NIP formulas and theories

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T is a complete theory with infinite models, \( L \) is its language and \( C \) is its monster model.

1 Formulas with IP

**Definition 1.1** \( \varphi(x,y) \) has IP (the independence property) if there are \((a_i : i < \omega), (b_I : I \subseteq \omega)\) such that

\[ \models \varphi(a_i, b_I) \iff i \in I \]

If \( \varphi \) does not have IP we say it has NIP. It is said that \( T \) has IP if some formula has IP in \( T \), and otherwise it is said that \( T \) has NIP.

**Remark 1.2**
1. If \( \varphi \) has IP, then for every set \( X \) there are \((a_i : i \in X), (b_I : I \subseteq X)\) such that

\[ \models \varphi(a_i, b_I) \iff i \in I \]

2. If for arbitrarily large \( n < \omega \) there are \((a_i : i < n)\) such that for all \( I \subseteq n \),

\[ \{\varphi(a_i, y) : i \in I\} \cup \{\neg \varphi(a_i, y) : i \in n \setminus I\} \]

is consistent, then \( \varphi(x, y) \) has IP.

3. If \( \varphi(x, y, z) \in L \) and \( \varphi(x, y, a) \) has IP in \( T(a) \), then \( \varphi(x; y, z) \) has IP in \( T \).

**Lemma 1.3** If \( \varphi(x, y) \) has IP, then \( \varphi^{-1}(y, x) \) has IP.

**Proof:** Let \( n < \omega \). There are \((a_X : X \in \mathcal{P}(n)), (b_I : I \subseteq \mathcal{P}(n))\) such that \( \models \varphi(a_X, b_I) \iff X \in I \). Let \( U_i := \{X : i \in X\} \) for \( i < n \) and let \( c_i := b_{U_i} \). Then \( \models \varphi^{-1}(c_i, a_X) \iff i \in X \).

**Definition 1.4** The alternation number of \( \varphi(x, y) \), \( \text{alt}(\varphi) \), is the maximal \( n \) such that for some indiscernible sequence \((a_i : i < \omega)\), for some \( b \), \( \omega \) can be decomposed in consecutive segments \( I_1, \ldots, I_n \), and \( \varphi(a_i, b) \) has constant truth value for \( i \) in the same segment and opposite truth value to \( \varphi(a_j, b) \) if \( i, j \) are in consecutive segments. If the maximal \( n \) does not exists we put \( \text{alt}(\varphi) = \infty \).

**Remark 1.5**
1. If \( \text{alt}(\varphi) = \infty \), then for every ordinal \( \alpha \) there is an indiscernible sequence \((a_i : i < \alpha)\) such that for some \( b \), for all \( i < \alpha \), \( \models \varphi(a_i, b) \iff \neg \varphi(a_{i+1}, b) \).

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2. If \( \text{alt}(\varphi) < \infty \), then for all limit ordinals \( \alpha \), for all indiscernible sequences \( (a_i : i < \alpha) \), for all \( b \), there is some \( j < \alpha \) such that \( \models \varphi(a_i, b) \) for all \( i > j \) or \( \models \neg \varphi(a_i, b) \) for all \( i > j \).

**Proposition 1.6** \( \varphi \) has IP if and only if \( \text{alt}(\varphi) = \infty \)

**Proof:** Assume \( \varphi(x, y) \) has IP. There is some indiscernible sequence \( (a_i : i < \omega) \) such that for all \( I \subseteq \omega \) the set \( \{ \varphi(a_i, y) : i \in I \} \cup \{ \neg \varphi(a_i, y) : i \in \omega \setminus I \} \) is consistent. In particular, \( \{ \varphi(a_2, y) : i < \omega \} \cup \{ \neg \varphi(a_{2+1}, y) : i < \omega \} \) is consistent, which clearly implies \( \text{alt}(\varphi) = \infty \).

For the other direction, assume \( \text{alt}(\varphi) = \infty \) and choose an indiscernible sequence \( (a_i : i < \omega) \) such that \( \{ \varphi(a_2, y) : i < \omega \} \cup \{ \neg \varphi(a_{2+1}, y) : i < \omega \} \) is consistent. We claim that \( \{ \varphi(a_i, y) : i \in I \} \cup \{ \neg \varphi(a_i, y) : i \in \omega \setminus I \} \) is consistent for all \( I \subseteq \omega \). It is enough to check that for any finite disjoint \( I, J \subseteq \omega \), \( \Sigma(y) := \{ \varphi(a_i, y) : i \in I \} \cup \{ \neg \varphi(a_i, y) : i \in J \} \) is consistent. Let \( m_1 < \ldots < m_n \) and \( k_1 < \ldots < k_j \) be respective enumerations of \( X \) and \( Y \) and choose even numbers \( m'_1 < \ldots < m'_n \) and odd numbers \( k'_1 < \ldots < k'_j \) such that \( m_1, \ldots, m_n, k_1, \ldots, k_j \) have the same order type. By assumption \( \{ \varphi(a_i, y) : i = m'_1, \ldots, m'_n \} \cup \{ \neg \varphi(a_i, y) : i = k'_1, \ldots, k'_j \} \) is consistent. By indiscernibility \( \Sigma(y) \) is consistent. \( \Box \)

**Remark 1.7** Every boolean combination \( \varphi(x_1, \ldots, x_n; y_1, \ldots, y_n) \) of formulas \( \varphi_i(x_i, y_i) \) with NIP has also NIP. The tuple \( x_i \) may have elements in common with \( x_j \) but it is disjoint with \( y_j \).

**Proof:** We may assume \( x_i = x_j \) and \( y_i = y_j \) for all \( i, j \). It is clear that \( \neg \varphi(x,y) \) has IP if and only if \( \varphi(x,y) \) has IP. On the other hand, an easy argument shows that if \( \varphi(x,y) := \varphi_1(x,y) \land \varphi_2(x,y) \) has infinite alternation number then one of the formulas \( \varphi_1(x,y) \) has also infinite alternation number. \( \Box \)

**Proposition 1.8** Let \( y \) be a fixed \( n \)-tuple of variables. The following are equivalent:

1. No formula \( \varphi(x,y) \) has IP.

2. If \( \alpha \) has cofinality \( \geq |T|^{+} \), \( (a_i : i < \alpha) \) is an indiscernible sequence, and \( B \) is a set of \( \leq n \) elements, then for some \( j < \alpha \), \( (a_i : j < i < \alpha) \) is \( B \)-indiscernible.

**Proof:** \( 1 \Rightarrow 2 \) It is enough to prove, assuming \( 1 \), that for each \( \varphi(x_1, \ldots, x_m; y) \in L \), for each limit ordinal \( \alpha \), for each indiscernible sequence \( (a_i : i < \alpha) \), for each \( n \)-tuple \( b \), there is some \( j < \alpha \) such that for all tuples \( j < i_1 \ldots < i_m < \alpha \), for all \( j < l_1 \ldots < l_m < \alpha \), \( \models \varphi(a_{i_1}, \ldots, a_{i_m}; b) \leftrightarrow \varphi(a_{l_1}, \ldots, a_{l_m}; b) \). And this is the case, since otherwise the indiscernible sequence \( (b_i : i < \omega) \) with \( b_i := a_{0,i}, \ldots, a_{(m-1),i} \) would witness that \( \text{alt}(\varphi(x_1, \ldots, x_m; y)) = \infty \).

\[ 2 \Rightarrow 1 \] is clear by point 1 in Remark 1.5. \( \Box \)

**Proposition 1.9** If some formula has IP in \( T \), there is some \( \varphi(x,y) \) with IP where \( x \) is a single variable.

**Proof:** By Lemma 1.3 it suffices to find some IP formula \( \varphi(x,y) \) where \( y \) is a single variable. This follows from Proposition 1.8 since point 2 for all \( |B| \leq n \) implies point 2 for all \( |B| \leq n + 1 \). We check this. Assume \( B = \{ b_1, \ldots, b_{n+1} \} \), \( \alpha \) has cofinality \( \geq |T|^{+} \) and \( (a_i : i < \alpha) \) is an indiscernible sequence such that for no \( j < \alpha \) the sequence \( (a_i : j < i < \alpha) \) is \( B \)-indiscernible. Choose \( j < \alpha \) such that \( (a_i : j < i < \alpha) \) is \( B_{\alpha+1} \)-indiscernible. Then \( (a_i b_{n+1} : j < i < \alpha) \) is indiscernible and we can choose now some \( l < \alpha \) such that \( j \leq l \) and \( (a_i b_{n+1} : l < i < \alpha) \) is \( \{ b_1, \ldots, b_n \} \)-indiscernible. It follows that \( (a_i : l < i < \alpha) \) is \( B \)-indiscernible. \( \Box \)
2 Number of types

Definition 2.1 For any cardinal κ, ded(κ) is the supremum of the number of branches of a tree of cardinality κ.

Remark 2.2 1. ded(κ) is the supremum of the cardinalities of linearly ordered sets having a dense subset of cardinality κ.

2. κω ≤ ded(κ) ≤ 2κ.

Proof: 1. Given a tree, consider the lexicographic order of nodes and branches. Given a linearly ordered set construct a convenient set of closed intervals with endpoints in the dense set.

2. For κω ≤ ded(κ) note that κω can be identified with the set of branches of the tree κ<ω.

Lemma 2.3 If F ⊆ 2λ and |F| > ded(λ), then for each n < ω there is some I ⊆ λ such that |I| = n and F ∩ I = 2I.

Proof: Assume F, λ are a counterexample, with λ minimal. Note that F can be naturally identified with a set of branches of the tree ∪i<λ F↾i. By minimality of λ, we may assume that for each i < λ, |F↾i| ≤ ded(λ).

For each f ∈ F ∩ i, let F(f) = {g ∈ F : f ⊆ g}, let Gi = {f ∈ F ∩ i : |F(f)| > ded(λ), and let G = {f ∈ F : f ∩ i ∈ Gi} for all i < λ. Then G ⊆ F is a set of branches of the tree ∪i<λ Gi. Note that F ∩ G = ∪i<λ ∪g∈F↾i<λ G↾i F↾i and hence |F ∩ G| ≤ ded(λ), and |G| > ded(λ). Therefore, we can assume G = F. In other terms, we can assume that for each i < λ, |F(f)| > ded(λ).

Now we prove by induction on n, that for each n < ω, for each f ∈ ∪i<λ F↾i there is some I ⊆ λ such that |I| = n and F↾I = 2I. This is clear for n = 0 since F↾i = ∅. Let us consider the case n + 1. By definition of ded, since F↾i is a set of branches of the tree ∪i<λ F↾i, this tree has cardinality > λ and therefore |F↾i| > λ for some i < λ. By the induction hypothesis, for each g ∈ F↾i there is some I↾g ⊆ λ such that |I↾g| = n and F↾I↾g = 2I↾g. By cardinality reasons, there are two different g, h ∈ F↾i such that I↾g = I↾h. Choose j < i such that h↾j = g↾j. Then j ∉ I↾g = I↾h. Fix j : I↾j = I↾j. Then F↾j I↾j = 2I↾j.

Proposition 2.4 1. If ϕ has IP, then for each cardinal κ there is a set A of cardinality κ such that |Sϕ(A)| = 2κ.

2. If ϕ has NIP, then for each cardinal κ: if |A| = κ, then |Sϕ(A)| ≤ ded(κ).

Proof: 1 is clear. For 2 we use Lemma 2.3. Assume |A| = κ and |Sϕ(A)| > ded(κ). Let ϕ = ϕ(x, y) and let l be the length of y. Fix an enumeration (ai : i < κ) of A. For each p(x) ∈ Sϕ(A), let fp ∈ 2n be defined by fp(i) = 0 iff ϕ(x, ai) ∈ p. Let F = {fp : p ∈ Sϕ(A)}. Since |F| > ded(κ), for each n < ω there is some I ⊆ κ such that |I| = n and F ∩ I = 2I. For each X ⊆ I, {ϕ(x, ai) : i ∈ X} ∪ {¬ϕ(x, ai) : i ∈ I ∖ X} is consistent. Hence ϕ(x,y) has IP.

Corollary 2.5 1. If T has IP, then for each cardinal κ ≥ |T| there is a set A of cardinality κ such that |S1(A)| = 2κ.
2. If $\varphi$ has NIP, then for each cardinal $\kappa \geq \omega$: if $|A| = \kappa$, then $|S_\kappa(A)| \leq \text{ded}(\kappa)|T|$.

**Remark 2.6** $\kappa < \text{ded}(\kappa)$ for all infinite $\kappa$.

**Proof:** Assume $\kappa = \text{ded}(\kappa)$. This implies every NIP formula is stable, which is not true. If $\varphi(x, y)$ is NIP, then for each set $A$ of cardinality $\leq \kappa$, $|S_\kappa(A)| \leq \text{ded}(\kappa) = \kappa$ and hence $\varphi$ is $\kappa$-stable and therefore stable. \qed

### 3 Stability and simplicity

The reader is assumed to be familiar with the following definitions and the facts concerning stability and simplicity stated thereafter. See [7] for details.

**Definition 3.1** (Reminding)

1. $\varphi(x, y)$ is **stable** if for all infinite $\lambda$, for all $A$, if $|A| \leq \lambda$, then $|S_\lambda(A)| \leq \lambda$. Otherwise it is **unstable**.

2. $\varphi(x, y)$ has the **order property** if there are $(a_i : i < \omega)$ and $(b_i : i < \omega)$ such that
   
   $\models \varphi(a_i, b_j) \iff i < j$

3. $\varphi(x, y)$ has the **strict order property** if there are $(a_i : i < \omega)$ such that $\varphi(\mathcal{C}, a_i) \subsetneq \varphi(\mathcal{C}, a_{i+1})$.

4. $T$ is **stable** if all formulas are stable in $T$. Otherwise it is **unstable**.

**Fact 3.2** (Reminding)

1. Stable formulas are NIP.

2. If $T$ is unstable, there is an unstable formula $\varphi(x, y)$ where $x$ is a single variable.

3. $\varphi(x, y)$ is stable if and only if it does not have the order property.

4. If $\varphi$ is stable, then also $\varphi^{-1}$ is stable.

5. Let $y$ be a $n$-tuple of variables. If $\varphi(x, y)$ has the strict order property, then
   
   $\psi(y_1, y_2) := \forall x(\varphi(x, y_1) \rightarrow \varphi(x, y_2)) \land \exists x(\varphi(x, y_2) \land \lnot \varphi(x, y_1))$

   defines a partial order of $\mathcal{C}^n$ which has infinite chains.

6. If $\varphi(x, y)$ is unstable and it is NIP, then
   
   (a) Some conjunction of $\varphi(x, y)$ with formulas of the form $\varphi(x, a)$ and $\lnot \varphi(x, a)$ has the strict order property.

   (b) For some $n < \omega$, for some $s \in 2^n$, the formula $\psi(x; y_1, \ldots, y_n) = \bigwedge_{i<n} \varphi(x, y_{i+1})^{s(i)}$

   has the strict order property (where $\varphi^0 = \varphi$ and $\varphi^1 = \lnot \varphi$).

7. $T$ is unstable if and only if it has IP or it has the strict order property.
**Proposition 3.3**  Assume $T$ is an unstable NIP theory. Then there is a definable partial order of the universe $\mathcal{C}$ with infinite chains.

**Proof:** Choose an unstable $\varphi(x, y)$ where $y$ is a single variable and apply points 6 (a) and 5 of Proposition 3.2. \(\square\)

**Definition 3.4** (Reminding)

1. $\varphi(x, y)$ has the $k$–tree property if there is a tree $(a_s : s \in \omega < \omega)$ such that $\{\varphi(x, a_{f|n}) : n < \omega\}$ is consistent for every $f \in \omega^\omega$, and $\{\varphi(x, a_{s-i}) : i < \omega\}$ is $k$-inconsistent for every $s \in \omega < \omega$.

2. $T$ is simple if no formula has the $k$–tree property in $T$ for any $k < \omega$.

**Fact 3.5** (Reminding)

1. If $T$ is not simple, some formula $\varphi(x, y)$, where $x$ is a single variable, has the 2-tree property in $T$.

2. Stable formulas do not have the $k$-tree property.

3. Stable theories are simple.

4. If $\varphi(x, y)$ has the strict order property, then

   $$\psi(x; y_1 y_2) := \neg \varphi(x, y_1) \land \varphi(x, y_2)$$

   has the 2-tree property.

5. Simple theories do not have the strict order property.

6. Simple unstable theories have the IP.

7. Stable theories are just those simple theories that have NIP.

4  O-minimality

**Definition 4.1** $T$ is o-minimal if the language of $T$ contains a binary predicate $<$ which is interpreted as a linear order of the universe and every definable set is a finite union of open intervals $( (a, b), (-\infty, b), (a, +\infty), (-\infty, +\infty) )$ and points.

**Proposition 4.2** Every o-minimal theory is NIP.

**Proof:** By Proposition 1.9 it is enough to prove that all formulas of the form $\varphi(x, y)$, where $x$ is a single variable, are NIP. Assume $\varphi(x, y)$ is a counterexample to this. By Proposition 1.6, there is some indiscernible sequence of elements $(a_i : i < \omega)$ and some tuple $b$ such that $\models \varphi(a_i, b) \iff \neg \varphi(a_{i+1}, b)$. By o-minimality $\varphi(x, b)$ defines a finite union of interval and points. Hence, for some boolean combination $\psi(x, z)$ of formulas $x < z$, for some tuple $c$, $\varphi(x, b)$ is equivalent to $\psi(x, c)$. Since $\models \psi(a_i, c) \iff \neg \psi(a_{i+1}, c)$, $\psi(x, z)$ is IP too. By Remark 1.7, $x < z$ is IP. But this contradicts point 1 of Proposition 2.4 since for each finite $A$, $|S_{x<y}(A)| \leq 2 \cdot |A| + 1$. \(\square\)
Proposition 4.3 Assume $T$ is o-minimal. If $f$ is a definable function defined on an open interval $(a, b)$, there is a finite sequence $a = a_0 < a_1 < \ldots < a_n = b$ such that in every interval $(a_i, a_{i+1})$ $f$ is constant or strictly increasing or strictly decreasing.

Proof: See, for instance, Théorème 4.6 in [12] or Theorem 4.2 in [14].

Proposition 4.4 If $T$ is o-minimal, then the operator $acl = dcl$ has the exchange property and therefore it defines a pregeometry.

Proof: The order $<$ allows us to define over $A$ all elements of $acl(A)$ and hence $dcl(A) = acl(A)$. We check the exchange property. Assume $b \in acl(A) \setminus acl(A)$. Then $b \in dcl(Ac)$ and for some $n$ there is a $0$-definable mapping $f : \mathbb{C}^n \to \mathbb{C}$ and some $a_1, \ldots, a_n$ such that $f(a_1, \ldots, a_n) = b$. Since $b \notin acl(A)$, $c = a_i$ for some $i$. Without loss of generality, $i = 1$. We may assume there is an open interval $(c_1, c_2)$ containing $c$. By Proposition 4.3, there are $c_1 = d_0 < \ldots < d_m = c_2$ such that $f$ is constant or strictly increasing or strictly decreasing in every interval $(d_i, d_{i+1})$. We may assume $f$ is not constant nor strictly increasing or decreasing in $(d_i, d_{i+2})$ and therefore each $d_i$ is definable over $A$. It follows that $c \neq d_i$ for all $i$ and hence $c \in (d_i, d_{i+1})$ for some $i$. If $f$ is constant in $(d_i, d_{i+1})$ then $b$ is $A$-definable. Assume $f$ is strictly increasing or decreasing in $(d_i, d_{i+1})$. Then $f$ is one-to-one and hence $c$ is definable over $Ab$ and therefore $c \in acl(Ab)$. □

5 TP₁ and TP₂

Definition 5.1 $\varphi(x, y)$ has the tree property of the first kind (TP₁) if there is a tree $(a_s : s \in \omega^{<\omega})$ such that \{\varphi(x, a_{f[n]} : n < \omega\} is consistent for all $f \in \omega^\omega$, and $\varphi(x, a_s) \land \varphi(x, a_t)$ is inconsistent if $s, t \in \omega^{<\omega}$ are incomparable in the lexicographic order. We say that $\varphi(x, y)$ has NTP₁ if it does not have TP₁. The theory $T$ has TP₁ if some formula has TP₁ in $T$. Otherwise $T$ has NTP₁.

Let $2 \leq k < \omega$. We say that $\varphi(x, y)$ has the $k$-tree property of the second kind if there are $a_j^i$ ($i, j < \omega$) such that $\{\varphi(x, a_{f[i]}^j) : i < \omega\}$ is consistent for all $f \in \omega^\omega$, and $\{\varphi(x, a_j^i) : j < \omega\}$ is $k$-inconsistent for all $i < \omega$. The formula $\varphi(x, y)$ has TP₂ if it has the 2-tree property of the second kind. Otherwise it has NTP₂. The theory $T$ has TP₂ if some formula has TP₂ in $T$. Otherwise it has NTP₂.

Proposition 5.2 1. If $\varphi(x, y)$ has TP₁, then it has the 2-tree property.

