Lascar strong types and forking in NIP theories

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This is an updated and slightly expanded version of a tutorial given in the *Mini-Course in Model Theory*, Torino, February 9-11, 2011. Some parts were previously exposed in the Model Theory Seminar of Barcelona. The main goals were to clarify the relation of forking with some versions of splitting in NIP theories and to present the known results on G-compactness, including a full proof of a theorem of E. Hrushovski and A. Pillay on G-compactness of NIP theories over extension bases. The tutorial was given in parallel to a tutorial of H. Adler on forking and dividing over models in NTP_2 theories, now written in [2]. Thanks are due to D. Zambella, organizer of the course.

The context is the standard in Model Theory. T is a complete theory with infinite models, L is its language and $\mathfrak{C}$ is its monster model. Generally x is a tuple of variables and a a tuple of parameters. A set is small if its cardinality is smaller than the cardinality of the monster model. For any other issue concerning notation and terminology, consult [4].

1 Strong types and G-compactness

Definition 1.1 Let A be a small set of parameters. A relation R on the monster model is *A-invariant* if it is invariant under the group $\mathrm{Aut}(\mathfrak{C}/A)$ of all automorphisms of $\mathfrak{C}$ pointwise fixing A, that is, $f(R) = R$ for all $f \in \mathrm{Aut}(\mathfrak{C}/A)$. It is *type-definable over A* if it is the solution set $R = \pi(\mathfrak{C})$ of a set $\pi(x)$ of formulas over A. It is *definable over A* if $\pi(x)$ consists of a single formula $\varphi(x) \in L(A)$. If $A = \emptyset$ we say that R is *0-type-definable* or *0-definable.*

Clearly, if R is definable over A, it is type-definable over A and this implies it is A-invariant. In general, these implications can not be reversed.

Remark 1.2

1. *R is type-definable over A iff R and its complement are type-definable over A.*
2. *If R is type-definable over A and it is B-invariant, then it is type-definable over B.*
3. *R is A-invariant iff it is a union $R = \bigcup_{i\in I} R_i$ of relations R_i which are type-definable over A.*

Proof: See Lemma 1.4 of [4]. □

Definition 1.3 An equivalence relation E between tuples of the same length in the monster model is *bounded* if the number of its equivalence classes is small. It is called *finite* if this number is finite.

Remark 1.4 *Let $(E_i : i \in I)$ be a family of equivalence relations on the monster model.*

1. *If all E_i are A-invariant, $\bigcap_{i\in I} E_i$ is A-invariant.*
2. *If all E_i are type-definable over A, $\bigcap_{i\in I} E_i$ is type-definable over A.*
3. *If all E_i are bounded, $\bigcap_{i\in I} E_i$ is bounded.*

Proof: For item *3* see Remark 9.2 in [4]. $\square$

Definition 1.5 Let us fix a set of parameters A and an ordinal α.

1. The intersection of all bounded A-invariant equivalence relations on α-tuples is the smallest bounded A-invariant equivalence relation on these tuples. It is called the *Lascar equivalence relation over A* and it is denoted by $\stackrel{\text{Ls}}{\equiv}_A$. We say that the tuples a, b have the same *Lascar strong type over A* if $a \stackrel{\text{Ls}}{\equiv}_A b$.
2. The intersection of all bounded type-definable over A equivalence relations on α-tuples is the smallest bounded type-definable over A equivalence relation on I-tuples, it is called the *Kim-Pillay equivalence relation over A* and it is denoted by $\stackrel{\text{KP}}{\equiv}_A$. Two tuples a, b have the same *KP-type over A* if $a \stackrel{\text{KP}}{\equiv}_A b$.
3. The intersection of all finite A-definable equivalence relations on α-tuples is a bounded type-definable over A equivalence relation sometimes called the *Shelah equivalence relation* and denoted by $\stackrel{\text{s}}{\equiv}_A$. We say that a, b have the same *strong type over A* if $a \stackrel{\text{s}}{\equiv}_A b$.
4. As usual, we write $a \equiv_A b$ for equality of type over A: $\text{tp}(a/A) = \text{tp}(b/A)$. It is a bounded A-type-definable equivalence relation.

Remark 1.6 $a \stackrel{\text{Ls}}{\equiv}_A b \Rightarrow a \stackrel{\text{KP}}{\equiv}_A b \Rightarrow a \stackrel{\text{s}}{\equiv}_A b \Rightarrow a \equiv_A b$.

Definition 1.7 We define the *distance over A*, $\mathrm{d}_A(a, b) \in \omega \cup \{\infty\}$, of two tuples a, b of the same length. There are three cases. If $a = b$ we set $\mathrm{d}_A(a, b) = 0$. If there is a natural number $n \geq 1$ for which there are infinite A-indiscernible sequences $I_1, \ldots, I_n$ and tuples $a_1, \ldots, a_{n+1}$ such that $a = a_1$, $b = a_{n+1}$ and $a_i, a_{i+1} \in I_i$ for all $i = 1, \ldots, n$, we define $\mathrm{d}_A(a, b)$ as the least such number n. If $a \neq b$ and there is no such n we put $\mathrm{d}_A(a, b) = \infty$.

Remark 1.8 $a \stackrel{\text{Ls}}{\equiv}_A b$ *iff* $\mathrm{d}_A(a, b) < \infty$.

