More on NIP and related topics

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These lecture notes continue [6].

1 M-dividing and thorn-forking

Definition 1.1 The relation of M-*dividing independence* is defined as follows: $A \perp^{\mathrm{M}}_{C} B$ if and only if $A \perp^{\mathrm{a}}_{D} B$ for all D such that $C \subseteq D \subseteq \mathrm{acl}(BC)$. *Thorn-forking independence* (þ-independence) is the relation $\perp^{þ}$ defined as $\perp^{þ} = (\perp^{\mathrm{M}})^{*}$.

Lemma 1.2 $\perp^{\mathrm{M}}$ *has strong finite character.*

Proof: [1] Assume $A \not\perp^{\mathrm{M}}_{C} B$. Then $A \not\perp^{\mathrm{a}}_{D} B$ for some D such that $C \subseteq D \subseteq \mathrm{acl}(BC)$. Let a enumerate A and choose some element $e \in \mathrm{acl}(AD) \cap \mathrm{acl}(BD) \smallsetminus \mathrm{acl}(D)$. Fix some finite tuple $d \in D$ and some formula $\alpha(u, x, v) \in L$ such that $\models \alpha(e, a, d)$ and $\alpha(u, a, d)$ is algebraic. Me may assume that for some $k < \omega$, $\alpha(u, x, v) \vdash \exists^{\leq k} u \alpha(u, x, v)$. Since $e \in \mathrm{acl}(BD) \subseteq \mathrm{acl}(BC)$, we may choose an algebraic formula $\beta(u) \in L(BC)$ such that $\models \beta(e)$. Let $e_1, \ldots, e_n$ be the realizations of $\beta(u)$ in $\mathrm{acl}(D)$. We may assume they are algebraic over the finite tuple d and we can choose some $\chi(u, v) \in L$ such that $\chi(u, d)$ is algebraic and every e_i realizes $\chi(u, d)$. Let $\delta(v) \in L(BC)$ be an algebraic formula isolating $\mathrm{tp}(d/BC)$. Finally we define

$$\varphi(x) = \exists u \exists v (\beta(u) \wedge \delta(v) \wedge \alpha(u, x, v) \wedge \neg\chi(u, v)).$$

Clearly, $\models \varphi(a)$. We claim that if $\models \varphi(a')$, then $a' \not\perp^{\mathrm{M}}_{C} B$. Assume $\models \varphi(a')$ and choose e', d' such that $\models \beta(e') \wedge \delta(d') \wedge \alpha(e', a', d') \wedge \chi(e', d')$. Let $D' = Cd'$. Then $C \subseteq D' \subseteq \mathrm{acl}(BC)$. We check that $a' \not\perp^{\mathrm{a}}_{D'} B$. Since $\models \alpha(e', a', d')$, $e' \in \mathrm{acl}(AD')$. Since $\models \beta(e')$, $e' \in \mathrm{acl}(BC)$ and then $e' \in \mathrm{acl}(BD')$. By choice of χ, if e'' realizes $\beta(u)$ and $\neg\chi(u, d)$, then $e'' \notin \mathrm{acl}(D)$ and hence $e'' \notin \mathrm{acl}(Cd)$. Since $d \equiv_{BC} d'$, if e'' realizes $\beta(u)$ and $\neg\chi(u, d')$, then $e'' \notin \mathrm{acl}(Cd')$. But $\models \beta(e') \wedge \neg\chi(e', d')$ and therefore $e' \notin \mathrm{acl}(D')$. □

Proposition 1.3 $\perp^{\mathrm{M}}$ *is a preindependence relation. It is stronger than* $\perp^{\mathrm{a}}$ *and weaker than* $\perp^{\mathrm{d}}$*. It satisfies all algebraicity properties and anti-reflexivity. Moreover,* $\perp^{\mathrm{M}} = \perp^{\mathrm{a}}$ *if and only if* $\perp^{\mathrm{a}}$ *satisfies right base monotonicity.*

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[1]A simplified version of this proof shows that $\perp^{\mathrm{a}}$ has strong finite character

Proof: By anti-reflexivity of $\mathop{\smile\!\!\!\!\!\!\!\mid}^{d}$ it is clear that $\mathop{\smile\!\!\!\!\!\!\!\mid}^{d}$ is stronger than $\mathop{\smile\!\!\!\!\!\!\!\mid}^{a}$. Since $\mathop{\smile\!\!\!\!\!\!\!\mid}^{d}$ has right base monotonicity and right algebraicity, it follows easily that it is also stronger than $\mathop{\smile\!\!\!\!\!\!\!\mid}^{M}$. Clearly, $\mathop{\smile\!\!\!\!\!\!\!\mid}^{M}$ is stronger than $\mathop{\smile\!\!\!\!\!\!\!\mid}^{a}$. It is easy to check that the basic axioms transfer from $\mathop{\smile\!\!\!\!\!\!\!\mid}^{a}$ to $\mathop{\smile\!\!\!\!\!\!\!\mid}^{M}$. Strong finite character has been proven in Lemma 1.2. Anti-reflexivity and left algebraicity are clear and also the last statement is straightforward. □

Remark 1.4 *Summing-up what we know, in any theory:*

$$\mathop{\smile\!\!\!\!\!\!\!\mid}^{u} \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^{i} \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^{f} \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^{d} \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^{M} \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^{a}$$

Definition 1.5 An independence relation is *strict* if it satisfies anti-reflexivity.

Proposition 1.6 $\mathop{\smile\!\!\!\!\!\!\!\mid}^{þ}$ *is a preindependence relation, and it satisfies all algebraicity properties and anti-reflexivity. It is weaker than any strict independence relation.*

Proof: The first assertion follows from Prop 1.3 and Proposition 17.4 of [6]. Now let $\mathop{\smile\!\!\!\!\!\!\!\mid}$ be a strict independence relation. Since $\mathop{\smile\!\!\!\!\!\!\!\mid}$ is stronger than $\mathop{\smile\!\!\!\!\!\!\!\mid}^{a}$ and has right base monotonicity, it is also stronger than $\mathop{\smile\!\!\!\!\!\!\!\mid}^{M}$. Then $\mathop{\smile\!\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\!\mid}^{*} \Rightarrow (\mathop{\smile\!\!\!\!\!\!\!\mid}^{M})^{*} = \mathop{\smile\!\!\!\!\!\!\!\mid}^{þ}$. □

Proposition 1.7 *The following are equivalent.*

1. $\mathop{\smile\!\!\!\!\!\!\!\mid}^{þ}$ *is a strict independence relation.*
2. $\mathop{\smile\!\!\!\!\!\!\!\mid}^{þ}$ *has local character.*
3. *In T there is some strict independence relation.*

Proof: *1* ⇔ *2* follows from Proposition 1.3 and point *4* in Proposition 17.4 of [6]. For *3* ⇒ *2* note that if $\mathop{\smile\!\!\!\!\!\!\!\mid}$ is an independence relation then (by Proposition 1.6) $\mathop{\smile\!\!\!\!\!\!\!\mid} \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^{þ}$ and therefore local character transfers from $\mathop{\smile\!\!\!\!\!\!\!\mid}$ to $\mathop{\smile\!\!\!\!\!\!\!\mid}^{þ}$. □

Proposition 1.8 *If* acl *has the exchange property and therefore defines a pregeometry (in particular in an o-minimal and in a strongly minimal theory),* $\mathop{\smile\!\!\!\!\!\!\!\mid}^{þ} = \mathop{\smile\!\!\!\!\!\!\!\mid}^{M} = \mathop{\smile\!\!\!\!\!\!\!\mid}^{\dim}$.

Proof: By Corollary 19.5 of [6], $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\dim}$ is a strict independence relation in T and hence by Proposition 1.6 $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\dim} \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^{þ} \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^{M}$. We show now that $A \mathop{\smile\!\!\!\!\!\!\!\mid}^{M}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!\!\mid}^{\dim}_{C} B$. We use symmetry of $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\dim}$ and Fact 19.4 of [6]. Assume $A \not\mathop{\smile\!\!\!\!\!\!\!\mid}^{\dim}_{C} B$. Then for some $D \subseteq B$, for some element $b \in B$, Db is algebraically independent over C, but $b \in \mathrm{acl}(ACD)$. Hence b witnesses that $\mathrm{acl}(ACD) \cap \mathrm{acl}(BC) \not\subseteq \mathrm{acl}(CD)$, which implies $A \not\mathop{\smile\!\!\!\!\!\!\!\mid}^{a}_{CD} B$. Therefore $A \not\mathop{\smile\!\!\!\!\!\!\!\mid}^{M}_{C} B$. □

Definition 1.9 Let $\varphi(x, y) \in L$, let $k < \omega$, and let a be a finite tuple. The formula $\varphi(x, a)$ *strongly divides over A with respect to k* if $a \notin \mathrm{acl}(A)$ and $\{\varphi(x, b) : b \equiv_A a\}$ is k-inconsistent. The formula $\varphi(x, a)$ þ-*divides over A* if for some tuple c, $\varphi(x, a)$ strongly divides over Ac (with respect to some k). The formula $\varphi(x, a)$ þ-*forks over A* if $\varphi(x, a)$ implies a disjunction $\varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi_i(x, a_i)$ þ-divides over A. A partial type $\pi(x)$ þ-*divides over A* (þ-*forks over A*) if it implies a formula $\varphi(x, a)$ that þ-divides over A (respectively, þ-forks over A).

Remark 1.10 *If* $\varphi(x,a)$ þ-divides over A, *then for some finite tuple* c, $\varphi(x,a)$ *strongly divides over* Ac

Proof: Let c be a tuple such that $\varphi(x,a)$ strongly divides over Ac with respect to some k. Then $a \notin \mathrm{acl}(Ac)$ and $\{\varphi(x,b) : b \equiv_A a\}$ is k-inconsistent. If $p(y) = \mathrm{tp}(a/Ac)$, then $p(y_1) \cup \ldots \cup p(y_k) \vdash \neg\exists x \bigwedge_{i=1}^{k} \varphi(x,y_i)$. By compactness, there is some $\psi(y) \in p$ such that $\psi(y_1) \wedge \ldots \wedge \psi(y_k) \vdash \neg\exists x \bigwedge_{i=1}^{k} \varphi(x,y_i)$. If c' is a finite subtuple of c such that $\psi(y) \in L(Ac')$, then $\varphi(x,a)$ strongly divides over Ac' with respect to k. □

Lemma 1.11 *Assume* $M \supseteq C$ *is* ω*-saturated over* C. *For any* A *the following are equivalent:*

1. $A \mathop{\smile\!\!\!\!\!|}^{\mathrm{M}}_{C} M$
2. *For every* C' *such that* $C \subseteq C' \subseteq M$, $\mathrm{acl}(AC') \cap M = \mathrm{acl}(C')$.
3. *For every (finite) tuple* $a \in A$, *if* $\varphi(x) \in L(M)$ *and* $\models \varphi(a)$, *then* $\varphi(x)$ *does not strongly divide over any* C' *such that* $C \subseteq C' \subseteq M$.
4. *For every (finite) tuple* $a \in A$, $\mathrm{tp}(a/M)$ *does not* þ*-divide over* C.
5. *For every (finite) tuple* $a \in A$, $\mathrm{tp}(a/M)$ *does not* þ*-fork over* C.

Proof: *1* ⇒ *2* is obvious and *3* ⇒ *4* ⇒ *5* follows easily from the ω-saturation of M over C.

2 ⇒ *3*. Assume $a \in A$, $b \in M$, $\varphi(x,y) \in L$, $\models \varphi(a,b)$, and $\varphi(x,b)$ strongly divides over some $C' \subseteq M$ extending C. Then $b \notin \mathrm{acl}(C')$ but $b \in \mathrm{acl}(aC')$, which shows that $A \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{a}}_{C'} M$ and hence $A \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{M}}_{C} M$.

5 ⇒ *1*. Assume $A \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{M}}_{C} M$. This means $A \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{a}}_{C'} M$ for some $C' \subseteq M$ containing C. Fix some element $b \in \mathrm{acl}(AC') \cap M \smallsetminus \mathrm{acl}(C')$. Then for some formula $\varphi(x,y,z) \in L$, for some finite tuple $a \in A$, some finite tuple $c \in C'$, and some $k < \omega$, $\models \varphi(a,b,c)$ and $\models \exists^{\leq k} y \varphi(a,y,c)$. Put

$$\varphi'(x,y,z) = \varphi(x,y,z) \wedge \exists^{\leq k} y \varphi(x,y,z)).$$

Clearly, $\models \varphi'(a,b,c)$ and $\varphi'(x,b,c)$ strongly divides over Cc. Therefore $\mathrm{tp}(a/M)$ þ-forks over C. □

Lemma 1.12 *Assume* $A \subseteq B \subseteq C$. *If* $p(x) \in S(B)$ *does not* þ*-fork over* A, *then some extension* $q(x) \in S(C)$ *of* p *does not* þ*-fork over* A.

Proof: $p(x) \cup \{\neg\varphi(x) : \varphi(x) \in L(C)$ þ-forks over $A\}$ is consistent. □

Corollary 1.13 $A \mathop{\smile\!\!\!\!\!|}^{\text{þ}}_{C} B$ *if and only if for all (finite) tuples* $a \in A$, $\mathrm{tp}(a/BC)$ *does not* þ*-fork over* C.

Proof: It follows from lemmas 1.11 and 1.12. □

2 Externally definable sets

Definition 2.1 A relation $R \subseteq M^n$ is *externally definable* if for some $\varphi(x) \in L(\mathfrak{C})$, $R = \varphi(M)$. A formula $\varphi(x) \in L(\mathfrak{C})$ is called an *honest definition* of R if $\varphi(M) = R$ and for each $\psi(x) \in L(M)$, if $\varphi(M) \subseteq \psi(M)$, then $\varphi(\mathfrak{C}) \subseteq \psi(\mathfrak{C})$, that is, $\varphi(x) \vdash \psi(x)$.

Remark 2.2 *If $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$, then for any b there is some $b' \equiv_{Ma} b$ such that $ab' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$. In other words, the inverse of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}$ has the extension property over models.*

Proof: $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$ means that $\mathrm{tp}(c/Ma)$ is a heir of $\mathrm{tp}(c/M)$. Since heirs can be extended, there is some $c' \equiv_{Ma} c$ such that $\mathrm{tp}(c'/Mab)$ is a heir of $\mathrm{tp}(c'/M)$, that is, $ab \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c'$. Now choose b' such that $cb' \equiv_{Ma} c'b$. It follows that $ab' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$. □

Lemma 2.3 *If $\varphi(x, y)$ has* nip, then for each model M, for each finite tuple c, there is some tuple $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$ and some formula $\theta(x) \in L(a)$ such that $\theta(M) = \varphi(M, c)$ and $\models \theta(b) \to \varphi(b, c)$ for every b such that $ba \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$.

Proof: Let $Q = \{\mathfrak{q}(x) \in S(\mathfrak{C}) : \mathfrak{q}$ coheirs from M and $\varphi(x, c) \in \mathfrak{q}\}$. It is closed in $S(\mathfrak{C})$ and it is bounded. Let $Q = \{\mathfrak{q}_\alpha(x) : \alpha < \lambda\}$. We construct inductively a sequence $(a_\alpha : \alpha < \lambda)$ of finite tuples a_α and corresponding formulas $\theta_\alpha(x) \in \mathfrak{q}_\alpha \restriction a_{\leq\alpha}$ such that

1. $a_{<\alpha} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$
2. $\models \theta_\alpha(b) \to \varphi(b, c)$ for every b such that $ba_{\leq\alpha} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$.

To obtain a_α we first observe that there is no sequence $(b_i : i < \omega)$ such that $b_i \models \mathfrak{q}_\alpha \restriction Ma_{<\alpha}b_{<i}c$ for even i and $b_i \models \mathfrak{q}_\alpha \restriction Ma_{<\alpha}b_{<i} \cup \{\neg\varphi(x, c)\}$ for odd i since such a sequence would be indiscernible and would verify $\varphi(b_i, c) \Leftrightarrow i$ is even, contradicting the nipof $\varphi(x, y)$. The construction must fail at some odd i. Choose a minimal odd i for which there is no $b_i \models \mathfrak{q}_\alpha \restriction Ma_{<\alpha}b_{<i} \cup \{\neg\varphi(x, c)\}$ such that $b_{\leq i}a_{<\alpha} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$. By induction hypothesis $a_{<\alpha} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$. Then $b_{<i}a_{<\alpha} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$. Notice that $\{b : bb_{<i}a_{<\alpha} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c\}$ is type-definable over $Mb_{<i}a_{<\alpha}c$ by $\{\neg\psi(x, b_{<i}a_{<\alpha}) : \psi(x, y_{<i}z_{<\alpha}) \in L(Mc)$ is not satisfiable in $M\}$. By compactness there is some formula $\theta_\alpha(x) \in \mathfrak{q}_\alpha \restriction Ma_{<\alpha}b_{<i}$ such that for every b such that $ba_{<\alpha}b_{<i} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$, $\models \theta_\alpha(b) \to \varphi(b, c)$. Put $a_\alpha = b_{<i}$ and observe that $a_{\leq\alpha} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$.

Since Q is closed, by compactness, there are $\alpha_1, \ldots, \alpha_n$ such that $\theta_{\alpha_1}, \ldots, \theta_{\alpha_n}$ cover Q. Let $\theta(x) = \theta_{\alpha_1}(x) \vee \ldots \vee \theta_{\alpha_n}(x)$. Then $\theta(x) \in \mathfrak{q}$ for every $\mathfrak{q} \in Q$. We check that θ and $a = a_{<\lambda}$ satisfy our requirements. Assume $ba \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$ and $\models \theta(b)$. By point 2, $\models \varphi(b, c)$. It remains to show that $\theta(M) = \varphi(M, c)$. Let $m \in M$ be such that $\models \varphi(m, c)$. Then $\mathrm{tp}(m/\mathfrak{C}) \in Q$ and therefore $\theta(x) \in \mathrm{tp}(a/\mathfrak{C})$ and $\models \theta(m)$. On the other hand, if $m \in M$ then $ma \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$ and hence $\models \theta(m) \to \varphi(m, c)$. □

Proposition 2.4 *If $\varphi(x, y) \in L$ has* nip, then for every model M, for every tuple c, $\varphi(M, c)$ has an honest definition.

Proof: Find $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$ and $\theta(x) \in L(a)$ as in Lemma 2.3. We claim that $\theta(x)$ is an honest definition of $\varphi(M, c)$. Let $\psi(x) \in L(M)$ be such that $\theta(M) \subseteq \psi(M)$. If there is some $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$ such that $\models \varphi(b, c) \wedge \neg\psi(b)$, then $\models \theta(m) \wedge \neg\psi(m)$ for some $m \in M$. Hence $\models \varphi(b, c) \to \psi(b)$ for all $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$. We show that $\theta(x) \vdash \psi(x)$. Assume $\models \theta(b)$. By Remark 2.2 there is some $b' \equiv_{Ma} b$ such that $b'a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$. Then $\models \theta(b')$. By Lemma 2.3, $\models \varphi(b', c)$. Since $b' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{u}}_M c$, we obtain $\models \psi(b')$. Since $b \equiv_M b'$, $\models \psi(b)$. □

Theorem 2.5 *If T has* nip, then the projection $\pi(R) \subseteq M^n$ in M of any externally definable relation $R \subseteq M^{n+1}$ is externally definable. Therefore the theory of M expanded with all externally definable relations has elimination of quantifiers.