2. If $\varphi(x, y)$ has the k-tree property of the second kind, then it has the k-tree property.

3. Simple theories have NTP₁ and NTP₂.

4. If $\varphi(x, y)$ has TP₂, then $\varphi(x, y)$ has IP.

5. NIP theories have NTP₂.

Proof: 1 is clear. For 2 put $b_0 = a_0^0$ and $b_s = a_s^{n+1}$ if $s \in \omega^{n+1}$. 3 follows from 1 and 2.

4. Let $(a_i^j : i < \omega)$ witness TP₂ of $\varphi(x, y)$. We check that $(a_i^0 : i < \omega)$ witnesses IP of $\varphi(x, y)$. Let $X \subseteq \omega$, and let $f : \omega \to \omega$ be defined by $f(n) = 0$ if $n \in X$ and $f(n) = 1$ otherwise. Since $\{\varphi(x, a_{f[i]}^j) : i < \omega\}$ is consistent, $\{\varphi(x, a_0^0) : n \in X\} \cup \{\neg \varphi(x, a_0^0) : n \in \omega \setminus X\}$ is also consistent. 5 follows from 4. □

Proposition 5.3 $T$ is simple if and only if it has NTP₁ and NTP₂.

Proof: Later.
6 Limit and average types

Definition 6.1 For any sequence $a = (a_i : i \in I)$ of tuples of the same length, if $F$ is a proper filter over $I$, then we define

$$\lim_F(a/A) := \{ \varphi(x) \in L(A) : \{ i \in I : \models \varphi(a_i) \in F \} \}.$$

It is a partial type over $A$ and it is finitely satisfiable in $\{ a_i : i \in I \}$. If $F$ is an ultrafilter, it is complete: $\lim_F(a/A) \in S(A)$.

If $I$ is a linearly ordered set without last element, $\operatorname{Av}(a/A)$ is $\lim_F(a/A)$ where $F$ is the proper filter over $I$ generated by all (nonempty) final segments. It is called the average type of $a$ over $A$.

There is another use of the notion of average type. If $I$ is an infinite indiscernible set (not just a sequence!), then $\operatorname{Av}(I/A)$ is defined as the partial type over $A$ consisting in all formulas $\varphi(x) \in L(A)$ which are true of almost all $a \in I$, i.e., $|\{ a \in I : \models \neg \varphi(a) \}| < \omega$. Note that in this sense $\operatorname{Av}(I/A) = \lim_F(a/A)$ if $a$ is a one-to-one enumeration of $I$ and $F$ is the filter of all cofinite subsets of the index set. We will see that in NIP theories these two notions of average type are compatible.

All this applies also when $A = \mathbb{C}$.

Proposition 6.2 Assume $T$ has NIP.

1. Let $I$ be a linearly ordered set without last element, and let $a = (a_i : i \in I)$ be indiscernible. Then $\operatorname{Av}(a/A) \in S(A)$.

2. Assume $I$ is an infinite indiscernible set. Then $\operatorname{Av}(I/A) \in S(A)$. If $a$ is an enumeration of $I$ with an index set linearly ordered without last element, then $\operatorname{Av}(I/A) = \operatorname{Av}(a/A)$.

Proof: 1 follows from Proposition 1.6. 2. Assume first $|I| = \omega$ and let $a$ be an enumeration of $I$ of order type $\omega$. In this case it is clear, by definition, that $\operatorname{Av}(I/A) = \operatorname{Av}(a/A)$. From this it follows (by considering a suitable countable subset) that for any infinite indiscernible set $I$, $\operatorname{Av}(I/A) \in S(A)$. Now let $a$ be an enumeration of $I$ by a linear ordering without last element. Clearly $\operatorname{Av}(I/A) \subseteq \operatorname{Av}(a/A)$. Since they are complete types, they coincide. \(\square\)

Remark 6.3 If $\varphi(x,y)$ has NIP, then there is some $k < \omega$ such that for every $b$, for every infinite indiscernible set $I$, either $|\{ a \in I : \models \varphi(a,b) \}| < k$ or $|\{ a \in I : \models \neg \varphi(a,b) \}| < k$.

Remark 6.4 Assume $T$ has NIP, let $I$ be a linearly ordered set without last element, and let $a = (a_i : i \in I)$ be $A$-indiscernible. Then $\operatorname{Av}(a/C)$ is finitely satisfiable in $\{ a_i : i \in I \}$ and hence it does not fork over $\{ a_i : i \in I \}$. Moreover $a_i \models \operatorname{Av}(a/Aa_{<i})$ for all $i \in I$.

Remark 6.5 Assume $T$ has NIP, let $I$ be a linearly ordered set without last element, and let $a = (a_i : i \in I)$ be $A$-indiscernible. Then $b \models \operatorname{Av}(a/Aa)$ if and only if $a^-\models \operatorname{Av}(a/A)$.

7 Splitting

Definition 7.1 (Reminding) Let $A \subseteq B$, and let $p(x) \in S(B)$. We say that $p$ splits over $A$ if for some formula $\varphi(x,y) \in L$ there are tuples $a,b \in B$ such that $a \equiv_A b$, $\varphi(x,a) \in p$,
and \( \neg \varphi(x, b) \in p \). The definition applies also to \( B = \mathfrak{C} \). Note that if \( p \in S(\mathfrak{C}) \), then \( p \) does not split over \( A \) if and only if \( p^f = p \) for all automorphisms \( f \in \text{Aut}(\mathfrak{C}/A) \), that is, if and only if \( p \) is \( A \)-invariant.

**Remark 7.2 (Reminding)** If \( p(x) \in S_n(A) \), the number of global extensions of \( p \) that do not split over \( A \) is at most \( 2^{2^{\omega(A)}} \).

**Remark 7.3** Let \( A \subseteq B \), assume \( p(x) \in S(B) \) does not split over \( A \), \( I \) is a totally ordered set, and \((a_i : i \in I)\) is a sequence of tuples \( a_i \in B \) such that \( a_i \models p \upharpoonright Aa_{<i} \). Then \((a_i : i \in I)\) is \( A \)-indiscernible.

**Proof**: By induction on \( n \) it is easy to see that if \( i_1 < \ldots < i_n \) and \( j_i < \ldots < j_n \), then \( a_{i_1}, \ldots, a_{i_n} \equiv_A a_{j_1}, \ldots, a_{j_n} \).

**Lemma 7.4** Assume \( T \) has NIP. Let \( p_1, p_2 \in S(\mathfrak{C}) \) be global types that do not split over \( A \) and assume \( p_1 \upharpoonright A = p_2 \upharpoonright A \). If there is a sequence \((a_i : i < \omega)\) such that \( a_i \models p_1 \upharpoonright Aa_{<i} \) and \( a_i \models p_2 \upharpoonright Aa_{<i} \), then \( p_1 = p_2 \).

**Proof**: Assume \( \varphi(x, y) \in L \) and \( \varphi(x, b) \in p_1 \) and let us check that \( \varphi(x, b) \in p_2 \). Let \((c_i : i < \omega)\) be chosen in such a way that \( c_{i+1} \models p_1 \upharpoonright Abc_{<i+1} \) and \( c_{i+1} \models p_2 \upharpoonright Abc_{<i+1} \). We claim that \( (c_i : i < \omega) \equiv_A (a_i : i < \omega) \). Assume, inductively, that \( c_0, \ldots, c_{n-1} \equiv_A a_0, \ldots, a_{n-1} \) and suppose \( \models \psi(c_0, \ldots, c_{n+1}) \) where \( \psi(x_0, \ldots, x_{n+1}) \in L(\mathfrak{A}) \). Then \( \models \psi(c_0, \ldots, c_{n+1}, x) \in p_2 \) and by non-splitting over \( A \), \( \models \psi(a_0, \ldots, a_{n+1}, x) \in p_2 \). Hence \( \models \psi(a_0, \ldots, a_{n+1}, x) \in p_2 \). The odd case is analogous. By the claim, \((c_i : i < \omega)\) is \( A \)-indiscernible. Since \( T \) has NIP, and \( \models \varphi(c_{i+1}, b) \) for all \( i < \omega \), also \( \models \varphi(c_{i+1}, b) \) for all \( i < \omega \). Hence \( \varphi(x, b) \in p_2 \).

**Proposition 7.5** Assume \( T \) has NIP. Let \( p_1, p_2 \in S(\mathfrak{C}) \) be global types that do not split over \( A \) and assume \( p_1 \upharpoonright A = p_2 \upharpoonright A \). If there are sequences \((a_i : i < \omega)\), \((b_i : i < \omega)\) such that \( a_i \models p_1 \upharpoonright Aa_{<i} \) and \( b_i \models p_2 \upharpoonright Aa_{<i} \) for all \( i < \omega \), and moreover \( (a_i : i < \omega) \equiv_A (b_i : i < \omega) \), then \( p_1 = p_2 \).

**Proof**: Let \( f \in \text{Aut}(\mathfrak{C}/A) \) be such that \( f(a_i : i < \omega) = (b_i : i < \omega) \). Since \( p_1 \) does not split over \( A \), \( p_1^f = p_1 \). But then \( b_i \models p_1 \upharpoonright Aa_{<i} \) and by Lemma 7.4, \( p_1 = p_2 \).

**Corollary 7.6** Assume \( T \) has NIP. If \( p(x) \in S_n(A) \), the number of global extensions of \( p \) that do not split over \( A \) is \( \leq 2^{2^{\omega(A)}} \).

**Proof**: By Proposition 7.5 each global non-splitting extension of \( p \) is determined by the type over \( A \) of an \( \omega \)-sequence of realizations of \( p \).

**Corollary 7.7** Assume \( T \) has NIP. Let \( p \in S(\mathfrak{C}) \) a global type that does not split over \( A \) and assume \( a_i \models p \upharpoonright Aa_{<i} \) for all \( i < \omega \). If \( \text{Av}(\{a_i : i < \omega\}/\mathfrak{C}) \) does not split over \( A \), then \( p = \text{Av}(\{a_i : i < \omega\}/\mathfrak{C}) \).

**Proof**: By Lemma 7.4 and Remark 6.4.

### 8 Coheirs

**Definition 8.1** Let \( M \subseteq A \), let \( p(x) \in S(M) \) and let \( p(x) \subseteq q(x) \in S(A) \). We say that \( q \) is a coheir of \( p \) if it is finitely satisfiable in \( M \). The definition applies also to \( A = \mathfrak{C} \). Coheirs are a particular case of non-splitting extensions.

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Proposition 8.2 1. If $T$ has IP, then for each $\lambda \geq |T|$ there is some model $M$ of cardinality $\lambda$ and some $p(x) \in S_1(M)$ having $2^{(2^\lambda)}$ coheirs over $\mathcal{C}$.

2. If $T$ has NIP, then for each $\lambda \geq |T|$, for each model $M$ of cardinality $\lambda$, for each $p(x) \in S_n(M)$, $p(x)$ has at most $2^\lambda$ coheirs over $\mathcal{C}$.

Proof: 1. By IP, there are $\varphi(x,y) \in L$, $(a_i : i < \lambda)$, and $(b_X : X \subseteq \lambda)$ such that $\models \varphi(a_i, b_X) \iff i \in X$. Let $M \supseteq \{a_i : i < \lambda\}$ be a model of cardinality $\lambda$ and for each ultrafilter $U$ over $\lambda$, let $p_U := \lim_U((a_i : i < \lambda)/\mathcal{C})$. If $U \neq V$, then there is some $X \in U \setminus V$, which implies $\varphi(x, b_X) \in p_U$ and $\neg \varphi(x, b_X) \in p_V$, and hence $p_U \neq p_V$. Every $p_U$ is finitely satisfiable in $M$ and therefore it is a coheir of $P_U \upharpoonright M$. There are $2^{2^\lambda}$ ultrafilters over $U$ and there are only $2^\lambda$ complete 1-types over $M$. Hence, for some $p(x) \in S_1(M)$ there are $2^{(2^\lambda)}$ types $p_U$ extending $p$.

2. It follows from 7.6 since every coheir is a non-splitting extension. $\square$

9 Forking and Lascar splitting

Definition 9.1 (Reminding) A partial type $\pi(x,a)$ (where $\pi(x,y)$ is over the empty set) divides over $A$ if for some $A$-indiscernible sequence $(a_i : i < \omega)$ of tuples $a_i \equiv_A a$, $\bigcup_{i < \omega} \pi(x,a_i)$ is inconsistent. The type $\pi(x,a)$ forking over $A$ if $\pi(x,a)$ implies a disjunction $\varphi_1(x,a_1) \lor \cdots \lor \varphi_n(x,a_n)$ where every $\varphi_i(x,a_i)$ divides over $A$. For a global type, forking and dividing are always equivalent.

Definition 9.2 (Reminding) The group $\text{Aut}(\mathcal{C}/A)$ of strong automorphisms over $A$ is the subgroup of $\text{Aut}(\mathcal{C}/A)$ generated by all automorphisms fixing some model containing $A$. Two tuples $a, b$ have same Lascar strong type over $A$, written $a \equiv^L_A b$, if they are in the same orbit under $\text{Aut}(\mathcal{C}/A)$. The relation of being two members of an infinite $A$-indiscernible sequence is type-definable over $A$ by a type $\text{nc}(x,y)$. Remind that equality of Lascar strong type over $A$ is the transitive closure of the relation defined by $\text{nc}(x,y)$.

Remark 9.3 If $(a_i : i \in I)$ is an infinite $A$-indiscernible sequence, then for any $i_0 < \cdots < i_n \in I$, and $j_0 < \cdots < j_n \in I$,

$$a_{i_0} \cdots a_{i_n} \equiv_A a_{j_0} \cdots a_{j_n}$$

Proof: One can assume $i_n < j_0$ and this case is easy since $\models \text{nc}(a_{i_0} \cdots a_{i_n}; a_{j_0} \cdots a_{j_n})$. $\square$

Definition 9.4 Let $A \subseteq B$, and let $p(x) \in S(B)$. We say that $p$ strongly splits over $A$ if for some $\varphi(x,y) \in L$ there are tuples $a, b \in B$ such that $\models \text{nc}(a, b), \varphi(x,a) \in p$, and $\neg \varphi(x,b) \in p$. We say that $p$ Lascar-splits over $A$ if for some formula $\varphi(x,y) \in L$ there are tuples $a, b \in B$ such that $a \equiv^L_A b, \varphi(x,a) \in p$, and $\neg \varphi(x,b) \in p$. If $A$ is a model, Lascar-splitting over $A$ is equivalent to splitting over $A$. These definitions apply also to $B = \mathcal{C}$. Note that if $p \in S(\mathcal{C})$, then $p$ does not Lascar-split over $A$ if and only if $p^f = p$ for all strong automorphisms $f \in \text{Aut}(\mathcal{C}/A)$.

Remark 9.5 Let $A \subseteq B$, and let $p(x) \in S(B)$.

1. If $p(x)$ strongly splits over $A$, then $p(x)$ Lascar-splits over $A$. 9
2. If $p(x)$ Lascar-splits over $A$, then $p$ splits over $A$.

Proposition 9.6 Assume $T$ has NIP. A global type $p \in S(\mathcal{C})$ does not fork over $A$ if and only if it does not Lascar-split over $A$.

Proof: Assume first $p$ does not fork over $A$. Let $\varphi(x, y) \in L$. It is enough to check that $\varphi(x, b) \in p$ whenever $\varphi(x, a) \in p$ and $\models nc_A(a, b)$. Let $(a_i : i < \omega)$ be some $A$-indiscernible sequence with $a = a_0$ and $b = a_1$. If $\varphi(x, a) \in p$ but $\varphi(x, b) \not\in p$ then since $p$ does not divide over $A$ and the sequence $(a_{2i}, a_{2i+1} : i < \omega)$ is $A$-indiscernible, it follows that \( \{\varphi(x, a_{2i}) \land \neg \varphi(x, a_{2i+1}) : i < \omega\} \) is consistent, which implies that $\text{alt}(\varphi(x, y)) = \infty$.

Assume now $p$ does not Lascar-split over $A$. We will check that no formula in $p$ divides over $A$. Let $\varphi(x, y) \in L$ and assume that $\varphi(x, a) \in p$ and $(a_i : i < \omega)$ is an $A$-indiscernible sequence with $a = a_0$. Then $a_i \equiv_A a$ and therefore $\varphi(x, a_i) \in p$ for all $i < \omega$. This shows that \( \{\varphi(x, a_i) : i < \omega\} \) is consistent. \(\square\)

Definition 9.7 Extending slightly terminology of [11], we say that $B$ is complete over $A$ if $A \subseteq B$, and every $n$-type over $A$ is realized in $B$. We also say that $B$ is $\omega$-saturated over $A$ if $A \subseteq B$ and for each finite $B_0 \subseteq B$ every $n$-type over $AB_0$ is realized in $B$. This last condition implies that $B$ is an $\omega$-saturated model. Obviously, if $M$ is $\omega$-saturated over $A$, and $M$ is complete over $A$. In particular, the monster model $\mathcal{C}$ is complete over any set $A$. This notions can also be extended to Lascar strong types. For instance, we say that $B$ is Lascar-complete over $A$ if $A \subseteq B$, and every finitary Lascar strong type over $A$ is realized in $B$.

Remark 9.8 If $M$ is $\omega$-saturated over $A$ and $p(x) \in S(M)$, then

1. $p$ forks over $A$ if and only if $p$ divides over $A$.
2. $p$ strongly splits over $A$ if and only if $p$ Lascar-splits over $A$.

Remark 9.9 If $B$ is $A$-complete and $p(x) \in S(B)$ forks over $A$, then $p$ splits over $A$ or $p$ divides over $A$.

Proof: Assume $p(x)$ forks over $A$ but does not split over $A$. There are some formulas $\theta(x, z), \varphi_1(x, y_1), \ldots, \varphi_n(x, y_n) \in L$, some tuple $c \in B$ and some tuples $a_1, \ldots, a_n$ such that $\theta(x, c) \in p(x)$, $\varphi_i(x, a_1) \lor \ldots \lor \varphi_n(x, a_n)$, and each $\varphi(x, a_i)$ divides over $A$. By $A$-completeness of $B$ we can choose $d, b_1, \ldots, b_n \in B$ such that $ca_1 \ldots a_n \equiv_A db_1 \ldots b_n$. Then $\theta(x, d) \lor \varphi_1(x, b_1) \lor \ldots \lor \varphi_n(x, b_n)$ and $\theta(x, d) \in p(x)$. Hence $\varphi_i(x, b_i) \in p$ for some $i$. Since $\varphi_i(x, b_i)$ divides over $A$, $p(x)$ divides over $A$. \(\square\)

Remark 9.10 A careful reading of the proof of Proposition 9.6 shows that if $A \subseteq B$ and $p(x) \in S(B)$, then:

1. If $T$ has NIP and $p$ does not divide over $A$, then $p$ does not strongly split over $A$.
2. If $B$ is $\omega$-saturated over $A$, and $p$ does not strongly split over $A$, then $p$ does not divide over $A$.

Hence, if $B$ is $\omega$-saturated over $A$, then over $A$

\[\text{forking} = \text{dividing} \Rightarrow \text{strongly splitting} = \text{Lascar splitting} \Rightarrow \text{splitting}\]
and if moreover \( T \) has NIP, then forking, dividing, strongly splitting, and Lascar splitting over \( A \) are all the same. If additionally \( A \) is a model, then splitting over \( A \) is also equivalent to all these properties.

**Remark 9.11** Assume \( T \) has NIP. If \( p(x) \in S(B) \) does not fork over \( A \subseteq B \), then \( p \) does not Lascar-split over \( A \).

**Proof:** If \( p \) does not fork over \( A \), \( p \) has a global extension \( p \) that does not fork over \( A \). By Proposition 9.6, \( p \) does not Lascar split over \( A \). Clearly, \( p \) does not Lascar-split over \( A \) neither.

**Proposition 10.1** Let \( M \) be \( \omega \)-saturated over \( A \subseteq M \). If \( a \downarrow_A M \) and \( b \downarrow_{Aa} M \), then \( ab \downarrow_A M \).

**Proof:** A well-known property of dividing is: if \( \text{tp}(a/M) \) does not divide over \( A \) and \( \text{tp}(b/Ma) \) does not divide over \( Aa \), then \( \text{tp}(ab/M) \) does not divide over \( A \). Since \( M \) is \( \omega \)-saturated over \( A \) forking and dividing over \( A \) for types over \( M \) are equivalent.

**Corollary 9.13** If \( a \downarrow_A A \) and \( b \downarrow_{Aa} A \), then \( ab \downarrow_A A \).

**Proof:** Choose \( M \supseteq A \), a model \( \omega \)-saturated over \( A \). Let \( a' \equiv_A a \) such that \( a' \downarrow_A M \). Choose \( b' \) such that \( ab \equiv_A a'b' \). Since \( b' \downarrow_{Aa'} A \), we may choose \( b'' \) such that \( b'' \downarrow_{Aa''} M \) and \( b'' \equiv_{Aa''} b' \). By Proposition 9.12, \( a'b'' \downarrow_A M \). In particular \( a'b'' \downarrow_A A \). Since \( ab \equiv_A a'b'' \), \( ab \downarrow_A A \).

