p) contains an element generic over A. Then the partial type over A defining Stab(p) is generic and by Proposition 5.4 $\mathrm{G}^0 \leq$ Stab(p). For the other direction, assume $\mathrm{G}^0 \leq$ Stab(p). Again by Proposition 5.4 there is some $g \in$ Stab(p) generic over A. Since $g * p = p$, there is some $a \models p$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A g$ and $g \cdot a \models p$. Then g is also generic over Aa and by point *3* of Corollary 8.10 $g \cdot a$ is generic over A. Therefore p is generic. □

Proposition 8.14 *If T is stable and* G *is connected, for each A there is only one generic type $p(x) \in S(A)$ such that $p(x) \vdash S(x)$.*

Proof: Let $p(x), q(x)$ be generic types over A extending S(x). By transitivity of the action of G on the generic types, there is some $g \in$ G such that $g*p = q$. By connectedness, $\mathrm{G} = \mathrm{G}^0$. By genericity (Proposition 8.13) $\mathrm{G}^0 \subseteq$ Stab(p). Hence $g \in$ Stab(p) and $q = g * p = p$. □

Remark 8.15 *Consider the transitive action of left translation* $(g, h) \mapsto g \cdot h$ *of* G *on* G*. An element* $g \in G$ *is generic over* A *in the sense of the group* G *if and only if it is generic over* A *in the sense of the action. Similarly for right translation.*

Proposition 8.16 *In a supersimple theory there exist types of maximal* SU*-rank among the complete types over* A *extending* $\mathrm{S}(x)$ *and they are precisely the generic types over* A *in* S*. The same is true in a superstable theory with respect to the* U*-rank. Moreover if* T *is superstable and* $p(x) \in S(A)$ *is a type extending the partial type* $\mathrm{S}(x)$*, then* p *is generic if and only if* $R^\infty(p) = R^\infty(\mathrm{S}(x))$*. If* T *is totally trascendental a type* $p(x) \in S(A)$ *is generic if and only if* $\mathrm{RM}(p) = \mathrm{RM}(\mathrm{S}(x))$*. By the previous remark, a similar characterization can be given for the generics of the group* G*.*

Proof: Let T be supersimple. Let $a \in \mathrm{S}$ and let $g \in \mathrm{G}$ be such that $a \mathop{\smile\!\!\!\!\!\!|}_A g$. Then

$$\mathrm{SU}(a/A) = \mathrm{SU}(a/Ag) = \mathrm{SU}(g \cdot a/Ag) \leq \mathrm{SU}(g \cdot a/A)$$

and if $\mathrm{SU}(a/A)$ is maximal we get the equality $\mathrm{SU}(g \cdot a/Ag) = \mathrm{SU}(g \cdot a/A)$ and hence $g \cdot a \mathop{\smile\!\!\!\!\!\!|}_A g$.

Now, let $a \in \mathrm{S}$ be arbitrary and let us check that $\mathrm{SU}(a/A) \leq \mathrm{SU}(c/A)$ for some $c \in \mathrm{S}$ generic over A. First take $b \equiv a$ such that $b \mathop{\smile\!\!\!\!\!\!|} A$. Then $\mathrm{SU}(a/A) \leq \mathrm{SU}(a) = \mathrm{SU}(b) = \mathrm{SU}(b/A)$. Now let $g \in \mathrm{G}$ be generic over Ab. By Proposition 8.3 $c = g \cdot b$ is generic over A and as showed above $\mathrm{SU}(b/A) \leq \mathrm{SU}(c/A)$.

If $a, b \in \mathrm{S}$ are independent generics over A then, by Proposition 8.5, $g \cdot a = b$ for some $g \in \mathrm{G}$ generic over Aa and we see that $\mathrm{SU}(a/A) \leq \mathrm{SU}(b/A)$. It follows that all generics over A have the same SU-rank.

Finally, if $a \in S$ has maximal SU-rank over A and $g \in \mathrm{G}$ is such that $g \mathop{\smile\!\!\!\!\!\!|}_A a$, then, as indicated above, $g \cdot a \mathop{\smile\!\!\!\!\!\!|}_A g$. Moreover since $\mathrm{SU}(a/A) \leq \mathrm{SU}(g \cdot a/A)$, $\mathrm{SU}(g \cdot a/A)$ is also maximal and therefore $\mathrm{SU}(g \cdot a/A) = \mathrm{SU}(g \cdot a)$, that is, $g \cdot a \mathop{\smile\!\!\!\!\!\!|} A$. We conclude that $g \cdot a \mathop{\smile\!\!\!\!\!\!|} A, g$ and hence that a is generic over A.