Proof: Let $\pi(R) = \{m \in M^n : R(m,a) \text{ for some } a \in M\}$. Choose with Proposition 2.4 an honest definition $\theta(x,y)$ for R. Let $\psi(x,y)$ be the formula $y \neq y$. Then for any $m \in M^n$:

$$\begin{array}{rcl} m \in \pi(R) & \Leftrightarrow & R(m,a) \text{ for some } a \in M \\ & \Leftrightarrow & \theta(m,M) \neq \emptyset \\ & \Leftrightarrow & \theta(m,M) \not\subseteq \psi(M) \\ & \Leftrightarrow & \theta(m,\mathfrak{C}) \not\subseteq \psi(\mathfrak{C}) \\ & \Leftrightarrow & \models \exists y \theta(m,y) \end{array}$$

and therefore $\exists y \theta(x,y)$ defines externally $\pi(R)$. □

3 Broom lemma

Proposition 3.1 *Let $\mathop{\smile\!\!\!\!\!\mid}$ be a ternary relation that satisfies the basic axioms and extension and is stronger than $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$. If T is* NTP$_2$*, A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}$, and $\varphi(x,a)$ divides over A, then there are some model $M \supseteq A$ and a $\mathop{\smile\!\!\!\!\!\mid}$-free extension $\mathfrak{p}(x) \in S(\mathfrak{C})$ of* tp(a/M) *such that $\{\varphi(x,a_i) : i < \omega\}$ is inconsistent for every sequence $(a_i : i < \omega)$ generated by $\mathfrak{p}$ over M with the rule $a_i \models \mathfrak{p} \restriction Ma_{<i}$.*

Proof: Let $I = (a_i : i < \omega)$ be an A-indiscernible sequence witnessing that $\varphi(x,a)$ k-divides over A, that is, $\{\varphi(x,a_i) : i < \omega\}$ is k-inconsistent and $a_i \equiv_A a$ for all $i < \omega$. Choose a model $M \supseteq A$ and choose a $|M|^+$-saturated model $N \supseteq M$. Let $\lambda = (2^{|N|})^+$ and extend the sequence I to an A-indiscernible sequence $I' = (a_i : i < \lambda)$. Since $I' \mathop{\smile\!\!\!\!\!\mid}_A A$, by the extension property of $\mathop{\smile\!\!\!\!\!\mid}$ we can assume that $I' \mathop{\smile\!\!\!\!\!\mid}_A N$. By choice of λ, infinitely many a_i have the same type over N. Without loss of generality, $a_i \equiv_N a_j$ for all $i,j < \omega$. Since $a_0 \mathop{\smile\!\!\!\!\!\mid}_A N$, $p(x) = \mathrm{tp}(a_0/N)$ has a global extension $\mathfrak{p}(x)$ which is $\mathop{\smile\!\!\!\!\!\mid}$-free over A. Similarly $q(x_i : i < \omega) = \mathrm{tp}((a_i : i < \omega)/N)$ has a global extension $\mathfrak{q}(x_i : i < \omega)$ which is $\mathop{\smile\!\!\!\!\!\mid}$-free over A. Since $\mathop{\smile\!\!\!\!\!\mid} \Rightarrow \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$, these global extensions do not Lascar-split over A and therefore they do not split over M. Since N is $|M|^+$-saturated a type over N has at most one global extension that does not split over M (if $\mathfrak{p}_1$, $\mathfrak{p}_2$ are two such global extensions of some type $r(x) \in S(N)$ and $\psi(x,b) \in \mathfrak{p}_1$, some $b' \in M$ realizes $\mathrm{tp}(b/M)$ and then $\psi(x,b') \in r$ and $\psi(x,b) \in \mathfrak{p}_2$.). In particular $\mathfrak{p}$ and $\mathfrak{q}$ are the only $\mathop{\smile\!\!\!\!\!\mid}$-free over A global extensions of p and q respectively. It follows that $\mathfrak{q} \restriction x_i = \mathfrak{p}(x_i)$. Now construct the sequence $(d^n : n < \omega)$ choosing $d^n \models \mathfrak{q} \restriction Md^{<n}$. Then $d^n = (d^n_i : i < \omega) \equiv_A (a_i : i < \omega)$ and therefore $\{\varphi(x,d^n_i) : i < \omega\}$ is k-inconsistent. Since T is NTP$_2$, for some $f : \omega \to \omega$, $\{\varphi(x,d^n_{f(n)}) : n < \omega\}$ is inconsistent. Notice that $d^n_{f(n)} \models \mathfrak{p} \restriction M(d^i_{f(i)} : i < n)$. Since $\mathfrak{p}$ does not split over M, if $(c_n : n < \omega)$ satisfies $c_n \models \mathfrak{p} \restriction Mc_{<n}$ for all $n < \omega$ then $(c_n : n < \omega) \equiv_M (d^n_{f(n)} : n < \omega)$. Hence, the type $\mathfrak{p}$ and the model M satisfy our requirements excepts, perhaps, the condition $\mathrm{tp}(a/M) \subseteq \mathfrak{p}$. To repair this, choose $a' \models \mathfrak{p} \restriction M$ and choose an automorphism $f \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $f(a') = a$. Then $f(M)$ and $\mathfrak{p}^f$ have the right properties for a. □

Definition 3.2 The formula $\varphi(x,a)$ *quasi divides* over A if for some $n < \omega$ there is a sequence $(a_i : i < n)$ such that $a_i \equiv_A a$ for all $i < n$ and $\{\varphi(x,a) : i < n\}$ is inconsistent. Clearly, dividing implies quasi dividing.

Lemma 3.3 (Broom Lemma) *Let $\mathop{\smile\hskip -0.9em ^| \ }$ be a ternary relation that satisfies the basic axioms, extension and is stronger than $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{i}}$. Assume $\mathop{\smile\hskip -0.9em ^| \ }$ satisfies left extension over A, $\alpha(x,e) \vdash \psi(x,c) \vee \bigvee_{i<n} \varphi_i(x,a_i)$, and for each $i < n$, $\varphi_i(x,a_i)$ divides over A. Assume additionally that some A-indiscernible sequences $a^i = (a^i_j : j < \omega)$ witnessing dividing over A of each $\varphi_i(x,a_i)$ satisfy:*

1. $a^i_0 = a_i$
2. $a^i_j \mathop{\smile\hskip -0.9em ^| \ }_A a^i_{<j} a^{<i}$ *if* $j > 0$.
3. $c \mathop{\smile\hskip -0.9em ^| \ }_A a^{<n}$.

Then there is a sequence $(e_i : i < m)$ of A-conjugates of e such that

$$\bigwedge_{i<m} \alpha(x,e_i) \vdash \psi(x,c).$$

Notice that if $\psi(x,c)$ is $x \neq x$, the conclusion is that $\alpha(x,e)$ quasi divides over A.

Proof: By induction on n. The case $n = 0$ is clear, with $m = 1$ and $e_0 = e$. Assume inductively that the result holds for n and suppose $\alpha(x,e) \vdash \psi(x,c) \vee \bigvee_{i \leq n} \varphi_i(x,a_i)$ with exemplifying sequences of dividing $a^i = (a^i_j : j < \omega)$ as indicated above for $i \leq n$. We can assume dividing is with respect to the same $k < \omega$.

Claim: There are sequences $a^{<n,0}, \ldots, a^{<n,k-1}$, where each $a^{<n,l}$ is of the form $a^{<n,l} = a^{0,l}, \ldots, a^{n-1,l}$ and each $a^{i,l}$ is an ω-sequence $a^{i,l} = (a^{i,l}_j : j < \omega)$, such that:

1. $a^{<n,0} = a^{<n}$
2. $a^{<n,l} a^n_l \equiv_{Ac} a^{<n} a^n_0$ for all $l < k$.
3. $c a^{<n,k-1} a^{<n,k-2} \ldots a^{<n,l+1} \mathop{\smile\hskip -0.9em ^| \ }_A a^{<n,l}$ for all $l < k-1$.

Note that 2 implies $c \mathop{\smile\hskip -0.9em ^| \ }_A a^{<n,k-1}$

We first show how the claim helps to prove the lemma. Choose $(e_l : l < k)$ such that $e_0 = e$ and for $l \geq 1$, $e_l a^{<n,l} a^n_l \equiv_{Ac} e a^{<n} a^n_0$. Then

$$\alpha(x,e_l) \vdash \psi(x,c) \vee \bigvee_{i<n} \varphi_i(x,a^{i,l}_0) \vee \varphi_n(x,a^n_l).$$

Let $\alpha_0 = \bigwedge_{l<k} \alpha(x,e_l)$. Since $a^{<n}$ witnesses k-dividing of $\varphi_n(x,a_n)$,

$$\alpha_0(x) \vdash \psi(x,c) \vee \bigvee_{i<n} \bigvee_{l<k} \varphi_i(x,a^{i,l}_0)$$

We now define for $r \leq k$

$$\psi_r(x,c_r) = \psi(x,c) \vee \bigvee_{i<n} \bigvee_{r \leq l < k} \varphi_i(x,a^{i,l}_0)$$

and we prove by induction on r that there is some formula $\alpha_r(x)$ which is a conjunction of A-conjugates of $\alpha(x,e)$ and such that $\alpha_r(x) \vdash \psi_r(x,c_r)$. If we apply this to $r = k$ we obtain $\alpha_k(x) \vdash \psi(x,c)$ and the proof finishes.

For the starting case $r = 0$ we have found $\alpha_0(x)$. Assume inductively we have obtained $\alpha_r(x)$. Then

$$\alpha_r(x) \vdash \psi_{r+1}(x, c_{r+1}) \vee \bigvee_{i<n} \varphi_i(x, a_0^{i,r})$$

The assumptions of the Lemma hold for this implication (with $c = c^{r+1}$ and $a^i = a^{i,r}$) and therefore we can apply the induction hypothesis in order to obtain α_{r+1}, a conjunction of A-conjugates of $\alpha_r(x)$ (hence of $\alpha(x, e)$) such that $\alpha_{r+1}(x) \vdash \psi_{r+1}(x, c^{r+1})$.

Proof of the Claim:

Since $\mathop{\smile\!\!\!\!\!\!|}$ is stronger than $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$ and $c \mathop{\smile\!\!\!\!\!\!|}_A a^0, \ldots, a^n$, each a^i is in fact Ac-indiscernible. If $b_0 = a_0^n \ldots a_{k-2}^n$ and $b_1 = a_1^n \ldots a_{k-1}^n$ we have then $b_0 \equiv_{Ac} b_1$. By induction on $j < k$ we now construct finite sequences of ω-sequences

$$a^{<n,l,j} = a^{0,l,j}, \ldots, a^{n-1,l,j}$$

for $1 \leq l \leq j$ such that

1. $a^{<n,0,j} = a^{<n}$
2. $a^{<n,l,j} a_l^n \equiv_{Ac} a^{<n} a_0^n$ for all $l \leq j$.
3. $ca^{<n,j,j} a^{<n,j-1,j} \ldots a^{<n,l+1,j} \mathop{\smile\!\!\!\!\!\!|}_A a^{<n,l,j}$ for $l < j$ and $c \mathop{\smile\!\!\!\!\!\!|}_A a^{<n,j,j}$.

The starting point is simply $a^{<n,0,0} = a^{<n}$. Now assuming we have obtained $a^{<n,l,j}$ for all l such that $1 \leq l \leq j$, we set $a^{<n,0,j+1} = a^{<n}$ and we construct $a^{<n,l,j+1}$ for $0 < l \leq j+1$ as follows: since $b_1 \equiv_{Ac} b_0$, there are $d_{j+1}, \ldots, d_1$ such that

$$d_{j+1}, d_j, \ldots, d_1, b_1 \equiv_{Ac} a^{<n,j,j}, a^{<n,j-1,j}, \ldots, a^{<n,0,j}, b_0 \tag{1}$$

and since (by left transitivity) $cb_1 \mathop{\smile\!\!\!\!\!\!|}_A a_0^n a^{<n}$ we can find by left extension $a^{<n,l,j+1}$ for $1 \leq l \leq j+1$ such that

$$a^{<n,j+1,j+1}, a^{<n,j,j+1}, \ldots, a^{<n,1,j+1} b_1 \equiv_{Ac} d_{j+1}, d_j, \ldots, d_1, b_1 \tag{2}$$

and

$$a^{<n,j+1,j+1} a^{<n,j,j+1} \ldots a^{<n,1,j+1} cb_1 \mathop{\smile\!\!\!\!\!\!|}_A a_0^n a^{<n}. \tag{3}$$

We check point 2. By (1) and (2), $a^{<n} a_0^n \equiv_{Ac} d_1 a_1^n \equiv_{Ac} a^{<n,1,j+1} a_1^n$. Let $1 \leq l \leq j$. By the induction hypothesis on j, $a^{<n} a_0^n \equiv_{Ac} a^{<n,l,j} a_l^n$. Again by (1) and (2), $a^{<n,l,j} a_l^n \equiv_{Ac} d_{l+1} a_{l+1}^n \equiv_{Ac} a^{<n,l+1,j+1} a_{l+1}^n$. Hence $a^{<n,l+1,j+1} a_{l+1}^n \equiv_{Ac} a^{<n} a_0^n$. Point 3 follows from (3) for the case $l = 0$ and by induction hypothesis on j and invariance of $\mathop{\smile\!\!\!\!\!\!|}$ (and again (1) and (2)) for the case $l \geq 1$ (the last part of 3 is a consequence of 2 since $c \mathop{\smile\!\!\!\!\!\!|}_A a^{<n}$).

Once the construction has been completed, we use the last step

$$a^{<n,0}, \ldots, a^{<n,k-1} = a^{<n,0,k-1}, \ldots, a^{<n,k-1,k-1}.$$

□

Proposition 3.4 *Assume T is* NTP$_2$. *Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation that satisfies the basic axioms, extension and is stronger than $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$. If A is an extension base for $\mathop{\smile\!\!\!\!\!\!|}$ and $\mathop{\smile\!\!\!\!\!\!|}$ satisfies left extension over A, then if a formula forks over A, it quasi divides over A. Since $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{u}}$ satisfies all these conditions in the case $A = M$ is a model, any formula that forks over a model M quasi divides over M.*

Proof: Assume $\varphi(x,a) \vdash \bigvee_{i<n} \varphi_i(x,a_i)$ where every $\varphi_i(x,a_i)$ divides over A. By Proposition 3.1, for every $i<n$ there is some global type $\mathfrak{p}_i$ which is $\downarrow$-free over A and some model $M_i \supseteq A$ such that $\operatorname{tp}(a_i/M_i) \subseteq \mathfrak{p}_i$ and $\{\varphi_i(x,c_j) : j<\omega\}$ is inconsistent for every sequence $(c_j : j<\omega)$ such that $c_j \models \mathfrak{p}_i \upharpoonright M_i c_{<j}$. Let $a^i = (a^i_j : j<\omega)$ be chosen in such a way that $a^i_0 = a_i$ and $a^i_j \models \mathfrak{p}_i \upharpoonright M_i a^{<i} a^i_{<j}$ for $j>0$. We can apply the Broom Lemma 3.3 obtaining that $\varphi(x,a)$ quasi divides over A. $\square$

Corollary 3.5 *Assume T is* NTP_2 *and $p(x) \in S(M)$. If $p(x) \vdash \psi_i(x,c) \vee \varphi(x,a)$ and $\varphi(x,a)$ forks over M, then for some $m<\omega$ there are $(c_j : j<m)$ such that $c_j \equiv_M c$ for every $j<m$ and $p(x) \vdash \bigvee_{j<m} \psi(x,c_j)$*

Proof: Choose $\alpha(x,e) \in p$ such that $\alpha(x,e) \vdash \psi_i(x,c) \vee \varphi(x,a)$. Then $\alpha(x,e) \wedge \neg\psi(x,c) \vdash \varphi(x,a)$ and therefore $\alpha(x,e) \wedge \neg\psi(x,c)$ forks over M. By Proposition 3.4, $\alpha(x,e) \wedge \neg\psi(x,c)$ quasi divides over M, and this gives the result. $\square$

Proposition 3.6 *Assume T is* NTP_2*. Every type $p(x) \in S(M)$ has a global nonforking heir $\mathfrak{p}(x) \in S(\mathfrak{C})$.*

Proof: We must check the consistency of $p(x) \cup \{\neg\varphi(x) : \varphi(x) \in L(\mathfrak{C})$ forks over $A\} \cup \{\neg\psi(x,c) : \psi(x,y) \in L(M), c \in \mathfrak{C}$ and for no $m \in M$, $\psi(x,m) \in p\}$. If it is inconsistent, then $p(x) \vdash \psi(x,c) \vee \varphi(x,a)$ for some formula $\varphi(x,a)$ that forks over M, some $c \in \mathfrak{C}$, and some formula $\psi(x,y) \in L(M)$ such that $\psi(x,m) \notin p$ for all $m \in M$. By Corollary 3.5, for some $m<\omega$ there are $(c_j : j<\omega)$ such that and $c_j \equiv_M c$ for all $j<m$ and $p(x) \vdash \bigvee_{j<m} \psi(x,c_j)$. This contradicts the fact that $p(x)$ has a heir over $M(c_j : j<m)$. $\square$

4 Forking and dividing over a model

Definition 4.1 We introduce a new ternary relation, the relation of *strict invariance* $\downarrow^{\mathrm{ist}}$: $A \downarrow^{\mathrm{ist}}_C B$ if and only if for every $a \in A$ there is some global type $\mathfrak{p}(x) \in S(\mathfrak{C})$ extending $\operatorname{tp}(a/CB)$ such that $\mathfrak{p}$ does not Lascar split over C and for every set $D \supseteq C$, for every $d \models \mathfrak{p} \upharpoonright D$, $D \downarrow^{\mathrm{f}}_C d$.