## 10 Nonsplitting extensions and products

**Proposition 10.2** Assume \( B \) is complete over \( A \) and \( p(x) \in S(B) \). If \( p \) does not split over \( A \), then for every \( C \supseteq B \) there is a unique \( q(x) \in S(C) \) extending \( p \) that does not split over \( A \). Similarly for Lascar-splitting if \( B \) is Lascar-complete over \( A \).

**Proof:** \( q(x) := p(x) \cup \bigcup_{\varphi(x,y) \in \mathcal{L}} \{ \varphi(x,a) : a \in C \text{ and } \varphi(x,a') \in p \text{ for some } a' \equiv_A a \} \) and write \( a' \models_A a \) in the second case.

**Definition 10.3** Assume \( B \) is complete over \( A \) and \( p(x) \in S(B) \) does not split over \( A \). For any set \( C \supseteq B \), \( p|_A C \) is the only complete extension of \( p \) to \( C \) that does not split over \( A \).

**Remark 10.3** Assume \( B \) is complete over \( A \) and \( p(x) \in S(B) \) does not split over \( A \).

1. For any \( D \supseteq C \supseteq B \), \( (p|_A C)|_A D = p|_A D \).
2. For any \( C \supseteq B \), if the sequence \( (a_i : i < \omega) \) is chosen in such a way that \( a_i \models p|_A Ca_i < i \), then \( (a_i : i < \omega) \) is \( C \)-indiscernible.

**Remark 10.4** Assume \( B \) is \( AA' \)-complete and \( p(x) \in S(B) \) does not split over \( A \) nor over \( A' \). Then for any \( C \supseteq B \), \( p|_A C = p|_A C \).

**Proof:** Assume \( \varphi(x,y) \in \mathcal{L} \), \( a \in C \) and \( \varphi(x,a) \in p|_A C \). Choose \( b \in B \) such that \( a \equiv_{AA'} b \). Since \( a \equiv_A b \), \( \varphi(x,b) \in p \subseteq p|_A C \). Since \( a \equiv_{AA'} b \), \( \varphi(x,a) \in p|_A C \).
Definition 10.5 Assume \( p(x) \in S(B) \) does not split over some \( A \subseteq B \) of cardinality \( \kappa \geq \omega \) and \( B \) is \( A' \)-complete for all \( A' \subseteq B \) of cardinality \( \kappa \). Then for any \( C \supseteq B \), \( p_C \) is the unique extension of \( p \) over \( C \) that does not split over a subset of \( B \) of cardinality \( \kappa \). It is independent of the choice of \( \kappa \).

Definition 10.6 Assume \( B \) is complete over \( A \), \( p(x), q(y) \in S(B) \) and \( q(y) \) does not split over \( A \). We define the product\(^1\) \( p \otimes_A q \) as \( \text{tp}(ab/B) \) where \( a \models p \) and \( b \models q \). It is independent of the choice of \( a, b \). If \( \kappa = |A| \) and \( B \) is \( A' \)-complete for all \( A' \subseteq B \) of cardinality \( \kappa \) (for instance, \( B \) is a \( \kappa^+ + \omega \)-saturated model), then it is also independent of the choice of \( A \) (and \( \kappa \)) and we denote the product by \( p \otimes q \).

Lemma 10.7 Assume \( B \) is complete over \( A \), and \( p(x), q(y) \in S(B) \) do not split over \( A \). Then \( p \otimes_A q \) does not split over \( A \).

Proof: Assume \( \varphi(x, y, z) \in L, c \in B \), and \( \varphi(x, y, c) \in p \otimes_A q \). Choose \( a \models p \), choose \( b \models q \), and let \( c' \in B \) be such that \( c \models_A c' \). Since \( p \) does not split over \( A \), \( ac \models_A ac' \) and hence \( \varphi(a, y, c') \in q \). \( Ba \) and \( \varphi(a, b, c') \). It follows that \( \varphi(x, y, c') \in p \otimes_A q \). \( \square \)

Proposition 10.8 Assume \( B \) is complete over \( A \) and \( p(x), q(y), r(z) \in S(B) \) do not split over \( A \).

1. For any \( C \supseteq B \), \( (p|AC \otimes_A q|AC) = (p \otimes_A q)|AC \).

2. \( (p \otimes_A q) \otimes_A r = p \otimes_A (q \otimes_A r) \).

Proof: 1. \( (p|AC \otimes_A q|AC) \) is an extension of \( p \otimes_A q \) over \( C \) and it does not split over \( A \).

2. Take \( a \models p \), \( b \models q \), and \( c \models r \). Clearly, \( abc \models (p \otimes q) \otimes_A r \). On the other hand, \( bc \models (q|BA \otimes_A r|BA) \) and by 1 \( bc \models (q \otimes_A r)|BA \). Hence \( abc \models p \otimes_A (q \otimes_A r) \). \( \square \)

Definition 10.9 Assume \( B \) is complete over \( A \) and \( p(x) \in S(B) \) does not split over \( A \). The \( n \)-th power \( p(x_1, \ldots, x_n)^{(n)} \) is defined for \( n \geq 1 \) as the product \( p(x_1) \otimes_A \cdots \otimes_A p(x_n) \). By associativity, it is well-defined. It is a complete type over \( B \) and it does not split over \( A \). We define the \( \omega \)-power of \( p \) as \( p^{(\omega)}(x_i : i < \omega) = \bigcup_{i < \omega} p^{(i+1)}(x_0, \ldots, x_i) \). Again, it is a complete type over \( B \) and it does not split over \( A \). If \( \kappa = |A| \) and \( B \) is \( A' \)-complete for all \( A' \subseteq B \) of cardinality \( \kappa \) then powers of \( p \) are independent of the choice of \( A \) and we can write \( p^{(\kappa)} \) and \( p^{(\omega)} \).

Remark 10.10 Assume \( B \) is complete over \( A \) and \( p(x) \in S(B) \) does not split over \( A \).

1. \( (a_i : i < \omega) \models p^{(i)} \) if and only if \( a_i \models p|BA_{a < i} \) for all \( i < \omega \).

2. If \( (a_i : i < \omega) \models p^{(\omega)} \), then \( (a_i : i < \omega) \) is indiscernible over \( B \).

3. If \( (a_i : i < \omega) \models p^{(\omega)} \), then \( (a_i : j \leq i < \omega) \models p^{(\omega)}|BA_{a < j} \).

Proof: 3. Clear since by (1) \( p^{(\omega)}|BA_{a < j} = (p|BA_{a < j})^{(\omega)} \).

\(^1\)We have changed the order of \( p \) and \( q \) in the definition of product with the purpose of making the definition of the power easier to understand.
11 Lascar strong types and KP-splitting

The results in Section 10 can be generalized to non Lascar-splitting extensions and Lascar-complete sets. In particular:

**Remark 11.1** Let $B$ be Lascar-complete over $A$ and let $p(x) \in S(B)$.

1. If $p(x)$ does not Lascar-split over $A$, then for any $C \supseteq B$, we also denote $p|_{AC}$ the unique extension of $p$ to $C$ that does not Lascar-split over $A$.

2. If $q(y) \in S(B)$ does not Lascar-split over $A$, we also denote $p(x) \otimes_A q(y)$ the type of a tuple $ab$ such that $a \models p$ and $b \models q|_{A}Ba$ (in the new sense).

**Proposition 11.2** Assume $T$ has NIP and $B \supseteq A$ is Lascar-complete over $A$ and $p(x) \in S(B)$. Then $p$ does not fork over $A$ if and only if $p$ does not Lascar-split over $A$.

**Proof:** One direction follows from Remark 9.11. Now, if $p$ does not Lascar-split over $A$, then $p$ has a global extension $p$ that does not Lascar-split over $A$. By Proposition 9.6, $p$ does not fork over $A$. Hence $p$ does not fork over $A$. \( \square \)

**Proposition 11.3** Assume $T$ has NIP.\(^2\) Let $B \supseteq A$ be Lascar-complete over $A$. If the types $p(x), q(y) \in S(B)$ do not Lascar-split over $A$, then $p(x) \otimes_A q(y)$ does not Lascar-split over $A$.

**Proof:** Extend $B$ to an $\omega$-saturated over $A$ model $M \supseteq B$. Assume $p'(x), q'(y) \in S(M)$ do not Lascar-split over $A$. Let $a \models p'$ and $b \models q'|_A Ma$. Then $p'(x) \otimes_A q'(y) = tp(ab/M)$. Since $a \downarrow_A M$ and $b \downarrow_{A} M$, by Proposition 9.12, $ab \downarrow_A M$. Hence $tp(ab/M)$ does not Lascar-split over $A$. Now consider $p(x) \otimes q(y)$ for $p, q \in S(B)$. We have shown that $p|_{AB}M \otimes q|_{B}M$ does not Lascar-split over $A$. Since this type extends $p(x) \otimes q(y)$, it follows that $p(x) \otimes q(y)$ does not Lascar-split over $A$. \( \square \)

**Proposition 11.4** Assume $T$ has NIP. Assume $B \supseteq A$ is Lascar-complete over $A$ and $p(x), q(y), r(z) \in S(B)$ do not Lascar split over $A$.

1. For any $C \supseteq B$, $(p|_{AC} \otimes_A q|_A C) = (p \otimes_A q)|_A C$.

2. $(p \otimes_A q) \otimes_A r = p \otimes_A (q \otimes_A r)$.

**Proof:** Like the proof of proposition 10.8, but using now Proposition 11.3. \( \square \)

**Remark 11.5** Assume $T$ has NIP. Let $B \supseteq A$ be Lascar-complete over $A$. Assuming $p(x) \in S(B)$ does not Lascar-split over $A$, the powers $p^{(n)\lambda}$ and $p^{(\omega)\lambda}$ are defined in analogous way as we did in the nonsplitting case, using associativity of the product. It follows from Proposition 11.3 that the powers $p^{(n)\lambda}$ and $p^{(\omega)\lambda}$ do not Lascar-split over $A$.

**Lemma 11.6** Assume $T$ has NIP. Let $B \supseteq A$ be Lascar-complete over $A$, and assume $p(x) \in S(B)$ does not Lascar-split over $A$.

1. $(a_i : i < \omega) \models p^{(\omega)\lambda}$ if and only if $a_i \models p|_{ABa_{<\omega}}$ for all $i < \omega$.

\(^2\)In fact the assumption of NIP is unnecessary since one can use left transitivity of $\downarrow^i$ (see Section 18). The same applies to Proposition 11.4, Remark 11.5 and Lemma 11.6.
2. If \( a, b \models p \) then \( a \equiv_A b \).

3. If \( a_1 \ldots a_n \models p^{(n)}_A \) and \( b_1 \ldots b_n \models p^{(n)}_A \), then \( a_1 \ldots a_n \equiv_A b_1 \ldots b_n \).

4. If \( (a_i : i < \omega) \models p^{(\omega)}_A \upharpoonright A \), then \( (a_i : i < \omega) \) is indiscernible over \( A \).

5. If \( (a_i : i < \omega) \models p^{(\omega)}_A \), then \( (a_i : j \leq i < \omega) \models p^{(\omega)}_A \upharpoonright A a_{<j} \).

**Proof:** 1. Clear.

2. Choose \( a' \in B \) such that \( a \equiv_A a' \) and choose \( n \) such that \( \models n c^A_3(a, a') \). Then \( \models n c^A_3(b, a') \) and hence \( b \equiv_A a' \equiv_A a \).

3. By 2, since \( p^{(n)}_A \) does not Lascar-split over \( A \).

4. We may assume \( (a_i : i < \omega) \models p^{(\omega)}_A \) and then we use 3 since \( a_1, \ldots, a_n \models p^{(n)}_A \) whenever \( i_0 < \ldots < i_n \).

5. Like in Remark 10.10.

**Proposition 11.7** Let \( p(x) \in S(A) \) and assume there is a global extension \( p \in S(\mathcal{C}) \) of \( p \) that does not Lascar split over \( A \). For any \( c, d \models p \) the following are equivalent:

1. \( c \equiv_A d \)

2. For some Lascar \( A \)-complete \( B \) there is a non Lascar-splitting extension \( q(x) \in S(B) \) of \( p \) and some \( (a_i : i < \omega) \) such that both \( c^-(a_i : i < \omega) \) and \( d^-(a_i : i < \omega) \) realize \( q^{(\omega)}_A \upharpoonright A \).

3. \( \models n c^A_3(c, d) \)

**Proof:** 1 \( \Rightarrow \) 2. Choose \( B \supseteq A \) Lascar complete over \( A \). By point 1 of Lemma 11.6, \( a \equiv_A b \) whenever \( a, b \models p \upharpoonright B \). We can assume \( c \models p \upharpoonright B \) and hence \( p \upharpoonright B \models Lstp(c/A) \) (otherwise we choose \( a \models p \upharpoonright B \) and some \( f \in Aut(\mathcal{C}/A) \) such that \( f(a) = c \), and we replace \( p \) and \( B \) by \( f(p) \) and \( f(B) \)). Now let \( q = p \upharpoonright Bcd \), let \( (a_i : i < \omega) \) be a realization of the power \( q^{(\omega)}_A \) and let \( \tilde{a} = (a_i : 0 < i < \omega) \). Since \( c \equiv_A d \) and \( tp(\tilde{a}/Bcd) = tp((a_i : i < \omega)/Bcd) \) does not Lascar-split over \( A \), \( c^A \equiv_A \tilde{d} \). By point 1 of Lemma 11.6, \( \tilde{a} \models q^{(\omega)}_A \upharpoonright Bcd a_0 \) (a type that does not Lascar-split over \( A \)) and therefore \( a_0 \equiv_A c \) implies \( a_0 \tilde{a} \equiv_A c \tilde{a} \). Since \( a_0 \tilde{a} \equiv q^{(\omega)}_A \) we conclude that \( c \tilde{a} \) and \( \tilde{d} \tilde{a} \) realize \( q^{(\omega)}_A \upharpoonright A \).

2 \( \Rightarrow \) 3. Clear since (By point 4 Lemma 11.6) any realization of \( q^{(\omega)}_A \upharpoonright A \) is \( A \)-indiscernible.

3 \( \Rightarrow \) 1 is obvious.

**Definition 11.8** (Reminding) For any given length of tuples, for any set \( A \), there is a least bounded type-definable over \( A \) equivalence relation \( E_{KP_A} \), the Kim-Pillay relation. It is defined by \( E_{LA} \), the Lascar relation, which is the least bounded \( A \)-invariant equivalence relation. If \( E_{LA} \) is type-definable, then \( E_{LA} = E_{KP_A} \). In any case, \( E_{LA} \) is equality of Lascar strong type over \( A \). Similarly, \( E_{KP_A} \) is equality of \( KP \)-type over \( A \). We write \( E_{KP_A}(a, b) \Leftrightarrow a \equiv_A b \). The \( KP \)-type over \( A \) of a tuple \( a \) is \( tp(a/bdd(A)) \), where \( bdd(A) \) is the class of all hyperimaginaries that have a bounded \( A \)-orbit, that is, \( a \equiv_A b \) iff \( E(a, b) \) for all bounded \( A \)-type-definable equivalence relation \( E \) iff \( tp(a/bdd(A)) = tp(b/bdd(A)) \).

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This can be proved directly if one defines \( p^{(n)}_A \) by left nesting, since this would be the type of each subtuple of length \( n \).
Corollary 11.9 Assume $T$ has NIP and $p(x) \in S(A)$ does not fork over $A$. For any $c, d \models p$,
\[ c \equiv_A d \text{ if and only if } \models \psi^2_A(c, d) \]
Hence, in $p$, equality of Lascar strong type over $A$, $\equiv_A$, is type definable over $A$ and coincides with $\equiv_A$, equality of KP-type over $A$.

Proof: If $p$ does not fork over $A$, $p$ has a global nonforking extension $p$. By Proposition 9.6, $p$ does not Lascar-split over $A$. The rest follows from Proposition 11.7.

Proposition 11.10 Assume $T$ has NIP. Let $p_1, p_2 \in S(\mathcal{C})$ be global types that do not Lascar-split over $A$ and assume $p_1 \models A = p_2 \models A$. Let $B \supseteq A$ be Lascar-complete over $A$, let $p = p_1 \upharpoonright B$ and let $a = (a_i : i < \omega) \models p^{(\omega)}_A$. If $p_1 \upharpoonright Aa = p_2 \upharpoonright Aa$, then $p_1 = p_2$.

Proof: We claim that if $\alpha' = (a_i : i < \alpha)$ is an $A$-indiscernible sequence extending $a$, then $a'c$ is also $A$-indiscernible for any $c \models p_1 \upharpoonright Aa'$ or $c \models p_2 \upharpoonright Aa'$. Consider the case $c \models p_1 \upharpoonright Aa'$. Assume $i_0 < \ldots < i_n < \alpha$, $\psi(x_0, \ldots, x_n, y) \in L(A)$ and $\models \psi(a_{i_0}, \ldots, a_{i_n}, c)$. Then $\psi(a_{i_0}, \ldots, a_n, y) \models p_1$. Since $p_1$ does not Lascar-split over $A$ and $a_0, \ldots, a_n \models A a_0, \ldots, a_n, y \models p_1$. Then $a_n + 1 \models c \models p_1 \upharpoonright Aa$. The case $c \models p_1 \upharpoonright Aa'$ is similar but one needs the assumption $p_1 \upharpoonright Aa = p_2 \upharpoonright Aa$.

Now assume $\varphi(x, y) \in L, \varphi(x, b) \models p_1$ and $\neg \varphi(x, b) \models p_2$. Construct $(c_1 : i < \omega)$ in such a way that $c_2 \models p_1 \upharpoonright Aa$ and $c_{2n+1} \models p_2 \upharpoonright Aa$. Note that $a$ is $A$-indiscernible. By the claim $a^\omega(c_1 : i < \omega)$ is also $A$-indiscernible. Since $\models \varphi(a_2, b)$ and $\models \varphi(a_2, b)$ we see that $\text{alt}(\varphi) = \infty$, contradicting NIP of $T$.

Definition 11.11 We say that $p(x) \in S(B)$ KP-splits over $A \subseteq B$ if there are tuples $a, b \in B$ and $\varphi(x, y) \in L$ such that $\varphi(x, a) \in p$, $\neg \varphi(x, b) \in p$, and $a \equiv_A b$. Note that Lascar-splitting implies KP-splitting and KP-splitting implies splitting. Note also that a global type $p$ does not KP-split over $A$ if and only if it is bdd$(A)$-invariant, that is, $p^f = p$ for all $f \in \text{Aut}(\mathcal{C}/\text{bdd}(A))$.

Lemma 11.12 Assume $T$ has NIP. Let $f \in \text{Aut}(\mathcal{C}/A)$ an let $p$ be a global type that does not Lascar-split over $A$. If for each $n < \omega$, for each $a \models p^{(n)} \upharpoonright A$, $a \equiv_A f(a)$, then $p^f = p$.

Proof: Clearly, $p \upharpoonright A = p^f \upharpoonright A$. Choose $B$ Lascar-complete over $A$ and let $p = p \upharpoonright B$ and $a = (a_i : i < \omega) \models p^{(\omega)}_A$. By Proposition 11.10 it will suffice to prove $p \upharpoonright Aa = p^f \upharpoonright Aa$. By Corollary 11.9, if $a_{<n} \equiv_A f(a_{<n})$ for all $n < \omega$, then $a \equiv_A f(a)$. Let $\varphi(x, y) \in L(A)$ and assume $\varphi(x, a) \in p$. Since $p$ does not Lascar-split over $A$ and $a \equiv_A f^{-1}(a)$, $\varphi(x, f^{-1}(a)) \in p$. Then $\varphi(x, a) \in p^f$. It follows that $p \upharpoonright Aa = p^f \upharpoonright Aa$.

Proposition 11.13 Assume $T$ has NIP. Let $B$ be Lascar-complete over $A$ and $p(x) \in S(B)$. Then $p$ Lascar-splits over $A$ if and only if $p$ KP-splits over $A$.

Proof: It is enough to check that a type does not KP-split if it does not Lascar-split and it is enough to consider the case of a global type $p$. Assume $p$ does not Lascar-split over $A$, and let $f \in \text{Aut}(\mathcal{C}/\text{bdd}(A))$. We can check that $p^f = p$ using Lemma 11.12. since $p^{(n)}_A$ does not Lascar-split over $A$ and $a$ and $f(a)$ are realizations of $p^{(n)}_A \upharpoonright A$ such that $a \equiv_A f(a)$, and then, by Proposition 11.7, $a \equiv_A f(a)$.
12 Morley sequences

Definition 12.1 Assume the index set $I$ is linearly ordered by $<$. We say that $(a_i : i \in I)$ is a Morley sequence over $A$ if it is $A$-indiscernible and it is $A$-independent in the sense of forking: for all $i \in I$, $tp(a_i/Aa_{<i})$ does not fork over $A$. If $p(x)$ is the common type $tp(a_i/A)$ of all $a_i$, we say that $(a_i : i \in I)$ is a Morley sequence in $p$.