The case superstable with rank U or R^∞ and the case totally trascendental with Morley rank are similar or even simpler since we can use point *2* of Corollary 8.10. □

9 One-based groups

Notation 9.1 *Assume* T *is stable and let* H *be a type-definable subgroup of* G*. Using Corollary 7.15, we fix for each coset* $H \cdot b$ *of* H *a corresponding sequence of imaginaries* $[H \cdot b]$ *such that any automorphism of* $\mathfrak{C}$ *fixes setwise* $H \cdot b$ *if and only if it fixes pointwise* $[H \cdot b]$*. Similarly for left cosets.*

Definition 9.2 *Let* T *be stable.* G *is* one-based *if and only if for each* A*, for each* $a \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{G})$*,* $\mathrm{Cb}(a/A) \subseteq \mathrm{acl}^{\mathrm{eq}}(a)$*. If for each* A*, for each* $n < \omega$*, for each* $a \in \mathrm{G}^n$*,* $\mathrm{Cb}(a/A) \subseteq \mathrm{acl}^{\mathrm{eq}}(a)$*, then* G *is one based. Clearly, if* G *is one-based then for each* $n < \omega$*,* G^n *is one-based. Moreover every subgroup of* G *is one-based.*

Remark 9.3 *Let* T *be stable,* $a \in \mathrm{G}$*,* $A = \mathrm{acl}^{\mathrm{eq}}(A)$*,* $p(x) = \mathrm{tp}(a/A)$ *and* $H = \mathrm{Stab}(p)$*. Assume* $H \cdot a$ *is type-definable over* A*. Consider the transitive action (over* A*) of* H *on* $H \cdot a$ *given by* $(g, c) \mapsto g \cdot c$*. Then* $p(x)$ *is a type of the space* $H \cdot a$ *and* H *is also the stabilizer of* p *in the action, that is,* $H = \{g \in H : g * p = p\}$*.*

Proof: Since T is stable, $H = \mathrm{Stab}(p) = S(p) = \{g \in \mathrm{G} : g \cdot p \cup p$ does not fork over $A\}$. Let $g \in H$. We will show that $g * p = p$. By Lemma 6.2 there is some $h \models p$ such that $g \cdot h \models p$, $g \cdot h \mathop{\smile\!\!\!\!\!\!\;|}_A g$ and $h \mathop{\smile\!\!\!\!\!\!\;|}_A g$. Then $g \cdot h$ is a common realization of p and $g * p$ and therefore $p = g * p$. □

Lemma 9.4 *Let T be stable, $a \in \mathrm{G}$, $A = \mathrm{acl}^{\mathrm{eq}}(A)$, $p(x) = \mathrm{tp}(a/A)$ and $H = \mathrm{Stab}(p)$. Assume $H \cdot a$ is type-definable over A. Then H is connected and $p(x)$ is the generic type of the coset $H \cdot a$ in the transitive action (over A) of H on $H \cdot a$ given by $(g, c) \mapsto g \cdot c$.*

Proof: We first check that $H^0 \cdot a$ is also type-definable over A. If not, then $[H_0 \cdot a] \not\subseteq A$ ant therefore $H_0 \cdot a$ has unboundedly many A-conjugates. These conjugates are cosets of H_0 and are contained in $H \cdot a$ (because $H \cdot a$ is A-invariant). By translation we see that the index of H^0 in H is unbounded, a contradiction.

Since H^0 is a generic subgroup of H, we can choose $b \in H^0$ generic over Aa in H. Then $b \mathop{\smile\!\!\!\!\!\!\;|}_A a$ and hence $\mathrm{tp}(b \cdot a/A) = b * p = p$ (because $b \in \mathrm{Stab}(p)$). By point *3* of Corollary 8.10 $b \cdot a$ is generic over A as element of $H \cdot a$ in the action of H.

Since $H^0 \cdot a$ is over A and $b \cdot a \in H^0 \cdot a$ and $b \cdot a \models p$, it follows that $p(x) \vdash H^0 \cdot a(x)$. Since H, H^0 are type-definable over A, if $H^0 \neq H$ we can find $c \in H \smallsetminus H^0$ such that $c \mathop{\smile\!\!\!\!\!\!\;|}_A a$. Then $c \in \mathrm{Stab}(p)$ and hence $c \cdot a \models c * p = p$. As $p(x) \vdash H^0 \cdot a(x)$, $c \cdot a \in H^0 \cdot a$ and hence $c \in H^0$, a contradiction. □

Notation 9.5 $\mathrm{Stab}(a/A)$ *will be a shorthand for* $\mathrm{Stab}(\mathrm{stp}(a/A))$*, like* $Cb(a/A)$ *is a shorthand for* $Cb(\mathrm{stp}(a/A))$.