Remark 4.2 $\downarrow^{\mathrm{ist}}$ *is invariant, monotone, and has the extension property.*

Remark 4.3 *Assume $\mathfrak{p}(x) \in S(\mathfrak{C})$. The following are equivalent for any set C:*

1. *$\mathfrak{p}$ is $\downarrow^{\mathrm{ist}}$-free over C, that is, $a \downarrow^{\mathrm{ist}}_C B$ for every $B \supseteq C$, for every $a \models \mathfrak{p} \upharpoonright B$.*
2. *$\mathfrak{p}$ does not Lascar-split over C and $B \downarrow^{\mathrm{f}}_C a$ for every $B \supseteq C$, for every $a \models \mathfrak{p} \upharpoonright B$.*

Proposition 4.4 *Assume T is* NTP_2*. If $\varphi(x,a)$ divides over A, $\mathfrak{p}(x) \in S(\mathfrak{C})$ is a $\downarrow^{\mathrm{ist}}$-free extension of $\operatorname{tp}(a/A)$ and $M \supseteq A$ is a model, then $\{\varphi(x,a_i) : i<\omega\}$ is inconsistent for every sequence $(a_i : i<\omega)$ generated by $\mathfrak{p}$ over M with the rule $a_i \models \mathfrak{p} \upharpoonright Ma_{<i}$.*

Proof: Let $M \supseteq A$ be a model and choose a $|M|^+$-saturated model $N \supseteq M$ and some realization $b \models \mathfrak{p} \upharpoonright N$. Since $a \equiv_A b$, $\varphi(x,b)$ divides over A and there is an A-indiscernible sequence $(a_i : i<\omega)$ starting with $a_0 = b$ such that $\{\varphi(x,a_i) : i<\omega\}$ is k-inconsistent. Since $b \downarrow^{\mathrm{ist}}_A N$, it follows that $N \downarrow^{\mathrm{f}}_A b$ and $N \downarrow^{\mathrm{d}}_A b$. Therefore there is an N-indiscernible sequence $(b_i : i<\omega)$ such that $(b_i : i<\omega) \equiv_{Ab} (a_i : i<\omega)$. Note that $b_0 = b$ and hence

$b_i \models \mathfrak{p} \restriction N$. Let $q = \mathrm{tp}((b_i : i < \omega)/N)$ and let $(d^n : n < \omega)$ a sequence constructed in N with the rule $d^n \models q \restriction Md^{<n}$. Write $d^n = (d^n_i : i < \omega)$. If $f : \omega \to \omega$, then $d^n_{f(n)} \models \mathfrak{p} \restriction M(d^i_{f(i)} : i < n)$, because $q \restriction x_n = \mathfrak{p}(x_n) \restriction N$. As in the proof of Proposition 3.1, $\{\varphi(x, d^n_i) : i < \omega\}$ is k-inconsistent and by NTP_2 for some $f : \omega \to \omega$, $\{\varphi(x, d^n_{f(n)}) : n < \omega\}$ is inconsistent. Since $\mathfrak{p}$ does not split over M, for every sequence $(c_i : i < \omega)$ such that $c_i \models \mathfrak{p} \restriction Mc_{<i}$ we have $(c_i : i < \omega) \equiv_M (d^n_{f(n)} : n < \omega)$ and then $\{\varphi(x, c_i) : i < \omega\}$ is inconsistent. □

Proposition 4.5 *Assume T is* NTP_2*. If A is an extension base for $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}$, then $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{f}}_A = \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_A$.*

Proof: Assume $\varphi(x, b) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi_i(x, a_i)$ divides over A and let $\bar{a} = ba_1 \ldots a_n$. Let $M \supseteq A$ be a model. Since A is an extension base, $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}_A A$ and by extension we can assume $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}_A M$. Let $\mathfrak{p}$ be a global $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}$-free over A extension of $\mathrm{tp}(\bar{a}/M)$. Construct now a sequence $(\bar{a}^i : i < \omega)$ such that $\bar{a}^i \models \mathfrak{p} \restriction M\bar{a}^{<i}$. Then $\bar{a}^i = b_i a^i_1 \ldots a^i_n$ and $\mathfrak{p}_j = \mathfrak{p} \restriction x_j$ is a global type extending $\mathrm{tp}(a_j/M)$ which is $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}$-free over A. By Proposition 4.4 $\{\varphi_j(x, a^i_j) : i < \omega\}$ is inconsistent. We claim that $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent, which will show that $\varphi(x, b)$ divides over A. Assume not, and let c be a realization of $\{\varphi(x, b_i) : i < \omega\}$. For each $i < \omega$ there is some j such that $\models \varphi_j(c, a^i_j)$. Then for some j there is an infinite subset $I \subseteq \omega$ such that $\models \varphi_j(c, a^i_j)$ for every $i \in I$. By indiscernibility $\{\varphi_j(x, a^i_j) : i < \omega\}$ is consistent, a contradiction. □

Proposition 4.6 *Assume T is* NTP_2*. Let $\mathop{\smile\!\!\!\!\!\!\;|}$ be a ternary relation that satisfies the basic axioms, extension and is stronger than $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{i}}$. If A is an extension base for $\mathop{\smile\!\!\!\!\!\!\;|}$ and $\mathop{\smile\!\!\!\!\!\!\;|}$ satisfies left extension over A, then:*

1. *A is an extension base for $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}$.*
2. *$\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{f}}_A = \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_A$.*

Proof: *1.* In order to prove that $a \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}_A A$ let us check the consistency of $p(x) \cup \Sigma_1(x) \cup \Sigma_2(x)$, where

1. $p(x) = \mathrm{tp}(a/A)$
2. $\Sigma_1(x) = \{\varphi(x, b) \leftrightarrow \varphi(x, b') : b \overset{\mathrm{Ls}}{\equiv}_A b',\ \varphi(x, y) \in L(A),\ b, b' \in \mathfrak{C}\}$
3. $\Sigma_2(x) = \{\neg\psi(x, d) : \psi(a, y) \text{ forks over } A,\ \psi(x, y) \in L(A), d \in \mathfrak{C}\}$

Note that a global type $\mathfrak{p}$ extending $p(x) \cup \Sigma_1(x) \cup \Sigma_2(x)$ does not Lascar-split over A; moreover, for every set $D \supseteq A$, for every $a' \models \mathfrak{p} \restriction D$, if $\psi(x, y) \in L(A)$ and $\psi(x, d) \in \mathfrak{p} \restriction D$, then $\psi(a', y)$ does not fork over A since otherwise $\psi(a, y)$ forks over A and then $\neg\psi(x, d) \in \Sigma_2(x) \subseteq \mathfrak{p}$.

If the set is not consistent, then $p(x) \vdash \bigvee_{i<m} \neg(\varphi_i(x, b_i) \leftrightarrow \varphi_i(x, b'_i)) \vee \psi(x, d)$ where $(\varphi_i(x, b_i) \leftrightarrow \varphi_i(x, b'_i)) \in \Sigma_1$ and $\neg\psi(x, d) \in \Sigma_2$. By Proposition 3.4, $\psi(a, y)$ quasi divides over A, that is, $\{\psi(x, a_i) : i = 1, \ldots, n\}$ is inconsistent for some $a_1, \ldots, a_n$ with $a_i \equiv_A a$. Let $q(x_1, \ldots, x_n) = \mathrm{tp}(a_1, \ldots, a_n/A)$. Then $q \restriction x_j = p(x_j)$ and $p(x_j) \vdash \bigvee_{i<m} \neg(\varphi_i(x_j, b_i) \leftrightarrow \varphi_i(x_j, b'_i)) \vee \psi(x_j, d)$ for each $j = 1, \ldots, n$. It follows that

$$q(x_1, \ldots, x_n) \vdash \bigwedge_{j=1}^{n} \bigvee_{i<m} \neg(\varphi_i(x_j, b_i) \leftrightarrow \varphi_i(x_j, b'_i)) \vee \psi(x_j, d).$$

Since $\neg\exists y(\psi(x_1,y)\wedge\ldots\wedge\psi(x_n,y))\in q(x_1,\ldots,x_n)$, we see that

$$q(x_1,\ldots,x_n)\vdash\bigvee_{j=1}^{n}\bigvee_{i<m}\neg(\varphi_i(x_j,b_i)\leftrightarrow\varphi_i(x_j,b_i'))$$

which contradicts the fact that $a_1,\ldots,a_n \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_A A$ and q has therefore a global extension Lascar-invariant over A.

2 follows from *1* and Proposition 4.5. □

Theorem 4.7 *If T is* NTP$_2$*, then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_M=\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_M$ for every model M.*

Proof: By Proposition 4.6 since $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{u}}$ satisfies all the requirements on $\mathop{\smile\!\!\!\!\!\!|}$. □

Lemma 4.8 *If $\varphi(x,a)$ forks over A and $B\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A a$, then $\varphi(x,a)$ forks over AB.*

Proof: Let $\varphi(x,a)\vdash\varphi_1(x,a_1)\vee\ldots\vee\varphi_n(x,a_n)$ where every $\varphi_i(x,a_i)$ divides over A. Choose $B'\equiv_{Aa}B$ such that $B'\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A aa_1,\ldots,a_n$ and therefore $B'\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_A aa_1,\ldots,a_n$. By Proposition 4.9 in [6], every $\varphi_i(x,a_i)$ divides over AB'. Then $\varphi(x,a)$ forks over AB' and also over AB. □

Proposition 4.9 *1. Assume T is* NTP$_2$*. If A is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ or $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has left extension over A, then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A=\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_A$.*

2. If $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A=\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_A$, then A is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has left extension over A.

Proof: *1.* Suppose $\varphi(x,a)$ forks over A and choose a model $M\supseteq A$. If A is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ then $M\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A A$ and by extension we may assume $M\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A a$. If $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has left extension over A we reach the same conclusion since $A\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A a$. By Lemma 4.8, $\varphi(x,a)$ forks over M. By Theorem 4.7, $\varphi(x,a)$ divides over M. Then, clearly, $\varphi(x,a)$ divides over A.

2. Assume $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A=\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_A$. It is clear that A is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ since every set is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$. We prove now that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has left extension over A. Suppose $a\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A b$ and let c be given. We seek some $c'\equiv_{Aa}c$ such that $c'a\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A b$. Let $p(x)=\mathrm{tp}(c/Aa)$. It suffices to show the consistency of

$$p(x)\cup\{\neg\varphi(x,a,b):\varphi(x,y,z)\in L(A)\text{ and }\varphi(x,y,b)\text{ forks over }A\}.$$

Assume it is inconsistent. Then $p(x)\vdash\bigvee_{i<m}\varphi_i(x,a,b)$ for some formulas $\varphi_i(x,y,z)\in L(A)$ such that $\varphi_i(x,y,b)$ forks over A. Let $\varphi(x,y,z)=\bigvee_{i<m}\varphi_i(x,y,z)$. Then $\varphi(x,y,b)$ forks over A and by assumption it divides over A. Since $a\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_A b$, it also divides Aa (see Proposition 4.9 in [6]). Then $p(x)$ divides over Aa, which is impossible since $p(x)\in S(Aa)$. □

Corollary 4.10 *If T is* NTP$_2$*, then the following are equivalent for any A:*

1. *A is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$.*
2. *$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has left extension over A.*
3. *$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A=\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_A$.*

Proof: By Proposition 4.9. □

Proposition 4.11 *Assume T is* NTP_2. *A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ if and only if A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}$.*

Proof: Since $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}} \Rightarrow \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$, it is clear that every extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}$ is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ in any context. Now assume NTP_2 and let A be an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$. Since $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}} \Rightarrow \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$, A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ and by 4.10 $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_A = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}_A$. In particular forking over A implies quasi dividing over A. Then the proof of point *2* of Proposition 4.6 can be reproduced here to ensure that A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}$. □

Corollary 4.12 *If T has* nip, *then the following are equivalent for any A:*

1. *A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$.*
2. *$\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ has left extension over A.*
3. $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_A = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_A$.
4. *A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}$.*
5. *A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$.*

Proof: It follows immediately from propositions 4.11 and 4.10 since in a nip theory $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ (see Proposition 18.12 of [6]) □

Question 4.13 *Is in a simple theory every set A an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$?*

5 Bounded forking

Lemma 5.1 *Assume B is A-complete (i.e., every finitary type over A is realized in B) and $p(x) \in S(B)$ is finitely satisfiable in A. Then $p(x)$ does not split over A, and for every $C \supseteq B$, $p|_A C$ is also finitely satisfiable in A. Hence if $q(y) \in S(B)$ is another type finitely satisfiable in A, then the product $p(x) \otimes_A q(y)$ is finitely satisfiable in A.*

Proof: See sections 9 and 10 of [6] for notation and terminology. Assume $\varphi(x, y) \in L$, $c \in C$ is a tuple and $\varphi(x, c) \in p|_A C$. Choose $c' \in B$ such that $c \equiv_A c'$. Since $p|_A C$ does not split over A, $\varphi(x, c') \in p$ and therefore $\models \varphi(a, c')$ for some $a \in A$. Since $c \equiv_A c'$, $\models \varphi(a, c)$. Now let us consider the product $p \otimes_A q$. Assume $\varphi(x, y) \in p \otimes_A q$ and choose $ab \models p \otimes_A q$. Then $a \models p$ and $b \models q|_A Ba$. Since $\varphi(a, y) \in q|_A Ba$, there is some $b' \in A$ such that $\models \varphi(a, b')$. Then $\varphi(x, b') \in p(x)$ and therefore $\models \varphi(a', b')$ for some $a' \in A$. □

Proposition 5.2 *T is* nip *if and only if T is* NTP_2 *and $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ is bounded.*

Proof: By Proposition 18.12 of [6], we know that in a nip theory $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ is bounded. Assume now T is NTP_2 and $\varphi(x, y) \in L$ has the independence property in T. We may assume that y is a single variable. We will show that there is a global type $\mathfrak{p}(x)$ which does not fork over a model M and it is not M-invariant. Then $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}} \neq \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ and again by Proposition 18.12 of [6], $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ is unbounded.

In order to obtain $\mathfrak{p}(x)$, we start with an infinite set A such that for each subset $X \subseteq A$, $\{\varphi(x, a) : a \in X\} \cup \{\neg\varphi(x, a) : a \in A \smallsetminus X\}$ is consistent. The partial type $r(y) =$

$\{y \neq a : a \in A\}$ is finitely satisfiable in A and hence it can be extended to a global type $\mathfrak{r}(y)$ which is finitely satisfiable in A. Choose an A-complete set $B \supseteq A$ and let $p(y) = \mathfrak{r} \restriction B$. The power $p^{(\omega)_A}$ is well-defined and by Lemma 5.1 it is finitely satisfiable in A. Let $\psi(x; y, z) = \varphi(x, y) \wedge \neg\varphi(x, z)$, and for every sequence $a = (a_i : i < \omega)$ of tuples a_i of the right length, let $\Gamma_a = \{\psi(x; a_i) : i < \omega\}$.

We claim that there is a set C such that for every indiscernible sequence $a = (a_i : i < \omega)$ such that $a_i \models (p \otimes_A p)|BC$, the corresponding set Γ_a is consistent. Otherwise we can build a sequence $(a^j : j < \omega_1)$ of indiscernible sequences $a^j = (a^i_j : i < \omega)$ such that $a^j_i \models (p \otimes_A p)|_A Ba^{<j}$ and $\Gamma_j = \Gamma_{a^j}$ is inconsistent. For some $k < \omega$ there is an infinite subset $I \subseteq \omega_1$ such that for every $j \in I$, Γ_j is k-inconsistent. Without loss of generality, $I = \omega$. We will show now that the array $(\psi(x; a^j_i) : i, j < \omega)$ witnesses that $\psi(x; y, z)$ has TP_2. Each row $\{\psi(x, a^j_i) : i < \omega\} = \Gamma_j$ is k-inconsistent. For each $f : \omega \to \omega$, $a^j_{f(j)} \models (p \otimes_A p)|_A B(a^i_{f(i)} : i < j)$ and therefore all sequences $(a^j_{f(j)} : j < \omega)$ have the same type over B, the type $(p \otimes_A p)^{(\omega)_A}$. By Lemma 5.1, this type is finitely satisfiable in A. To check the consistency of $\{\psi(x; a^j_{f(j)}) : j < \omega\}$ we may assume $f(j) = 0$ for all j. If it is not consistent, then for some $n < \omega$, $\models \neg\exists x \bigwedge_{j<n} \psi(x; a^j_0)$. Each a^j_0 is of the form $a^j_0 = b_j c_j$ where $b_j \models p$ and $c_j \models p|_A Bb_j$. Since p extends r and is finite satisfiable in A, $p(y) \otimes_A p(z) \vdash y \neq z$. Consequently, $b_j \neq c_j$. By similar reasons, $b_j \neq b_l$ and $c_j \neq c_l$ for $j \neq l$. If $b_j = c_l$ for some $j \neq l$, then, by indiscernibility, $b_j = c_j$. Hence $b_j \neq c_l$ for all j, l. By finite satisfiability there are $(b_j c_j : j < n)$ in A such that $\models \neg\exists x \bigwedge_{j<n} \psi(x; b_j, c_j)$ and $b_j \neq c_l$ for all j, l and $b_j \neq b_l$ and $c_j \neq c_l$ for all $j \neq l$. But by choice of φ, $\models \exists x (\bigwedge_{j<n} \varphi(x, b_j) \wedge \bigwedge_{j<n} \neg\varphi(x, c_j))$.

Now we finish the proof using our set C. Let M be a model such that $BC \subseteq M$ and choose $ab \models (p \otimes_A p)|_A M$. By the claim, $\psi(x; a, b)$ does not divide over M. By Theorem 4.7, $\psi(x; a, b)$ does not fork over M. Hence $\psi(x; a, b) \in \mathfrak{p}(x)$ for some global type $\mathfrak{p}(x)$ which does not fork over M. Since $a \equiv_M b$ and $\varphi(x, a) \in \mathfrak{p}$, if $\mathfrak{p}$ were M-invariant, then $\varphi(x, b) \in \mathfrak{p}$ but this is not possible since $\neg\varphi(x, b) \in \mathfrak{p}$. □

6 Thorn-forking in simple theories

The proofs in this section have been explained to us by Hans Adler. We assume knowledge of hyperimaginaries as exposed in [7].

Definition 6.1 T has *weak elimination of hyperimaginaries* if for every hyperimaginary e there is some sequence $e' = (e_i : i \in I)$ of imaginaries e_i such that $\mathrm{bdd}(e) = \mathrm{bdd}(e')$.

Definition 6.2 Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation. Given $p(x) \in S(B)$, we say that a set $C \subseteq \mathrm{acl}(B)$ is a *weak canonical base* of p (with respect to $\mathop{\smile\!\!\!\!\!\!|}$) if for every $a \models p$, $a \mathop{\smile\!\!\!\!\!\!|}_C B$ and for every set $C' \subseteq \mathrm{acl}(B)$: if $a \mathop{\smile\!\!\!\!\!\!|}_{C'} B$, then $C \subseteq \mathrm{acl}(C')$. Clearly, if we require additionally $C = \mathrm{acl}(C)$, then C is unique and then we call it *the weak canonical base of* p. We say that $\mathop{\smile\!\!\!\!\!\!|}$ *has weak canonical bases* if every type has a weak canonical base with respect to $\mathop{\smile\!\!\!\!\!\!|}$.