Definition 1.9 An automorphism $f \in \text{Aut}(\mathfrak{C}/A)$ is *strong over A* if it is a finite product $f = f_1 \circ \ldots \circ f_n$ of automorphisms $f_i \in \text{Aut}(\mathfrak{C}/M_i)$ where each M_i is a model containing A. The strong automorphisms over A form a normal subgroup $\text{Autf}(\mathfrak{C}/A)$ of $\text{Aut}(\mathfrak{C}/A)$.

The group of strong automorphisms was introduced by D. Lascar in [15]. This group acts on the tuples of $\mathfrak{C}$. The orbits of the action are now called Lascar strong types.

Remark 1.10

1. *If $\mathrm{d}_A(a, b) \leq 1$, then $a \equiv_M b$ for some model $M \supseteq A$.*
2. *$a \equiv_M b$ for some $M \supseteq A$ implies $\mathrm{d}_A(a, b) \leq 2$.*

3. $a \overset{\text{Ls}}{\equiv}_A b$ *iff* $f(a) = b$ *for some* $f \in \mathrm{Autf}(\mathfrak{C}/A)$.

Proof: See Proposition 9.12 and Corollary 9.15 of [4]. □

Definition 1.11 T is *G-compact over A* if $\overset{\text{Ls}}{\equiv}_A = \overset{\text{KP}}{\equiv}_A$ for all possible lengths of tuples.

Remark 1.12 *The following are equivalent:*

1. *T is G-compact over A*
2. *$\overset{\text{Ls}}{\equiv}_A$ is type-definable over A.*
3. *For some $n < \omega$, for all tuples a, b: $a \overset{\text{Ls}}{\equiv}_A b$ iff $\mathrm{d}_A(a, b) \leq n$.*

Proof: See Remark 10.16 in [4]. The equivalence with item *3* was first proven by L. Newelski in [16]. □

Definition 1.13 Let E be a 0-type-definable equivalence relation. A *hyperimaginary of sort E* is an equivalence class a_E. Note that $\mathrm{Aut}(\mathfrak{C}/A)$ acts on the hyperimaginaries of sort E by $f(a_E) = f(a)_E$. The hyperimaginary a_E is *A-bounded* if it has a small orbit in this action. The class of all A-bounded hyperimaginaries is $\mathrm{bdd}(A)$.

Note that whenever E is a bounded 0-type-definable equivalence relation, then $a_E \in \mathrm{bdd}(\emptyset)$. Moreover (see Proposition 15.27 in [4]) if $a_E \in \mathrm{bdd}(\emptyset)$, then one can find some bounded 0-type-definable equivalence relation F such that $a_E = a_F$.

Definition 1.14 *Let A be a class of hyperimaginaries. We write $a \equiv_A b$ to mean that there is an automorphism $f \in \mathrm{Aut}(\mathfrak{C})$ fixing all hyperimaginaries of A and such that $f(a) = b$.*

Remark 1.15 $a \overset{\text{KP}}{\equiv}_A b$ *iff* $a \equiv_{\mathrm{bdd}(A)} b$.

Proof: We sketch the proof. For details see Proposition 15.21 in [4]. Working in $T(A)$ and using Lemma 15.20 from [4], we may assume $A = \emptyset$. From right to left: clear since $a_{\overset{\text{KP}}{\equiv}} \in \mathrm{bdd}(\emptyset)$. For the other direction, one should check that $\equiv_{\mathrm{bdd}(\emptyset)}$ is bounded and 0-type-definable. There is a single hyperimaginary $e \in \mathrm{bdd}(\emptyset)$ such that $\equiv_{\mathrm{bdd}(\emptyset)} = \equiv_e$. Let e be of sort E and let c be such that $c_E = e$. Then $\equiv_e$ is type definable over c by: $x \equiv_e y \Leftrightarrow \exists z(E(z, c) \wedge xc \equiv yz)$. Since it is invariant over $\emptyset$ and type-definable, it is type-definable over $\emptyset$. The mapping sending each equivalence class $a_{\equiv_e}$ to $\mathrm{tp}(a/c)$ is one-to-one and therefore $\equiv_e$ is bounded. □

There is a well-known version of this last result for the case $\overset{\text{s}}{\equiv}_A$ in terms of imaginaries:

Fact 1.16 $a \overset{\text{s}}{\equiv}_A b$ *iff* $a \equiv_{\mathrm{acl}^{\mathrm{eq}}(A)} b$.

2 NIP, simple and NTP$_2$ theories

Definition 2.1 The formula $\varphi(x, y) \in L$ has the *tree property (with respect to $k < \omega$)* if there is a tree $(a_s : s \in \omega^{<\omega})$ of tuples a_s such that

- $\{\varphi(x, a_{s^\frown n}) : n < \omega\}$ is k-inconsistent for all $s \in \omega^{<\omega}$.

- $\{\varphi(x, a_{f\restriction n}) : n < \omega\}$ is consistent for all $f \in \omega^\omega$.

The theory T is *simple* if no formula has the tree property in T.

Simple theories were first defined by S. Shelah in [18], but the most relevant facts were found later by B. Kim and A. Pillay. There are some expository books and our main reference is [4].