Proposition 9.6 *Let T be stable and let G be one-based, let $a \in \mathrm{G}$, $A = \mathrm{acl}^{\mathrm{eq}}(A)$, $p(x) = \mathrm{tp}(a/A)$ and $H = \mathrm{Stab}(p)$. Then H is connected and type-definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, $H \cdot a$ is type-definable over A and $p(x)$ is the generic type of $H \cdot a$ in the transitive action of H on $H \cdot a$ given by $(g, c) \mapsto g \cdot c$.*

Proof: To prove H is connected and $p(x)$ is the generic of $H \cdot a$ we will prove first that $H \cdot a$ is type-definable over A and then Lemma 9.4 will be applied. Let us choose $b \in \mathrm{G}$ generic over Aa. Let $\kappa > |A| + |T|$ and let $M \supseteq Ab$ be a κ-saturated and strongly κ-homogeneous model such that $a \mathop{\smile\!\!\!\!\!\!\;|}_{Ab} M$. Since $a \mathop{\smile\!\!\!\!\!\!\;|}_A b$, we have $a \mathop{\smile\!\!\!\!\!\!\;|}_A M$.

In M (as in any other model) G determines a group $\mathrm{G}^M = M \cap \mathrm{G}$ and this group acts on the types $q(x) \in S(M)$ extending the partial type $\mathrm{G}(x)$ by $(g, q) \mapsto g \cdot q$. The stabilizer of q in this action is $M \cap \mathrm{Stab}(q)$. Observe that $c \cdot p = \mathrm{tp}(c \cdot a/M)$ for all $c \in M$. Therefore, for any $f \in \mathrm{Aut}(M/A)$ the following conditions are all equivalent

1. $\mathrm{tp}(b \cdot a/M)^f = \mathrm{tp}(b \cdot a/M)$
2. $f(b) \cdot \mathrm{tp}(a/M) = b \cdot \mathrm{tp}(a/M)$ (notice that $p \in S(A)$ and hence $p^f = p$ and $\mathrm{tp}(a/M)^f = \mathrm{tp}(a/M)$)
3. $b^{-1} \cdot f(b) \cdot \mathrm{tp}(a/M) = \mathrm{tp}(a/M)$
4. $b^{-1} \cdot f(b) \in \mathrm{Stab}(\mathrm{tp}(a/M)) = \mathrm{Stab}(p)$ $(= H)$
5. $b \cdot H = f(b) \cdot H$.
6. $f \in \mathrm{Aut}(M/[b \cdot H])$ (notice that $[b \cdot H]$ is contained in M since $b \cdot H$ is type-definable over Ab)

By Lemma 1.2 we conclude that

$$Cb(b \cdot a/M) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A, [b \cdot H]) \tag{1}$$

and that in $T(A)$:

$$Cb(b \cdot a/M) = [b \cdot H]$$

Since G is also one-based in $T(A)$ we get $[b \cdot H] \subseteq \mathrm{acl}^{\mathrm{eq}}(A, b \cdot a)$. By genericity $b \cdot a \perp_A a$ and hence $a \perp_A b \cdot a, [b \cdot H]$ and in particular

$$b \cdot a \underset{A[b \cdot H]}{\perp} a$$

Since also (by 1) $b \cdot a \perp_{A[b \cdot H]} M$, the types $\mathrm{tp}(b \cdot a/M)$ and $\mathrm{stp}(b \cdot a/Aa[b \cdot H])$ are parallel and therefore they have a common nonforking extension $\mathfrak{q} \in S(\mathfrak{C})$. We use now this common nonforking extension to show that $H \cdot a$ is type-definable over M. Assume $f \in \mathrm{Aut}(\mathfrak{C}/M)$. Since f fixes $\mathrm{tp}(b \cdot a/M)$, $\mathfrak{q}^f = \mathfrak{q}$. Since $\mathfrak{q}$ contains $\mathrm{stp}(b \cdot a/Aa[b \cdot H])$ and the partial type $b \cdot H \cdot a(x)$ is over $a[b \cdot H]$ and $b \cdot a \in b \cdot H \cdot a$, we get that

$$\mathfrak{q}(x) \vdash b \cdot H \cdot a(x)$$

Now observe that $b \cdot H$ is type-definable over $Ab \subseteq M$ and therefore it is setwise fixed by f. If we apply f (remembering $\mathfrak{q}^f = \mathfrak{q}$) we get