Proposition 6.3 *In a simple theory T, T has weak elimination of hyperimaginaries if and only if $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has weak canonical bases in T^{eq}.*

Proof: Weak elimination of hyperimaginaries clearly provides weak canonical bases for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ in T^{eq}. Assume now $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has weak canonical bases in T^{eq}. Let $e = a_E$ be a hyperimaginary and choose b such that $a \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{f}}_{e} b$ and $b \equiv_e a$. Let $\mathfrak{p}(x)$ be a global nonforking extension of $\mathrm{tp}(a/b)$ and let e' be the canonical base of $\mathfrak{p}$. Since $\mathfrak{p}$ does not fork over e, $e' \in \mathrm{bdd}(e)$. Since $a \downarrow_{e'} e$ and $e \in \mathrm{dcl}^{\mathrm{eq}}(a)$, $e \in \mathrm{bdd}(e')$ (in fact, e is definable over e'; see Proposition 18.7 in [7]). Hence $\mathrm{bdd}(e) = \mathrm{bdd}(e')$. Now let e'' be a sequence of imaginaries enumerating the weak canonical base of $\mathfrak{p} \restriction ab$. Since $\mathfrak{p}$ does not fork over ab, it does not fork over e'' and therefore $e' \in \mathrm{bdd}(e'')$. Since $\mathfrak{p} \restriction ab$ does not fork over a and it does not fork over b, $e'' \in \mathrm{bdd}(a) \cap \mathrm{bdd}(b) = \mathrm{bdd}(e)$. We conclude $\mathrm{bdd}(e) = \mathrm{bdd}(e'')$. □

Lemma 6.4 *If $\downarrow$ is an independence relation, then $a \downarrow_C b$ if and only if there is a $\downarrow$-free sequence $(b_i : i < \omega)$ starting with $b_0 = b$ which is Ca-indiscernible.*

Proof: Let $a \downarrow_C b$. It is easy to obtain a sequence $(b_i : i < \omega)$ such that $b_0 = b$, $b_n \equiv_{Ca} b$, and $b_n \downarrow_C ab_{<n}$. We can extend the sequence and apply Erdős-Rado to produce another sequence with the same properties and which is moreover Ca-indiscernible.

For the other direction, assume the sequence $(b_i : i < \omega)$ starts with $b_0 = b$, is $\downarrow$-free and is Ca-indiscernible. Let κ be an upper bound for local character in $\downarrow$ and extend the sequence to a Ca-indiscernible sequence $(b_i : i < \kappa^+)$. Choose $D \subseteq C\{b_i : i < \kappa^+\}$ of cardinality $\leq \kappa$ such that $a \downarrow_D Cb_{<\kappa^+}$. There is some $j < \kappa^+$ such that $D \subseteq b_{<j}C$. Then $a \downarrow_{Cb_{<j}} b_{<\kappa^+}$ and therefore $a \downarrow_{Cb_{<j}} b_j$. Since $b_j \downarrow_C b_{<j}$, we conclude $a \downarrow_C b_j$ and therefore $a \downarrow_C b$. □

Remark 6.5 *For any independence relation $\downarrow$ with weak canonical bases: if $C_1, C_2 \subseteq B$, $A \downarrow_{C_1} B$, and $A \downarrow_{C_2} B$, then $A \downarrow_{\mathrm{acl}(C_1)\cap\mathrm{acl}(C_2)} B$.*

Proof: Clear, since for any tuple $a \in A$ the weak canonical base of $\mathrm{tp}(a/B)$ belongs to $\mathrm{acl}(C_1) \cap \mathrm{acl}(C_2)$. □

Proposition 6.6 *If $\downarrow$ is a strict independence relation with weak canonical bases, then: $a \downarrow_C B$ if and only if for some infinite linearly ordered set I there is a BC-indiscernible sequence $(a_i : i \in I)$ of realizations of $\mathrm{tp}(a/BC)$ such that $\mathrm{acl}(a_{<i}) \cap \mathrm{acl}(a_{\geq i}) \subseteq \mathrm{acl}(C)$ for every $i \in I$.*

Proof: $\Rightarrow$. Choose with Lemma 6.4 a BC-indiscernible and $\downarrow$-free sequence $(a_i : i < \omega)$ such that $a_0 = a$. Since $a_{<i} \downarrow_C a_{\geq i}$ and $\downarrow$ satisfies anti-reflexivity, $\mathrm{acl}(a_{<i}) \cap \mathrm{acl}(a_{\geq i}) \subseteq \mathrm{acl}(C)$.

$\Leftarrow$. Let κ be a regular cardinal greater than the upper bound for local character, and let I be an ordered set with order type $\kappa{+}1{+}\kappa^*$, say $I = I^- \cup \{c\} \cup I^+$ where I^- has order type κ and I^+ has the inverse order type κ^*. We may assume this is the order type of the sequence $a_I = (a_i : i \in I)$. Find $D \subseteq \{a_i : i \in I^-\}$ such that $BC \downarrow_D (a_i : i \in I^-)$ and $|D| < \kappa$. We can assume D is a proper initial segment of $(a_i : i \in I^-)$ and then by indiscernibility $BC \downarrow_D a_I$. It follows that $BC \downarrow_{a_{I^-}} a_I$. By a similar argument, $BC \downarrow_{a_{I^+}} a_I$ and therefore $BC \downarrow_{a_c a_{I^+}} a_I$. By Remark 6.5 $BC \downarrow_A a_I$ for $A = \mathrm{acl}(a_{I^-}) \cap \mathrm{acl}(a_c a_{I^+})$ and then, by assumption, $BC \downarrow_C a_I$. Therefore $a \downarrow_C B$. □

Theorem 6.7 *If T is simple and has weak elimination of hyperimaginaries, then $\downarrow^{\text{þ}} = \downarrow^{\mathrm{f}}$ in T^{eq}.*

Proof: We can assume $T = T^{\mathrm{eq}}$. By Proposition 1.6 it is enough to prove that $\downarrow^{\mathrm{f}}$ is weaker than any strict independence relation $\downarrow$. Assume $a \downarrow_C b$. By Proposition 6.4, there is a $\downarrow$-free sequence $(b_i : i < \omega)$ starting with $b_0 = b$ which is Ca-indiscernible. Since

$\downarrow$ is strict and $a_{<i} \downarrow_C a_{\geq i}$, it follows that $\mathrm{acl}(a_{<i}) \cap \mathrm{acl}(a_{\geq i}) \subseteq \mathrm{acl}(C)$. By Proposition 6.3, $\downarrow^{\mathrm{f}}$ has weak canonical bases and by Proposition 6.6, $a \downarrow^{\mathrm{f}}_C b$. □

Definition 6.8 We say that T has *stable forking* if whenever $B \supseteq C$ and $a \not\downarrow^{\mathrm{f}}_C B$, then there is some stable formula $\varphi(x,y) \in L$ and some tuple $b \in BC$ such that $\models \varphi(a,b)$ and $\varphi(x,b)$ forks over C.

Question 6.9 *Does stable forking of T imply stable forking of T^{eq} ?*

Proposition 6.10 *Every simple theory with stable forking has weak elimination of hyperimaginaries.*

Proof: By Proposition 6.3 it suffices to show that $\downarrow^{\mathrm{f}}$ has weak canonical bases in T^{eq}. In fact the proof of this proposition shows that it is enough to find weak canonical bases for types of real elements over real parameters. Let $p(x) \in S(B)$ be such a type, let $\mathfrak{p}(x) \in S(\mathfrak{C})$ be a global nonforking extension of p and for each stable $\varphi(x,y) \in L$ let $e_\varphi \in \mathfrak{C}^{\mathrm{eq}}$ be the canonical parameter of a definition of $\mathfrak{p} \upharpoonright \varphi$. Then $\mathfrak{p} \upharpoonright \varphi$ does not fork over A if and only if $e_\varphi \in \mathrm{acl}^{\mathrm{eq}}(A)$. If $C = \{e_\varphi : \varphi(x,y) \in L \text{ is stable }\}$, then $C \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$ and by stable forking, $p(x)$ does not fork over $D \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$ if and only if $C \subseteq \mathrm{acl}^{\mathrm{eq}}(D)$. □

Corollary 6.11 *If T is a simple theory with elimination of hyperimaginaries or with stable forking, then $\downarrow^{\mathrm{b}} = \downarrow^{\mathrm{f}}$ in T^{eq}. In particular, this happens in stable and in supersimple theories.*

Proof: By Proposition 6.10 and Theorem 6.7. For elimination of hyperimaginaries in supersimple theories see [7]. □

7 Simple types

Definition 7.1 A partial type $\pi(x)$ is *simple* if $\mathrm{D}(\pi,\Delta,k) < \omega$ for all Δ, k. [2]

Remark 7.2 *Any extension of a simple type is simple. Moreover, T is simple if and only if in T all types are simple.*

Hart, Kim and Pillay take point 3 in the next proposition as a definition of simple type (see [11]). They seem to consider only complete types. The equivalence with point 4 is from Adler (see [5])

Proposition 7.3 *The following are equivalent for any partial finitary type $\pi(x)$ over A:*

1. *$\pi(x)$ is simple.*
2. *There is no formula $\varphi(x,y) \in L$, natural number k and sequences $(a_f : f \in \omega^\omega)$, $(b_s : s \in \omega^{<\omega})$ such that $a_f \models \pi \cup \{\varphi(x, b_{f\upharpoonright n}) : n < \omega\}$ for all f and $\{\varphi(x, b_{s^\frown n}) : n < \omega\}$ is k-inconsistent for all s.*
3. *For any completion $p(x) \in S(B)$ of $\pi(x)$ there is some $C \subseteq B$ such that $|C| \leq |T|$ and $p(x)$ does not fork over C.*

[2] See chapter 3 of [7] for more information on this rank.

4. *If $A \subseteq C$, $a \models \pi$ and $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_C a$, then $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_C b$.*

Proof: 1 ⇔ 2 is standard. 1 ⇒ 3. Let $p(x) \in S(B)$ be a completion of $\pi(x)$. We can choose $C \subseteq B$ of cardinality $\leq |T|$ such that for each $\varphi(x,y) \in L$, for each $k < \omega$, $\mathrm{D}(p,\varphi,k) = \mathrm{D}(p \restriction C, \varphi, k)$. Clearly, p does not fork over C. For instance, see Lemma 4.14 in [7].

3 ⇒ 4. We can easily adapt propositions 12.4 and 12.5 of [7]. Without loss of generality, b is finite. Assume $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_C a$ and find $(b_i : i < \omega)$, an aC-indiscernible sequence such that $b_0 = b$, $b_i \equiv_{aC} b$ and $b_{<i} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_C b_i$. Extend the sequence to an aC-indiscernible sequence $(b_i : i < \kappa)$ with $\kappa > |T|$. By simplicity, there is some $D \subseteq Cb_{<j}$ for some $j < \kappa$ such that $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_D Cb_{<\kappa}$. It follows that $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_C b_j$ and therefore that $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_C b$.

4 ⇒ 1. Assume $\mathrm{D}(\pi,\varphi,k) = \omega$ for some φ and some k. By Proposition 2.23 in [7], for some sequence $(a_i : i \leq \omega)$ such that each $\varphi(x,a_i)$ k-divides over $Aa_{<i}$ there is some $c \models \pi(x) \cup \{\varphi(x,a_i) : i \leq \omega\}$ such that $(a_i : i \leq \omega)$ is Ac-indiscernible. Then $\mathrm{tp}(a_\omega/Aa_{<\omega}c)$ is finitely satisfiable in $a_{<\omega}$ and therefore $a_\omega \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{Aa_{<\omega}} c$ but $\mathrm{tp}(c/Aa_{\leq\omega})$ divides over $a_{<\omega}$ and therefore $c \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{Aa_{<\omega}} a_\omega$. □

Remark 7.4 *Proposition 7.3 holds also for non finitary types if 3 is reformulated as: for any completion $p(x) \in S(B)$ of $\pi(x)$ there is some $C \subseteq B$ such that $|C| \leq |T| + |x|$ and $p(x)$ does not fork over C.*

Remark 7.5 1. *If $\pi(x,y)$ is simple, then the type $\exists y \pi(x,y)$ is simple. Therefore, if $\pi(x,y)$ is a simple partial type over A, then $\pi \restriction x = \{\varphi(x) \in L(A) : \pi(x,y) \vdash \varphi(x)\}$ is simple.*

2. *$\mathrm{tp}(a/A)$ and $\mathrm{tp}(b/Aa)$ are simple if and only if $\mathrm{tp}(ab/A)$ is simple. More generally, $\mathrm{tp}(a_i/Aa_{<i})$ is simple for all $i < \alpha$ if and only if $\mathrm{tp}((a_i : i < \alpha)/A)$ is simple.*

3. *If the types $\pi_i(x_i)$ are simple for all $i \in I$, then $\bigcup_{i\in I} \pi_i(x_i)$ is simple.*

4. *If the types $\pi_1(x), \pi_2(x)$ are simple, then the type $\pi_1(x) \vee \pi_2(x)$ is simple.*

Proof: 1. Note that if $a \models \exists y \pi(x,y)$ then $ab \models \pi$ for some b.

2. This follows easily using, for instance, point 3 in Proposition 7.3.

3 follows from 2. □

Question 7.6 1. *Assume $p(x) \in S(B)$ is simple and does not fork over $A \subseteq B$. Is $p \restriction A$ simple ?* [3]

2. *Assume π is simple. Does forking equal dividing for realizations of π ?*

3. *Let P be the collection of all simple complete 1-types over A and let D be the union of all $p(\mathfrak{C})$ for $p \in P$. Is D with its induced structure over A simple ?*

8 nip-types

Definition 8.1 A partial type $\pi(x)$ is nip or *dependent* if there are no sequences $(a_i : i < \omega)$ and $(b_X : X \subseteq \omega)$ and formula $\varphi(x,y) \in L$ such that $a_i \models \pi$ and $\models \varphi(a_i, b_X) \Leftrightarrow i \in X$.

[3]Posed by H. Adler.

The definition of nip type and many results in this section are based on [5].

Remark 8.2 *1. Any extension of a* nip *type is* nip.

2. T is nip *if and only if all types are* nip.

Lemma 8.3 *The following are equivalent for any partial type $\pi(x)$:*

1. *$\pi(x)$ is* nip.
2. *There are no sequences $(a_i : i < \omega)$ and $(b_X : X \subseteq \omega)$ and formula $\varphi(x, y) \in L$ such that $b_X \models \pi$ and $\models \varphi(b_X, a_i) \Leftrightarrow i \in X$.*
3. *For each $\varphi(x, y) \in L$ there is some conjunction $\psi(x)$ of formulas in $\pi(x)$ such that $\varphi(x, y) \wedge \psi(x)$ does not have the independence property.*
4. *For each $\varphi(y, x) \in L$ there is some conjunction $\psi(x)$ of formulas in $\pi(x)$ such that $\varphi(y, x) \wedge \psi(x)$ does not have the independence property.*

Proof: *1* $\Leftrightarrow$ *2* is as the proof of Lemma 1.3 of [6]. *1* $\Leftrightarrow$ *3* and *2* $\Leftrightarrow$ *4* are proved by compactness. □

Proposition 8.4 *The following are equivalent for any partial type $\pi(x)$ over A:*

1. *$\pi(x)$ is* nip.
2. *If $a \models \pi$, $\varphi(x, y) \in L(A)$, $\alpha \geq \omega$ is a limit ordinal and $(b_i : i < \alpha)$ is A-indiscernible, then for some $\beta < \alpha$, for all $i, j \geq \beta$, $\models \varphi(a, b_i) \leftrightarrow \varphi(a, b_j)$.*
3. *If $\varphi(y, x) \in L$, $\alpha \geq \omega$ is a limit ordinal and $(b_i : i < \alpha)$ is an A-indiscernible sequence of realizations b_i of π, then for any tuple a there is some $\beta < \alpha$, for all $i, j \geq \beta$, $\models \varphi(a, b_i) \leftrightarrow \varphi(a, b_j)$.*
4. *If $a \models \pi$, α is a limit ordinal of cofinality $\geq (|T| + |a| + \kappa)^+$ and $(b_i : i < \alpha)$ is an A-indiscernible sequence of tuples b_i of length $\leq \kappa$, then for some $\beta < \alpha$, $(b_i : \beta \leq i < \alpha)$ is Aa-indiscernible.*
5. *For any tuple a, if α is a limit ordinal of cofinality $\geq (|T| + |a| + |x|)^+$ and $(b_i : i < \alpha)$ is an A-indiscernible sequence of realizations b_i of π, then for some $\beta < \alpha$, $(b_i : \beta \leq i < \alpha)$ is Aa-indiscernible.*

Proof: *1* $\Leftrightarrow$ *2* $\Leftrightarrow$ *3*: use Lemma 8.3, compactness and the fact that a formula $\varphi(x, y)$ is nip iff $\mathrm{alt}(\varphi) < \omega$. *2* $\Rightarrow$ *4*: as in the proof of Proposition 1.8 of [6]. *3* $\Rightarrow$ *5*: similar except that now we also need to use point *2* of the next remark (no vicious circle arises) that makes sure that $\pi(x_1) \wedge \ldots \wedge \pi(x_n)$ is nip. *4* $\Rightarrow$ *2* and *5* $\Rightarrow$ *3*: note that if *2* or *3* fail for some φ and some α, then by compactness they also fail for every limit $\alpha \geq \omega$. □

Remark 8.5 *1. If $\pi(x, y)$ is* nip*, then the type $\exists y \pi(x, y)$ is* nip. *Therefore, if $\pi(x, y)$ is a* nip *partial type over A, then $\pi \restriction x = \{\varphi(x) \in L(A) : \pi(x, y) \vdash \varphi(x)\}$ is* nip.

2. If the types $\pi_i(x_i)$ are nip *for all $i \in I$, then $\bigcup_{i \in I} \pi_i(x_i)$ is* nip.

3. If the types $\pi_1(x), \pi_2(x)$ are nip*, then the type $\pi_1(x) \vee \pi_2(x)$ is* nip.

4. $\mathrm{tp}(ab/A)$ *is* nip *if and only if* $\mathrm{tp}(b/A)$ *and* $\mathrm{tp}(a/Ab)$ *are* nip.

Proof: *1* is clear by definition of nip type.

2. Similar to the proof of Proposition 1.9 in [6]. We use point *4* of Proposition 8.4. It is enough to prove it for finite $|I|$ and since it can be done by induction, it suffices to consider the case of two nip types $\pi_1(x_1)$, $\pi_2(x_2)$ over A. $a_1 \models \pi_1$, $a_2 \models \pi_2$ and $(b_i : i < \alpha)$ is an A-indiscernible sequence (for the right ordinal α). Then for some $\beta < \alpha$, $(b_i : \beta \leq i < \alpha)$ is Aa_1-indiscernible. This means that $(a_1 b_i : \beta \leq i < \alpha)$ is A-indiscernible. Again, there is some $\gamma < \alpha$ such that $(a_1 b_i : \gamma \leq i < \alpha)$ is Aa_2-indiscernible. Then $(b_i : \gamma \leq i < \alpha)$ is Aa_1a_2-indiscernible.