Fact 2.2 *Simple theories are G-compact over any set, with distance 2, that is, in a simple theory for any tuples a, b: $a \stackrel{\mathrm{Ls}}{\equiv}_A b$ iff $\mathrm{d}_A(a, b) \leq 2$.*

Proof: See Proposition 10.12 of [4]. This was first proven by B. Kim and A. Pillay in [13]. □

Definition 2.3 The formula $\varphi(x, y) \in L$ has the *independence property* if there is a sequence of parameters $(a_n : n < \omega)$ such that for every $X \subseteq \omega$ the following is consistent:

$$\{\varphi(x, a_n) : n \in X\} \cup \{\neg\varphi(x, a_n) : n \notin X\}$$

The theory is NIP (or *dependent*) if no formula has the independence property in T.

The independence property is studied by S. Shelah in [19] and by B. Poizat in [17]. The most recent developments in the model theory of NIP theories are due to S. Shelah, E. Hrushovski and A. Pillay among others. For an exposition we refer to H. Adler [1] and P. Simon [20].

Both in the definition of NIP and of simplicity we may allow the formula $\varphi(x, y)$ to contain some parameters since they can be added to the nodes a_s of the tree or to the tuples a_n.

Remark 2.4 *The formula $\varphi(x, y) \in L$ has the independence property iff for some indiscernible sequence $(a_n : n < \omega)$ there is some c such that $\models \varphi(c, a_n)$ iff n is even.*

Proof: Assume $\varphi(x, y)$ has the independence property. By compactness we may assume that the sequence $(a_n : n < \omega)$ witnessing the independence property is indiscernible. But $\{\varphi(x, a_{2n}) : n < \omega\} \cup \{\neg\varphi(x, a_{2n+1}) : n < \omega\}$ is consistent.

For the other direction, assume there is an indiscernible sequence $(a_n : n < \omega)$ such that $\{\varphi(x, a_{2n}) : n < \omega\} \cup \{\neg\varphi(x, a_{2n+1}) : n < \omega\}$ is consistent. We claim that $\{\varphi(x, a_n) : n \in X\} \cup \{\neg\varphi(x, a_n) : n \in \omega \smallsetminus X\}$ is consistent for all $X \subseteq \omega$. It is enough to check that for any finite disjoint $X, Y \subseteq \omega$, $\Sigma(x) = \{\varphi(x, a_n) : n \in X\} \cup \{\neg\varphi(x, a_n) : n \in Y\}$ is consistent. Let $m_1 < \ldots < m_i$ and $k_1 < \ldots < k_j$ be respective enumerations of X and Y and choose even numbers $m'_1 < \ldots < m'_i$ and odd numbers $k'_1 < \ldots < k'_j$ such that $m_1, \ldots, m_i, k_1, \ldots, k_j$ and $m'_1, \ldots, m'_i, k'_1, \ldots, k'_j$ have the same order type. By assumption $\{\varphi(x, a_n) : n = m'_1, \ldots, m'_i\} \cup \{\neg\varphi(x, a_n) : n = k'_1, \ldots, k'_j\}$ is consistent. By indiscernibility $\Sigma(x)$ is consistent. □

A theory is stable if and only if it is simple and NIP. An important class of unstable NIP theories are the o-minimal theories.

Fact 2.5 *In an o-minimal theory all automorphims are strong. If two tuples have the same type over A, then they have the same type over some model containing A. Hence, all o-minimal theories are G-compact over any set, with distance 2.*

Proof: See Lemma 24 in [21]. $\square$

Definition 2.6 The formula $\varphi(x, y) \in L$ has TP_2, *the tree property of the second kind*, if there is an array of tuples $(a_{ij} : i, j < \omega)$ and some $k < \omega$ such that:

- $\{\varphi(x, a_{ij}) : j < \omega\}$ is k-inconsistent for every $i < \omega$.
- $\{\varphi(x, a_{if(i)}) : i < \omega\}$ is consistent for every $f : \omega \to \omega$.

The theory T is TP_2 if some formula has the tree property of the second kind in T, otherwise it is NTP_2.

Lemma 2.7 *If T has* TP_2*, then some formula $\varphi(x, y)$ has* TP_2 *in T with respect to $k = 2$.*

Proof: Start with an array $(a_{i,j} : i, j < \omega)$, a natural number k and a formula $\varphi(x, y)$ witnessing TP_2, We may assume k is minimal, that is, no array and formula witness TP_2 of T with a smaller number. We may also assume that the rows of the array are mutually indiscernible, in the sense that each row is indiscernible over the other rows. For more details on this see, for instance, Lemma 1.2 of [7]. Consider the set $\{\varphi(x, a_{i,0}), \varphi(x, a_{i,1}) : i < \omega\}$. If it is consistent, then (by indiscernibility of the array) the formula $\psi(x; y_0y_1) = \varphi(x, y_0) \wedge \varphi(x, y_1)$ and the array $(a_{i,2j}a_{i,2j+1} : i < \omega, j < \omega)$ witness TP_2 with a smaller number. If it is inconsistent, we choose $n < \omega$ such that $\{\varphi(x, a_{i,0}), \varphi(x, a_{i,1}) : i < n\}$ is inconsistent. Then the formula $\psi(x; y_0, \ldots, y_{n-1}) = \bigwedge_{i<n} \varphi(x, y_i)$ together with the array $(a_{ni,j}, \ldots, a_{n(i+1)-1,j} : i < \omega, j < \omega)$ witnesses TP_2 with $k = 2$. $\square$

Remark 2.8 *Simple and* NIP *theories are* NTP_2*.*

Proof: Assume $\varphi(x, y)$ has TP_2, witnessed by the array $(a_{ij} : i, j < \omega)$ and the number $k < \omega$. If we put $b_\emptyset = a_{00}$ and $b_s = a_{n+1,s(n)}$ for $s \in \omega^{n+1}$, then the tree $(b_s : s \in \omega^{<\omega})$ witnesses that $\varphi(x, y)$ has the tree property with respect to k and, therefore, T is not simple. On the other hand, by Lemma 2.7 we can assume that $\varphi(x, y)$ has TP_2 with respect to $k = 2$, and then the sequence $(a_{i0} : i < \omega)$ witnesses that $\varphi(x, y)$ has the independence property. $\square$

NTP_2 theories were defined by S. Shelah in [19] and they are being systematically investigated in the last few years. See A. Chernikov's exposition in [7].