$$\mathfrak{q}(x) \vdash b \cdot H \cdot f(a)(x)$$

which implies that $b \cdot H \cdot a(x) \cup b \cdot H \cdot f(a)(x)$ is consistent and hence $b \cdot H \cdot a \cap b \cdot H \cdot f(a) \neq \emptyset$ and finally $H \cdot a = H \cdot f(a) = f(H \cdot a)$. Thus

$$H \cdot a \text{ is type-definable over } M.$$

But $H \cdot a$ is also type-definable over Aa. Hence $[H \cdot a] \subseteq M$ and $[H \cdot a] \subseteq \mathrm{dcl}^{\mathrm{eq}}(Aa)$. Since $a \perp_A M$ we get $[H \cdot a] \perp_A [H \cdot a]$ and therefore $[H \cdot a] \subseteq \mathrm{acl}^{\mathrm{eq}}(A) = A$.

This way we have proven that $H \cdot a$ is type-definable over A and by Lemma 9.4 we conclude that H is connected. It only remains to check that H is in fact type-definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

Since $a \perp_A b$, $H = \mathrm{Stab}(a/Ab)$. We claim now that

$$H = \mathrm{Stab}(a \cdot b/Ab).$$

To check this, let us assume that $g \in \mathrm{Stab}(a/Ab)$. Then for some $c \equiv_{\mathrm{acl}^{\mathrm{eq}}(Ab)} a$, $g \perp_{Ab} c$ and $g \cdot c \equiv_{\mathrm{acl}^{\mathrm{eq}}(Ab)} c$. Clearly $c \cdot b \equiv_{\mathrm{acl}^{\mathrm{eq}}(Ab)} a \cdot b$, $g \perp_{Ab} c \cdot b$ and $g \cdot c \cdot b \equiv_{\mathrm{acl}^{\mathrm{eq}}(Ab)} c \cdot b$. Therefore $g \in \mathrm{Stab}(a \cdot b/Ab)$. The argument for the other inclusion is similar.

It follows then that H is type-definable over $Cb(a \cdot b/Ab)$. By one-basedness, $Cb(a \cdot b/Ab) \subseteq \mathrm{acl}^{\mathrm{eq}}(a \cdot b)$. Hence $[H] \subseteq \mathrm{acl}^{\mathrm{eq}}(a \cdot b)$. On the other hand, H is type-definable over A and then $[H] \subseteq A$. Since $a \cdot b$ is generic over A, $a \cdot b \perp A$ and therefore $[H] \perp [H]$ and $[H] \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Hence H is type-definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. □

Lemma 9.7 *Let T be stable and let* G *be one-based. Any connected type-definable subgroup $H \leq G$ is type-definable over* $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

Proof: Fix $A = \mathrm{acl}^{\mathrm{eq}}(A)$ such that H is type-definable over A. By Remark 7.11, if $p(x) \in S(A)$ is the generic of H, then $H = \mathrm{Stab}(p)$. By Proposition 9.6 H is type-definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. □

Theorem 9.8 *Let T be stable and let* G *be one-based. The connected component* G^0 *is abelian and can be extended to a relatively definable abelian normal subgroup of* G *of finite index. Hence* G *is abelian by finite.*

Proof: Once it has been established that G^0 is abelian, we obtain the promised relatively definable abelian normal subgroup of finite index as $H = Z(C_G(G^0))$, the center of the centralizer in G of the connected component G^0. Observe that

$$C_G(G^0) = \bigcap_{g \in G^0} \{a \in G : a \cdot g = g \cdot a\}$$

is the intersection of a family of relatively φ-definable (where $\varphi(x, y)$ is $x \cdot y = y \cdot x$) subgroups and hence by Proposition 2.3 it is the intersection of finitely many of them. Thus $C_G(G^0)$ is relatively definable. Since it extends G^0 it has bounded index. Being relatively definable of bounded index, it has finite index. Its center is again relatively definable of finite index, contains G^0, and it is abelian.