3. Use point *4* of Proposition 8.4.

4 We only need to check that $\mathrm{tp}(ab/A)$ is nip if $\mathrm{tp}(b/A)$ and $\mathrm{tp}(a/Ab)$ are nip. The proof is like in case *2*. □

Proposition 8.6 *If $p(x) \in S(B)$ is* nip *and it does not fork over $A \subseteq B$, then $p(x)$ does not Lascar-split over A.*

Proof: Like Proposition 9.6 of [6]. Let $\mathfrak{p}(x)$ be a global nonforking extension of p assume it does Lascar-split over A. Then for some tuples a, b, $\models \mathrm{nc}_A(a, b)$ and $\varphi(x, a) \in \mathfrak{p}(x)$ but $\neg\varphi(x, b) \in \mathfrak{p}(x)$. Choose $\theta(x) \in p$ such that $\psi(x, y) = \varphi(x, y) \wedge \theta(x)$ does not have the independence property. We have an A-indiscernible sequence $(a_i : i < \omega)$ such that $a_0 = a$, $a_1 = b$ and (since $\mathfrak{p}$ does not fork over A) there is a realization c of $\{\psi(x, a_{2i}) \wedge \neg\psi(x, a_{2i+1}) : i < \omega\}$. But then c witnesses that $\varphi(x, y) \wedge \psi(x)$ has infinite alternation number. □

Corollary 8.7 *If $\pi(x)$ is a* nip *partial type over A, then there is a bounded number of types $\mathfrak{p} \in S(\mathfrak{C})$ extending π that do not fork over A. In fact the number is $\leq 2^{2^{|T|+|A|+|x|}}$.*

Proposition 8.8 *Let $\pi(x)$ be* nip*. For any indiscernible sequence a without last element and for any infinite indiscernible set I, if they consist of realizations of π, then for any set B, $\mathrm{Av}(a/B) \in S(B)$ and $\mathrm{Av}(I/B) \in S(B)$.*

Proof: Clear since for any $\varphi(x, y) \in L$ there is some conjunction $\psi(x)$ of formulas of π such that $\mathrm{alt}(\varphi(x, y) \wedge \psi(x)) < \omega$. Check section 6 of [6] for details. □

Definition 8.9 A nip-*sequence over A* is a Morley sequence over A (A-indiscernible and A-independent in the sense of nonforking) of realizations of a nip-type $p(x) \in S(A)$.

Remark 8.10 *If B is Lascar-complete over $A \subseteq B$, $\mathfrak{p}(x) \in S(\mathfrak{C})$ does not Lascar-split over A, and the sequence $(a_i : i < \alpha)$ is constructed with the rule $a_i \models \mathfrak{p} \restriction Ba_{<i}$, then a is B-indiscernible.*

Proof: If $i_0 < \ldots i_n < \alpha$ and $p(x) = \mathfrak{p} \restriction B$, then $\mathrm{tp}(a_{i_0}, \ldots, a_{i_n}/B) = p^{(n)_A}$. □

Lemma 8.11 *Let $\alpha \geq \omega$, let $a = (a_i : i < \alpha)$ be an A-indiscernible sequence, and let $\mathfrak{p}(x)$ be a global type which does not Lascar-split over A and such that $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i < \omega$. If $b \models \mathfrak{p} \restriction Aa$ then $a^\frown(b)$ is A-indiscernible.*

Proof: Assume $i_0 < \ldots < i_n < \alpha$, $\varphi(x_0, \ldots, x_n, y) \in L(A)$ and $\models \varphi(a_{i_0}, \ldots, a_{i_n}, b)$. Since $b \models \mathfrak{p} \restriction Aa$, $\varphi(a_{i_0}, \ldots, a_{i_n}, x) \in \mathfrak{p}(x)$. Notice that $a_{i_0}, \ldots, a_{i_n} \overset{\mathrm{Ls}}{\equiv}_A a_0, \ldots, a_n$. Since $\mathfrak{p}$ does not Lascar-split over A, then $\varphi(a_0, \ldots, a_n, x) \in \mathfrak{p}$. Hence, $\models \varphi(a_0, \ldots, a_n, a_{n+1})$. □

Lemma 8.12 *For each* nip *sequence $a = (a_i : i < \alpha)$ over A with $\alpha \geq \omega$ there is a unique global type $\mathfrak{p}$ that generates a in the sense that $\mathfrak{p}$ does not fork over A and $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i < \alpha$.*

Proof: Existence of $\mathfrak{p}$ does not need nip nor indiscernibility and it follows from the fact that if $p_i(x) = \mathrm{tp}(a_i/Aa_{<i})$ then $\bigcup_{i<\alpha} p_i$ does not fork over A and can be extended to some $\mathfrak{p} \in S(\mathfrak{C})$ nonforking over A.

Uniqueness is as in the proof of Proposition 11.10 of [6]: assuming $\mathfrak{p}_1, \mathfrak{p}_2$ generate a, we first claim that any A-indiscernible extension $a' = (a_i : i < \beta)$ of a can be extended again adding a realization of $\mathfrak{p}_1 \restriction Aa'$ or a realization of $\mathfrak{p}_2 \restriction Aa'$. This follows from Proposition 8.6 and Lemma 8.11. Now assume $\varphi(x, b) \in \mathfrak{p}_1$ and $\neg\varphi(x, b) \in \mathfrak{p}_2$ and construct a sequence $c = (c_i : i < \omega)$ such that $c_{2\cdot i} \models \mathfrak{p}_1 \restriction Aabc_{<2\cdot i}$ and $c_{2\cdot i+1} \models \mathfrak{p}_2 \restriction Aabc_{<2\cdot i+1}$. Using the claim we see that $a^\frown c$ is A-indiscernible, which contradicts nip of p. □

Lemma 8.13 *Let $\alpha \geq \omega$ and let $a = (a_i : i < \alpha)$ a* nip*-sequence over A with global type $\mathfrak{p}(x)$. If $b = (b_i : i < \beta)$ and $b_i \models \mathfrak{p} \restriction Aab_{<i}$ for all $i < \beta$, then $a^\frown b$ is* nip *over A.*

Proof: It is enough to check that $a^\frown b$ is indiscernible over A. We may assume $\beta = n < \omega$ and the proof can be done by induction and for this Lemma 8.11 is enough. □

Proposition 8.14 *If $\alpha \geq \omega$ and $a = (a_i : i < \alpha)$ and $b = (b_i : i < \alpha)$ are* nip*-sequences over A and have the same Lascar strong-type over A, then they have the same global type.*

Proof: Let $f \in \mathrm{Autf}(\mathfrak{C}/A)$ be such that $f(a) = b$. Let $\mathfrak{p}$ be the global type of a. Then $\mathfrak{p}^f$ is the global type of b. Since $\mathfrak{p}$ does not Lascar-split over A, $\mathfrak{p}^f = \mathfrak{p}$. □

Remark 8.15 *If $p(x) \in S(A)$ is* nip *and it does not fork over A, then for all $a, b \models p$:*

$$a \overset{\mathrm{Ls}}{\equiv}_A b \text{ if and only if } \models \mathrm{nc}^2_A(a, b)$$

and therefore $\overset{\mathrm{Ls}}{\equiv}_A = \overset{\mathrm{KP}}{\equiv}_A$ for realizations of p.

Proof: By Proposition 11.7 of [6]. □

Proposition 8.16 *If $\mathfrak{p} \restriction A$ is* nip*, then $\mathfrak{p}$ Lascar-splits over A if and only if $\mathfrak{p}$ KP-splits over A.*

Proof: Like Proposition 11.13 of [6]. Assume $\mathfrak{p}$ does not Lascar-split over A, let $f \in \mathrm{Autf}(\mathfrak{C}/\mathrm{bdd}(A))$, choose $B \supseteq A$ Lascar-complete over A, let $p(x) = \mathfrak{p} \restriction B$, and let a realize the product $p^{(\omega)_A}$. Since $a \overset{\mathrm{KP}}{\equiv}_A f(a)$ and $p^{(\omega)_A}$ is a nip type that does not fork over A, $a \overset{\mathrm{Ls}}{\equiv}_A f(a)$. Then, these two Morley sequences have the same global type and $\mathfrak{p} = \mathfrak{p}^f$. □

Corollary 8.17 *Let $\mathfrak{p}$ be definable and assume $\mathfrak{p} \restriction A$ is* nip*. Then $\mathfrak{p}$ does not fork over A if and only if $\mathfrak{p}$ is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.

Proof: If $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$, it does not fork over $\mathrm{acl}^{\mathrm{eq}}(A)$ and therefore it does not fork over A. For the other direction, Assume $\mathfrak{p}$ does not fork over A. By Proposition 8.16, it does not KP-split over A. Hence $\mathfrak{p}$ is $\mathrm{bdd}(A)$-invariant. Let $\varphi(x, y) \in L$ and let c_φ be a canonical parameter for $\{a : \varphi(x, a) \in \mathfrak{p}\}$. If $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(A))$, $\mathfrak{p}^f = \mathfrak{p}$ and therefore $f(c_\varphi) = c_\varphi$. Hence $c_\varphi \in \mathrm{bdd}(A)$. Since c_φ is an imaginary, $c_\varphi \in \mathrm{acl}^{\mathrm{eq}}(A)$. This means that $\mathfrak{p} \restriction \varphi$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. □

Proposition 8.18 *If $a = (a_i : i < \alpha)$ with $\alpha \geq \omega$ is a* nip*-sequence over A and it is totally indiscernible over A, then its global type is* $\mathrm{Av}(a/\mathfrak{C})$.

Proof: Since it is clear that $a_i \models \mathrm{Av}(a/Aa_{<i})$, we only need to check that $\mathrm{Av}(a/\mathfrak{C})$ does not Lascar-split over A. Assume f is a strong automorphism over A. Then a and $f(a)$ have the same Lascar strong-type over A and by Proposition 8.14 they have the same global type $\mathfrak{p}$ and therefore we can construct a sequence $c = (c_i : i < \omega)$ such that $a^\frown c$ and $f(a)^\frown c$ are A-indiscernible (it suffices to choose $c_i \models \mathfrak{p} \restriction Aaf(a)c_{<i}$). It follows that $\mathrm{Av}(a/\mathfrak{C}) = \mathrm{Av}(a^\frown c/\mathfrak{C}) = \mathrm{Av}(c/\mathfrak{C}) = \mathrm{Av}(f(a)^\frown c/\mathfrak{C}) = \mathrm{Av}(f(a)/\mathfrak{C})$ and therefore $\mathrm{Av}(a/\mathfrak{C})^f = \mathrm{Av}(a/\mathfrak{C})$. □

9 Generically stable types revisited

Definition 9.1 *Generically stable types* have been defined in section 15 of [6] and they have been discussed under the assumption that the theory is nip. We now redefine the notion to make it compatible with [20]: a global type $\mathfrak{p}(x)$ is *generically stable over A* if and only if it is A-invariant and for every $\alpha \geq \omega$, for every $\varphi(x) \in L(\mathfrak{C})$, for every Morley sequence $a = (a_i : i < \alpha)$ generated by $\mathfrak{p}$ over A, the set $\{i < \alpha : \models \varphi(a_i)\}$ is finite or cofinite. Since all Morley α-sequences over A generated by an A-invariant type $\mathfrak{p}$ have the same type over A, it is enough to check the condition for *some* Morley α-sequence over A generated by $\mathfrak{p}$. A global type $\mathfrak{p}$ is *generically stable* if it is generically stable over some A. A type $p(x) \in S(A)$ is *generically stable* if some global extension of p is generically stable over A.

Remark 9.2 *For every infinite sequence $a = (a_i : i \in I)$, the average type $\mathrm{Av}(a/\mathfrak{C}) = \{\varphi(x) \in L(\mathfrak{C}) : \{i \in I : \not\models \varphi(a_i)\}$ is finite $\}$ is a partial type over A. Notice that a global type $\mathfrak{p}(x)$ is generically stable over A iff it is A-invariant and for every $\alpha \geq \omega$, for every Morley sequence $a = (a_i : i < \alpha)$ generated by $\mathfrak{p}$ over A, $\mathrm{Av}(a/\mathfrak{C})$ is a complete type.*

Remark 9.3 *1. If $\mathfrak{p}$ is generically stable and it is B-invariant, then it is generically stable over B.*

2. Assuming the type $\mathfrak{p}$ is A-invariant, the following are equivalent:

(a) $\mathfrak{p}$ is generically stable over A.

(b) For every Morley sequence $a = (a_i : i < \omega + \omega)$ generated by $\mathfrak{p}$ over A, $\mathrm{Av}(a/\mathfrak{C})$ is a complete type.

(c) For every $\varphi(x, y) \in L$ there is some $n_\varphi < \omega$ such that for every $\alpha \geq \omega$, for every Morley sequence $a = (a_i : i < \alpha)$ generated by $\mathfrak{p}$ over A, for every b, if $\{i < \alpha : \models \varphi(a_i, b)\}$ has cardinality $\geq n_\varphi$, then it is cofinite.

3. If $\mathfrak{p}$ is generically stable over A and $a = (a_i : i < \alpha)$ is a Morley sequence generated by $\mathfrak{p}$ over A, then $\mathrm{Av}(a/\mathfrak{C}) = \mathfrak{p}$.

Proof: *1.* Assume $\mathfrak{p}$ is generically stable over A. Let $a = (a_i : i < \alpha)$ be a Morley sequence generated by $\mathfrak{p}$ over B and let $\varphi(x) \in L(\mathfrak{C})$. Choose a Morley sequence $b = (b_i : i < \alpha)$ generated by $\mathfrak{p}$ over AB. Then b is generated by $\mathfrak{p}$ over B (and hence $a \equiv_B b$) and over A. Since $\{i < \alpha : \models \varphi(b_i)\}$ is finite or cofinite, the same happens with $\{i < \alpha : \models \varphi(a_i)\}$.

2. Only *(b)* $\Rightarrow$ *(c)* needs to be checked. It is enough to prove *(c)* for all limit $\alpha \geq \omega$. Assume for some $\varphi(x, y) \in L$ for all $n < \omega$ there is some limit ordinal $\alpha \geq \omega$ for which there is some b and some sequence $(a_i : i < \alpha)$ generated by $\mathfrak{p}$ over A such that $\{i < \alpha : \models \varphi(a_i, b)\}$ is of cardinality $\geq n$ and $\{i < \alpha : \models \neg\varphi(a_i, b)\}$ is infinite. There are two possible cases:

either $\{i < \alpha :\models \varphi(a_i, b)\}$ is cofinal in α or it is not. We may assume that for every n the first case holds or for every n the second case holds. In the first case, $\{\neg\varphi(x_i, y) : i < \omega\} \cup \{\varphi(x_{\omega+i}, y) : i < \omega\}$ is consistent with the power $\mathfrak{p}^{(\omega+\omega)_A} \restriction A(x_i : i < \omega + \omega)$, in contradiction with *(b)*. In the second case $\{\varphi(x_i, y) : i < \omega\} \cup \{\neg\varphi(x_{\omega+i}, y) : i < \omega\}$ is consistent with the power $\mathfrak{p}^{(\omega+\omega)_A} \restriction A(x_i : i < \omega + \omega)$, again in contradiction with *(b)*.

3. It is enough to show that $\mathfrak{p} \subseteq \mathrm{Av}(a/\mathfrak{C})$. Let $\varphi(x, y) \in L$ and $c \in \mathfrak{C}$ be such that $\varphi(x, c) \in \mathfrak{p}(x)$. Choose $b = (b_i : i < \alpha)$ a Morley sequence generated by $\mathfrak{p}$ over Ac. Then $\models \varphi(b_i, c)$ for all $i < \alpha$ and therefore $\varphi(x, c) \in \mathrm{Av}(b/\mathfrak{C})$. Since b is generated by $\mathfrak{p}$ over A, $a \equiv_A b$ and hence $\mathrm{Av}(a/\mathfrak{C}) = \mathrm{Av}(b/\mathfrak{C})$. □

Proposition 9.4 *If $a = (a_i : i < \alpha)$ with $\alpha \geq \omega$ is a sequence generated over M by the type $\mathfrak{p}(x) \in S(\mathfrak{C})$ and $\mathfrak{p}$ is M-definable and coinherits from M, then a is totally indiscernible over M.*

Proof: This is Proposition 15.2 of [6]. We give anyway a shorter proof based on Théorème 12.15 of [21]. We may assume $\alpha = \omega$ and it is enough to prove that

$$a_0 \dots a_{i-1} a_i a_{i+1} a_{i+2} \dots a_n \equiv_M a_0 \dots a_{i-1} a_{i+1} a_i a_{i+2} \dots a_n.$$

Take a model $N \supseteq Ma$ which is complete over M. Then $\mathfrak{p}$ is N-definable and coinherits from N. The unique N-definable extension of $p(x) = \mathfrak{p} \restriction N$ over any set $B \supseteq N$ is $\mathfrak{p} \restriction B$, which moreover is the unique heir of p over B (see Proposition 7.6 of [7]) and the unique nonsplitting extension $p|_M B$ of p over B. Let $b_0, \dots, b_m$ be such that $b_i \models \mathfrak{p} \restriction N b_{<i}$ for all $i \leq m$. Note that $\mathrm{tp}(b_1/Mb_0)$ is a coheir of $\mathrm{tp}(b_1/N)$ and therefore $\mathrm{tp}(b_0/Nb_1)$ is the unique heir of $\mathrm{tp}(b_0/N) = p$ over Nb_1. Hence $b_1 \models \mathfrak{p} \restriction N$ and $b_0 \models \mathfrak{p} \restriction Nb_1$. Since $\mathfrak{p}$ does not split over M, $\mathrm{tp}(b_0 b_1/N) = p \otimes_M p = \mathrm{tp}(b_1 b_0/N)$. And moreover $\mathrm{tp}(b_0 b_1 b_2 \dots b_m/N) = p^{(m+1)_M} = \mathrm{tp}(b_1 b_0 b_2 \dots b_m/N)$. Since $a^\frown(b_0, \dots, b_m)$ is M-indiscernible, we obtain

$$\begin{array}{rllll} a_0 \dots a_{i-1} a_i a_{i+1} a_{i+2} \dots a_n & \equiv_M & a_0 \dots a_{i-1} & b_0 b_1 & b_2 \dots b_{n-i} \\ & \equiv_M & a_0 \dots a_{i-1} & b_1 b_0 & b_2 \dots b_{n-i} \\ & \equiv_M & a_0 \dots a_{i-1} & a_{i+1} a_i & a_{i+2} \dots a_n \end{array}$$

□

Remark 9.5 *Assume $\mathfrak{p}(x) \in S(\mathfrak{C})$. If for some $M \supseteq A$, $\mathfrak{p}$ is M-definable and coinherits from M, then (according to Proposition 15.1 of [6]), for every $M \supseteq A$, $\mathfrak{p}$ is M-definable and coinherits from M.*

Proposition 9.6 *Let $\mathfrak{p}(x) \in S(\mathfrak{C})$ be A-invariant. Then 1 ⇒ 2 ⇒ 3 ⇒ 4, where:*

1. *$\mathfrak{p}$ is generically stable.*
2. *For each $\varphi(x, y) \in L$ there is some $n_\varphi < \omega$ such that for any Morley sequence $a = (a_i : i < \omega)$ generated by $\mathfrak{p}$ over A, $\mathfrak{p}$ is definable over a by:*

$$\varphi(x, c) \in \mathfrak{p}(x) \ \Leftrightarrow \models \bigvee_{w \subseteq 2 \cdot n_\varphi,\, |w| = n_\varphi} \bigwedge_{i \in w} \varphi(a_i, c)$$

3. *$\mathfrak{p}$ is definable over A and coinherits from some (every) $M \supseteq A$.*
4. *Any Morley sequence generated by $\mathfrak{p}$ over A is totally indiscernible over A.*

Proof: *1* $\Rightarrow$ *2*. By point 3 of Remark 9.3 $\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$. Point 2 of Remark 9.3 gives the numbers $n_\varphi < \omega$.