In section 6 we will show that NIP theories are G-compact over extension bases (to be defined in Section 6), with distance 2. This result is due to E. Hrushovski and A. Pillay (see Lemma 2.9 in [10]) and has been recently generalized by I. Ben-Yaacov and A. Chernikov to all NTP_2 theories, but with distance 3 (see Corollary 3.6 of [3]). It is unknown whether it can be improved to distance 2.

3 Dividing, forking and splitting

Definition 3.1 Let $\varphi(x, y) \in L$.

1. $\varphi(x, a)$ *divides over A* if there is an A-indiscernible sequence $(a_i : i < \omega)$ such that $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent and $a \equiv_A a_0$.
2. $\varphi(x, a)$ *forks over A* if it implies a disjuntion of formulas that divide over A.

3. A partial type divides (forks) over A if it implies a formula that divides (forks) over A.

Dividing implies forking. B. Kim proved in [12] that in simple theories they coincide (see also Proposition 5.17 in [4]). A basic property of dividing is that a partial type over A does not divide over A. This is false for forking in some cases. But forking has the extension property: if a type $p(x)$ over B does not fork over $A \subseteq B$ and $C \supseteq B$ is given, then some complete extension of $p(x)$ over C does not fork over A. This extension property is sometimes false for dividing.

Definition 3.2 A type $p(x) \in S(B)$ *splits over* $A \subseteq B$ if there is a formula $\varphi(x,y) \in L(A)$ and finite tuples $a \equiv_A b$ such that $\varphi(x,a) \in p(x)$ and $\neg\varphi(x,b) \in p(x)$. If $a \overset{\mathrm{KP}}{\equiv}_A b$ we say that *KP-splits over* A and if $a \overset{\mathrm{Ls}}{\equiv}_A b$, we say that $p(x)$ *Lascar splits over* A. If $\mathrm{d}_A(a,b) \leq 1$ we say that $p(x)$ *strongly splits over* A . The type $p(x)$ is *finitely satisfiable in* A if every finite subset of $p(x)$ is realized in A. In case A is a model this means that $p(x)$ is a *coheir* of $p \restriction A$.

We will discuss all these splitting notions and their relation to dividing and forking. We only consider the case of complete types and the results should not be extrapolated to formulas.

Remark 3.3 *In general for all* $p(x) \in S(B)$ *(over* $A \subseteq B$*)*

$$\text{dividing} \Rightarrow \text{forking} \Rightarrow \text{not finitely satisfiable in } A$$

and

$$\text{strongly splitting} \;\Rightarrow\; \text{Lascar splitting} \;\Rightarrow\; \text{KP-splitting} \Rightarrow\; \text{splitting} \;\Rightarrow \text{not fin. sat. in } A$$

Proof: Only the implications *splitting over* $A \Rightarrow$ *not finitely satisfiable in* A and *forking over* $A \Rightarrow$ *not finitely satisfiable in* A need some explanation. Assume $p(x) \in S(B)$ is finitely satisfiable in A. If $p(x)$ forks over A, then $p(x) \vdash \psi_1(x,a_1) \vee \ldots \vee \psi_n(x,a_n)$ for some formulas $\psi_i(x,y_i) \in L$ and tuples a_i such that $\psi_i(x,a_i)$ divides over A. By finite satisfiability some $\psi_i(x,a_i)$ is satisfied in A, which is incompatible with dividing over A.

If now $p(x)$ splits over A, then for some tuples $a,b \in B$, for some formula $\varphi(x,y) \in L(A)$, $a \equiv_A b$, $\varphi(x,a) \in p(x)$ and $\neg\varphi(x,b) \in p(x)$. Then $\varphi(x,a) \wedge \neg\varphi(x,b)$ is satisfiable in A, contradicting the fact that $a \equiv_A b$. □

We can obtain better results making B large over A or making A a model.

Definition 3.4 A model N is *ω-saturated over* $A \subseteq N$ if for every finite tuple $b \in N$ every n-type over Ab is realized in N.