Remark 12.2 If $B$ is complete over $A$, and $p(x) \in S(B)$ does not split over $A$, then any $(a_i : i < \omega) \models p^{(\omega)\lambda} \upharpoonright A$ is a Morley sequence in $p \upharpoonright A$.

Lemma 12.3 Assume $T$ has NIP. Let $p_1, p_2$ be global types that do not fork over $A$. Let $I$ be infinite and linearly ordered by $<$. Let $a = (a_i : i \in I)$ be a Morley sequence in $p(x) = p_1 \upharpoonright A = p_2 \upharpoonright A$. If $a_i \models p_1 \upharpoonright Aa_{<i} = p_2 \upharpoonright Aa_{<i}$ for all $i \in I$, then $p_1 = p_2$.

Proof: Note first that, by compactness, we can assume that $I = \omega$ with its standard ordering. The rest is like the proof of Proposition 11.10.

Proposition 12.4 Assume $T$ has NIP and $I$ is a linearly ordered set without last element.

1. If $a = (a_i : i \in I)$ is a Morley sequence in $p(x) \in S(A)$, then there is a unique global type $\lambda \models p$ such that $\lambda$ does not fork over $A$ and $a_i \models \lambda \upharpoonright Aa_{<i}$ for all $i \in I$. Moreover $\lambda \upharpoonright A = Av(a/Aa)$.

2. If $p \in S(\mathcal{C})$ does not fork over $A$, there is a Morley sequence $(a_i : i < \omega)$ in $p = p \upharpoonright A$ whose associated global type as in the previous point is $p$.

Proof: 1. If $p_i(x) = tp(a_i/Aa_{<i})$, then $\bigcup_{i \in I} p_i(x) = Av(a/Aa)$ does not fork over $A$ and therefore it has a global nonforking (over $A$) extension $p(x)$. Then $p(x)$ does not Lascar-split over $A$ and $a_i \models p \upharpoonright Aa_{<i}$ for all $i \in I$. Uniqueness follows from Lemma 12.3.

2. Let $M \supseteq A$ be a model. Then $p$ does not split over $M$. Choose $(a_i : i < \omega)$ in such a way that $a_i \models p \upharpoonright Ma_{<i}$. Then $(a_i : i < \omega)$ is $M$-indiscernible and hence $A$-indiscernible. It is therefore a Morley sequence in $p \upharpoonright A$.

Proposition 12.5 Assume $I$ is a linearly ordered set. If $a = (a_i : i \in I)$ is $A$-independent, then $a \nsubseteq_A A$.

Proof: We may assume $I$ is finite. The we can proceed by induction in $|I|$ using Corollary 9.13.

Proposition 12.6 Assume $T$ has NIP and $p$ does not split over $A$. The sequence $a = (a_i : i < \omega)$ is a Morley sequence over $A$ with global type $p$ if and only if $a \models p^{(\omega)\lambda} \upharpoonright A$.

Proof: Let $M \supseteq A$ be Lascar-complete over $A$ and let $p = p \upharpoonright M$. Note that $p^{(\omega)\lambda} \upharpoonright \mathcal{C} = (p(\mathcal{C})^{(\omega)\lambda} = p^{(\omega)}$ and hence $p^{(\omega)\lambda} \upharpoonright A = p^\omega \upharpoonright A$. Now, assume $a = (a_i : i < \omega)$ is a Morley sequence over $A$ with global type $p$ and let $b = (b_i : i < \omega) \models p^{(\omega)\lambda}$. By induction it is easy to see that $a_{<i} \equiv A b_{<i}$ for all $i < \omega$. The other direction follows from Remark 12.2.

Question 12.7 Does Proposition 12.6 hold assuming only that $p$ does not fork over $A$? The problem is with the direction from left to right.
13 Special sequences and eventual types

Definition 13.1 An infinite sequence \( a = (a_i : i \in I) \) is A-special if it is A-indiscernible and every \( b = (b_i : i \in I) \equiv_A (a_i : i \in I) \) can be extended to an A-indiscernible sequence \( b^-c \) by adding a new tuple \( c \) such that also \( a^-c \) is A-indiscernible.

Lemma 13.2 Let \( a, b \) be infinite A-indiscernible sequences and assume \( a \) and \( b \) have the same Ehrenfeucht-Mostowski set over \( A \). Then \( a \) is A-special if and only if \( b \) is A-special.

Proof: Let \( \Phi(x : i < \omega) \) be the Ehrenfeucht-Mostowski set over \( A \) of the A-indiscernible sequence \( a = (a_i : i \in I) \). By definition, for any \( \varphi(x_1, \ldots, x_n) \in L(A), \varphi \in \Phi \) if and only if \( \models \varphi(a_1, \ldots, a_n) \) for all (for some) \( i_1 < \ldots < i_n \) in \( I \). Assume \( b = (b_j : j \in J) \) is A-indiscernible, with the same Ehrenfeucht-Mostowski set, and assume \( b \equiv_A b' = (b'_j : j \in J) \). Let \( \varphi(x_1, \ldots, x_n, x_{n+1}) \in \Phi \) and \( j_1 < \ldots < j_n \) in \( J \). It is enough to check that \( \varphi(b_{j_1}, \ldots, b_{j_n}, x_{n+1}) \land \varphi(b'_{j_1}, \ldots, b'_{j_n}, x_{n+1}) \) is consistent. Let \( p(x_1, \ldots, x_n) = \text{tp}(b_j, \ldots, b_{j_n}/A) \) and choose \( i_1 < \ldots < i_n \) in \( I \). Since \( a \) is A-special and also \( p = \text{tp}(a_1, \ldots, a_n/A), p(x_1, \ldots, x_n) \vdash \exists x_{n+1}(\varphi(a_1, \ldots, a_n, x_{n+1}) \land \varphi(x_1, \ldots, x_n, x_{n+1})) \). It follows that \( p(x_1, \ldots, x_n) \vdash \exists x_{n+1}(\varphi(b_{j_1}, \ldots, b_{j_n}, x_{n+1}) \land \varphi(x_1, \ldots, x_n, x_{n+1})) \) and therefore \( \models \exists x_{n+1}(\varphi(b_{j_1}, \ldots, b_{j_n}, x_{n+1}) \land \varphi(b'_{j_1}, \ldots, b'_{j_n}, x_{n+1})) \).

Lemma 13.3 Assume \( T \) has NIP and \( a \) is A-special. Let \( n < \omega \) and suppose \( a_i \equiv_A a \) for all \( i < n \). Then for some tuple \( b \) all the sequences \( a^-i(b) \) are A-indiscernible.

Proof: Since \( a \) is A-special, we can construct a sequence \( d = (d_i : i < \omega) \) such that \( a^-d \) is A-indiscernible and for each \( i < n \), if \( d' = (d_{j+n} : j < \omega) \), then \( a_i^-d \) is A-indiscernible. Now if \( b \models \text{Av}(d/A(a_i : i < n)) = \text{Av}(d'/A(a_i : i < n)) \), then \( a_i^-d \) is A-indiscernible for all \( i < n \).

Proposition 13.4 Assume \( T \) has NIP. If \( a = (a_i : i \in I) \) is A-special then for any family \( (b^i : i < \lambda) \) where \( b^i \equiv_A a \), for any linearly ordered set \( J \) there is some sequence \( c = (c_j : j \in J) \) such that every \( b^i^-c \) is A-indiscernible.

Proof: Notice that if \( a \) is A-special and we extend it to an A-indiscernible sequence adding finitely many tuples \( c_1, \ldots, c_n \) at the end of \( a \), then the extended sequence \( a^-c \) has the same Ehrenfeucht-Mostowski set over \( A \) and it is therefore A-special. Using this and compactness, it is easily seen that it suffices to apply Lemma 13.3.

Definition 13.5 Assume \( I \) is an infinite linearly ordered set. Let \( a = (a_i : i \in I) \) be A-special. The eventual type of \( a \) over \( B \supseteq A \), \( \text{Ev}_A(a/B) \), is the set of formulas \( \varphi(x) \in L(B) \) such that for any \( b \equiv_A a \) there is some \( \omega \)-sequence \( c \) such that \( b^-c \) is A-indiscernible and \( \varphi(x) \in \text{Av}(b^-c/B) \). Usually \( A \) is clear from the context, and we can omit it and write \( \text{Ev}(a/B) \).

Remark 13.6 If \( a \) is A-special, and \( C \supseteq B \supseteq A \), then \( \text{Ev}(a/B) \subseteq \text{Ev}(a/C) \).

Remark 13.7 Let \( \varphi(x, y) \in L \) and assume \( \text{alt}(\varphi) \) is finite. For any tuple \( b \), for any set \( A \), for any Ehrenfeucht-Mostowski set \( \Phi \) we may choose the least \( k_{\varphi, b} < \omega \) such that in any A-indiscernible infinite sequence with Ehrenfeucht-Mostowski set \( \Phi \), \( \varphi(x, b) \) has at most \( k_{\varphi, b} \) alternations. This number can always be realized in any order type of an infinite sequence: for any infinite linearly ordered set \( I \) there is an A-indiscernible sequence \( a = (a_i : i \in I) \) with Ehrenfeucht-Mostowski set \( \Phi \) such that \( \varphi(x, b) \) has \( k_{\varphi, b} \) alternations in \( a \).
Proof: By compactness. If \( p(x) = \text{tp}(b/A) \), \( \psi(x) \in p \), and \( i_1 < \ldots < i_{k_{\varphi,b}} \) it is enough to find some \( A\)-indiscernible sequence \( (a_i : i \in I) \) with Ehrenfeucht-Mostowski set \( \Phi \) and some \( c \models p \) such that \( \models \varphi(a_i,c) \leftrightarrow \neg \varphi(a_{i+1},c) \) for all \( j = 1, \ldots, k_{\varphi,b} - 1 \) and this is clearly possible since the formula

\[
\exists x(\psi(x) \land \bigwedge_{1 \leq j < k_{\varphi,b}} \varphi(x_j, x) \leftrightarrow \neg \varphi(x_{j+1}, x))
\]

belongs to \( \Phi \).

\( \square \)

Lemma 13.8 Assume \( T \) has NIP and let \( I \) be linearly ordered, without last element. Let \( a = (a_i : i \in I) \) be \( A \)-special and let \( B \supseteq A \). For any \( \varphi(x,y) \in L \), for any \( b \in B \), since \( \text{alt}(\varphi) < \infty \) we may choose the least \( k_{\varphi,b} < \omega \) such that in any \( a' \equiv_A a \), \( \varphi(x,b) \) has at most \( k_{\varphi,b} \) alternations. Then \( \varphi(x,b) \in \text{Ev}(a/B) \) if and only if there is some \( a'' \equiv_A a \) such that \( \varphi(x,b) \) has \( k_{\varphi,b} \) alternations in \( a' \) and \( \varphi(x,b) \in \text{Av}(a''/B) \).

Proof: Choose \( \varphi(x,b) \in \text{Ev}(a/B) \). Choose \( a' \equiv_A a \) with biggest possible \( k_{\varphi,b} \). There is some \( \omega \)-sequence \( c \) such that \( a'' = c \) is \( A \)-indiscernible and \( \varphi(x,b) \in \text{Av}(a''/c/B) \). By choice of \( a' \) (and Remark 13.7), \( \varphi(x,b) \in \text{Av}(a'/B) \). For the other direction, let \( a'' \equiv_A a \) such that \( \varphi(x,b) \) has \( k_{\varphi,b} \) alternations in \( a' \) and \( \varphi(x,b) \in \text{Av}(a''/B) \). Let \( d \equiv_A a \). Since \( a' \equiv_A d \) and \( d' \) is \( A \)-special, there is some \( \omega \)-sequence \( c \) such that \( a' \sim c \) and \( d' \) are \( A \)-indiscernible. Then \( \varphi(x,b) \in \text{Av}(a''/c/A) = \text{Av}(c/A) = \text{Av}(d'/c/A) \).

\( \square \)

Lemma 13.9 Assume \( T \) has NIP and let \( I \) be linearly ordered, without last element. If \( a = (a_i : i \in I) \) is \( A \)-special, \( \varphi(x) \in \text{Ev}(a/B) \) and \( \psi(x) \in \text{Ev}(a/B) \), then \( (\varphi(x) \land \psi(x)) \in \text{Ev}(a/B) \).

Proof: Choose \( a' \equiv_A a \) with maximal alternation number for \( \varphi(x) \), \( a'' \equiv_A a \) with maximal alternation number for \( \psi(x) \), and \( a''' \equiv_A a \) with maximal alternation number for \( (\varphi(x) \land \psi(x)) \). By Proposition 13.4, there is some \( \omega \)-sequence \( c \) such that \( a' \sim c \), \( a'' \sim c \), and \( a''' \sim c \) are indiscernible over \( A \). Since \( \varphi(x), \psi(x) \in \text{Av}(c/B) \), \( (\varphi(x) \land \psi(x)) \in \text{Av}(a''/c/B) \) and hence \( (\varphi(x) \land \psi(x)) \in \text{Ev}(a/B) \).

\( \square \)

Proposition 13.10 Assume \( T \) has NIP and let \( I \) be linearly ordered, without last element. For any \( A \)-special \( a = (a_i : i \in I) \), for any \( B \supseteq A \), \( \text{Ev}(a/B) \subseteq S(B) \).

Proof: Let \( \varphi(x,y) \in L \) and \( b \in B \). By Lemma 13.8, \( \varphi(x,b) \in \text{Ev}(a/B) \) or \( \neg \varphi(x,b) \in \text{Ev}(a/B) \). Now assume \( \varphi(x,b) \in \text{Ev}(a/B) \) and \( \neg \varphi(x,b) \in \text{Ev}(a/B) \). Choose some \( a' \equiv_A a \) with biggest possible alternation number \( k_{\varphi,b} \). Without loss of generality, \( \varphi(x,b) \) holds in a final segment of \( a' \). Since \( \neg \varphi(x,b) \in \text{Ev}(a/B) \), there is some \( \omega \)-sequence \( c \) such that \( a'' \sim c \) is \( A \)-indiscernible and \( \neg \varphi(x,b) \in \text{Av}(a''/c/A) \). By Remark 13.7, this contradicts the choice of \( k_{\varphi,b} \).

\( \square \)

Proposition 13.11 Assume \( T \) has NIP and let \( I \) be linearly ordered, without last element. Let \( a = (a_i : i \in I) \) be \( A \)-special. Then \( \text{Ev}(a/\mathfrak{C}) \) does not split over \( A \) and \( a_i \models \text{Ev}(a/A_{<i}) \) for all \( i \in I \).

Proof: Let \( \varphi(x,y) \in L(A) \) and let \( b \equiv_A b' \) be tuples such that \( \varphi(x,b) \in \text{Ev}(a/\mathfrak{C}) \). Choose \( c \equiv_A a \) with a maximal number of alternations of \( \varphi(x,b) \) and choose \( c' \) such that \( b \equiv_A b' \). Then \( c' \equiv_A a \) and it has a maximal number of alternations of \( \varphi(x,b') \). By Lemma 13.8, \( \varphi(x,b) \in \text{Av}(c/Ab) \). Hence \( \varphi(x,b') \in \text{Av}(c'/Ab') \) and by Lemma 13.8 \( \varphi(x,b') \in \text{Ev}(a/Ab') \).

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Now assume $\varphi(x_1, \ldots, x_n) \in L(A)$, $i_1 < \ldots < i_n < i$ and $\varphi(a_{i_1}, \ldots, a_{i_n}, x) \in \text{Ev}(a/Aa_{<i})$. By definition of eventual type, there is some $c = (c_j : j < \omega)$ such that $a^{-c}$ is $A$-indiscernible and $\varphi(a_{i_1}, \ldots, a_{i_n}, x) \in \text{Av}(a^{-c}/Aa_{<i})$. Then $\models \varphi(a_{i_1}, \ldots, a_{i_n}, c_j)$ for some $j$ and by indiscernibility $\models \varphi(a_{i_1}, \ldots, a_{i_n}, a_i)$. This shows that $a_i \models \text{Ev}(a/Aa_{<i})$.

\[ \square \]

**Corollary 13.12** Assume $T$ has NIP and let $I$ be linearly ordered, without last element. A sequence $a = (a_i : i \in I)$ is $A$-special if and only if there is a global type $p$ that does not split over $A$ and $a_i \models p \mid Aa_{<i}$ for all $i \in I$. The global type $p$ is $\text{Ev}(a/A)$. 

**Proof:** From left to right use Proposition 13.11 with $p = \text{Ev}(a/C)$. For the other direction, it is straightforward that $a$ is $A$-indiscernible. Assume $a' \equiv_A a$ and let $c \models p \mid Aa'$. Then $p \restriction a = \text{Av}(a/Aa)$ and $p \restriction a' = \text{Av}(a'/Aa')$, and hence $a^{-c} (c)$ and $a'^{-c} (c)$ are $A$-indiscernible.

\[ \square \]

**Corollary 13.13** (Strong Borel Definability) Assume $T$ has NIP and the global type $p$ does not split over $A$. For each $\varphi(x,y) \in L$, $\{b : \varphi(x,y) \in p\}$ is a finite boolean combination of $A$-type-definable subsets.

**Proof:** Let $a = (a_i : i < \omega)$ be a Morley sequence over $A$ with global type $p$. By Corollary 13.12 $p = \text{Ev}(a/C)$. Let $n_\varphi$ be the alternation number of $\varphi(x,y) \in L$. By Lemma 13.8, $\varphi(x,y) \in \text{Ev}(a/C)$ if and only if for some $n \leq n_\varphi$ there are $(a'_1, \ldots, a'_n) \models p^{(n)} \restriction A$ such that $\models \bigwedge_{1 \leq i < n} (\varphi(a'_i, b) \leftrightarrow \neg \varphi(a'_{i+1}, b))$ and $\models \varphi(a'_n, b)$ and there are not $(a'_1, \ldots, a'_{n+1}) \models p^{(n+1)} \restriction A$ such that $\models \bigwedge_{1 \leq i < n+1} (\varphi(a'_i, b) \leftrightarrow \neg \varphi(a'_{i+1}, b))$.

\[ \square \]

### 14 Weakly special sequences

**Definition 14.1** An infinite sequence $a = (a_i : i \in I)$ is weakly $A$-special if it is $A$-indiscernible and every $b = (b_i : i \in I) \equiv_A (a_i : i \in I)$ can be extended to an $A$-indiscernible sequence $b'\cdot c$ by adding an $\omega$-sequence $c$ such that also $a^{-c}$ is $A$-indiscernible.

**Lemma 14.2** Let $a, c$ be infinite sequences and assume $a^{-c}$ is $A$-indiscernible. If $a$ is weakly $A$-special, then also $a^{-c}$ is weakly $A$-special.

**Proof:** It is a modification of the proof of Lemma 13.2. Let $a = (a_i : i \in I)$ and $c = (c_j : j \in J)$, let $b = a^{-c} = (b_i : i \in I \cup J)$ and let $\Phi(x_i : i < \omega)$ be the Ehrenfeucht-Mostowski set over $A$ of the $A$-indiscernible sequence $a = (a_i : i \in I)$. Assume $b \equiv_A b' = (b'_j : j \in I \cup J)$. Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m) \in \Phi$ and $j_1 < \ldots < j_n$ in $I \cup J$. It is enough to check that $\varphi(b_{j_1}, \ldots, b_{j_n}, y_1, \ldots, y_m) \Leftrightarrow \varphi(b'_{j_1}, \ldots, b'_{j_n}, y_1, \ldots, y_m)$ is consistent. Choose $i_1 < \ldots < i_n$ in $I$. Notice that $a_{i_1}, \ldots, a_{i_n} \equiv_A b_{j_1}, \ldots, b_{j_n}$. Since $a$ is weakly $A$-special, whenever $a_{i_1}, \ldots, a_{i_n} \equiv_A d_1, \ldots, d_n$ then

$$\models \exists y_1, \ldots, y_m (\varphi(a_{i_1}, \ldots, a_{i_n}, y_1, \ldots, y_m) \Leftrightarrow \varphi(d_1, \ldots, d_n, y_1, \ldots, y_m)).$$

By using some $f \in \text{Aut}(C/A)$ sending $a_{i_1}, \ldots, a_{i_n}$ to $b_{j_1}, \ldots, b_{j_n}$, we see that whenever $b_{j_1}, \ldots, b_{j_n} \equiv_A d_1, \ldots, d_n$ then also

$$\models \exists y_1, \ldots, y_m (\varphi(b_{j_1}, \ldots, b_{j_n}, y_1, \ldots, y_m) \Leftrightarrow \varphi(d_1, \ldots, d_n, y_1, \ldots, y_m)).$$

In particular, $\models \exists y_1, \ldots, y_m (\varphi(b_{j_1}, \ldots, b_{j_n}, y_1, \ldots, y_m) \Leftrightarrow \varphi(b'_{j_1}, \ldots, b'_{j_n}, y_1, \ldots, y_m))$.

\[ \square \]
Fact 14.3 (Reminding) $KP$-equivalence is finitary in the following sense: if $a = (a_i : i \in I)$, and $b = (b_i : i \in I)$, then $a \equiv_{KP} b$ if and only if $a \upharpoonright I_0 \equiv_{KP} b \upharpoonright I_0$ for all finite $I_0 \subseteq I$.