2 $\Rightarrow$ *3*. Let a be a any Morley sequence generated by $\mathfrak{p}$ over A. Since $\mathfrak{p}$ is definable over a and it is A-invariant, it is definable over A. Let $M \supseteq Ma$. Since $\mathfrak{p}$ is finitely satisfiable in a, it coinherits from M. By Remark 9.5 this is also true for any $M \supseteq A$.

3 $\Rightarrow$ *4*. By Proposition 9.3. $\square$

Proposition 9.7 *If $\mathfrak{p}$ is generically stable over A, then $\mathfrak{p} \restriction A$ is stationary and $\mathfrak{p}$ is its only global nonforking extension.*

Proof: Assume $\mathfrak{q}(x)$ is a global nonforking extension of $p(x) = \mathfrak{p} \restriction A$. We claim that for every $n < \omega$, if $a_i \models \mathfrak{q} \restriction Aa_{<i}$ for all $i \leq n$, then $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i \leq n$. We prove it by induction on n. It is clear for $n = 0$. Assume it holds for n and assume $a_i \models \mathfrak{q} \restriction Aa_{<i}$ for all $i \leq n+1$. By induction hypothesis, $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i \leq n$. Let $a' = (a'_i : i < \omega)$ be a Morley sequence generated by $\mathfrak{p}$ over A with $a'_i = a_i$ for $i \leq n$ and let $b \models \mathfrak{q} \restriction Aa'$. Note that $b \equiv_{Aa_{\leq n}} a_{n+1}$. We will prove now that $b \equiv_{Aa_{\leq n}} a'_{n+1}$. This will imply that $a_{n+1} \models \mathfrak{p} \restriction Aa_{\leq n}$ and this will finish the proof of the claim. Let $\varphi(y, z, x) \in L(A)$ be such that $\models \varphi(a_{<n}, a_n, b)$. Then $\varphi(a_{<n}, a_n, x) \in \mathfrak{q}$. If there is some $i > n$ such that $\neg\varphi(a_{<n}, a'_i, x) \in \mathfrak{q}$, then, since $\mathfrak{q}$ does not divide over A, and $(a_{<n}a'_{2\cdot j}a'_{2\cdot j+1} : 2 \cdot j > i)$ is an A-indiscernible sequence of realizations of $\mathrm{tp}(a_{<n}a_n, a'_i/A)$, we see that

$$\{\varphi(a_{<n}, a'_{2\cdot j}, x) \wedge \neg\varphi(a_{<n}, a'_{2\cdot j+1}, x) : 2 \cdot j > i\}$$

is consistent, which contradicts the fact that $(a'_j : j < \omega)$ is a Morley sequence over A generated by a type $\mathfrak{p}$ which is generically stable over A. Hence $\varphi(a_{<n}, a_i, x) \in \mathfrak{q}$ for all $i > n$ and therefore $\models \varphi(a_{<n}, a_i, b)$ for all $i > n$. Since $\mathfrak{p} = \mathrm{Av}(a'/\mathfrak{C})$, $\varphi(a_{<n}, x, b) \in \mathfrak{p}$. Since $\mathfrak{p}$ is A-invariant and $a_{<n}a_n \equiv_A a_{<n}b$, $\varphi(a_{<n}, x, a_n) \in \mathfrak{p}$ and then $\models \varphi(a_{<n}, a'_{n+1}, a_n)$. Since a' is totally indiscernible over A, $\models \varphi(a_{<n}, a_n, a'_{n+1})$.

With the claim we now finish the proof. We check that $\mathfrak{q} \subseteq \mathfrak{p}$. Assume $\varphi(x, c) \in \mathfrak{q}$. Let $a = (a_i : i < \omega)$ be such that $a_i \models \mathfrak{q} \restriction Aca_{<i}$. Then $\models \varphi(a_i, c)$ for all $i < \omega$. By the claim $a_i \models \mathfrak{p} \restriction Aa_{<i}$ and therefore $\mathrm{Av}(a/\mathfrak{C}) = \mathfrak{p}$. It follows that $\varphi(x, c) \in \mathfrak{p}$. $\square$

Proposition 9.8 [4] *Assume $\mathfrak{p}$ is generically stable over A. For any $a \models \mathfrak{p} \restriction A$, for any b such that $b \mathop{\downarrow}^{\mathrm{f}}_{A} A$:*

$$a \mathop{\downarrow}^{\mathrm{f}}_{A} b \;\Leftrightarrow\; b \mathop{\downarrow}^{\mathrm{f}}_{A} a$$

Proof: Let $p(x) = \mathfrak{p} \restriction A = \mathrm{tp}(a/A)$ and let $q(y) = \mathrm{tp}(b/A)$. Recall that $\mathfrak{p}$ is definable over A. We claim that for any $\varphi(x, y) \in L(A)$, for any global nonforking extension $\mathfrak{q}$ of q: $\varphi(x, b) \in \mathfrak{p}$ if and only if $\varphi(a, y) \in \mathfrak{q}$. We prove first the claim. Assume $\varphi(x, b) \in \mathfrak{p}$. Let $\psi(y) \in L(A)$ be a definition of $\mathfrak{p} \restriction \varphi(x, y)$. Let $\mathfrak{q}$ be any global nonforking extension of q, and suppose $\neg\varphi(a, y) \in \mathfrak{q}$. Since $\models \psi(b)$, $\neg\varphi(a, y) \wedge \psi(y) \in \mathfrak{q}$. Let $a = (a_i : i < \omega)$ be a Morley sequence generated by $\mathfrak{p}$ over A. Since $\mathfrak{q}$ does not fork over A, $\{\neg\varphi(a_i, y) \wedge \psi(y) : i < \omega\}$ is consistent. Let b' realize this partial type formula. Since $\models \psi(b')$, $\varphi(x, b') \in \mathfrak{p}$. Since $\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$, $\neg\varphi(x, b') \in \mathfrak{p}$, a contradiction. For the other direction, if $\varphi(x, b) \notin \mathfrak{p}$, then $\neg\varphi(x, b) \in \mathfrak{p}$ and therefore $\neg\varphi(a, y)$ and then $\varphi(a, y) \notin \mathfrak{q}$.

Note that the claim implies that q has a unique nonforking extension over Aa. Now we use the claim for the proof. Assume $a \mathop{\downarrow}^{\mathrm{f}}_{A} b$ and $\models \varphi(a, b)$. We check that $\varphi(a, y)$ does

[4] Communicated personally by A. Pillay in June 2010

not fork over A. This is clear by the claim since $\varphi(x, b) \in \mathfrak{p}$ and q has a global nonforking extension. For the other direction, assume $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{A} a$ and $\models \varphi(a, b)$. We check now that $\varphi(x, b)$ does not fork over A. Since $\varphi(a, y)$ is in the unique nonforking extension of q over Aa, by the claim $\varphi(x, b) \in \mathfrak{p}$. Hence the result. $\square$

Proposition 9.9 *If $a = (a_i : i < \omega)$ is a* nip*-sequence over A with global type $\mathfrak{p}(x)$, the following are equivalent:*

1. *For some (all) $M \supseteq A$, $\mathfrak{p}$ is definable over M and coinherits from M.*
2. *a is totally indiscernible over A.*
3. *For each $\varphi(x, y) \in L$ there is some $n_\varphi < \omega$ such that $\mathfrak{p}$ is definable over a by:*

$$\varphi(x, c) \in \mathfrak{p} \Leftrightarrow \models \bigvee_{w \subseteq 2 \cdot n_\varphi, |w| = n_\varphi} \bigwedge_{i \in w} \varphi(a_i, c)$$

Proof: *1* $\Rightarrow$ *2*. By Proposition 9.4.

2 $\Rightarrow$ *3*. By Proposition 8.18 $\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$ and this is a definition of $\mathrm{Av}(a/\mathfrak{C})$ if the sequence is totally indiscernible and n_φ is the alternation number of φ relative to $\mathfrak{p} \upharpoonright A$.

3 $\Rightarrow$ *1*. If M is a model containing Aa, then $\mathfrak{p}$ is M-definable and a coheir of $\mathfrak{p} \upharpoonright M$. $\square$

Corollary 9.10 *Let $p(x) \in S(A)$ be* nip *and let $\mathfrak{p}$ be a global nonforking extension of p. The following are equivalent.*

1. *$\mathfrak{p}$ is generically stable.*
2. *$\mathfrak{p}$ is generically stable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.
3. *For some (all) $M \supseteq A$, $\mathfrak{p}$ is definable over M and coinherits from M.*
4. *Every Morley sequence $a = (a_i : i < \omega)$ generated by $\mathfrak{p}$ over A is totally indiscernible over A.*

Proof: *1* $\Rightarrow$ *2*. By Proposition 9.6, $\mathfrak{p}$ is definable. By Corollary 8.17, it is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Hence it is $\mathrm{acl}^{\mathrm{eq}}(A)$-invariant. By Point *1* of Remark 9.3, it is generically stable over $\mathrm{acl}^{\mathrm{eq}}(A)$.

2 $\Rightarrow$ *3*. By Proposition 9.6.

3 $\Rightarrow$ *4*. By Proposition 9.9.

4 $\Rightarrow$ *1*. Notice that $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ and therefore it is $\mathrm{acl}^{\mathrm{eq}}(A)$-invariant. Let $a = (a_i : i < \omega)$ be a Morley sequence generated by $\mathfrak{p}$ over $\mathrm{acl}^{\mathrm{eq}}(A)$. Note that a is totally indiscernible over $\mathrm{acl}^{\mathrm{eq}}(A)$. Since p is nip, a satisfies the requirements of the definition of generically stable. $\square$

Corollary 9.11 *Let $p(x) \in S(A)$. If some extension of $p(x)$ over* $\mathrm{acl}^{\mathrm{eq}}(A)$ *is generically stable, then all extensions of $p(x)$ over* $\mathrm{acl}^{\mathrm{eq}}(A)$ *are stationary and generically stable and all global nonforking extensions of $p(x)$ are generically stable over* $\mathrm{acl}^{\mathrm{eq}}(A)$ *and are A-conjugate. Moreover, if p is* nip*, then the following are equivalent.*

1. *Some global nonforking extension of p is generically stable.*

2. *Some extension of p over $\mathrm{acl}^{\mathrm{eq}}(A)$ is generically stable.*

3. *$p(x)$ does not fork over A and every Morley sequence over A of realizations of p is totally indiscernible over A.*

4. *Some Morley sequence over A of realizations of p is totally indiscernible over A.*

Proof: All extensions of $p(x)$ over $\mathrm{acl}^{\mathrm{eq}}(A)$ are A-conjugate. If some extension is generically stable then, by Proposition 9.7, it is stationary and, similarly, all other extensions are stationary and their nonforking extension are A-conjugate.

Assume now p is nip.

1 $\Leftrightarrow$ *2*. By Corollary 9.10.

2 $\Rightarrow$ *3*. Let $\mathfrak{p} \supseteq p$ be generically stable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Since $\mathfrak{p}$ does not fork over A, p does not fork over A. A Morley sequence over A in p is in fact a Morley sequence over $\mathrm{acl}^{\mathrm{eq}}(A)$ in some extension of p; since the extension is generically stable, the Morley sequence is totally indiscernible.

3 $\Rightarrow$ *4*. Since p does not fork over A, there are Morley sequences over A in p.

4 $\Rightarrow$ *2*. We have assumed p is nip. Let $a = (a_i : i < \omega)$ be a Morley sequence over A in p, totally indiscernible over A. Let $\mathfrak{p}$ be a nonforking extension of p that generates a over A. By Proposition 8.18, $\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$ and it is definable over a. By Proposition 8.17 it is $\mathrm{acl}^{\mathrm{eq}}(A)$-invariant. Hence $\mathfrak{p}$ is generically stable over $\mathrm{acl}^{\mathrm{eq}}(A)$. □

Proposition 9.12 *If $\mathfrak{p}$ is generically stable over B and does not fork over $A \subseteq B$, then $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.*

Proof: Let $a = (a_i : i < \omega)$ be a Morley sequence generated by $\mathfrak{p}$ over B. By Proposition 9.6, $\mathfrak{p}$ is definable over a. Since it is B-invariant, it is also definable over B. Since $\mathfrak{p}$ does not fork over A, $a_i \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_A B a_{<i}$ for all $i < \omega$. By induction one sees that $a_{<i} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_A B$ and therefore $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_A B$. Let $\varphi(x, y) \in L$ and let c_φ be the canonical parameter of a definition of $\mathfrak{p} \restriction \varphi$. Then $c_\varphi \in \mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(B)$ and hence $c_\varphi \in \mathrm{acl}^{\mathrm{eq}}(A)$. □

10 Stable types

Definition 10.1 A partial type $\pi(x)$ is *stable* if all its completions are definable. [5]

Remark 10.2 1. *Any extension of a stable type is stable.*

2. *T is stable if and only if all types are stable.*

The notion of stable type is discussed in the literature in different places. Here we look at [5], [8], [12], [19], [18], [13] and also at Poizat and Lascar's works mentioned in the footnote.

Proposition 10.3 *The following are equivalent for any partial finitary type $\pi(x)$ over A:*

1. *$\pi(x)$ is stable.*

[5]Definition of Lascar and Poizat in [16], and also in the books [15] and [21].

2. *For any* $\varphi(x,y) \in L$, $\bigcup_{f\in 2^\omega} \pi(x_f) \cup \bigcup_{f\in 2^\omega}\{\varphi(x_f, y_{f\restriction i})^{f(i)} : i < \omega\}$ *is inconsistent, where* $\varphi^0 = \varphi$, $\varphi^1 = \neg\varphi$.

3. *For every* $\varphi(x,y) \in L$, *for every* $B \supseteq A$, *every* $p(x) \in S_\varphi(B)$ *consistent with* $\pi(x)$ *is definable (by a formula of the form*

$$\psi(y) = \exists x_1 \dots x_n \exists y_1 \dots y_m \chi(y, x_1, \dots, x_n, y_1, \dots y_m)$$

where χ *is a conjunction of formulas of the form* $\varphi(x_i, y_j)$, $\neg\varphi(x_i, y_j)$, $\varphi(x_i, y)$, *some* $\varphi(x_i, y)$*-formulas over* B *and some formulas of* $\pi(x)$.*)*

4. *For every* $\varphi(x,y) \in L$, *for every infinite cardinal* λ, *for every* $B \supseteq A$ *with* $|B| \le \lambda$, *there are at most* λ *types* $p(x) \in S_\varphi(B)$ *consistent with* $\pi(x)$.

5. *For every* $B \supseteq A$ *there are at most* $|B|^{|T|}$ *types* $p(x) \in S(B)$ *extending* $\pi(x)$.

6. *For some cardinal* λ, *for every* $B \supseteq A$ *with* $|B| \le \lambda$, *there are at most* λ *types* $p(x) \in S(B)$ *extending* $\pi(x)$.

7. *For some cardinal* λ, *for every* $\varphi(x,y) \in L$, *for every* $B \supseteq A$ *with* $|B| \le \lambda$, *there are at most* λ *types* $p(x) \in S_\varphi(B)$ *consistent with* $\pi(x)$.

8. *For every* $\varphi(x,y) \in L$ *there is some infinite cardinal* λ *such that for every* $B \supseteq A$ *with* $|B| \le \lambda$, *there are at most* λ *types* $p(x) \in S_\varphi(B)$ *consistent with* $\pi(x)$.

Proof: One can easily adapt the proofs of the unrelativized version. It is clear that the stronger version of 3 implies 4, that $3 \Rightarrow 1$, that $1 \Rightarrow 5$ and that $4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8$. For $8 \Rightarrow 2 \Rightarrow 3$ (strong version) see, for example, Proposition 2.6 of [7]. □

Lemma 10.4 *Let* $\varphi(x,y) \in L$ *and let* $\pi(x)$ *be a partial type over* A. *Assume there are no sequences* $(a_i : i < \omega)$ *and* $(b_i : i < \omega)$ *such* $a_i \models \pi$ *for all* $i < \omega$ *and* $\models \varphi(a_i, b_j) \Leftrightarrow i < j$ *for all* $i, j < \omega$. *Then any type* $p(x) \in S_\varphi(A)$ *such that for any* $\psi(x) \in p$, $\pi(x) \cup \{\psi(x)\}$ *is satisfiable in* A *is definable by a positive boolean combination of formulas of the form* $\varphi(a, y)$ *with* $a \in A$.

Proof: Same proof as Lemma 2.10 in [7] but taking care of choosing $c_i \models \pi$. □

Proposition 10.5 *Let* $\pi(x)$ *be a partial type. The following are equivalent:*

1. $\pi(x)$ *is stable.*

2. *There are no sequences* $(a_i : i < \omega)$ *and* $(b_i : i < \omega)$ *such* $a_i \models \pi$ *for all* $i < \omega$ *and for some* $\varphi(x,y) \in L$, $\models \varphi(a_i, b_j) \Leftrightarrow i < j$ *for all* $i, j < \omega$.

3. *For each* $\varphi(x,y) \in L$, *there is some conjunction* $\psi(x)$ *of formulas in* $\pi(x)$ *such that* $\varphi(x,y) \wedge \psi(x)$ *does not have the order property.*

Proof: $2 \Leftrightarrow 3$ is just compactness. $1 \Rightarrow 2$ is a standard argument (see Proposition 2.11 in [7]), but it needs to be adapted to $\kappa = |A| + |T|$ using $\mathrm{ded}(\kappa)$. $2 \Rightarrow 1$: if $\pi(x)$ is over A, it is enough to check that any extension of π over a $(|A| + |T|)^+$-saturated model $M \supseteq A$ is definable, and this follows from Lemma 10.4. □

Remark 10.6 1. *If* $\pi(x,y)$ *is stable, then the type* $\exists y \pi(x,y)$ *is stable. Therefore, if* $\pi(x,y)$ *is a stable partial type over* A, *then* $\pi \restriction x = \{\varphi(x) \in L(A) : \pi(x,y) \vdash \varphi(x)\}$ *is stable.*

2. $\mathrm{tp}(a/A)$ *and* $\mathrm{tp}(b/Aa)$ *are stable if and only if* $\mathrm{tp}(ab/A)$ *is stable. More generally,* $\mathrm{tp}(a_i/Aa_{<i})$ *is stable for each* $i < \alpha$ *iff* $\mathrm{tp}((a_i : i < \alpha)/A)$ *is stable.*

3. *If the types* $\pi_i(x_i)$ *are stable for all* $i \in I$*, then* $\bigcup_{i\in I} \pi_i(x_i)$ *is stable.*

4. *If the types* $\pi_1(x), \pi_2(x)$ *are stable, then the type* $\pi_1(x) \vee \pi_2(x)$ *is stable.*

Proof: For 2 just count types. □

Proposition 10.7 *Assume* $\pi(x)$ *is* nip *and unstable.*

1. *Let* $\mathfrak{D} = \pi(\mathfrak{C})$*. Then for some* $n < \omega$ *there is a formula* $\psi = \psi(x_1, \ldots, x_n; y_1, \ldots, y_n) \in L$ *which defines a partial order in* $\mathfrak{C}^n$ *with infinite chains in* $\mathfrak{D}^n$*.*

2. *There is a formula* $\psi = \psi(x, y) \in L(\mathfrak{C})$ *which defines a partial order in* $\mathfrak{C}$ *with infinite chains in* $\mathfrak{D} = \pi(\mathfrak{C})$*.*

Proof: It is an adaptation of Proposition 2.18 of [7]. □

Proposition 10.8 *A partial type is stable if and only if it is simple and* nip.