Proposition 3.5 *If* N *is* ω*-saturated over* $A \subseteq N$*, then for all* $p(x) \in S(N)$ *(over* A*):*

$$\text{dividing} = \text{forking} \Rightarrow \text{strongly splitting} = \text{Lascar splitting} \Rightarrow \text{KP-splitting} \Rightarrow \text{splitting}$$

Proof: *forking* $\Rightarrow$ *dividing*. Assume $p(x) \in S(N)$ and $p(x) \vdash \psi_1(x,a_1) \vee \ldots \vee \psi_n(x,a_n)$ where each $\psi_i(x,a_i)$ divides over A. There is some tuple $a \in N$ and some formula $\varphi(x,y) \in L$ such that $\varphi(x,a) \in p(x)$ and $\varphi(x,a) \vdash \psi_1(x,a_1) \vee \ldots \vee \psi_n(x,a_n)$. By saturation of N, we can find $b_1,\ldots,b_n \in N$ such that $a_1,\ldots,a_n \equiv_{Aa} b_1,\ldots,b_n$. Then $\varphi(x,a) \vdash \psi_1(x,b_1) \vee$

$\ldots \vee \psi_n(x, b_n)$ and each $\psi_i(x, b_i)$ divides over A. It follows that $\psi(x, b_i) \in p(x)$ for some i, and this implies that $p(x)$ divides over A.

dividing $\Rightarrow$ strongly splitting. Assume the formula $\varphi(x, a) \in p(x)$ divides over A, witnessed by the A-indiscernible sequence $(a_i : i < \omega)$. We can assume $a = a_0$. Let $n < \omega$ be such that $\{\varphi(x, a_i) : i \leq n\}$ is inconsistent and find, by saturation, some tuples $a'_1, \ldots, a'_n \in B$ such that $a_1, \ldots, a_n \equiv_{Aa} a'_1, \ldots, a'_n$. Since $p(x)$ is complete and consistent, $\neg\varphi(x, a'_i) \in p(x)$ for some $i \leq n$. Since $\mathrm{d}_A(a, a'_i) \leq 1$, this shows that $p(x)$ strongly splits over A.

Lascar splitting $\Rightarrow$ strongly splitting. Assume $p(x) \in S(N)$ does not strongly split over A, $\varphi(x, y) \in L(A)$, $\varphi(x, a) \in p(x)$, b is a tuple of N and $b \overset{\mathrm{Ls}}{\equiv}_A a$. We want to check that $\varphi(x, b) \in p(x)$. For some $n < \omega$, $d_A(a, b) \leq n$. Choose tuples $a_1, \ldots, a_{n+1}$ and infinite indiscernible sequences $I_1, \ldots, I_n$ such that $a = a_1$, $b = a_{n+1}$ and $a_i, a_{i+1} \in I$ for all i. By saturation of N, there are $b_1, \ldots, b_n \in N$ such that $a_1, \ldots, a_n \equiv_{Aab} b_1, \ldots, b_n$. Then $a = b_1$, $b = b_n$ and $d_A(b_i, b_{i+1}) \leq 1$ for all i. Inductively we see that $\varphi(x, b_i) \in p(x)$ for all i. Hence $\varphi(x, b) \in p(x)$. $\square$

Proposition 3.6 *If $A = M$ is a model and $M \subseteq B$, then for any $p(x) \in S(B)$ (over M):*

$$\text{Lascar splitting} = \text{KP-splitting} = \text{splitting} \Rightarrow \text{not coheir}$$

Proof: We need to check *splitting over M $\Rightarrow$ Lascar splitting over M*. But this is clear since $a \equiv_M b$ implies $a \overset{\mathrm{Ls}}{\equiv}_M b$ since any automorphism $f \in \mathrm{Aut}(\mathfrak{C}/M)$ is strong. $\square$

If T is NIP we can obtain more information:

Proposition 3.7 *Assume T is* NIP. *Strongly splitting implies dividing and Lascar splitting implies forking. Hence for any $p(x) \in S(B)$ (over $A \subseteq B$):*

$$\begin{array}{ccccccc}
\text{strongly splitting} & \Rightarrow & \text{Lascar splitting} & \Rightarrow & \text{KP-splitting} & \Rightarrow & \text{splitting} \\
\Downarrow & & \Downarrow & & & & \Downarrow \\
\text{dividing} & \Rightarrow & \text{forking} & & \Rightarrow & & \text{not finitely sat. in } A
\end{array}$$

If N is ω-saturated over A then for all $p(x) \in S(N)$ (over A):

$$\text{strongly splitting} = \text{dividing} = \text{forking} = \text{Lascar splitting} = \text{KP-splitting} \Rightarrow \text{splitting}$$

If $A = M \subseteq B$ is a model, then for all $p(x) \in S(B)$ (over M):

$$\begin{array}{ccccccc}
\text{strongly splitting} & \Rightarrow & \text{Lascar splitting} & = & \text{KP-splitting} & = & \text{splitting} \\
\Downarrow & & \Downarrow & & & & \Downarrow \\
\text{dividing} & = & \text{forking} & & \Rightarrow & & \text{not coheir}
\end{array}$$

Hence, in a NIP *theory if N is ω-saturated over a model $M \subseteq N$, for any $p(x) \in S(N)$ all these notions coincide, although they are stronger than not being a coheir.*

Proof: Let T be NIP. We first prove *strongly splitting over $A \Rightarrow$ dividing over A*. Assume $p(x) \in S(B)$ does not divide over $A \subseteq B$ but it strongly splits over A. For some tuples $a, b \in B$, for some formula $\varphi(x, y) \in L(A)$, $\mathrm{d}_A(a, b) \leq 1$, $\varphi(x, a) \in p(x)$ and $\neg\varphi(x, b) \in p(x)$. We may assume that there is an A-indiscernible sequence $(a_i : i < \omega)$ with $a_0 = a$ and $a_1 = b$. Then $(a_{2i}a_{2i+1} : i < \omega)$ is A-indiscernible and, since $\varphi(x, a) \wedge \neg\varphi(x, b)$ does not divide over A, $\{\varphi(x, a_{2i}) \wedge \neg\varphi(x, a_{2i+1}) : i < \omega\}$ is consistent. Let c realize this set of formulas. Then $\models \varphi(c, a_i)$ iff i is even. By Remark 2.4, this implies that $\varphi(x, y)$ has the independence property.