Definition 14.4 Assume $I$ is an infinite linearly ordered set. Let $a = (a_i : i \in I)$ be weakly $A$-special. The eventual type of a over $B \supseteq A$, $Ev_A(a/B)$, is the set of formulas $\varphi(x) \in L(B)$ such that for any $b \equiv_{A} a$ there is some $\omega$-sequence $c$ such that $b^c$ is $A$-indiscernible and $\varphi(x) \in Av(b^c/B)$. We will see that this definition coincides with the older one if $a$ is $A$-special. As in the previous case, usually we omit $A$ and write $Ev(a/B)$.

Remark 14.5 Let $\varphi(x,y) \in L$ and assume $\text{alt}(\varphi)$ is finite. Let $b$ be an $n$-tuple, let any set $A$, and assume $a = (a_i : i \in I)$ is $A$-indiscernible. We may choose the maximal $k_{\varphi,b} < \omega$ such that for any sequence $c$ such that $a^c$ is $A$-indiscernible $\varphi(x,b)$ has $k_{\varphi,b}$ alternations. This number can always be realized in a sequence $a' \equiv_{A} a$: there is some $a' \equiv_{A} a$ such that $\varphi(x,b)$ has $k_{\varphi,b}$ alternations in $a'$.

Proof: Choose a model $M \supseteq A$. Let $p(x) = tp(b/M)$ and let $q(x_i : i \in I) = tp(a/M)$. If $\psi(x) \in p$, and $i_1 < \ldots < i_{k_{\varphi,b}} \in I$ then
\[
\{\psi(x) \wedge \bigwedge_{1 \leq j < k_{\varphi,b}} \varphi(x_{i_j},x) \implies \neg \varphi(x_{i_{j+1}},x)\} \cup q(x_i : i \in I)
\]
is consistent. Hence for some $b' \equiv_{M} b$, for some $a' \equiv_{M} a$, $\models \bigwedge_{1 \leq j < k_{\varphi,b}} \varphi(a'_{i_j},b') \iff \neg \varphi(a'_{i_{j+1}},b')$. Let $a''$ be such that $b'a' \equiv_{M} ba''$. Then $a \equiv_{A} a''$ and $\varphi(x,b)$ has $k_{\varphi,b}$ alternations in $a''$.

Lemma 14.6 Assume $T$ has NIP, and let $I$ be linearly ordered, without last element. Let $a = (a_i : i \in I)$ be weakly $A$-special and let $B \supseteq A$. For any $\varphi(x,y) \in L$, for any $b \in B$, since $\text{alt}(\varphi) < \omega$ we may choose the maximal $k_{\varphi,b} < \omega$ such that in some $a' \equiv_{A} a$, $\varphi(x,b)$ has $k_{\varphi,b}$ alternations. Then $\varphi(x,b) \in Ev(a/B)$ if and only if there is some $a' \equiv_{A} a$ such that $\varphi(x,b)$ has $k_{\varphi,b}$ alternations in $a'$ and $\varphi(x,b) \in Av(a'/B)$.

Proof: Assume $\varphi(x,b) \in Ev(a/B)$. Choose $a' \equiv_{A} a$ with biggest possible $k_{\varphi,b}$. There is some $\omega$-sequence $c$ such that $a'^c$ is $A$-indiscernible and $\varphi(x,b) \in Av(a'^c/B)$. By choice of $a'$ (and Remark 14.5), $\varphi(x,b) \in Av(a'/B)$. For the other direction, let $a' \equiv_{A} a$ such that $\varphi(x,b)$ has $k_{\varphi,b}$ alternations in $a'$ and $\varphi(x,b) \in Av(a'/B)$. Let $d \equiv_{A} a$. Since $a' \equiv_{A} a$ and $a'$ is weakly $A$-special, there is some $\omega$-sequence $c$ such that $a'^c$ and $d'^c$ are $A$-indiscernible. Then $\varphi(x,b) \in Av(d'^c/A) = Av(c/A) = Av(d'^c/A)$.

Proposition 14.7 Assume $T$ has NIP. Let $I$ be linearly ordered, without last element. For any weakly $A$-special $a = (a_i : i \in I)$, for any $B \supseteq A$, $Ev(a/B) \subseteq S(B)$.

Proof: Like Proposition 13.10 but using now Lemma 14.6 and Remark 14.5.

Remark 14.8 Assume $T$ has NIP. If $a$ is $A$-special, then its eventual type $Ev_s(a/B)$ as special sequence and its eventual type $Ev_{ws}(a/B)$ as weakly special sequence coincide.

Proof: By Proposition 14.7, since clearly $Ev_{s}(a/B) \subseteq Ev_{ws}(a/B)$.

Proposition 14.9 Assume $T$ has NIP, and let $I$ be linearly ordered without last element. Let $a = (a_i : i \in I)$ be weakly $A$-special. Then $Ev(a/C)$ does not Lascar split over $A$ and $a_i \models Ev(a/Aa_{<i})$ for all $i \in I$. 

Proof: Like the proof of Proposition 13.11 but using Lascar strong types.

Lemma 14.10 Assume $T$ has NIP, and let $I$ be linearly ordered. If $a = (a_i : i \in I)$ is a Morley sequence over $A$ and $b = (b_i : i \in I) \equiv_A (a_i : i \in I)$, then there is some tuple $c$ such that $a^-c$ and $b^-c$ are $A$-indiscernible.

Proof: We may assume $I$ does not have a last element. Let $f \in Aut(\mathfrak{C}/A)$ be such that $f(a) = b$. Let $p$ be the global type associated to $a$ as in Proposition 12.4 and let $q$ be the corresponding type for $b$. Then $p^f = q$. Since $p$ does not fork over $A$, it does not Lascar-split over $A$ and therefore $p^f = p$. Hence $p = q$. Let $c \equiv p \upharpoonright Aa$. Since $p \upharpoonright Aa = Av(a/Aa)$ and $p \upharpoonright Ab = Av(b/Ab)$, by By Remark 6.5 $a^-c$ and $b^-c$ are $A$-indiscernible.

Lemma 14.11 Assume $T$ has NIP, and let $I$ be linearly ordered. Assume $a = (a_i : i \in I)$ is an infinite Morley sequence over $A$. If $b = (b_i : i \in I) \equiv_A (a_i : i \in I)$ and $c$ is a tuple such that $a^-c$ and $b^-c$ are $A$-indiscernible, then $a^-c \equiv_A b^-c$.

Proof: Since $a^-c$ and $b^-c$ are again Morley sequences over $A$, by Proposition 12.5, $a^-c \subseteq A$ and $b^-c \subseteq A$, and by Corollary 11.9 it is enough to show that $a^-c \forall p \equiv_A b^-c$. As stated in Fact 14.3, it suffices to show that all finite subsequences have the same $KP$-type over $A$ and this is clear since we can find corresponding finite tuples in $a$ and $b$ with same $KP$-type over $A$.

Proposition 14.12 Assume $T$ has NIP, and let $I$ be linearly ordered. If $a = (a_i : i \in I)$ is a Morley sequence over $A$, then $a$ is weakly special over $A$.

Proof: Let $b \equiv_A a$. By Lemma 14.10 there is some tuple $c$ such that $a^-c$ and $b^-c$ are $A$-indiscernible. By Lemma 14.11, $a^-c \equiv_A b^-c$. Since $a^-c$ and $b^-c$ are Morley sequences over $A$, the process can be iterated and we can obtain an $\omega$-sequence $c$ such that $a^-c$ and $b^-c$ are $A$-indiscernible.

Corollary 14.13 Assume $T$ has NIP. If $a = (a_i : i \in I)$ is a Morley sequence over $A$, then the global type of $a$ is $Ev(a/\mathfrak{C})$.

Proposition 14.14 If $a = (a_i : i < \omega)$ is a Morley sequence over $A$ and $B \supseteq A$, there is a Morley sequence $b = (b_i : i < \omega)$ over $B$ such that $a \equiv_A b$.

Proof: Let $\alpha$ be the length of each $a_i$, let $\kappa = |B| + |T| + |a|$ and $\lambda = \beth(2^\kappa)$. Extend $a$ to an $A$-indiscernible sequence $(a_i : i < \lambda)$. It is also a Morley sequence over $A$. Construct inductively a sequence $(a_i' : i < \lambda)$ such that for all $i < \lambda$, $a_{<i} \equiv_A a_{<i}'$ and $a_i' \downarrow A Ba_{<i}'. To obtain $a_i'$ we choose some $f \in Aut(\mathfrak{C}/A)$ such that $f(a_{<i}) = a_{<i}'$. Since $p(x) = tp(a_i/Aa_{<i})$ does not fork over $A$, its conjugate $p'(x) \in S(Aa_{<i}')$ does not fork over $A$ and hence it has an extension $q(x) \in S(Ba_{<i}')$ which does not fork over $A$. We take as $a_i'$ a realization of $q$. Then $a_{<i}a_{<i} \equiv_A a_{<i}'a_{<i}'$ and $a_i' \downarrow A Ba_{<i}'$. There is a $B$-indiscernible sequence $b = (b_i : i < \omega)$ such that for each $n < \omega$ there are $i_0 < \ldots < i_n < \lambda$ such that $b_{i_0}, \ldots, b_n \equiv_B a_{i_0}', \ldots, a_{i_n}'$. Then $b_0, \ldots, b_n \equiv_A a_{i_0}', \ldots, a_{i_n}' \equiv_A a_{i_0}, \ldots, a_{i_n}$ and therefore $a \equiv_A b$. Since $a_i' \downarrow_A Ba_{<i}'$, we see that $b_n \downarrow_A Bb_{<n}$ and thus $b_n \downarrow_B b_{<n}$. This shows that $b$ is a Morley sequence over $B$.  

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Proposition 14.15 If \( a = (a_i : i < \omega) \) is a Morley sequence over \( A \) and \( B \supseteq A \), there is a Morley sequence \( b = (b_i : i < \omega) \) over \( B \) such that \( a \equiv_A b \). Assuming NIP we can obtain \( a \overset{Ls}{=} b \).

Proof: It is an elaboration of the proof of Proposition 14.14, so we only point out the modifications. We extend \((a_i : i < \omega)\) to the \( A \)-indiscernible sequence \((a_i : i < \lambda)\), we choose a model \( M \supseteq B \), and we construct inductively the sequence \((a'_i : i < \lambda)\) in such a way that \( a_{<\lambda} \equiv_A a'_{<\lambda} \) and \( a'_i \downarrow_A Ma'_{<\lambda} \). We explain how to obtain \( a'_i \). Since \( a_i \downarrow_A a_{<\lambda} \), \( \text{tp}(a_i/Aa_{<\lambda}) \) has an extension over \( \text{bdd}(Aa_{<\lambda}) \) that does not fork over \( A \). Since all extensions of \( \text{tp}(a_i/Aa_{<\lambda}) \) over \( \text{bdd}(Aa_{<\lambda}) \) are \( Aa_{<\lambda} \)-conjugate, no such extension forks over \( A \). Hence \( a_i \downarrow_A \text{bdd}(Aa_{<\lambda}) \) and, in particular, \( a_i \downarrow_A \text{bdd}(A)a_{<\lambda} \). By the induction hypothesis, \( a_{<\lambda} \equiv_A a'_{<\lambda} \) and hence \( a_{<\lambda} \equiv_{\text{bdd}(A)} a'_{<\lambda} \). Choose an automorphism \( f \in \text{Aut}(C/\text{bdd}(A)) \) sending \( a_{<\lambda} \) to \( a'_{<\lambda} \). If \( p(x) = \text{tp}(a_i/\text{bdd}(A)a_{<\lambda}) \), \( p^f \) has an extension over \( Ma'_{<\lambda} \) that does not fork over \( A \). We take as \( a'_i \) a realization of this extension.

Finally we obtain the \( M \)-indiscernible sequence \( b = (b_i : i < \omega) \). It is a Morley sequence over \( B \). For every \( n < \omega \), there are \( i_0 < \ldots < i_n < \lambda \) such that \( b_0, \ldots, b_n \equiv_M a'_{i_0}, \ldots, a'_{i_n} \). Therefore

\[
b_{i_0}, \ldots, b_n \overset{Ls}{=} A a'_{i_0}, \ldots, a'_{i_n} \equiv_A a_{i_0}, \ldots, a_{i_n} \overset{Ls}{=} A a_0, \ldots, a_n.
\]

Hence \( a \overset{Ls}{=} b \). If \( T \) is NIP, by Proposition 12.5, \( a \downarrow_A A \), and Corollary 11.9 gives \( a \overset{Ls}{=} b \).

\[
\square
\]

Theorem 14.16 Assume \( T \) has NIP. The following are equivalent for \( a = (a_i : i < \omega) \).

1. \( a \) is weakly special over \( A \).
2. \( a \) is \( A \)-indiscernible and there is a global type \( p \) that does not fork over \( A \) and \( a_i \models p \) for all \( i < \omega \).
3. \( a \) is a Morley sequence over \( A \).
4. For some Lascar-complete set \( B \) over \( A \) there is a type \( p(x) \in S(B) \) that does not fork over \( A \) and \( b \models p^{(\omega)\lambda} \), for some \( b \equiv_A a \).
5. For some Lascar-complete set \( B \) over \( A \) there is a type \( p(x) \in S(B) \) that does not fork over \( A \) and \( b \models p^{(\omega)\lambda} \), for some \( b \overset{Ls}{=} A a \).

Proof: \( 1 \Rightarrow 2 \). By Proposition 14.9.
\( 2 \Leftrightarrow 3 \). By Proposition 12.4.
\( 3 \Rightarrow 5 \). By Proposition 14.15.
\( 5 \Rightarrow 4 \). Clear.
\( 4 \Rightarrow 3 \). By Lemma 11.6.
\( 3 \Rightarrow 1 \). By Proposition 14.12.

Corollary 14.17 Assume \( T \) has NIP. If \( a = (a_i : i \in I) \) is a Morley sequence over \( A \), then for any family \( (b^i : i < \lambda) \) where \( b^i \overset{Ls}{=} A a \), for any linearly ordered set \( J \) there is some sequence \( c = (c_j : j \in J) \) such that every \( b^i \sim c \) is a Morley sequence over \( A \).

Proof: By Lemma 14.11 and compactness.
Example 14.18 1. A Morley sequence which is not special (in an $\omega$-stable theory). Let $T$ be the theory of an equivalence relation $E$ with exactly two classes, both infinite. Choose $(a_i : i < \omega)$, different elements in one $E$-class. It is Morley (over $\emptyset$) but not special.

2. Indiscernible sequences that are not Morley (again in an $\omega$-stable theory). Let $T$ be the theory of an equivalence relation $E$ with infinitely many classes, all infinite. Take $(a_i : i < \omega)$, a sequence of different elements in an $E$-class. It is indiscernible (over $\emptyset$) but not it is not Morley. If we take $(b_i : i < \omega)$ where each $b_i$ is in a different $E$-class, then it is special over $\emptyset$ (and hence Morley).

3. Eventual types and average types. Let $T$ be the theory of the dense linear order without endpoints. Let $a = (a_i : i < \omega)$ be a strictly increasing sequence and let $(X,Y)$ be the cut defined by $a$ in the monster model. The sequence is special over $\emptyset$. Then $\text{Av}(a/\mathcal{C})$ is the type of the cut $(X,Y)$ while $\text{Ev}(a/\mathcal{C})$ is the type $+\infty$ (the type of an element greater than every element of $\mathcal{C}$). Notice that if we choose $b \in Y$, then $a$ is $b$-special and $\text{Ev}(a/\mathcal{C})$ is now the type $b^-$ of the left part of the cut determined by $b$.

15 Generically stable types

Proposition 15.1 Assume $T$ has NIP and $p$ does not fork over $A$.

1. If $p$ is definable, then it is definable over $\text{acl}^{eq}(A)$.

2. If $p$ is finitely satisfiable in some $M \supseteq A$, then it is finitely satisfiable in every $M \supseteq A$.

Proof: 1. By Proposition 11.13, $p$ does not $KP$-split over $A$, that is, $p$ is $\text{bdd}(A)$-invariant. Let $\varphi(x,y) \in L$ and let $c \in \mathcal{C}^{eq}$ be the canonical parameter of a definition of $\{a : \varphi(x,a) \in p\}$. Since $c \in \text{bdd}(A)$ and $c$ is an imaginary, $c \in \text{acl}^{eq}(A)$.

2. Fix $N \supseteq A$ such that $p$ is finitely satisfiable in $N$ and let $M \supseteq A$. Let $\varphi(x,y) \in L$, and assume $\varphi(x,a) \in p$. Choose $N' \equiv_M N$ such that $\text{tp}(N'/Ma)$ coheirs from $M$. Since $p$ is $M$-invariant, $p$ is finitely satisfiable in $N'$. Then there is some $b \in N'$ such that $\models \varphi(b,a)$. It follows that for some $b' \in M$, $\models \varphi(b',a)$. \hfill $\square$

Proposition 15.2 If $p$ is finitely satisfiable in $M$ and it is definable over $M$ and the sequence $a = (a_i : i < \omega)$ is $M$-indiscernible and satisfies $a_i \models p \upharpoonright Ma_{<i}$ for all $i < \omega$, then $a$ is totally indiscernible over $M$.

Proof: It is enough to show that for every $n < \omega$ every permutation of $\{a_0, \ldots, a_n\}$ is elementary over $M$. Since a permutation is a product of transpositions of consecutive elements, it is enough to prove that for all $i < n$,

$$a_{<i}a_{i+1}a_{i+2}, \ldots, a_n \equiv_M a_{<i}a_{i+1}a_i, a_{i+2}, \ldots, a_n.$$  \hfill (1)

For this we will first prove that for all $i \leq n$,

$$a_i \models p \upharpoonright M a_{<i}a_{i+1} \ldots a_n$$  \hfill (2)

Let us check that (2) implies (1). Assume (2). Notice that $a_{n+1} \models p \upharpoonright M a_{<i}a_{j+1} \ldots a_n$ and hence

$$a_{<i}a_{i+1}a_{i+2}, \ldots, a_n \equiv_M a_{<i}a_{i+1}a_{i+2}, \ldots, a_n.$$
By indiscernibility over $M$:

$$a <_i a_i, a_{i + 1}, \ldots, a_n \equiv_M a <_i a_i, a_{i + 1}, \ldots, a_n.$$

Again by (2) and because $a_{n+1} \models p \mid Ma <_i a_{i+1}, \ldots, a_n$,

$$a <_i a_{i+1}, a_{i+1}, \ldots, a_n \equiv_M a <_i a_{i+1}, a_{i+1}, \ldots, a_n.$$  

Now we prove (2). Since $p$ is $M$-definable, $p \mid Ma <_i a_{i+1}, \ldots, a_n$ is the unique $M$-definable extension of $p \mid M$ over $Ma <_i a_{i+1}, \ldots, a_n$ and it is therefore the unique heir of $p \mid M$ over $Ma <_i a_{i+1}, \ldots, a_n$. We must check that $tp(a_i/\langle M_a <_i a_{i+1}, \ldots, a_n \rangle)$ is a heir of $tp(a_i/M)$ or, in other terms, that $tp(a_i/\langle a_{i+1}, \ldots, a_n/M_a \rangle)$ coheirs from $M$. We start checking that

$$tp(a_i, \ldots, a_n/M_a)$$  

coheirs from $M$. (3)

Let $\varphi(x_{i+1}, \ldots, x_n) \in L(Ma_{i+1})$ be such that $\models \varphi(a_{i+1}, \ldots, a_n)$. Since $p$ is finitely satisfiable in $M$ and $a_{i+1} \models pMa_{i+1}$, there is some $a''_n \in M$ such that $\models \varphi(a_{i+1}, \ldots, a_n-1, a''_n)$. By iteration we obtain $a''_i, \ldots, a''_n \in M$ such that $\models \varphi(a''_i, \ldots, a''_n)$.

Now we finish the proof checking that

$$tp(a_i, a_{i+1}, \ldots, a_n/M_a)$$  

coheirs from $M$. (4)

Let $\varphi(x_{i+1}, x_{i+1}, \ldots, x_n, x_i) \in L(M)$ be such that $\models \varphi(a_{i+1}, a_{i+1}, \ldots, a_n, a_i)$. By (3) there are $a'_i, \ldots, a'_n \in M$ such that $\models \varphi(a_i, a'_i, \ldots, a'_n, a_i)$. Since $tp(a_i/\langle M_a \rangle = p \mid M_a$ is definable over $M$, there is some $\theta(x_{i+1}, x_{i+1}, \ldots, x_n) \in L(M)$ such that for all $b_{i+1}, b_{i+1}, \ldots, b_n$ in $M_a$

$$\models \theta(b_i, \ldots, b_n)$$  

if and only if $\varphi(b_i, b_{i+1}, \ldots, b_n, x) \in tp(a_i/M_a)$. (5)

In particular

$$\models \theta(a_i, a'_i, \ldots, a'_n)$$  

if and only if $\varphi(a_i, a'_i, \ldots, a'_n, x) \in tp(a_i/M_a)$

and therefore $\models \theta(a_i, a'_i, \ldots, a'_n)$. It follows that

$$M \models \exists x_{i+1} \theta(x_{i+1}, a'_i, \ldots, a'_n)$$

and then there is some $a'_{i+1} \in M$ such that $\models \theta(a'_i, a'_i, \ldots, a'_n)$ and by (5)

$$\varphi(a'_i, a'_i, \ldots, a'_n, x) \in tp(a_i/M_a)$$

that is, $\models \varphi(a'_{i}, a'_i, \ldots, a'_n, a_i)$. 