Proof: From left to right it is a counting types argument: for simplicity it is like Proposition 2.21 of [7] and for nip like Remark 2.17 of [7]. From right to left use Proposition 10.7 and the fact that a type witnessing the strict order property is not simple. □

Proposition 10.9 *Let* $p(x) \in S(M)$ *be stable. The following are equivalent for any global extension* $\mathfrak{p}$ *of* p*:*

1. $\mathfrak{p}$ *does not fork over* M*.*

2. $\mathfrak{p}$ *is a heir of* p*.*

3. $\mathfrak{p}$ *is a coheir of* p*.*

4. $\mathfrak{p}$ *is the global* M*-definable extension of* p*.*

5. $\mathfrak{p}$ *does not split over* M*.*

Proof: The equivalence between *2*, *3* and *4* is stated in Corollary 7.14 of [7] and the equivalence between *4* and *5* in Proposition 7.12 of [7]. It is clear that *5* implies *1* (see, for instance, Proposition 7.6 of [7]. For *1* ⇒ *5* use Proposition 8.6. □

Corollary 10.10 1. *Stable types over models are stationary.*

2. *If* $p(x) \in S(A)$ *is stable, then any global extension of* p *is generically stable.*

3. *Stable types over* $\mathrm{acl}^{\mathrm{eq}}(A)$ *are stationary.*

4. *Any two global nonforking extensions of a stable type* $p(x) \in S(A)$ *are* A*-conjugate.*

Proof: *1* follows from Proposition 10.9 since the only nonforking extension of $p(x) \in S(M)$ is the only M-definable extension of p. For *2* note that a global extension $\mathfrak{p}$ of $p(x) \in S(A)$ does not fork over some model $M \supseteq A$ and then use Proposition 10.9 and Corollary 9.10. For *3* use *2*, Proposition 9.9 and Proposition 9.7. Of course, *3* gives a new proof of *1*.

4. Let $\mathfrak{p}, \mathfrak{q}$ be nonforking extensions of the stable type $p(x) \in S(A)$. Let $a \models \mathfrak{p} \restriction \mathrm{acl}^{\mathrm{eq}}(A)$ and $b \models \mathfrak{q} \restriction \mathrm{acl}^{\mathrm{eq}}(A)$. Since $a \equiv_A b$, there is some automorphism $f \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $f(a) = b$. Then $\mathfrak{p}^f$ is a nonforking extension of $\mathfrak{q} \restriction \mathrm{acl}^{\mathrm{eq}}(A)$. By *3*, $\mathfrak{p}^f = \mathfrak{q}$. □

Proposition 10.11 *If $p(x) \in S(B)$ is stable and does not fork over $A \subseteq B$, then $p(x)$ is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.

Proof: Let $\mathfrak{p}$ be a global extension of p that does not fork over A. By Corollary 10.10 $\mathfrak{p}$ is generically stable over some $C \supseteq B$. By Proposition 9.12 $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. In particular, p is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. □

Proposition 10.12 *For any partial type $\pi(x)$, the following are equivalent:*

1. *$\pi(x)$ is stable.*
2. *$\mathrm{CB}_\varphi(\pi(x)) < \infty$ for all $\varphi(x, y) \in L$.*[6]
3. *$\mathrm{CB}_\varphi(\pi(x)) < \omega$ for all $\varphi(x, y) \in L$.*

Proof: *2 ⇒ 1.* Assume $\pi(x)$ is unstable and use Proposition 10.3 to obtain some formula $\varphi = \varphi(x, y) \in L$ and some parameters $(a_s : s \in 2^{<\omega})$ such that for every $f \in 2^\omega$, $\pi(x) \cup \{\varphi(x, a_{f\restriction n})^{f(n)} : n < \omega\}$ is consistent. For every $s \in 2^n$ let $\psi_s = \bigwedge_{i<n} \varphi(x, a_{s\restriction i})^{s(i)}$ and choose $s \in 2^{<\omega}$ for which $\mathrm{CB}_\varphi(\pi(x) \cup \{\psi_s(x)\})$ is minimal and the corresponding degree is also minimal. Since $\psi_s \equiv \psi_{s^\frown 0}(x) \dot{\vee} \psi_{s^\frown 1}(x)$, one of $\pi(x) \cup \{\psi_{s^\frown i}(x)\}$ $(i = 0, 1)$ has smaller rank or both have same rank and one has smaller degree. In any case this contradicts the choice of s.

1 ⇒ 3. We use again Proposition 10.3. Assume $\mathrm{CB}_\varphi(\pi(x)) \geq n$ for every $n < \omega$. We will see that $\bigcup_{f\in 2^\omega} \pi(x_f) \cup \bigcup_{f \in 2^\omega} \{\varphi(x_f, y_{f\restriction i})^{f(i)} : i < \omega\}$ is consistent. We first claim that for any boolean combination $\psi(x)$ of φ-formulas such that $\mathrm{CB}_\varphi(\pi(x) \cup \{\psi(x)\}) \geq n+1$ there is some a such that $\mathrm{CB}_\varphi(\pi(x) \cup \{\psi(x) \wedge \varphi(x, a)\}) \geq n$ or $\mathrm{CB}_\varphi(\pi(x) \cup \{\psi(x) \wedge \neg\varphi(x, a)\}) \geq n$. To check this, let $X_{\pi,\psi} = \{\mathfrak{p}(x) \in S_\varphi(\mathfrak{C}) : \pi(x) \cup \{\psi(x)\}$ is consistent with $\mathfrak{p}(x)\}$. Since $X_{\pi,\psi}$ has Cantor-Bendixson rank $\geq n + 1$, it contains two different points $\mathfrak{p}_1, \mathfrak{p}_2$ of Cantor-Bendixson rank $\geq n$. Find a such that $\varphi(x, a) \in \mathfrak{p}_1$ and $\neg\varphi(x, a) \in \mathfrak{p}_2$. Then $\mathfrak{p}_1 \in X_{\pi,\psi(x)\wedge\varphi(x,a)}$ and $\mathfrak{p}_2 \in X_{\pi,\psi(x)\wedge\neg\varphi(x,a)}$ and these two sets have Cantor-Bendixson rank $\geq n$. With the claim we now can find $(a_s : s \in 2^n)$ such that $\pi(x) \cup \{\varphi(x, a_{s\restriction i})^{s(i)} : i < n\}$ is consistent for every $s \in 2^n$. The rest follows by compactness. □

Proposition 10.13 *Let $p(x) \in S(B)$, let $A \subseteq B$ and assume $p \restriction A$ is stable. Then $p(x)$ does not fork over A if and only if $\mathrm{CB}_\varphi(p) = \mathrm{CB}_\varphi(p \restriction A)$ for all φ.*

Proof: Given $\varphi(x, y) \in L$, let $X_{p,\varphi}$ be the set of all $\mathfrak{p} \in S(\mathfrak{C})$ consistent with p. Similarly, $X_{p\restriction A,\varphi}$ is the set of all $\mathfrak{p} \in S(\mathfrak{C})$ consistent with $p \restriction A$. Then $X_{p,\varphi} \subseteq X_{p\restriction A,\varphi}$ and these sets have ordinal (in fact finite) Cantor-Bendixson rank in $S_\varphi(\mathfrak{C})$. Note that every $\mathfrak{p} \in X_{p\restriction A,\varphi}$ of Cantor-Bendixson rank $\mathrm{CB}_\varphi(p \restriction A)$ is definable and has finitely many A-conjugates, and therefore it is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ and it does not fork over A.

Assume $\mathrm{CB}_\varphi(p) = \mathrm{CB}_\varphi(p \restriction A)$ for all φ. Fix some $\varphi(x, y) \in L$ and choose $\mathfrak{p} \in X_{p,\varphi}$ of maximal Cantor-Bendixson rank. Since $\mathfrak{p}$ does not fork over A, $p \restriction \varphi$ does not fork over A. Hence p does not fork over A.

Assume now p does not fork over A. By the proof of Proposition 10.11, there is some global extension $\mathfrak{p}$ of p definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Let $\varphi(x, y) \in L$ and choose $\mathfrak{q} \in X_{p\restriction A,\varphi}$ of maximal Cantor-Bendixson rank in $S_\varphi(\mathfrak{C})$. As said above, $\mathfrak{q}$ does not fork over A and then it can be extended to a complete global type $\mathfrak{q}'$ that does not fork over A. By Corollary 10.10, $\mathfrak{p}$ and $\mathfrak{q}'$ are A-conjugate. Hence $\mathfrak{p} \restriction \varphi$ is a point of X_p of Cantor-Bendixson rank $\mathrm{CB}_\varphi(p \restriction A)$, which shows that $\mathrm{CB}_\varphi(p) = \mathrm{CB}_\varphi(p \restriction A)$. □

[6] For more information on CB_φ see chapter 6 of [7]

11 Preservation of stability

Here we consider the following open problem, discussed by Hasson and Onshuus in [13]:

Question 11.1 *Assume $p(x) \in S(B)$ is stable and does not fork over $A \subseteq B$. Is $p \upharpoonright A$ stable?*

Proposition 11.2 *Assume $p(x) \in S(B)$ is stable and does not fork over $M \subseteq B$ then $p \upharpoonright M$ is stable.*

Proof: Let $\varphi(x, y) \in L$. We must check that for some $\psi(x) \in p \upharpoonright M$, $\varphi(x, y) \wedge \psi(x)$ does not have the order property. Since p is stable, for some $\theta(x, z) \in L$, for some tuple $a \in B$, $\theta(x, a) \in p$ and $\varphi(x, y) \wedge \theta(x, a)$ does not have the order property. By Proposition 10.11, p is definable over M. Let $\chi(z) \in L(M)$ be a definition of $p \upharpoonright \theta(x, z)$. By compactness there is a maximal $n < \omega$ for which there are $(a_i : i < n)$ and $(b_i : i < n)$ such that $\models \varphi(a_i, b_j) \wedge \theta(a_i, a)$ if and only if $i < j$. Fix some $\psi(z) \in \mathrm{tp}(a/M)$ such that for all $a' \models \psi$, n is an upper bound for such sequences with respect to $\varphi(x, y) \wedge \theta(x, a')$. Since $\psi(z) \wedge \chi(z)$ is consistent and it is over M, it is realized by some $a' \in M$. Since $\models \chi(a')$, $\theta(x, a') \in p \upharpoonright M$. Since $\models \psi(a')$, $\varphi(x, y) \wedge \theta(x, a')$ does not have the order property. $\square$

Proposition 11.3 *Assume $p(x) \in S(B)$ is stable and does not fork over $A \subseteq B$. If $p \upharpoonright A$ is* nip*, then it is stable.*

Proof: [7] Assume $p \upharpoonright A$ is unstable. By Proposition 10.7 and compactness, some $\varphi(x, y) \in L(\mathfrak{C})$ defines a (strict) partial ordering with an infinite chain $(a_i : i < \omega)$ of realizations a_i of $p \upharpoonright A$. Let d be the tuple of parameters of $\varphi(x, y)$. We may assume the sequence is Ad-indiscernible. Let us write $x < y$ for $\varphi(x, y)$. Take some $a \models p$. Since $a \equiv_A a_1$, we can assume $a = a_1$ and therefore $\models a_0 < a < a_2$. By deleting a_1 and renumbering the sequence, we can assume that $a_0 < x < a_1$ is consistent with $p(x)$. Since $p(x)$ is stable, there are only finitely many $n < \omega$ such that $a_n < x < a_{n+1}$ is consistent with $p(x)$. Hence $p(x) \vdash a_n < x < a_{n+m}$ for some $n, m \in \omega$. But then the sequence $(b_k : k < \omega)$ with $b_k = a_{n+2\cdot k\cdot m} a_{n+(2\cdot k+1)\cdot m} d$ is A-indiscernible and witnesses that the formula $a_n < x < a_{n+m}$ divides (with respect to 2) over A, contrarily to the assumption that p does not fork over A. $\square$

Definition 11.4 We say that T is *rosy* if there is some strict independence relation in T^{eq}. By Proposition 1.7 this is equivalent to $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ being an strict independence relation in T^{eq}. All simple theories are rosy because in them $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{f}}$ is (in T^{eq}) a strict independence relation.

Lemma 11.5 *If $p(x) \in S(B)$ is stable and does not* þ*-fork over $A \subseteq B$, then it is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.

Proof: Same as in Proposition 10.11, using $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ instead of $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{f}}$. $\square$

Proposition 11.6 *Assume T is rosy. If $p(x) \in S(B)$ is stable and does not fork over $A \subseteq B$, then $p \upharpoonright A$ is stable.*

[7] The proof is wrong, since the order may not be total. The statement is correct, by other proof supplied by Adler, Casanovas and Pillay which in fact solves affirmatively Question 11.1.

Proof: By a counting types argument (see Proposition 10.3) it is enough to prove that any extension q of $p \restriction A$ over a set $C = \mathrm{acl}^{\mathrm{eq}}(C) \supseteq A$ is definable. Let $a \models p$ and $b \models q$. Since $a \equiv_A b$ there is some B' such that $aB \equiv_A bB'$. Since $a \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathrm{f}}_A B$, we get $b \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathrm{f}}_A B'$ and, by rosyness, $B' \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\text{þ}}_A b$. Choose $B'' \equiv_{Ab} B'$ such that $B'' \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\text{þ}}_A Cb$. Then $b \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\text{þ}}_C B''$. Since $\mathrm{tp}(b/B''C)$ extends the stable type $\mathrm{tp}(b/B'')$, it is stable. By Lemma 11.5, $\mathrm{tp}(b/CB'')$ is definable over C. In particular, $q = \mathrm{tp}(b/C)$ is definable over C. □

12 Stably embedded types

Definition 12.1 Let $\mathfrak{D} \subseteq \mathfrak{C}$. We say that $\mathfrak{D}$ is *stably embedded over A* if for every $n < \omega$, every relatively definable relation $R \subseteq \mathfrak{D}^n$ is relatively definable over $A\mathfrak{D}$.

Proposition 12.2 *Let $\pi(x)$ be a partial 1-type over A and let $\mathfrak{D} = \pi(\mathfrak{C})$. The following are equivalent:*

1. *$\mathfrak{D}$ is stably embedded over A.*
2. *For every a, $\mathrm{tp}(a/\mathfrak{D})$ is definable over $A\mathfrak{D}$.*
3. *For every a, $\mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(Aa) \cap \mathrm{dcl}^{\mathrm{eq}}(A\mathfrak{D})) \vdash \mathrm{tp}(a/\mathfrak{D})$.*
4. *For every a there is some $D \subseteq A\mathfrak{D}$ such that $|D| \leq |T| + |A| + |a|$ and $\mathrm{tp}(a/D) \vdash \mathrm{tp}(a/\mathfrak{D})$.*
5. *For every a there is some $D \subseteq A\mathfrak{D}$ such that $|D| \leq |T| + |A| + |a|$ and $\mathrm{tp}(a/\mathrm{acl}(D)) \vdash \mathrm{tp}(a/\mathfrak{D})$.*
6. *Every A-elementary permutation $f : \mathfrak{D} \to \mathfrak{D}$ can be extended to an A-automorphism of $\mathfrak{C}$.*

Proof: *1* ⇔ *2* is clear.

2 ⇒ *3*. Let $\varphi(x, y) \in L$. Choose $\psi(y, z) \in L(A)$ and $c \in \mathfrak{D}$ such that $\psi(y, c)$ defines $\mathrm{tp}(a/\mathfrak{D}) \restriction \varphi$. By compactness, there is some $\theta(x) \in \pi(x)$ such that for all $b \models \theta(x)$, $\models \psi(b, c)$ if and only if $\varphi(x, b) \in \mathrm{tp}(a/\mathfrak{D})$. Let E be the equivalence relation defined by

$$E(c_1, c_2) \Leftrightarrow\ \models \forall y(\theta(y) \to (\psi(y, c_1) \leftrightarrow \psi(y, c_2))).$$

Notice that c_E is an A-imaginary definable over Aa and over $\mathfrak{D}$. There is some imaginary e such that any A-automorphism fixes c_E if and and only if it fixes e. Hence, $e \in \mathrm{dcl}^{\mathrm{eq}}(Aa) \cap \mathrm{dcl}^{\mathrm{eq}}(A\mathfrak{D})$. Let $a' \models \mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(Aa) \cap \mathrm{dcl}^{\mathrm{eq}}(A\mathfrak{D}))$. Choose some automorphism $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{dcl}^{\mathrm{eq}}(Aa) \cap (\mathrm{dcl}^{\mathrm{eq}}(A\mathfrak{D}))$ such that $f(a) = a'$. Since $f(c_E) = c_E$, for all $b \models \pi$, $\models \psi(b, c) \leftrightarrow \psi(b, f(c))$ and hence $\varphi(x, b) \in \mathrm{tp}(a/\mathfrak{D})$ iff $\varphi(x, b) \in \mathrm{tp}(a'/\mathfrak{D})$.

3 ⇒ *4*. For each element b of $\mathrm{dcl}^{\mathrm{eq}}(Aa) \cap \mathrm{dcl}^{\mathrm{eq}}(A\mathfrak{D})$ choose some finite $D_b \subseteq A\mathfrak{D}$ such that $b \in \mathrm{dcl}^{\mathrm{eq}}(D_b)$ and let D be the union of all the sets D_b.

4 ⇒ *5* is obvious.

5 ⇒ *2*. Let $\varphi(y, x) \in L$. We want to check first that $\mathrm{tp}(a/\mathfrak{D}) \restriction \varphi$ is definable over $\mathrm{acl}(D)$. Assume not. Then, by compactness, there are some tuples $b, b' \in \mathfrak{D}$ such that $b \equiv_{\mathrm{acl}(A)} b'$ and $\models \varphi(a, b) \leftrightarrow \neg\varphi(a, b')$. If we choose now a' such that $ab \equiv_{\mathrm{acl}(D)} a'b'$ we see that $a' \models \mathrm{tp}(a/\mathrm{acl}(D))$ but $a' \not\models \mathrm{tp}(a/\mathfrak{D})$, a contradiction with our assumption.