We show now that G^0 is abelian by proving that it is contained in its center. For $g \in G^0$ let Inn_g be the inner automorphism of G^0 of conjugation by g, $Inn_g(a) = a^g = g^{-1} \cdot a \cdot g$. Its graph $\{(a, a^g) : a \in G^0\}$ is a relatively definable subgroup of $G^0 \times G^0$ (a one-based group) and it is definably isomorphic to G^0. Hence it is connected. By Lemma 9.7, H is type-definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. From this it follows that if $g, h \in G^0$ and $\mathrm{stp}(g) = \mathrm{stp}(h)$ then $Inn_g = Inn_h$. Choose $g, h \in G^0$ generics over $\emptyset$ and such that $g \perp h$ and $\mathrm{stp}(g) = \mathrm{stp}(h)$. Since $Inn_g = Inn_h$, $a^g = a^h$ for all $a \in G^0$. Therefore $h \cdot g^{-1} \in Z(G^0)$. Since $h \cdot g^{-1}$ is generic over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, $\mathrm{stp}(h \cdot g^{-1})$ is the generic of G^0 in G. Thus for any generic $a \in G^0$, $a \in Z(G^0)$ too. Now, every element of $a \in G^0$ is the product of two generics: if $g \in G^0$ is generic over a then also $a \cdot g^{-1}$ is generic and $a = (a \cdot g^{-1}) \cdot g$. We conclude that any element of G^0 is in the center $Z(G^0)$.

Normality of $Z(C_G(G^0))$ follows from the fact that the centralizer is always normal in the normalizer (in G in our case, since G^0 is normal) and from the fact that if a subgroup is normal then also it center is normal. □

Theorem 9.9 *Let T be stable.* G *is one-based if and only if for each n, any relatively definable subset of* G^n *is a boolean combination of cosets of relatively* $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$*-definable subgroups of* G^n*.*

Proof: Assume G is one-based. Let $A = \mathrm{acl}^{\mathrm{eq}}(A)$. We show that for any $p(x), q(x) \in S_n(A)$ which imply $G^n(x)$, if for every relatively $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable $H \leq G^n$, for every $a \in G^n$, $p \vdash H \cdot a(x)$ if and only if $q \vdash H \cdot a(x)$, then $p = q$. By a standard argument from this follows that each formula $\varphi(x) \in L(A)$ consistent with $G^n(x)$ is a boolean combination of formulas defining cosets of relatively $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable subgroups of G^n

Let $a \models p$ and $b \models q$, and let $H_1 = \mathrm{Stab}(p)$ and $H_2 = \mathrm{Stab}(q)$. By Proposition 9.6, each H_i is connected and type-definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, $H_1 \cdot a$ and $H_2 \cdot b$ are type-definable over A and $p(x), q(x)$ are the generic types of $H_1 \cdot a$ and $H_2 \cdot b$ respectively in the transitive actions of H_1 on $H_1 \cdot a$ and of H_2 on $H_2 \cdot b$ given by $(g, c) \mapsto g \cdot c$. In particular this implies $p(x) \vdash H_1 \cdot a(x)$ and $q(x) \vdash H_2 \cdot b(x)$. By Theorem 7.14 each H_i is the intersection of a family $\mathcal{H}_i$ of relatively definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ subgroups of G^n. Then $p(x) \vdash H \cdot a(x)$ for all $H \in \mathcal{H}_1$ and $q(x) \vdash H \cdot b(x)$ for all $H \in \mathcal{H}_2$. By our hypothesis on p and q, $p(x) \vdash H_2 \cdot b(x)$ and $q(x) \vdash H_1 \cdot a(x)$.

We claim that $H_1 = H_2$. To verify the claim assume $c \in H_1$. We can assume $c \perp_A ab$. Since $c \in \mathrm{Stab}(p)$ and $\mathrm{Stab}(p)$ is also the stabilizer of p in the action of H_1 on $H_1 \cdot a$, we have $c * p = p$. Then $c \cdot a \models p$ and therefore $c \cdot a \in H_2 \cdot b$. On the other hand $a \models p$ and hence $a \in H_2 \cdot b$. It follows that $c \in H_2$. The other inclusion is proved in the same way.

Now we have $H_1 = H_2 = \mathrm{Stab}(p) = \mathrm{Stab}(q)$ and $p(x) \vdash H_1 \cdot a$ and $p(x) \vdash H_1 \cdot b$. It follows that $H_1 \cdot a = H_1 \cdot b = H_2 \cdot b$. Then q is also the generic of $H_1 \cdot a$ in the action of H_1 on $H_1 \cdot a$ and then (see Proposition 8.14) $p = q$.

For the other direction, assume that for every $n < \omega$ every relatively definable $X \subseteq \mathrm{G}^n$ is a boolean combination of cosets of relatively $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable subgroups of G^n. Let $a \in \mathrm{G}^n$ and let $A = \mathrm{acl}^{\mathrm{eq}}(A)$. Clearly, if a, b are in the same cosets of relatively $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable subgroups of G^n then $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. Since all these cosets are relatively definable over $\mathrm{acl}^{\mathrm{eq}}(a)$, $\mathrm{Cb}(a/A) \subseteq \mathrm{acl}^{\mathrm{eq}}(a)$. Hence G is one-based. □

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