\end{proof}

**Lemma 15.3** Let $a = (a_i : i < \omega)$ and $b = (b_i : i < \omega)$. If $a \equiv b$ is $A$-indiscernible and $a$ is totally indiscernible over $A$, then $a \equiv b$ is totally indiscernible over $A$. Moreover, if $T$ has NIP, then $Av(a \equiv b/\mathfrak{C}) = Av(a/\mathfrak{C})$.

**Proof:** Let $c, c'$ be finite subsequences of $a \equiv b$. Assume they have the same length and they do not contain repetitions. Find a subsequence $d$ of $a$ with the same order type as $c$ and a suptuple $d'$ of $d$ with the same order type as $c'$. Since $a \equiv b$ is $A$-indiscernible, $c \equiv_A d$ and $c' \equiv_A d'$ since $a$ is totally indiscernible over $A$, $d \equiv_A d'$. Hence $c \equiv_A c'$.

Assume now $T$ has NIP. By Proposition 6.2, if $c = (c_i : i < \omega)$ is totally indiscernible, then $\varphi(x) \in Av(c/\mathfrak{C})$ if and only if $\{i < \omega : \models \varphi(c_i)\}$ is infinite (equivalently, cofinite). Hence if $\varphi(x) \in Av(a/\mathfrak{C})$ then also $\varphi(x) \in Av(a \equiv b/\mathfrak{C})$. This shows $Av(a/\mathfrak{C}) \subseteq Av(a \equiv b/\mathfrak{C})$ and therefore $Av(a/\mathfrak{C}) = Av(a \equiv b/\mathfrak{C})$. 

\begin{proof}

\end{proof}
Proposition 15.4 Assume $T$ has NIP. Let $a = (a_i : i < \omega)$ be a Morley sequence over $A$. If $a$ is totally indiscernible, then $\text{Av}(a/\mathcal{C})$ does not fork over $A$ and it is the global type associated to $a$.

Proof: Let $\mathfrak{p}$ be the global type associated to $a$, that is, $\mathfrak{p}$ is the unique global type that does not fork over $A$ and satisfies $a_i \models \mathfrak{p} \upharpoonright Aa_{<i}$ for all $i < \omega$. Since $a_i \models \text{Av}(a/Aa_{<i})$, we only need to show that $\text{Av}(a/\mathcal{C})$ does not fork over $A$. Let $f \in \text{Aut}(\mathcal{C}/A)$ and let $f(a) = a' = (a'_i : i < \omega)$. It suffices to show that $\text{Av}(a/\mathcal{C}) = \text{Av}(a'/\mathcal{C})$. Since $a \equiv_A a'$ and $a$ is weakly special over $A$, there is an $\omega$ sequence $b = (b_i : i < \omega)$ such that $a^{-}b$ and $a'^{-}b$ are $A$-indiscernible. It is clear that $\text{Av}(a^{-}b/\mathcal{C}) = \text{Av}(b/\mathcal{C}) = \text{Av}(a'^{-}b/\mathcal{C})$. By Lemma 15.3, $\text{Av}(a/\mathcal{C}) = \text{Av}(a^{-}b/\mathcal{C})$ and $\text{Av}(a'/\mathcal{C}) = \text{Av}(a'^{-}b/\mathcal{C})$.

□

Lemma 15.5 Assume $T$ has NIP, and assume $a = (a_i : i < \omega)$ is totally indiscernible over $A$. Then $\text{Av}(a/\mathcal{C})$ is definable over $a$: if $\varphi(x,y) \in L$, there is a number $n_{\varphi} < \omega$ such that for all $c$,

$$\varphi(x,c) \in \text{Av}(a/\mathcal{C}) \text{ if and only if } \models \bigwedge_{w \subseteq 2^{n_{\varphi}}, |w| = n_{\varphi}} \varphi(a_i,c)$$

Proof: The number $n_{\varphi}$ is given by Remark 6.3. In fact $n_{\varphi} = \text{alt}(\varphi) + 2$.

□

Proposition 15.6 Assume $T$ has NIP. Let $a = (a_i : i < \omega)$ be a Morley sequence over $A$ and let $\mathfrak{p}$ be its associated global type. If for each $\varphi(x,y) \in L$ there is a number $n_{\varphi} < \omega$ such that $\mathfrak{p}$ is definable over $a$ by

$$\varphi(x,c) \in \mathfrak{p} \text{ if and only if } \models \bigwedge_{w \subseteq 2^{n_{\varphi}}, |w| = n_{\varphi}} \varphi(a_i,c)$$

then for every model $M \supseteq A$, $\mathfrak{p}$ is definable over $M$ and it is finitely satisfiable in $M$.

Proof: Since $\mathfrak{p}$ is finitely satisfiable in $a$, it is finitely satisfiable in some model $M \supseteq A$. By Proposition 15.1, it is finitely satisfiable in every model $M \supseteq A$. Since $\mathfrak{p}$ is definable, by Proposition 15.1 it is definable over $\text{acl}^{20}(A)$. Hence it is definable over every $M \supseteq A$.

□

Definition 15.7 A global type $\mathfrak{p}$ is generically stable over $A$ if for some model $M \supseteq A$, $\mathfrak{p}$ is definable over $M$ and it is finitely satisfiable in $M$.

Theorem 15.8 If $T$ has NIP and $\mathfrak{p}$ does not fork over $A$, the following are equivalent:

1. $\mathfrak{p}$ is generically stable over $A$.
2. For every model $M \supseteq A$, $\mathfrak{p}$ is definable over $M$ and finitely satisfiable in $M$.
3. For every model $M \supseteq A$, every Morley sequence $(a_i : i < \omega)$ over $M$ with associated global type $\mathfrak{p}$ is totally indiscernible over $M$.
4. Every (some) realization of $\mathfrak{p}(\omega) \upharpoonright A$ is totally indiscernible over $A$.
5. For every $\varphi(x,y) \in L$ there is some number $n_{\varphi} < \omega$ such that for every (some) Morley sequence $(a_i : i < \omega)$ over $A$ with global type $\mathfrak{p}$, $\mathfrak{p}$ is definable over $a$ by

$$\varphi(x,c) \in \mathfrak{p} \text{ if and only if } \models \bigwedge_{w \subseteq 2^{n_{\varphi}}, |w| = n_{\varphi}} \varphi(a_i,c)$$
**Proof:** \(1 \iff 2\). By Proposition 15.1.

2 \(\Rightarrow\) 3. By Proposition 15.2.

3 \(\Rightarrow\) 4. Choose \(M \supseteq A\) Lascar-complete over \(A\) and let \(p = p \upharpoonright M\). Then \(p\) does not fork over \(A\) and \(p^{(\omega)} \setminus p^{(\omega)} \upharpoonright M\). If \(a \models p^{(\omega)} \setminus A\), then \(a \equiv_A b\) for some \(b \models p^{(\omega)} \setminus A\), a Morley sequence over \(M\). The associated global type of \(b\) is \(p\). By 3 \(b\) is totally indiscernible over \(M\). Hence \(a\) is totally indiscernible over \(A\).

4 \(\Rightarrow\) 5. By Proposition 15.4 and Lemma 15.5.

5 \(\Rightarrow\) 1. By Proposition 15.6. \(\square\)

**Proposition 15.9** Assume \(T\) has NIP. If \(p\) is \(A\)-invariant and generically stable over \(A\), then \(p \upharpoonright A\) is stationary.

**Proof:** Let \(q\) be a nonforking extension of \(p \upharpoonright A\). We will show that \(p\) and \(q\) have a common Morley sequence over \(A\) and from this, by Lemma 12.3, it will follow that \(p = q\).

Let \(a = (a_i : i < \omega) \models p^{(\omega)} \setminus A\) and let \(b \models q \upharpoonright A\). Then \(a\) is a Morley sequence over \(A\) with global type \(p\). By Theorem 15.8 and Proposition 15.4, \(p = \text{Av}(a/\mathfrak{C})\). We claim that for all \(i < \omega\),

\[b \equiv_{Aa_{<i}} a_i.\]

We prove it by induction on \(i\). It is clear for \(i = 0\), since \(p \upharpoonright A = q \upharpoonright A\).

Let \(\varphi(x_0, \ldots, x_{i+1}) \in L(A)\) and assume \(\models \varphi(a_0, \ldots, a_j, b)\). Then \(\varphi(a_0, \ldots, a_i, x) \in q\). If \(j \geq i\), then \(a_{<j} a_i \equiv_A a_{<j} a_j\) and since \(q\) does not Lascar-split over \(A\), \(\varphi(a_0, \ldots, a_{i-1}, a_j, x) \in q\), that is \(\models \varphi(a_0, \ldots, a_{i-1}, a_j, b)\). Since \(p = \text{Av}(a/\mathfrak{C})\), \(\varphi(a_0, \ldots, a_{i-1}, x, b) \in p\). By the induction hypothesis and \(A\)-invariance of \(p\), \(\varphi(a_0, \ldots, a_{i-1}, x, a_i) \in p\). Then \(\models \varphi(a_0, \ldots, a_{i-1}, a_{i+1}, a_i)\). Since \(a\) is totally indiscernible over \(A\), \(\models \varphi(a_0, \ldots, a_{i-1}, a_i, a_{i+1})\). By the claim we get \(\text{tp}(a_i/Aa_{<i}) = \text{tp}(b/Aa_{<i}) = q \upharpoonright Aa_{<i}\) and hence \(q\) is the global type associated over \(A\) to the Morley sequence \(a\). \(\square\)

**Theorem 15.10** Assume \(T\) has NIP and \(p\) is \(A\)-invariant. The following are equivalent:

1. \(p\) is generically stable over \(A\).

2. For every \(B \supseteq A\), \(p \upharpoonright B\) is stationary.

3. For every \(n \geq 1\), for every \(B \supseteq A\), \(p^{(n)} \upharpoonright B\) is stationary.

**Proof:** Note that if \(p\) is generically stable over \(A\), then it is generically stable over any \(B \supseteq A\). Hence 1 \(\Rightarrow\) 2 follows from Proposition 15.9.

1 \(\Rightarrow\) 3. Since \(p\) is \(A\)-invariant, \(p^{(n)}\) is \(A\)-invariant too. Notice that, by associativity of the product, \((p^{(n)})^{(m)} = p^{(n-m)}\) and hence any realization of \((p^{(n)})^{(\omega)} \upharpoonright A\) (after elimination of brackets) a realization of \(p^{(\omega)} \upharpoonright A\). By point 4 of Theorem 15.8, \(p^{(n)}\) is generically stable over \(A\). By the previous paragraph, 3 follows from 1.

2 \(\Rightarrow\) 1. Let \(a = (a_i : i < \omega)\) be a Morley sequence over \(A\) with global type \(p\). By Proposition 12.4, \(p \upharpoonright Aa = \text{Av}(a/Aa)\). By Remark 6.4, \(\text{Av}(a/\mathfrak{C})\) does not fork over \(B = Aa\).

By stationarity, \(p = \text{Av}(a/\mathfrak{C})\). By Lemma 15.5 and point 5 of Theorem 15.8, \(p\) is generically stable over \(A\). \(\square\)

**Definition 15.11** If \(p(x, y) \in S(A)\) we define \(p^{-1}(y, x)\) as the type \(\text{tp}(ba/A)\) for any \(ab \models p\). This is only well-defined when the separation of variables \(x, y\) is fixed. Note that \(p\) and \(p^{-1}\) share almost all model-theoretical properties. This notation extends the more familiar notation used for formulas: \(\varphi^{-1}(y, x)\) is the formula \(\varphi(x, y)\) with opposite separation of variables. Note that \(p^{-1} = \{\varphi^{-1} : \varphi \in p\}\).
Lemma 15.12 \footnote{Suggested by Anand Pillay} Assume $T$ has NIP. Let $B$ be $A$-complete and assume $p(x) \in S(B)$ does not split over $A$. If $(p \otimes_A p)^{-1} = p \otimes_A p$, then every realization of $p^{(\omega)}_\lambda$ is totally indiscernible over $B$.

Proof: It is enough to prove that for every $n < \omega$, for every permutation $\pi$ of $\{1, \ldots, n\}$, $(a_1, \ldots, a_n) \models p^{(n)}_\lambda$ if and only if $(a_{\pi(1)}, \ldots, a_{\pi(n)}) \models p^{(n)}_\lambda$. Since every such permutation is a composition of transpositions of consecutive elements and since the product of types is associative, it suffices to check that for all $n < \omega$, $(a_1, \ldots, a_n, a_{n+1}) \models p^{(n+1)}_\lambda$ if and only if $(a_1, \ldots, a_{n-1}, a_{n+1}, a_n) \models p^{(n+1)}_\lambda$. But this is clear since

$$(a_1, \ldots, a_n, a_{n+1}) \models p^{(n+1)}_\lambda \iff a_{n} \models p^{(n-1)}_\lambda \text{ and } (a_n, a_{n+1}) \models (p \otimes_A p)|_{ABa_{<n}},$$

and

$$(a_n, a_{n+1}) \models (p \otimes_A p)|_{ABa_{<n}} \iff (a_{n+1}, a_n) \models ((p \otimes_A p)|_{ABa_{<n}})^{-1},$$

and

$$(p \otimes_A p)|_{ABa_{<n}}^{-1} = (p \otimes_A p)^{-1}|_{ABa_{<n}} = (p \otimes_A p)|_{Ba_{<n}}.$$

\[\square\]

Theorem 15.13 Assume $T$ has NIP, $p$ is $A$-invariant and no type over $A$ forks over $A$. Then $p$ is generically stable over $A$ if and only if for all $n \geq 1$, $p^{(n)}_\lambda | A$ is stationary.

Proof: One direction follows from Theorem 15.10. Assume then the right hand side and let us check that $p$ is generically stable over $A$. Choose $B \supseteq A$-complete and let $p = p \upharpoonright A$. By Theorem 15.8 and Lemma 15.12 it suffices to prove that $(p \otimes_A p)^{-1} = p \otimes_A p$. We need some preparation. For $n < \omega$, let $p^{(-n)}_\lambda$ be the tp$(a_n, \ldots, a_1/A)$ for $(a_1, \ldots, a_n) \models p^{(n)}_\lambda$ and let

$$p^{(\omega)}_\lambda = \bigcup_{n < \omega} p^{(-n)}_\lambda(x_0, \ldots, x_n).$$

Then $p^{(\omega)}_\lambda \in S(A)$ and

$$(a_i : i < \omega) \models p^{(\omega)}_\lambda \iff a_i \models p|_{A}Ba_{>i} \text{ for all } i < \omega.$$

Note that if $(a_i : i < \omega) \models p^{(\omega)}_\lambda$, then $a_i \downarrow_{A} Ba_{>i}$ and $(a_i : i < \omega)$ is $B$-indiscernible. By assumption each $p^{(n)}_\lambda | A$ is stationary. Then $p^{(-n)}_\lambda | A$ is also stationary and it follows that $r(x_i : i < \omega) = p^{(\omega)}_\lambda | A$ is stationary. Its unique global nonforking extension is $p^{(\omega)}$.

We claim that every realization of $r(x_i : i < \omega)$ (with the increasing order of $\omega$) is $A$-special. Let $a = (a_i : i < \omega) \models p^{(\omega)}_\lambda$ be such a realization an assume $b = (b_i : i < \omega) \models \equiv A a$. Choose $c \models p$. By assumption tp$(ab/A)$ does not fork over $A$ and hence there are $a'b' \equiv_A ab$ such that $a'b' \downarrow_A Bc$. Since $r$ is stationary, tp$(a'/Bc) = p^{(\omega)}_\lambda | Bc$. If $i_1 < \ldots < i_n < \omega$, then $(a'_{i_n}, \ldots, a'_{i_1}) \models p^{(n)}_\lambda|_{ABc}$ and hence $(c, a'_{i_n}, \ldots, a'_{i_1}) \models p^{(n+1)}_\lambda$. Therefore $a'^-(c)$ is $B$-indiscernible (if $a'$ is considered a decreasing sequence with order type $\omega^*$ we would say $(c)^-(a')$ is indiscernible). Similarly, $b'^-(c)$ is $B$-indiscernible. Now choose $c'$ such that $abc' \equiv_A a'b'c$. Clearly, $a'^-(c')$ and $b'^-(c')$ are $A$-indiscernible.

By Corollary 13.12 there is a global type $q$ that does not split over $A$ and $a_i \models q | Aa_{<i}$ for all $i < \omega$. By assumption $p | A$ is stationary and hence $p = q$. Let $a = (a_i : i < \omega) \models p^{(\omega)}_\lambda$ and let $c \models p \upharpoonright Ba$. Since $p$ does not split over $B$, and $a_i \models p \upharpoonright Ba_{>i}$, and $c \models p \upharpoonright Ba$, then by Remark 7.3 the sequence $(\ldots, a_1, a_0, c)$ is $B$-indiscernible, that
is, \((c)^-a = (a_0, a_1, \ldots)\) is \(B\)-indiscernible. Since \(q\) does not split over \(B\) and \(a_i \models q \upharpoonright Ba_{<i}\) and \(c \models p \upharpoonright Ba\), again by Remark 7.3 the sequence \((c)^-a = (a_0, a_1, \ldots)\) is \(B\)-indiscernible. Hence \((c)^-a\) and \((c)^-a\) are \(B\)-indiscernible. It follows that \(a_0 a_1 \equiv_B a_1 a_0\). Thus \((p \odot_A p)^{-1} = tp(a_0 a_1 / B) = tp(a_1 a_0 / B) = p \odot_A p\).

**Theorem 15.14** Assume \(T\) has NIP and \(p\) is \(A\)-invariant. The following are equivalent:

1. \(p\) is generically stable over \(A\).

2. \(p = Av(a / \mathfrak{C})\) for every Morley sequence \(a = (a_i : i < \omega)\) over any \(M \supseteq A\) with global type \(p\).

3. \((p \odot q)^{-1} = q \odot p\) for all \(B\)-invariant \(q\), for all \(B\).

4. \((p \odot p)^{-1} = p \odot p\)

**Proof:** 3 \(\Rightarrow\) 4 is obvious and 4 \(\Rightarrow\) 1 follows from Lemma 15.12 and Theorem 15.8. 1 \(\Rightarrow\) 2 follows from Proposition 15.4 and Theorem 15.8.

2 \(\Rightarrow\) 3. Let \(q\) be \(B\)-invariant. Let \(\varphi(x, y) \in p \odot q\). We will check that \(\varphi^{-1}(y, x) \in q \odot p\). Choose a model \(M \supseteq AB\) complete over \(AB\) and such that \(\varphi(x, y) \in L(M)\). Choose a Morley sequence \(a = (a_i : i < \omega)\) over \(M\) with global type \(p\). By 2 \(p = Av(a / \mathfrak{C})\). Let \(b \models q \upharpoonright Ma\). Then

\[(a_i, b) \models p \upharpoonright M \upharpoonright B q \upharpoonright M = (p \odot q) \upharpoonright M \text{ for all } i < \omega.\]

In particular \(\models \varphi(a_i, b)\) for all \(i < \omega\). This implies \(\varphi(x, b) \in Av(a / \mathfrak{C}) = p\). Choose now \(c \models p \upharpoonright Mb\). Then

\[(b, c) \models q \upharpoonright M \upharpoonright A p \upharpoonright M = (q \odot p) \upharpoonright M\]

and \(\models \varphi(c, b)\), that is \(\models \varphi^{-1}(b, c)\). Therefore \(\varphi^{-1}(y, x) \in q \odot p\). \(\square\)

### 16 Extension bases

**Definition 16.1** A set \(A\) is an **extension base** if no type over \(A\) forks over \(A\). In other terms, every \(p(x) \in S(A)\) has a global nonforking extension.

**Lemma 16.2** If \(p\) is \(A\)-invariant, then \(p \upharpoonright A + p \upharpoonright bdd(A)\).

**Proof:** Choose \(a \models p \upharpoonright bdd(A)\). Assume now \(b \models p \upharpoonright A\). We claim that \(a \equiv_{bdd(A)} b\). Since \(a \equiv_A b\), there is some \(f \in \text{Aut}(\mathfrak{C} / A)\) such that \(f(a) = b\). Note that \(f\) fixes setwise \(bdd(A)\).

By \(A\)-invariance \(p^f = p\). Hence \(b = f(a) \models (p \upharpoonright bdd(A))^f = p \upharpoonright bdd(A)\). \(\square\)

**Lemma 16.3** If for every finite subtuple \(a'\) of \(a\) there is a global \(A\)-invariant extension of \(tp(a' / A)\), then there is also a global \(A\)-invariant extension of \(tp(a / A)\).