Lascar splitting over $A \Rightarrow$ forking over A. Assume $p(x) \in S(B)$ does not fork over $A \subseteq B$ and choose some model $N \supseteq B$ which is ω-saturated over A. There is an extension $q(x) \in S(N)$ of $p(x)$ that does not fork over A. Hence $q(x)$ does not divide over A and (by NIP) it does not strongly split over A. Hence $q(x)$ does not Lascar split over A and the same can be said of $p(x)$.

The proof of *KP-splitting over $A \Rightarrow$ Lascar splitting over A* for $p(x) \in S(N)$ and N ω-saturated over $A \subseteq N$ is postponed to the last section. See Proposition 6.6.

The implication *forking over $M \Rightarrow$ dividing over M* holds more generally for all NTP_2 theories. This was first proven by A. Chernikov and I. Kaplan in [8]. For a simplified proof see [2]. □

The following diagram summarizes the implications in the general case:

$$\begin{tikzcd}
\text{strongly splitting} \arrow[r, shift left] & \text{Lascar splitting} \arrow[l, shift left, "B\ \omega\text{-sat.}/A"] \arrow[r, shift left] & \text{KP-splitting} \arrow[r] & \text{splitting} \arrow[d] \arrow[ll, shift left, "A \text{ a model}"] \\
\text{dividing} \arrow[u, "B\ \omega\text{-sat.}/A"] \arrow[r, shift left] & \text{forking} \arrow[l, shift left, "B\ \omega\text{-sat.}/A"] \arrow[rr] & & \text{not finitely satisfiable}
\end{tikzcd}$$

If moreover T is NIP, the diagram is as follows:

$$\begin{tikzcd}
\text{strongly splitting} \arrow[r, shift left] \arrow[d, shift left] & \text{Lascar splitting} \arrow[l, shift left, "B\ \omega\text{-sat.}/A"] \arrow[r, shift left] \arrow[d] & \text{KP-splitting} \arrow[l, shift left, "B\ \omega\text{-sat.}/A"] \arrow[r] & \text{splitting} \arrow[d] \arrow[ll, bend left, "A \text{ a model}"] \\
\text{dividing} \arrow[u, shift left, "B\ \omega\text{-sat.}/A"] \arrow[r, shift left] & \text{forking} \arrow[l, shift left, "B\ \omega\text{-sat.}/A \text{ or } A \text{ a model}"] \arrow[rr] & & \text{not finitely satisfiable}
\end{tikzcd}$$

4 Product of types

Definition 4.1 Following Lascar [15], we say that a set B is *complete* over $A \subseteq B$ if every n-type over A is realized in B. It is the right assumption to guarantee the existence of nonsplitting extensions.

Proposition 4.2 *If B is complete over $A \subseteq B$, $p(x) \in S(B)$ does not split over A and $B \subseteq C$, then there is a unique type $q(x) \in S(C)$ extending $p(x)$ that does not split over A.*

Proof: We define:

$$q(x) = p(x) \cup \{\varphi(x, a) : \varphi(x, y) \in L(A), a \in C \text{ and } \varphi(x, a') \in p(x) \text{ for some } a' \equiv_A a\}.$$

The type should extend $q(x)$. Since B is complete over A, $q(x)$ is a complete type over C (if consistent). This proves the uniqueness. We now check the consistency of $q(x)$. Assume, searching for a contradiction, that $p(x) \vdash \neg\varphi_1(x, a_1) \vee \ldots \vee \neg\varphi_n(x, a_n)$ where $a_i \in C$, $\varphi_i(x, y_i) \in L(A)$, $a_i \equiv_A a'_i$ and $\varphi_i(x, a'_i) \in p(x)$ for each i. Choose a formula $\theta(x, b) \in p(x)$ which implies the disjunction and choose tuples $b', a''_1, \ldots, a''_n$ in B such that $b', a''_1, \ldots, a''_n \equiv_A b, a_1, \ldots, a_n$. Then $\theta(x, b') \in p(x)$ and $\theta(x, b') \vdash \neg\varphi_1(x, a''_1) \vee \ldots \vee \neg\varphi_n(x, a''_n)$. For some i, $\neg\varphi_i(x, a''_i) \in p(x)$, which contradicts non-splitting of $p(x)$ over A since $a''_i \equiv_A a_i$. □

Definition 4.3 We will denote $p|_A C$ the unique nonsplitting extension of $p(x) \in S(B)$ over C as in the previous proposition. See the next remark to evaluate the dependency on A.