Let $\psi(x) \in L(\mathrm{acl}(D)$ define $\mathrm{tp}(a/\mathfrak{D}) \restriction \varphi$ and let $\psi_1(x), \ldots, \psi_n(x)$ be the list of all D-conjugates of $\psi(x)$. Let E be the equivalence relation defined by

$$E(x, z) \leftrightarrow \bigwedge_{1 \leq i \leq n} (\psi_i(x) \leftrightarrow \psi_i(z))$$

and let $b_0, \ldots, b_m \in \mathfrak{D}$ be representatives of all E-classes meeting $\mathfrak{D}$. Note that E is over D and therefore $\psi(\mathfrak{C}) \cap \mathfrak{D}$ is setwise fixed by $\mathrm{Aut}(\mathfrak{C}/ADb_0, \ldots, b_m)$. It follows that $\mathfrak{D} \cap \varphi(a, \mathfrak{C})$ is relatively definable over $ADb_0, \ldots, b_m$ and hence over $A\mathfrak{D}$.

4 ⇒ 6. Assume $f : A\mathfrak{D} \to A\mathfrak{D}$ is elementary, permutes $\mathfrak{D}$ and it is the identity on A and let g be an elementary mapping extending f. We check that for every element a there is some b such that $g \cup \{(a, b)\}$ is elementary. A standard back-and-forth argument shows that this is enough. Let c be a tuple enumerating $\mathrm{dom}(g) \setminus \mathfrak{D}$. By *4* there is some $D \subseteq A\mathfrak{D}$ of cardinality $\leq |T| + |A| + |c|$ such that $\mathrm{tp}(ac/D) \vdash \mathrm{tp}(ac/\mathfrak{D})$. Then $\mathrm{tp}(a/\mathrm{dom}(g)) \vdash \mathrm{tp}(a/\mathrm{dom}(g))$ and therefore $\mathrm{tp}(a/\mathrm{dom}(g))^g \vdash \mathrm{tp}(a/\mathrm{dom}(g))^g$. If b realizes $\mathrm{tp}(a/\mathrm{dom}(g))^g$, then $g \cup \{(a, b)\}$ is elementary.

6 ⇒ 1. Assume $R \subseteq \mathfrak{D}^n$ is relatively definable but not over $\mathfrak{D}$. Then, for every set $B \subseteq \mathfrak{D}$ there is some automorphism $f \in \mathrm{Aut}(\mathfrak{C}/AB)$ such that $f(R) \not\subseteq R$. Let $(R_\alpha : \alpha \in On)$ an enumeration of the orbit of R under $\mathrm{Aut}(\mathfrak{C}/A)$ and let $(d_\alpha : \alpha \in On)$ an enumeration of $\mathfrak{D}$. We inductively construct a continuous ascending chain $(f_\alpha : \alpha \in On)$ of A-elementary mappings f_α such that the domain and range of f_α are subsets of $\mathfrak{D}$, a_α belongs to the domain and range of $f_{\alpha+1}$ and $f_{\alpha+1}(R \cap \mathrm{dom}(f_{\alpha+1})) \not\subseteq R_\alpha$. We only need to specify how to get $f_{\alpha+1}$ from f_α. Assume there is an automorphism $g \in \mathrm{Aut}(\mathfrak{C}/A)$ extending f_α and such that $g(R) = R_\alpha$ (if there is not such g we omit this first step). By assumption there is an automorphism $f \in \mathrm{Aut}(\mathfrak{C}/A\mathrm{dom}(f_\alpha))$ such that $f(R) \not\subseteq R$. Then for some n-tuple $a \in R$, $\mathrm{tp}(a/A\mathrm{dom}(f_\alpha)) \cup \{\neg R(x)\}$ is consistent. Let us apply g to this type and let a' realize it. Then a' is an n-tuple of elements of $\mathfrak{D}$, $a' \equiv_{A\mathrm{dom}(f_\alpha)} a$ and $a' \notin g(R) = R_\alpha$. We obtain $f_{\alpha+1}$ by adding (a, a') to f_α and extending the result in a standard way to add a_α first to the domain and then to the range. The union of the chain is an A-elementary permutation of $\mathfrak{D}$ can not be extended to an automorphism of $\mathfrak{C}$. □

Proposition 12.3 *Let $\mathfrak{D}$ de defined over A by a partial 1-type $\pi(x)$ and assume it is stably embedded over A.*

1. *If $M \supseteq A$ is a saturated model of cardinality $> |T| + |A|$ then every A-elementary permutation of $D = \pi(M)$ can be extended to an A-automorphism of M.*
2. *If A is finite and $M \supseteq A$ is countable and saturated, then every A-elementary permutation of $D = \pi(M)$ can be extended to an A-automorphism of M.*

Proof: *1* is like *4 ⇒ 6* in the proof of Proposition 12.2.

2. It is again a back-and-forth argument, but we use now point *3* of Proposition 12.2. Assume $f : AD \to AD$ is elementary, permutes D and it is the identity on A, and let g be an elementary mapping in M such that $\mathrm{dom}(g) \setminus \mathrm{dom}(f)$ is finite. We will check that for every element $a \in M$ there is some element $b \in M$ such that $g \cup \{(a, b)\}$ is elementary. Let c be a tuple enumerating $\mathrm{dom}(g) \setminus D$. Then $\mathrm{tp}(ac/\mathrm{dcl}^{\mathrm{eq}}(Aac) \cap \mathrm{dcl}^{\mathrm{eq}}(ADc)) \vdash \mathrm{tp}(ac/D)$ and therefore $\mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(Aac) \cap \mathrm{dcl}^{\mathrm{eq}}(ADc)) \vdash \mathrm{tp}(a/\mathrm{dom}(g))$ and $\mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(Aac) \cap \mathrm{dcl}^{\mathrm{eq}}(ADc))^g \vdash \mathrm{tp}(a/\mathrm{dom}(g))^g$. Since $\mathrm{dcl}^{\mathrm{eq}}(Aac) \cap \mathrm{dcl}^{\mathrm{eq}}(ADc)$ is contained in the (imaginary) definable closure of a finite tuple, it can be realized by some $b \in M$. Then $g \cup \{(a, b)\}$ is elementary.
□

Lemma 12.4 *Let $\pi_1(x), \pi_2(x)$ be partial types over A. Let $M \supseteq A$ be $(|A|+|T|)^+$-saturated and let $p_1(x), p_2(x) \in S(M)$ be stable types extending $\pi_1(x)$ and $\pi_2(x)$ respectively. If $D = \pi_1(M) \cup \pi_2(M)$ and $p_1 \upharpoonright D = p_2 \upharpoonright D$, then $p_1 = p_2$.* [8]

Proof: Choose a model N such that $A \subseteq N \subseteq M$, $|N| \leq |A| + |T|$ and p_1, p_2 are definable over N. Construct Morley sequences over N, $a = (a_i : i < \omega)$ and $b = (b_i : i < \omega)$ using p_1 and p_2 respectively. Then $a_i, b_i \in D$ for all i. Add to a the Morley sequence $a' = (a'_i : i < \omega)$ where $a'_i \models p_1 \upharpoonright Naba'_{<i}$. Again, $a'_i \in D$ for all $i < \omega$. Then aa' is a Morley sequence generated by p_1 over N. By Proposition 11.2, $p_1 \upharpoonright N$ and $p_2 \upharpoonright N$ are stable. Hence aa' and b are totally indiscernible over N, $\mathrm{Av}(a/M) = p_1$ and $\mathrm{Av}(b/M) = p_2$. We claim that ba' is totally indiscernible. Assume inductively that $ba'_{<i}$ is totally indiscernible. Notice that $\mathrm{tp}(a'_i/ba'_{<i}) = p_1 \upharpoonright ba'_{<i} = p_2 \upharpoonright ba'_{<i} = \mathrm{Av}(b/ba'_{<i})$ and since $ba'_{<i}$ is totally indiscernible, this implies $ba'_{<i}a'_i$ is totally indiscernible. The claim implies $aa' \equiv ba'$. Now we observe that $p_1 = \mathrm{Av}(a/M) = \mathrm{Av}(aa'/M) = \mathrm{Av}(a'/M) \supseteq \mathrm{Av}(ba'/M) \subseteq \mathrm{Av}(b/M) = p_2$. Since $aa' \equiv ba'$, $\mathrm{Av}(ba'/\mathfrak{C})$ is a complete type over $\mathfrak{C}$ and therefore $\mathrm{Av}(a'/M) = \mathrm{Av}(ba'/M) = \mathrm{Av}(b/M)$ and hence $p_1 = p_2$. □

Proposition 12.5 *If $\mathfrak{D}$ is defined over A by a partial stable 1-type $\pi(x)$ then it is stably embedded over A.*

Proof: Let R be a relative definable n-ary relation on $\mathfrak{D}$. Let $\varphi(x, y) \in L$ be such that $\varphi(x, a)$ relatively defines R and choose an $(|A| + |T|)^+$-saturated model $M \supseteq A$ containing a. Let $D = \pi(M)$. We claim that R is relatively definable over AD. Let $f \in \mathrm{Aut}(\mathfrak{C}/AD)$ and let us check that f fixes R. Let c be a tuple in R, let $p_1(x) = \mathrm{tp}(c/M)$ and let $p_2(x) = \mathrm{tp}(f(c)/M)$. Then $\pi \subseteq p_1 \upharpoonright A = p_2 \upharpoonright A$ and $p_1 \upharpoonright D = p_2 \upharpoonright D$. Since p_1, p_2 are stable, by Lemma 12.4, $p_1 = p_2$, that is, $c \equiv_M f(c)$. Hence $f(c) \in R$. □

Definition 12.6 We say that $\mathfrak{D}$ is *uniformly stably embedded over A* if for any n, for any $\psi(x_1, \ldots, x_n; y) \in L$ there is some $\chi(x_1, \ldots, x_n; z) \in L(A)$ such that for any tuple b there is some tuple $c \in \mathfrak{D}$ such that for all $d_1, \ldots, d_n \in \mathfrak{D}$:

$$\models \psi(d_1, \ldots, d_n; b) \leftrightarrow \chi(d_1, \ldots, d_n; c)$$

Lemma 12.7 *If $\mathfrak{D}$ is definable over A and it is stably embedded over A, then it is uniformly stably embedded over A.*

Proof: Let $\varphi(x) \in L(A)$ define $\mathfrak{D}$, and assume $\psi(x_1, \ldots, x_n; y) \in L$. We first claim that there is a finite number of formulas $\chi_i(x_1, \ldots, x_n; z_i) \in L(A)$ $(i = 1, \ldots, m)$ such that for every tuple b there is some i and some tuple $c \in \mathfrak{D}$ such that for all $d_1, \ldots, d_n \in \mathfrak{D}$, $\models \psi(d_1, \ldots, d_n; b) \leftrightarrow \chi_i(d_1, \ldots, d_n; c)$. If not, then the following set $\Sigma(y)$ of formulas is consistent

$$\{\forall z(\varphi(z) \to \exists x_1 \ldots x_n(\varphi(x_1) \wedge \ldots \wedge \varphi(x_n) \wedge \neg(\psi(x_1, \ldots, x_n; y) \leftrightarrow \chi(x_1, \ldots, x_n; z))) : \chi \in L(A)\}$$

(where $\varphi(z) = \varphi(z_1) \wedge \ldots \wedge \varphi(z_k)$ if $z = z_1 \ldots z_k$ and this length depends on the formula χ) and this contradicts stable embeddability of $\mathfrak{D}$ over A. Using the claim we now define $\chi(x_1, \ldots, x_n; z, z_1, \ldots, z_m)$ as the formula

$$\bigvee_{i=1}^{m} z = z_i \wedge \bigwedge_{i=1}^{m} (z = z_i \to \chi_i(x_1, \ldots, x_n; z_i))$$

[8]Remerciements to Bruno Poizat for the statement and the proof.

(we may assume all z_i have the same length). It follows that for any tuple b there is some tuple $cc_1 \dots c_m \in \mathfrak{D}$ such that for all $d_1, \dots, d_n \in \mathfrak{D}$, $\models \psi(d_1, \dots, d_n; b) \leftrightarrow \chi(d_1, \dots, d_n; c, c_1, \dots, c_m)$. □

Definition 12.8 The *A-induced structure* on $\mathfrak{D} \subseteq \mathfrak{C}$ is the structure $\mathfrak{D}_{\mathrm{ind}(A)}$ with universe $\mathfrak{D}$ and a relation symbol R_ψ for any $\psi(x_1, \dots, x_n) \in L(A)$, which is interpreted as $\{(d_1, \dots, d_n) \in \mathfrak{D}^n :\models \varphi(d_1, \dots, d_n)\}$. We allow the case $A = \mathfrak{C}$ and for it we write $\mathfrak{D}_{\mathrm{ind}} = \mathfrak{D}_{\mathrm{ind}(\mathfrak{C})}$.

Proposition 12.9 *Assume $\mathfrak{D}$ is type-definable over A by a 1-type $\pi(x)$ and it is uniformly stably embedded over A.*

1. *If $\mathfrak{D}_{\mathrm{ind}}$ is stable, then $\pi(x)$ is stable.*
2. *If $\mathfrak{D}_{\mathrm{ind}}$ is* nip*, then $\pi(x)$ is* nip.
3. *If $\mathfrak{D}_{\mathrm{ind}}$ is simple, then $\pi(x)$ is simple.*

Proof: *1.* Assume $\varphi(x, y) \in L$ and there are $(a_i : i < \omega)$ and $(b_i : i < \omega)$ with $a_i \in \mathfrak{D}$ such that $\models \varphi(a_i, b_j)$ iff $i < j$. By uniform stable embeddability, there is some $\psi(x, z) \in L(A)$ such that for every b there is some $c \in \mathfrak{D}$ such that for every $d \in \mathfrak{D}$, $\models \psi(d, c) \leftrightarrow \varphi(d, b)$. Choose a corresponding $c_i \in \mathfrak{D}$ for every b_i. Then $\models \psi(a_i, c_j)$ iff $i < j$, which shows that $\mathfrak{D}_{\mathrm{ind}}$ is unstable.

2. Similar to *1.*

3. Assume $\varphi(x, y) \in L$ and there are $k < \omega$, $(a_f : f \in \omega^\omega)$ and $(b_s : s \in \omega^{<\omega})$ with $a_f \in \mathfrak{D}$ such that $\models \varphi(a_f, b_s)$ for every $s \subseteq f$ and $\{\varphi(x, b_{s^\frown n}) : n < \omega\}$ is k-inconsistent. Take some $\psi(x, z) \in L(A)$ such that for every b there is some $c \in \mathfrak{D}$ such that for every $d \in \mathfrak{D}$, $\models \psi(d, c) \leftrightarrow \varphi(d, b)$. Choose for every b_s some corresponding $c_s \in \mathfrak{D}$. Then $\models \psi(a_f, c_s)$ for all $s \subseteq f$. It follows that no subset of k elements $\{\psi(x, c_{s^\frown n}) : n < \omega\}$ is realized in $\mathfrak{D}$. Hence $\mathfrak{D}_{\mathrm{ind}}$ is not simple. □

Remark 12.10 *If $\mathfrak{D}$ is definable, then the induced structure $\mathfrak{D}_{\mathrm{ind}}$ has elimination of quantifiers.*

Proof: Note first that each quantifier-free formula is equivalent to an atomic formula. Now, it is enough to check that for any atomic $R_\varphi(x, y)$, the formula $\exists x R_\varphi(x, y)$ is equivalent to a quantifier-free formula. Let $\theta(x) \in L(\mathfrak{C})$ define $\mathfrak{D}$. Then $\exists x R_\varphi(x, y)$ is equivalent to $R_\psi(y)$ where $\psi(y) = \exists x(\theta(x) \wedge \varphi(x, y))$. □

Proposition 12.11 *Let $\mathfrak{D}$ be type-definable over A by a 1-type $\pi(x)$, and assume the induced structure $\mathfrak{D}_{\mathrm{ind}}$ has elimination of quantifiers.*

1. *If $\pi(x)$ is stable, then the structure $\mathfrak{D}_{\mathrm{ind}}$ is stable.*
2. *If $\pi(x)$ is* nip*, then the structure $\mathfrak{D}_{\mathrm{ind}}$ is* nip.
3. *If $\pi(x)$ is finite (that is, if $\mathfrak{D}$ is definable over A) and simple, then $\mathfrak{D}_{\mathrm{ind}}$ is simple.*

Proof: *1.* Note that every formula in $\mathfrak{D}_{\mathrm{ind}}$ is equivalent to an atomic formula $R_\varphi(x)$. If there are $a_i, b_i \in \mathfrak{D}$ such that $\mathfrak{D}_{\mathrm{ind}} \models R_\varphi(a_i, b_j)$ iff $i < j$, then $\models \varphi(a_i, b_j)$ iff $i < j$ and $\pi(x)$ is unstable.

2. Similar to *1.*

3. Let $\varphi(x)$ be a formula equivalent to $\pi(x)$. Suppose that for some ψ and $k < \omega$, there are $a_f, b_s \in \mathfrak{D}$ such that $\models \psi(a_f, b_s)$ whenever $s \subseteq f$ and $\{\psi(x, b_{s^\frown n}) : n < \omega\}$ is k-inconsistent in $\mathfrak{D}_{\text{ind}}$. Then $\{\varphi(x) \wedge \psi(x, b_{s^\frown n}) : n < \omega\}$ is k-inconsistent and $\varphi(x) \wedge \psi(x, y)$ has the tree property with respect to k. □

Question 12.12 *Does point* 3 *of Proposition 12.11 hold without assuming* $\pi(x)$ *is finite ?*

Proposition 12.13 *Let* $\mathfrak{D}$ *be definable by* $\varphi(x) \in L(A)$ *and assume it is stably embedded over* A*.*

1. $\varphi(x)$ *is stable iff the structure* $\mathfrak{D}_{\text{ind}}$ *is stable.*
2. $\varphi(x)$ *is* nip *iff the structure* $\mathfrak{D}_{\text{ind}}$ *is* nip.
3. $\varphi(x)$ *is simple iff the structure* $\mathfrak{D}_{\text{ind}}$ *is simple.*

Proof: By propositions 12.11, 12.9, Lemma 12.7 and Remark 12.10. □

Corollary 12.14 *Let* $\mathfrak{D}$ *be definable by* $\varphi(x) \in L(A)$*. Then* $\varphi(x)$ *is stable if and only if* $\mathfrak{D}$ *is stably embedded over* A *and* $\mathfrak{D}_{\text{ind}}$ *is stable.*

Proof: From left to right: Remark 12.10 and propositions 12.11 and 12.5. From right to left: Proposition 12.13. □

Example 12.15 *Let* $\mathfrak{C}$ *be the monster model of the (stable) theory of algebraically closed fields of characteristic zero. The type* $\pi(x) = \{x \neq q : q \in \mathbb{Q}\}$ *defines a class* I *such that* I_{ind} *has the strict order property.*

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