**Proof:** By compactness, since if \(p(x) = tp(a / A)\) it is enough to prove the consistency of

\[p(x) \cup \{\varphi(x, b) \leftrightarrow \varphi(x, c) : b \equiv_A c \text{ and } \varphi(x, y) \in L(A)\}\.

\(\square\)

**Lemma 16.4** Assume \(T\) has NIP, \(A = acl^{eq}(A)\) and \(c \in acl^{eq}(Aa)\). If \(tp(a / A)\) has a global \(A\)-invariant extension, then \(tp(ac / A)\) has a global \(A\)-invariant extension too.
Proof: Let $p(x) = \text{tp}(a/A)$ and let $p \supseteq p$ be a global $A$-invariant extension. Let $q(x, y) = \text{tp}(ac/A)$ and choose $q \supseteq q$, a global type such that $q \upharpoonright x = p$. We claim that $q$ is $A$-invariant. Choose $\delta(x, y) \in q$ such that for some $m < \omega$, $\delta(x, y) \vdash \exists y \delta(x, y)$.

We will prove that $q$ is $A$-invariant applying Lemma 11.12. It is enough to check that $q$ does not fork over $A$ and that for each $n < \omega$, $q^{(n)} \upharpoonright A$ is a Lascar strong type.

We first claim that $q$ does not fork over $A$. In order to check this, let $\varphi(x, y; z) \in L(A)$ and assume that $\varphi(x, y; b) \in q$ divides over $A$. There is an $A$-indiscernible sequence $(b_i : i < \omega)$ with $b = b_0$ such that $\{\varphi(x, y; b_i) : i \in \omega\}$ is inconsistent. Without loss of generality, $\varphi(x, y; z) \vdash \delta(x, y)$. Since $p$ does not fork over $A$, there is some $a'$ such that $\models \exists y \varphi(a', y; b_i)$ for all $i < \omega$. For each $i < \omega$, choose $e_i$ such that $\models \varphi(a', e_i)$. Since $\models \delta(a', e_i)$, for some infinite $I \subseteq \omega$, for all $i, j \in I$, $e_i = e_j$. Therefore if $j \in I$, then $\models \varphi(a', e_j, b_i)$ for all $i \in I$. By indiscernibility over $A$, $\{\varphi(x, y; b_i) : i < \omega\}$ is consistent, a contradiction.

Let $n < \omega$. Since $p^{(n)}$ is $A$-invariant, by Lemma 16.2 and Corollary 11.9, $p^{(n)} \upharpoonright A$ is a Lascar-strong type. We claim that $q^{(n)} \upharpoonright A$ is a Lascar strong type too. To begin with, we claim it gives rise only to finitely many Lascar strong types over $A$. Assume that, on the contrary,

$$\{\{\langle a_1^i, e_1^i \rangle, \ldots, \langle a_n^i, e_n^i \rangle \rangle : i < \omega\}$$

are realizations of $q^{(n)} \upharpoonright A$ with different Lascar strong type over $A$. Since $a_1^i, \ldots, a_n^i \equiv_A a_1^j, \ldots, a_n^j$, we may assume $a_1^i, \ldots, a_n^i = a_1^1, \ldots, a_n^1 = a_1, \ldots, a_n$ for all $i, j$. Since $e_1^i \equiv_{eq} a_1, e_1^j \equiv_{eq}$ for all $i, k \leq n$ and $i, k \neq n$.

Thus, $q^{(n)} \upharpoonright A$ has only finitely many classes on $q^{(n)} \upharpoonright A$. Since $q^{(n)}$ does not fork over $A$, by Corollary 11.9, $q^{(n)} \equiv_A A$ on $q^{(n)} \upharpoonright A$ is a bounded $A$-type-definable equivalence relation $E$. Let $b_1, \ldots, b_m$ be representatives of the different $E$-classes and for each two different $i, j \leq m$ choose a formula $\psi_i(x, y) \in E(x, y)$ such that $\models \varphi_i(a_i, a_j)$ and choose then some $\psi_i(x, y) \in E(x, y)$ such that

$$\psi_i(x, y) \land \psi_j(y, z) \land \psi_j(z, u) \vdash \varphi_i(x, u).$$

It is easy to check that $\psi(x, y) = \bigwedge_{i < j \leq m} \psi_i(x, y)$ defines $E$ on $q^{(n)} \upharpoon.right A$. We may assume $\psi(x, y) \in L(A)$ defines an equivalence relation $F$ with finitely many classes in the whole universe. Each $F$-class is interdefinable over $A$ with some element of acl$^q(A)$. Since $A = acl^q(A)$, this implies that $F$ (and hence also $q^{(n)} \equiv_A A$) has only one class in $q^{(n)} \upharpoon.right A$. \[ \square \]

**Proposition 16.5** Assume $T$ has NIP. The following are equivalent:

1. Every set $A$ is an extension base and $\equiv_A = \equiv_A$.

2. For any $A = acl^q(A)$, every $p(x) \in S_1(A)$ (in the home sort) has a global $A$-invariant extension.

3. For any $A = acl^q(A)$, every $p(x) \in S(A)$ has a global $A$-invariant extension.

**Proof:** 1 $\Rightarrow$ 2. Let $p(x) \in S_1(A)$, where $A = acl^q(A)$. Since $A$ is an extension base, there is a nonforking extension $p$ of $p$. Then $p$ does not Lascar-split over $A$. Since $A = acl^q(A)$ and $\equiv_A = \equiv_A$, $p$ does not split over $A$, that is, $p$ is $A$-invariant.

3 $\Rightarrow$ 1. Let $p(x) \in S(A)$ and let $q(x) \in S(acl^q(A))$ be some extension of $p$. By 3 there is an acl$^q(A)$-invariant global extension $p$ of $q$. Since $p$ does not fork over acl$^q(A)$, it does
not fork over \( A \). This shows that \( A \) is an extension base. Now assume \( a \overset{\text{eq}}{\equiv}_A b \), that is \( a \overset{\text{ac}(\mathcal{A})}{\equiv}_A b \). By \( 3 \) applied to \( \text{acl}^a(\mathcal{A}) \) and Lemma 16.2, \( a \overset{\text{ind}(\mathcal{A})}{\equiv}_A b \), that is \( a \overset{\text{ac}(\mathcal{A})}{\equiv} b \). By Corollary 11.9, \( a \overset{\text{eq}}{\equiv}_A b \).

2 \( \Rightarrow 3 \). By Lemma 16.3 it is enough to prove the result for finitary types \( p(x) \), and this can be done by induction on the length \( n \) of \( x \). Assume the result holds for types in \( n \) variables and let \( p(x_1,\ldots,x_{n+1}) \in \mathcal{S}(\mathcal{A}) \). Let \( M \supseteq A \) be a model complete over \( A \) and strongly \( \omega \)-homogeneous over \( A \). Let \( (a_1,\ldots,a_{n+1}) \models p \). Let \( e \) be a sequence of imaginaries enumerating \( \text{acl}^a(Ma_1,\ldots,a_n) \). By Lemmas 16.3 and 16.4 there is some \( e' \) such that \( \text{tp}(e'/M) \) does not split over \( A \) and extends \( \text{tp}(e/A) \). There are \( a_1',\ldots,a_n' \) such that \( a_1,\ldots,a_n,e \overset{A}{\equiv} a_1',\ldots,a_n',e' \). Choose \( B \) complete over \( M e' \). Since \( e' \overset{\text{ac}(b)}{\equiv} e \), by \( 2 \) and conjugation over \( A \), there is some \( a_{n+1}' \) such that \( a_{n+1}' e' \overset{A}{\equiv} a_{n+1}e \) and \( \text{tp}(a_{n+1}'/B) \) does not split over \( e' \).

We claim that \( q(x,y) = \text{tp}(a_{n+1}' e'/M) \) does not split over \( A \). To check this, consider some \( \varphi(x,y,z) \in L(\mathcal{A}) \) and some finite tuple \( b \in M \). Since \( \text{tp}(e'/M) \) does not split over \( A \) and \( M \) is strongly \( \omega \)-homogeneous over \( A \), for each \( \psi(y,z) \in L \), the set \( \{ b' \in M : \psi(y,z) \} \) is invariant under \( \text{Aut}(M/A) \). Hence also \( \{ b' \in M : \psi(y,z) \} \) is invariant under \( \text{Aut}(M/A) \). Since \( \text{tp}(a_{n+1}'/B) \) does not split over \( e' \), for all \( b' \in M \), \( e' b' \equiv e' b \) implies \( \varphi(x,y;b') \models q \Leftrightarrow \varphi(x,y;b') \models q \). If \( b' \in M \) and \( b \overset{A}{\equiv} b' \), then \( f(b) = b' \) for some \( f \in \text{Aut}(M/A) \). Hence \( e' b' \equiv e' b \) and therefore \( \varphi(x,y;b) \models q \Leftrightarrow \varphi(x,y;b') \models q \).

In particular, \( \text{tp}(a_1',\ldots,a_{n+1}'/M) \) does not split over \( A \) and it has a global \( A \)-invariant extension. Since \( a_1,\ldots,a_{n+1} \overset{A}{\equiv} a_1',\ldots,a_{n+1}', p(x_1,\ldots,x_{n+1}) \) has a global \( A \)-invariant extension too. \( \square \)

17 Abstract preindependence and independence relations

**Definition 17.1** Let \( \sqsubset \) be a ternary relation between sets. We consider a list of possible properties of \( \sqsubset \):

- **Invariance**: If \( A \sqsubset C \sqsubset B \) and \( f \in \text{Aut}(\mathcal{C}) \), then \( f(A) \sqsubset f(C) \sqsubset f(B) \).
- **Monotonicity**: If \( A \sqsubset C \sqsubset B, A' \subseteq A \), and \( B' \subseteq B \), then \( A' \sqsubset C \sqsubset B' \).
- **Right base monotonicity**: If \( A \sqsubset C \sqsubset B \) and \( C \subseteq D \subseteq B \), then \( A \sqsubset D \sqsubset B \).
- **Left base monotonicity**: If \( A \sqsubset C \sqsubset B \) and \( C \subseteq D \subseteq A \), then \( A \sqsubset D \sqsubset B \).
- **Right normality**: If \( A \sqsubset C \sqsubset B \), then \( A \sqsubset C \sqsubset CB \).
- **Left normality**: If \( A \sqsubset C \sqsubset B \), then \( AC \sqsubset C \sqsubset B \).
- **Right transitivity**: If \( C \subseteq B \subseteq D, A \sqsubset C \sqsubset B \), and \( A \sqsubset B \sqsubset D \), then \( A \sqsubset C \sqsubset D \).
- **Left transitivity**: If \( C \subseteq B \subseteq D, B \sqsubset C \sqsubset A \), and \( D \sqsubset B \sqsubset A \), then \( D \sqsubset C \sqsubset A \).
- **Symmetry**: If \( A \sqsubset C \sqsubset B \), then \( B \sqsubset C \sqsubset A \).
- **Right finite character**: If \( A \sqsubset C \sqsubset B \) for all finite \( B_0 \subseteq B \), then \( A \sqsubset C \sqsubset B \).
- **Left finite character**: If \( A_0 \sqsubset C \sqsubset B \) for all finite \( A_0 \subseteq A \), then \( A \sqsubset C \sqsubset B \).
**Strong finite character:** If \( A \nmid C B \), then there are finite tuples \( a \in A, b \in B \) and some formula \( \varphi(x,y) \in L(C) \) such that \( \models \varphi(a, b) \) and such that \( a' \nmid C B \) for all \( a' \models \varphi(x, b) \).

**Local character:** For every \( A \) there is a cardinal number \( \kappa(A) \) such that for any \( B \) there is some \( C \subseteq B \) such that \( |C| < \kappa(A) \) and \( A \nmid C B \).

**(Right) extension:** If \( A \nmid C B \) and \( B' \supseteq B \), then \( A' \nmid C B' \) for some \( A' \equiv_{BC} A \).

**Left extension:** If \( A \nmid C B \) and \( A' \supseteq A \), then \( A' \nmid C B' \) for some \( B' \equiv_{AC} B \).

**Anti-reflexivity:** If \( A \nmid C A \), then \( A \subseteq \acl(C) \).

**Right algebraicity:** If \( A \nmid C B \), then \( A \nmid C \acl(B) \).

**Left algebraicity:** If \( A \nmid C B \), then \( \acl(A) \nmid C B \).

**Base algebraicity:** If \( A \nmid C B \), then \( A \nmid C \acl(C) B \).

**Existence:** \( A \nmid C C \).

Let us call basic axioms to invariance, monotonicity, right base monotonicity, left transitivity, and left normality. A preindependence relation is a ternary relation \( \downarrow \) satisfying the basic axioms and strong finite character. An independence relation is a ternary relation \( \downarrow \) satisfying the basic axioms, left finite character, local character and extension. Note that invariance and extension imply right normality. Hence all independence relations satisfy right normality.

If \( A \nmid C B \) implies \( A \nmid^2 C B \) for all \( A, B, C \) we say that \( \nmid^1 \) is stronger than \( \nmid^2 \) and that \( \nmid^2 \) is weaker than \( \nmid^1 \).

**Fact 17.2** All independence relations are symmetric.

**Proof:** See [1], or [5], or [7].

**Definition 17.3** Given a ternary relation \( \downarrow \), we define the ternary relation \( \downarrow^* \) as follows:

\[
A \nmid^*_C B \text{ if and only if for all } B' \supseteq B \text{ there is some } A' \equiv_{BC} A \text{ such that } A' \nmid C B'.
\]

**Proposition 17.4**
1. If \( \downarrow \) is invariant, then \( \downarrow^* \) is invariant and stronger than \( \downarrow \).
2. If \( \downarrow \) satisfies monotonicity and invariance, then \( \downarrow^* \) satisfies extension.
3. Each basic axiom and also anti-reflexivity transfers from \( \downarrow \) to \( \downarrow^* \).
4. Assume \( \downarrow \) satisfies the basic axioms and left finite character. If \( \downarrow^* \) satisfies local character, then it is an independence relation.
5. Assume \( \downarrow \) satisfies monotonicity and invariance. Then, \( \downarrow = \downarrow^* \) if and only if \( \downarrow \) satisfies extension.
6. Assume \( \downarrow \) is invariant and satisfies monotonicity and strong finite character. Then \( \downarrow^* \) satisfies strong finite character.
7. If \( \downarrow \) satisfies left algebraicity, then \( \downarrow^* \) satisfies left algebraicity too.
8. If \( \downarrow \) satisfies monotonicity and invariance, \( \downarrow^* \) satisfies right algebraicity. If moreover \( \downarrow \) satisfies right base monotonicity, \( \downarrow^* \) satisfies also base algebraicity.

**Proof:** Everything (except 7 and 8) is from Adler [1]. Points 1–5 are also proved in [7]. 7 is straightforward. We prove now 6. Assume \( \downarrow \) has strong finite character and \( a \downarrow^* C B \). For some \( B' \supseteq B \), for all \( a' \equiv_{BC} a \), \( a' \downarrow^* C B' \). Let \( p(x) = \text{tp}(a/BC) \) and let

\[
\pi(x) = \{ \neg \varphi(x, b) : b \in B', \ \varphi(x, y) \in L(C), \ \text{and } a' \not\downarrow C b \text{ for all } a' \models \varphi(x, b) \}
\]

By strong finite character of \( \downarrow \), \( \pi(x) \cup p(x) \) is inconsistent and therefore for some \( \varphi(x, y) \in L(C) \), for some \( b \in B \), \( \varphi(x, b) \in p(x) \) and \( \pi(x) \cup \{ \varphi(x, b) \} \) is inconsistent. Note that \( \models \psi(a, b) \). We claim that for all \( a' \models \psi(x, b) \), \( a' \not\downarrow^* C b \). To check this, assume \( \models \psi(a', b) \) but \( a' \downarrow^* C b \). By definition of \( \downarrow^* \), there is some \( a'' \equiv_{C,b} a' \) such that \( a'' \not\downarrow C B' \). Then \( \models \psi(a'', b) \) and \( a'' \models \pi(x) \), a contradiction.

8. Assume \( a \downarrow^* C B \). By extension, there is some \( a' \equiv_{BC} a \) such that \( a' \not\downarrow^* C \text{acl}(B) \). Fix some \( f \in \text{Aut}(\mathcal{C}/BC) \) such that \( f(a') = a \). Since \( f(\text{acl}(BC)) = \text{acl}(BC) \), by invariance \( a \downarrow^* C \text{acl}(BC) \). By monotonicity \( a \downarrow^* C \text{acl}(B) \). On the other hand, by monotonicity and right base monotonicity, we conclude \( A \downarrow^*_{\text{acl}(C)} B \).

**Proposition 17.5** If \( \downarrow \) is a preindependence relation, then \( \downarrow^* \) is the weakest preindependence relation that satisfies extension and is stronger than \( \downarrow \).

**Proof:** By points 1, 2, 3, and 6 of Proposition 17.4, we know that \( \downarrow^* \) is a preindependence relation and satisfies extension. Note that if \( \downarrow_1 \) is stronger than \( \downarrow_2 \), then \( \downarrow_1^* \) is stronger than \( \downarrow_2^* \). Now assume \( \downarrow_1 \) is a preindependence relation with extension and it is stronger than \( \downarrow \). By point 5 of proposition 17.4, \( \downarrow_1 = \downarrow_1^* \) is stronger than \( \downarrow^* \). □

**Definition 16.6** \( \downarrow^f \) will denote nonforking independence: \( A \downarrow^f C B \) if and only if for every tuple \( a \in A \), \( \text{tp}(a/BC) \) does not fork over \( C \). In the previous chapters we have used \( \downarrow \) for this relation, but now \( \downarrow \) is used as an arbitrary ternary relation on sets. Nondividing independence can be defined similarly: \( A \downarrow^d C B \) if and only if \( \text{tp}(a/BC) \) does not divide over \( C \) for all tuples \( a \in A \).

**Fact 17.7** \( A \downarrow^d C B \) if and only if for any \( C \)-indiscernible sequence \( (b_i : i < \omega) \) with \( b_0 \in BC \) there is some \( AC \)-indiscernible sequence \( (b'_i : i < \omega) \equiv_{b_0} (b_i : i < \omega) \).

**Proof:** See, for instance, Chapter 4 of [7]. □

**Remark 17.8**
1. \( \downarrow^d \) is a preindependence relation and moreover it satisfies anti-reflexivity, right normality, existence, and left algebraicity.
2. \( (\downarrow^d)^* = \downarrow^f \).
3. \( \downarrow^f \) is a preindependence relation and moreover it satisfies anti-reflexivity, right normality, extension, and all algebraicity conditions.

**Proof:**
1. Left algebraicity of \( \downarrow^d \) follows from the fact that \( A \downarrow^d C B \Rightarrow \text{acl}(AC) \downarrow^d C B \), which can be easily checked using Fact 17.7.
2 is Proposition 12.14 of [7].

3. By Proposition 17.5 \( \downarrow^f \) is a preindependence relation and satisfies extension. The remaining points follow from Proposition 17.4. □
Fact 17.9 If $\downarrow$ is an independence relation, then $A \downarrow_C B \Rightarrow A \downarrow_C B$.

**Proof:** See Proposition 12.19 of [7].

Fact 17.10 If $T$ is simple, then $\downarrow^f = \downarrow^d$. Moreover, the following are equivalent:

1. $T$ is simple.
2. $\downarrow^f$ is an independence relation.
3. $\downarrow^d$ is an independence relation.
4. $\downarrow^f$ satisfies local character.
5. $\downarrow^d$ satisfies local character.
6. $\downarrow^f$ is symmetric.
7. $\downarrow^d$ is symmetric.
8. $\downarrow^f$ is right transitive.
9. $\downarrow^d$ is right transitive.

**Proof:** See Propositions 12.16 and 12.24 of [7].

Fact 17.11 $T$ is simple if and only if in $T$ there is an independence relation $\downarrow$ which satisfies the independence theorem over models: for any model $M$ for any $A, B \supseteq M$ such that $A \downarrow_M B$, if $a \downarrow_M A$, and $b \downarrow_M B$ and $a \equiv_M b$, then there is some $c$ such that $c \downarrow_M AB$, $c \equiv_A a$ and $c \equiv_B b$. Moreover, if $T$ is simple and $\downarrow$ is as indicated, then $\downarrow = \downarrow^d$.

**Proof:** See Theorem 12.21 of [7].

### 18 More preindependence relations

**Definition 18.1**

1. $A \downarrow^u_C B$ if and only if $\text{tp}(a/BC)$ is finitely satisfiable in $C$ for all tuples $a \in A$.
2. $A \downarrow^u_C B$ if and only if for all tuples $b_1, b_2 \in BC$, if $\models \text{nc}_C(b_1, b_2)$, then $\models \text{nc}_{AC}(b_1, b_2)$.
3. $A \downarrow^u_C B$ if and only if for each tuple $a \in A$ there is a global extension $p$ of $\text{tp}(a/BC)$ that does not Lascar-split over $C$.