Remark 4.4 *Assume B is AA'-complete and $p(x) \in S(B)$ does not split over A nor over A'. Then for every $C \supseteq B$, $p|_A C = p|_{A'} C$.*

Proof: Let $\varphi(x, y) \in L(A)$, $a \in C$ and $\varphi(x, a) \in p|_A C$. Choose a tuple $b \in B$ such that $a \equiv_{AA'} b$. Since $a \equiv_A b$, $\varphi(x, b) \in p \subseteq p|_{A'} C$. Since $a \equiv_{A'} b$, $\varphi(x, a) \in p|_{A'} C$. □

Remark 4.5 *If $A \subseteq B \subseteq C \subseteq D$, B is complete over A and $p(x) \in S(B)$ does not split over A, then:*

1. *$(p|_A C)|_A D = p|_A D$*
2. *If the sequence $(a_i : i < \omega)$ is such that $a_i \models p|_A Ba_{<i}$ for all $i < \omega$, then $(a_i : i < \omega)$ is B-indiscernible.*

Proof: *1.* Clear, since $(p|_A C)|_A D$ extends p and does not split over A.

2. By induction on $n < \omega$ we prove that for all $i_0 < \ldots < i_n < \omega$, $a_{i_0}, \ldots, a_{i_n} \equiv_B a_0, \ldots, a_n$. The case $n = 0$ is obvious. For the case $n + 1$, assume $\varphi(x_0, \ldots, x_{n+1}) \in L(B)$, $i_0 < \ldots < i_{n+1}$ and $\models \varphi(a_{i_0}, \ldots, a_{i_{n+1}})$. Then $\varphi(a_{i_0}, \ldots, a_{i_n}, x) \in p|_A Ba_{<i_{n+1}}$ and by nonsplitting and the inductive hypothesis, $\varphi(a_0, \ldots, a_n, x) \in p|_A Ba_{<n+1}$. Hence $\models \varphi(a_0, \ldots, a_{n+1})$. □

Definition 4.6 Let $B \supseteq A$ be complete over A and assume $q(y) \in S_y(B)$ does not split over A. For any $p(x) \in S_x(B)$ the *product* $p \otimes_A q$ is the only type in $S_{xy}(B)$ such that for all a, b:

$$ab \models p \otimes_A q \Leftrightarrow a \models p \text{ and } b \models q|_A Ba$$

Note that this is independent of the choice of a, b: if a', b' is another choice, then $ab \equiv_B a'b'$. Sometimes the product is defined in the reverse order: one assumes $p(x)$ does not split over A, takes $b \models q$ and $a \models p|_A Bb$ and sets $p \otimes_A q = \mathrm{tp}(ab/B)$.

Lemma 4.7 *Assume B is complete over $A \subseteq B$ and $p(x), q(y), r(z) \in S(B)$ and $q(y), r(z)$ do not split over A. Then:*

1. *$p \otimes_A q$ and $q \otimes_A r$ do not split over A.*
2. *$p \otimes_A (q \otimes_A r) = (p \otimes_A q) \otimes_A r$.*
3. *For any $C \supseteq B$, $(q \otimes_A r)|_A C = q|_A C \otimes_A r|_A C$*

Proof: *1.* We check this for $p \otimes_A q$. Assume $\varphi(x, y, u) \in L(A)$, c, c' are tuples of B such that $c \equiv_A c'$, $ab \models p \otimes_A q$ and $\models \varphi(a, b, c)$. Then $\varphi(a, y, c) \in q|_A Ba$, a type that does not split over A. Hence $\varphi(a, y, c') \in q|_A Ba$ and $\models \varphi(a, b, c')$.

2. If $a \models p$, $b \models q|_A a$ and $c \models r|_A ab$, then $ab \models p \otimes_A q$ and $abc \models (p \otimes_A q) \otimes_A r$.

3. Clear, since $q|_A C \otimes_A r|_A C$ is an extension of $q \otimes_A r$ that does not split over A. □

Definition 4.8 If B is complete over $A \subseteq B$ and $p(x) \in S(B)$ does not split over A, the *power* $p^{(n)_A}$ is defined as the n-times iterated product $p \otimes_A \ldots \otimes_A p$. More generally, we can define for any ordinal α the power $p^{(\alpha)_A}$ as the type over B of a sequence $(a_i : i < \alpha)$ such that $a_i \models p|_A a_{<i}$ for all $i < \alpha$. These types do not split over A.

Being complete over a subset A is a weaker condition than being a model ω-saturated over A. However, forking still implies splitting in this situation.

Remark 4.9 *Let B be complete over $A \subseteq B$ and assume $p(x) \in S(B)$ forks over A. Then $p(x)$ splits over A.*

Proof: Suppose $p(x)$ does not split over A. Let $N \supseteq B$ be ω-saturated over A. Let $q(x) = p(x)|_A N$. Since $q(x) \in S(N)$ does not split over A, by Proposition 3.5 it does not fork over A. Then $p(x)$ does not fork over A, a contradiction. □

A similar treatment of Lascar splitting is possible. We can define B to be Lascar complete over $A \subseteq B$ if for every finite tuple a there is some $a' \in B$ such that $a' \overset{\mathrm{Ls}}{\equiv}_A a$. An analogous of Proposition 4.2 holds: if B is Lascar complete over B and $C \supseteq B$, any type $p(x) \in S(B)$ that does not Lascar split over A has a unique extension $q(x) \in S(C)$ that does not Lascar split over A. Using this we can prove, as in Remark 4.9, that if $p(x) \in S(B)$ forks over $A \subseteq B$ and B is Lascar complete over A, then $p(x)$ Lascar splits over A.