**Proposition 18.2**

1. $\downarrow^u$ is a preindependence relation. Moreover it satisfies right normality and anti-reflexivity.
2. $\downarrow^u$ satisfies extension. Hence $(\downarrow^u)^* = \downarrow^u$ and it satisfies right and base algebraicity.
3. $A \downarrow^u_C B$ if and only if for every tuple $a \in A$ there is a sequence $b = (b_i : i \in I)$ in $C$ and some ultrafilter $U$ on $I$ such that $\text{tp}(a/BC) = \lim_U(b/BC)$. 

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Proof: 1 is clear.

2. By compactness, every type \( p(x) \in S(BC) \) finitely satisfiable in \( C \) can be extended to a complete type over \( BCD \) finitely satisfiable in \( C \).

3. Every \( p(x) \in S(BC) \) finitely satisfiable in \( C \) is in fact \( \lim_U (b/BC) \) for some sequence \( b \) of tuples in \( C \), for some ultrafilter \( U \) on \( I = p(x) \): the ultrafilter extends the set of all \( [\varphi] = \{ \psi \in p : \psi \equiv \varphi \} \) with \( \varphi \in p \) and the sequence is obtained by choosing some \( b_\varphi \models \varphi \) in \( C \) for every \( \varphi \in p \).

\[ \square \]

Remark 18.3 Notice that \( \models A \not\subseteq u \) for all \( A \neq \emptyset \). In stable \( T \), \( \models u_M = \models f_M \) for every model \( M \). In simple unstable \( T \), \( \models u_M \neq \models f_M \) for some model \( M \).

Proposition 18.4 \( \models s \) is a preindependence relation and satisfies right normality, left base monotonicity and left and base algebraicity.

Proof: Invariance, monotonicity, left and right normality, and left transitivity are straightforward.

Right base monotonicity. Assume \( A \not\subseteq s_C B \), and \( C \subseteq D \subseteq B \), and let us show that \( A \models s_D B \). Let \( b_1, b_2 \in BD = BC \) be such that \( \models nc_D(b_1, b_2) \) and let \( d \) enumerate \( D \). Then \( \models nc_C(b_1 d, b_2 d) \) and therefore \( \models nc_{AC}(b_1 d, b_2 d) \). It follows that \( \models nc_{AD}(b_1, b_2) \).

Strong finite character. Let \( A \not\subseteq s_C B \) and let \( b_1, b_2 \in BC \) be such \( \models nc_C(b_1, b_2) \) and \( \not\models nc_{AC}(b_1, b_2) \). For some tuple \( a \in A \), for some \( \theta(x, y, z) \in L(C) \), \( \theta(x, y, a) \in nc_{AC}(x, y) \) and \( \not\models \theta(b_1, b_2, a) \). We may assume that for every \( a' \), \( \theta(x, y, a') \) is a thick formula. If \( \models \models \not\models \theta(b_1, b_2, a') \), then \( a' \not\subseteq x_C b_1 b_2 \), because \( \theta(x, y, a') \in nc_{AC}(x, y) \).

Left base monotonicity: clear, since in the definition of \( \models s \) we may always assume that \( b_1, b_2 \in B \).

Finally, it is clear that \( A \not\subseteq s_C B \Rightarrow acl(AC) \not\subseteq s_C B \), and this implies left and base algebraicity.

\[ \square \]

Proposition 18.5 If \( M \supseteq C \) is \( \omega \)-saturated over \( C \), then the following are equivalent:

1. \( A \models s_C M \)
2. \( \text{tp}(a/M) \) does not strongly split over \( C \) for all tuples \( a \in A \).
3. \( \text{tp}(a/M) \) does not Lascar-split over \( C \) for all tuples \( a \in A \).

Proof: 1 \( \Rightarrow \) 2 is clear and does not need the assumption of \( \omega \)-saturation.

3 \( \Rightarrow \) 1. Assume \( b_0, b_1 \in M \) and \( \models nc_C(b_0, b_1) \). By \( \omega \)-saturation over \( C \), there is a \( C \)-indiscernible sequence \( (b_i : i < \omega) \) in \( M \). We claim that it is \( AC \)-indiscernible. Let \( a \in A \) be a tuple, let \( n < \omega \) and let \( i_0 < ... < i_n < \omega \). We must check that \( b_0, ..., b_n \equiv_{AC} b_{i_0}, ..., b_{i_n} \). But this is clear, since \( b_0, ..., b_n \equiv_{AC} b_{i_0}, ..., b_{i_n} \) and hence \( \models \varphi(a, b_0, ..., b_n) \leftrightarrow \varphi(a, b_{i_0}, ..., b_{i_n}) \) for all \( \varphi(x, y_0, ..., y_n) \in L(C) \).

2 \( \Leftrightarrow \) 3. By Remark 9.8.

\[ \square \]

Remark 18.6 Assume \( B \) is Lascar-complete over \( C \subseteq B \). The following are equivalent:

1. \( \text{tp}(a/B) \) does not Lascar-split over \( C \) for all tuples \( a \in A \).
2. $A \downarrow^1_C B$.

**Proof:** See Proposition 10.1.

**Proposition 18.7** $(\downarrow^*)^* = \downarrow^i$.

**Proof:** It is clear that $\downarrow^1$ is stronger than $(\downarrow^*)^*$. We prove $A(\downarrow^*)_C B \Rightarrow A \downarrow^i_C B$. Assume $A(\downarrow^*)_C B$ and choose a model $M \supseteq BC$ Lascar-complete over $C$ and $\omega$-saturated over $C$. There is some $A' \equiv_{BC} A$ such that $A' \downarrow^*_CM$. By Proposition 18.5 and Remark 18.6, $A' \downarrow^i_M$. It follows that $A \downarrow^i_C B$.

**Corollary 18.8** $\downarrow^i$ is a preindependence relation and it satisfies additionally extension, right-normality, anti-reflexivity, and all algebraicity conditions.

**Proof:** By propositions 18.7, 18.4, and 17.4.

**Proposition 18.9** $A \downarrow^u_C B \Rightarrow A \downarrow^i_C B \Rightarrow A \downarrow^f_C B$

**Proof:** $\downarrow^u$ has the extension property and a global type finitely satisfiable in $C$ does not split over $C$. Similarly, a global type does not fork over $C$ if it does not Lascar-split over $C$.

**Definition 18.10** Let $f$ be a function assigning a cardinal number to each cardinal number. We say that $\downarrow$ is bounded by $f$ if for all $C \subseteq B$ for every finitary type $p(x) \in S(C)$, there are at most $f(|T| + |C|)$ types $q(x) \in S(B)$ extending $p$ such that for any $a \models q$, $a \downarrow_C B$. We say that $\downarrow$ is bounded if it is bounded by some $f$.

**Proposition 18.11** $\downarrow^1$ is the weakest bounded preindependence relation that satisfies the extension axiom, and it is bounded by $f(\kappa) = 2^\kappa$.

**Proof:** For any finitary $p(x) \in S(C)$, the number of global types $p$ extending $p$ that do not Lascar-split over $C$ is bounded by $2^{2^{|T|+|C|}}$. Hence, $\downarrow^1$ is bounded by $f(\kappa) = 2^{2^\kappa}$. Now let $\downarrow$ be a bounded preindependence relation satisfying extension and assume $\downarrow^1$ is not weaker than $\downarrow$. There is a tuple $a$ and sets $C, B$ such that $a \downarrow_C B$ and $a \downarrow^1_B$. By extension, we may assume $B$ is a $\{C, |T|\}$-saturated model containing $C$. By Remark 18.6 and Proposition 18.5, $a \downarrow^*_B$. Hence, using saturation of $B$, there is a $C$-indiscernible sequence $b = (b_i : i < \omega)$ in $B$ such that for some $\varphi(x, y) \in L(C)$, $\models \varphi(a, b_0)$ and $\models \neg \varphi(a, b_1)$. In fact we obtain $i_1 < \ldots < i_n$ and $\psi(x, y_1, \ldots, y_n) \in L(C)$ such that $\models \psi(a, b_1, \ldots, b_n)$ and $\neg \psi(a, b_1, \ldots, b_n)$, but we may then assume $n < i_0$ and we can consider a derived $C$-indiscernible sequence of $n$-tuples of $b_0$’s giving the result. We may assume that $\models \varphi(a, b_i)$ for all $i \geq 2$ or $\models \neg \varphi(a, b_i)$ for all $i \geq 2$. Without loss of generality, we may assume we are in the second case. Note that $a \downarrow_C b$. Let $c = (c_i : i < \kappa)$ be a $C$-indiscernible sequence with same Ehrenfeucht-Mostowski set over $C$ as $b$. We claim that $p(x) = tp(a/C)$ has at least $\kappa$ extensions $q_i(x)$ over $C$ such that $\models a' \downarrow_C c$ for all $\models q_i$. Let $\pi(x)$ be the set of all formulas $\neg \psi(x)$ such that $\psi(x) \in L(C)$ and for all $\models \psi(x)$, $a' \downarrow_C c$. For each $i < \kappa$, $\pi(x) \cup p(x) \cup \{\varphi(x, c_j) \cup \{\neg \varphi(x, c_j) : j > i\}$ is consistent and we may choose a type $q_i(x) \in S(C)$ extending it. If $i < j$, then $q_i \neq q_j$ since $\neg \varphi(x, c_j) \in q_i$ while $\varphi(x, c_j) \in q_j$. If $a' \models q_i$, then, by strong finite character, $a' \downarrow_C c$. This contradicts boundedness of $\downarrow$.

**Proposition 18.12** The following are equivalent and they hold if $T$ has NIP.

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1. \( \downarrow^f \) is bounded.

2. \( \downarrow^f \) is bounded by \( f(\kappa) = 2^{2^\kappa} \).

3. \( \downarrow^f = \downarrow^i \).

**Proof:** Since \( \downarrow^i \) is stronger than \( \downarrow^f \), the conditions are equivalent by Proposition 18.11. By Proposition 9.6, 3 holds if \( T \) has NIP. \( \square \)

**Proposition 18.13** The following are equivalent.

1. \( T \) is stable.

2. \( T \) is simple and \( \downarrow^i = \downarrow^f \).

3. \( \downarrow^i \) has local character.

4. \( \downarrow^i \) is an independence relation.

5. \( \downarrow^i \) is symmetric.

**Proof:** 1 \( \Rightarrow \) 2. By Proposition 18.12, since stable theories are simple and have NIP.

2 \( \Rightarrow \) 3. Clear since in a simple theory \( \downarrow^f \) has local character (for instance, see Proposition 12.16 of [7]).

3 \( \Rightarrow \) 4. Clear, since by Corollary 18.8 \( \downarrow^i \) satisfies all the other conditions of independence.

4 \( \Rightarrow \) 5. By Fact 17.2.

5 \( \Rightarrow \) 3. Given \( A, B \) it is easy to find \( C \subseteq A \) of cardinality \( \leq |B| + |T| \) such that \( A \downarrow^u_C B \). By Proposition 18.9, \( A \downarrow^u_C B \) and by symmetry \( B \downarrow^C_A \).

4 \( \Rightarrow \) 1. By Proposition 18.11, \( \downarrow^i \) is bounded. Stable theories are characterized by the existence of a bounded independence relation. See, for instance, Theorem 12.22 of [7]. \( \square \)

**19  Algebraic independence**

**Definition 19.1** \( A \downarrow^a_C B \) if and only if \( acl(AC) \cap acl(BC) = acl(C) \).

**Proposition 19.2** \( \downarrow^a \) satisfies invariance, symmetry, transitivity, monotonicity, normality, finite character, local character, anti-reflexivity, algebraicity, existence, and extension. It satisfies all conditions of an independence relation except, perhaps, base monotonicity. Moreover it is weaker than \( \downarrow^i \).

**Proof:** Invariance, symmetry, transitivity, monotonicity, normality, finite character, anti-reflexivity, algebraicity and existence are easy to check.

Local character. Given \( A \) and \( B \), construct \( (C_i : i < \omega \) and \( (D_i : i < \omega \) as follows. Start with \( C_0 = D_0 = \emptyset \). Put \( D_{i+1} = acl(AC_i) \cap acl(B) \). For each \( d \in D_{i+1} \) choose a finite subset \( C_d \subseteq B \) such that \( d \in acl(C_d) \) and put \( C_{i+1} = \bigcup_{d \in D_{i+1}} C_d \). Then \( C = \bigcup_{i < \omega} C_i \) is a subset of \( B \) of cardinality \( \leq |A| + |T| \) and \( A \downarrow^a_C B \).
Extension. It is enough to prove that for all \( A, B, C \) there is some \( A' \equiv_C A \) such that \( A' \downarrow^a B \). Let \( a \) be an enumeration of \( \text{acl}(AC) \setminus \text{acl}(C) \). It suffices to show that for some \( a' \equiv_{\text{acl}(C)} a \), \( a' \cap \text{acl}(BC) = \emptyset \) since then we can obtain \( A' \) such that \( A'a' \equiv_{\text{acl}(C)} Aa \) and it follows that \( A' \downarrow^a B \). To obtain \( a' \) it is enough to prove that for any finite subtuple \( a_0 \) of \( a \) there is some \( a_0' \equiv_{\text{acl}(C)} a_0 \) such that \( a_0' \cap \text{acl}(BC) = \emptyset \), and this can be done by P.M. Neumann’s Lemma (see the Appendix) since \( a_0 \cap \text{acl}(C) = \emptyset \).

**Proposition 19.3** \( \downarrow^a \) satisfies base monotonicity if and only if the lattice of algebraically closed sets is modular. This is for all algebraically closed \( A, B, C \), if \( C \subseteq B \), then \( B \cap \text{acl}(AC) = \text{acl}((B \cap A)C) \).

**Proof:** Notice that if \( B \) is closed and contains \( C \), then \( \text{acl}(B \cap A)C \subseteq B \cap \text{acl}(AC) \). Now assume base monotonicity. Since \( A \downarrow_{A \cap B}^a B \), we get \( A \downarrow_{(A \cap B)C}^a B \) and therefore \( \text{acl}(AC) \cap B \subseteq \text{acl}((A \cap B)C) \). For the other direction, we assume modularity, \( A \downarrow_{C}^a B \) and \( C \subseteq D \subseteq B \). Then \( \text{acl}(AD) \cap B \subseteq \text{acl}((\text{acl}(B) \cap \text{acl}(A))D) \subseteq \text{acl}(\text{acl}(C)D) = \text{acl}(D) \). Hence \( A \downarrow_{D}^a B \).

**Fact 19.4** If \((\Omega, \text{cl})\) is a pregeometry, the ternary relation \( \downarrow^\text{dim} \) is defined on subsets of \( \Omega \) by

\[
A \downarrow_C \text{dim} B \iff \dim(A_0/C) = \dim(A_0/BC) \text{ for all finite } A_0 \subseteq A.
\]

It can also be defined by

\[
A \downarrow_C \text{dim} B \iff \text{every } X \subseteq A \text{ independent over } C \text{ is independent over } BC.
\]

It satisfies symmetry, transitivity, normality, monotonicity, base monotonicity, finite character, existence, and the following stronger form of local character: If \( A \) is finite, then for each \( B \) there is a finite \( C \subseteq B \) such that \( A \downarrow_C \text{dim} B \). Moreover \( A \downarrow_C \text{dim} \text{cl}(C) \) and the following version of anti-reflexivity holds: if \( A \downarrow_C \text{dim} A \), then \( A \subseteq \text{cl}(C) \). The pregeometry is modular if and only if

\[
A \downarrow_C \text{dim} B \iff \text{cl}(AC) \cap \text{cl}(BC) = \text{cl}(C).
\]

**Proof:** See [6].

**Proposition 19.5** Assume the algebraic closure operator \( \text{acl} \) has the exchange property and therefore defines a pregeometry in the universe. Let \( \downarrow^\text{dim} \) be\(^5\) the relation defined as in Fact 19.4. It satisfies invariance and all the properties stated in Fact 19.4. Moreover \( \downarrow^\text{dim} \) satisfies extension and strong finite character and hence it is an independence relation and also a preindependence relation. In this situation, \( \downarrow^\text{dim} = \downarrow^a \) if and only the pregeometry is modular.

**Proof:** We check the extension property, in fact in a strong sense that implies existence. Let \( a = (a_i : i < \alpha) \), and let \( C, B \) be sets. We prove inductively that for all \( \beta \leq \alpha \) there is some \( a'_{<\beta} \equiv_C a_{<\beta} \) such that \( a'_{<\beta} \downarrow_C \text{dim} B \). The limit case is clear since we extend the previous obtained sequences. Assume inductively this is the case for \( \beta \leq \alpha \) and let us consider the case of \( a_{\leq\beta} = a_{<\beta}a_\beta \). We have \( a'_{<\beta} \equiv_C a_{<\beta}a_\beta \) such that \( a'_{<\beta} \downarrow_C \text{dim} B \). Let \( b \) be

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\(^5\) The notation is \( \downarrow^9 \) in [2]
such that \( a' \leq b \equiv_A a \alpha \). It is enough to find \( b' \equiv_{C \setminus a} b \) such that \( b' \downarrow \dim B \). In other terms, we must check that for every element \( a \), for all sets \( B, C \) there is some \( a' \equiv_C a \), such that \( a \downarrow \dim B \). If \( a \in \text{acl}(C) \), we put \( a' = a \). If \( a \not\in \text{acl}(C) \), we may choose some \( a' \equiv_C a \) such that \( a' \not\in \text{acl}(BC) \), and it follows that \( a' \downarrow \dim B \).

Strong finite character. Since \( \downarrow \dim \) is symmetric, it is enough to check that the reverse of \( \downarrow \dim \) has the property. Assume \( A \not\subseteq_C B \). Then for some finite tuples \( a \in A \) and \( b = b_1, \ldots, b_n \in B \), \( b \) is algebraically independent over \( C \) but \( b_i \in \text{acl}(C, a, b_2, \ldots, b_n) \). Choose \( \varphi(x, y_1, \ldots, y_n) \in L(C) \) such that \( \models \varphi(a, b_1, \ldots, b_n) \) and such that \( b_i \in \text{acl}(C, a', b_2, \ldots, b_n) \) for every \( a' \) such that \( \models \varphi(a', b_1, \ldots, b_n) \). Then \( b \downarrow \dim a' \) for each such \( a' \).

**Corollary 19.6** In any \( o \)-minimal theory \( \downarrow \dim \) is an independence relation and also a preindependence relation. It satisfies anti-reflexivity and all algebraicity conditions.

## 20 Appendix

**Lemma 20.1 (P.M. Neumann)** Assume the group \( G \) acts on \( \Omega \) and all orbits are of size \( \geq \kappa \geq \omega \). If \( \Gamma \subseteq \Omega \) is finite and \( \Delta \subseteq \Omega \) satisfies \( |\Delta|^+ < \kappa \), then there exists some \( g \in G \) such that \( g\Gamma \cap \Delta = \emptyset \).

**Proof:** By induction on \( |\Gamma| \). It is obvious if \( |\Gamma| = 0 \). Assume \( |\Gamma| = n + 1 \). We can assume \( \Gamma \not\subseteq \Delta \) (otherwise choose \( g \in G \) with \( g\Gamma \not\subseteq \Delta \) and replace \( \Gamma \) by \( \Gamma' = g\Gamma \)). Fix \( \gamma_0 \in \Gamma \setminus \Delta \) and put \( \Gamma_0 = \Gamma \setminus \{\gamma_0\} \). Using the induction hypothesis we can construct inductively a sequence \( (g_i : i < |\Delta|^+) \) of elements of \( G \) such that

\[
g_i\Gamma_0 \cap (\Delta \cup \bigcup_{j<i} g_j\Delta) = \emptyset
\]

for all \( i < |\Delta|^+ \). Note that \( |\bigcup_{i<|\Delta|^+} g_i\Delta| \leq |\Delta|^+ < \kappa \). There are two cases. The first one consists in that \( g_i\gamma \not\in \Delta \) for some \( i < |\Delta|^+ \). Then \( g_i\Gamma \cap \Delta = \emptyset \). In the second case we have \( g_i\gamma \in \Delta \) for all \( i < |\Delta|^+ \). By cardinality reasons, \( g_i\gamma = g_j\gamma \) for some \( j < i < |\Delta|^+ \). Let \( g = g_j^{-1}g_i \). Note that \( g\gamma = \gamma \). Then \( g_i\Gamma_0 \cap g_i\Delta = \emptyset \) and therefore \( g\Gamma_0 \cap \Delta = \emptyset \). Hence \( g\Gamma \cap \Delta = \emptyset \).

**Corollary 20.2** Let \( a_1, \ldots, a_n \) be elements of the monster model such that \( a_i \not\in \text{acl}(A) \) for all \( i = 1, \ldots, n \). For any set \( B \) there are \( b_1, \ldots, b_n \) such that \( b_1 \ldots b_n \equiv a_1 \ldots a_n \) and \( b_i \not\in B \) for all \( i = 1, \ldots, n \).

**Proof:** By Lemma 20.1 with \( \Omega = \bigcup_{i=1}^n \{a : a \equiv_A a_i\}, \Gamma = \{a_1, \ldots, a_n\} \), \( \Delta = B \cap \Omega \), \( G = \text{Aut}(\mathcal{C}/A) \) and \( \kappa > |\Delta|^+ \).

## References