5 Global types

Definition 5.1 A *global type* is a type $\mathfrak{p}(x)$ over the monster model $\mathfrak{C}$. Every automorphism $f \in \mathrm{Aut}(\mathfrak{C})$ moves $\mathfrak{p}(x)$ to some *conjugate*

$$\mathfrak{p}^f(x) = \{\varphi(x, f(a)) : \varphi(x,y) \in L, \varphi(x,a) \in \mathfrak{p}(x)\}$$

We say that $\mathfrak{p}(x)$ is *A-invariant* if $\mathfrak{p}^f(x) = \mathfrak{p}(x)$ for all $f \in \mathrm{Aut}(\mathfrak{C}/A)$. Similarly, we say that it is *KP-invariant over A* or $\mathrm{bdd}(A)$*-invariant* if it is fixed under the action of $\mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(A))$ and we say that it is *Lascar invariant over A* if it is fixed under the action of $\mathrm{Autf}(\mathfrak{C}/A)$.

Remark 5.2
1. *$\mathfrak{p}(x)$ is A-invariant iff it does not split over A.*
2. *$\mathfrak{p}(x)$ is* $\mathrm{bdd}(A)$*-invariant iff it does not KP-split over A.*
3. *$\mathfrak{p}(x)$ is Lascar invariant over A iff it does not Lascar split over A.*

Remark 5.3
1. *Since the monster model is ω-saturated over A, for any global type $\mathfrak{p}(x)$, over any small set A:*

$$\text{dividing} = \text{forking} \Rightarrow \text{strongly splitting} = \text{Lascar splitting}$$

2. *If A is a model, then additionally:*

$$\text{invariant} = \text{KP-invariant} = \text{Lascar invariant}$$

3. *In a* NIP *theory, $\mathfrak{p}(x)$does not divide over A iff it does not fork over A iff it is Lascar invariant over A iff it is KP-invariant over A. If additionally A is a model, these conditions are also equivalent to being A-invariant.*

Recall that a global type $\mathfrak{p}(x)$ is A-definable if for each $\varphi(x,y) \in L$ the set $\{a : \varphi(x,a) \in \mathfrak{p}(x)\}$ is definable over A. The type definable if it is A-definable for some small set A.

Remark 5.4 *Assume $\mathfrak{p}(x)$ is definable. Then $\mathfrak{p}(x)$ is definable over A iff it is A-invariant.*

Proof: Let $X_\varphi = \{a : \varphi(x,a) \in \mathfrak{p}(x)\}$. If $\mathfrak{p}$ is A-definable, then $f(X_\varphi) = X_\varphi$ and hence $\mathfrak{p}^f = \mathfrak{p}$ for every $f \in \mathrm{Aut}(\mathfrak{C}/A)$. For the other direction use Remark 1.2. $\square$

Definition 5.5 A global type $\mathfrak{p}(x)$ is called *invariant* if it is A-invariant for some set A.

Remark 5.6 *If $\mathfrak{p}(x)$ is A-invariant, $B \supseteq A$ is complete over A, and $p(x) = \mathfrak{p} \restriction B$, then $p(x)|_A\mathfrak{C} = \mathfrak{p}(x)$.*

Proof: $\mathfrak{p}(x)$ is an extension of $p(x)$ that does not split over A. $\square$

Let $\mathfrak{q}$ be an invariant global type. There are different ways to define the product $\mathfrak{p} \otimes \mathfrak{q}$. One option is to step outside the monster model $\mathfrak{C}$ and work in another monster model $\mathfrak{C}'$ extending $\mathfrak{C}$ where every type over $\mathfrak{C}$ and any small subset of $\mathfrak{C}'$ is realized. Then we can realize $a \models \mathfrak{p}$ and $b \models \mathfrak{q}|\mathfrak{C}a$ in $\mathfrak{C}'$ and define $\mathfrak{p} \otimes \mathfrak{q} = \mathrm{tp}(ab/\mathfrak{C})$. By Remark 4.4, this is independent of the choice of the small set A over which $\mathfrak{q}$ is invariant. Another equivalent possibility is to choose a set $B \supseteq A$ which is complete over A and define $\mathfrak{p} \otimes \mathfrak{q} = (\mathfrak{p} \restriction B \otimes_A \mathfrak{q} \restriction B)|_A\mathfrak{C}$. There is a third option, described in the next result.

Remark 5.7 *Let $\mathfrak{q}(x)$ and $\mathfrak{q}(y)$ be global types, let A be a small set and assume $\mathfrak{q}(y)$ is A-invariant. For each $C \supseteq A$ let $r_C(x,y) = \mathrm{tp}(ab/C)$, where $a \models \mathfrak{p} \restriction C$ and $b \models \mathfrak{q} \restriction Ca$. Note that r_C is well-defined independently of the choice of a and b. Since $C \subseteq C'$ implies $r_C \subseteq r_{C'}$ the type $\mathfrak{r}(x,y) = \bigcup_{C \supseteq A} r_C$ is a global type. Then $\mathfrak{r} = \mathfrak{p} \otimes \mathfrak{q}$.*

Proof: We first check that $r_C(x,y)$ is well-defined. Assume $a_i \models \mathfrak{p} \restriction C$ and $b_i \models \mathfrak{q} \restriction Ca_i$ for $i = 1, 2$. Choose $f \in \mathrm{Aut}(\mathfrak{C}/C)$ such that $f(a_1) = a_2$ and let $b = f(b_1)$. Since $\mathfrak{q}^f = \mathfrak{q}$, $b \models \mathfrak{q} \restriction Ca_2$ and hence $a_1b_1 \equiv_C a_2b \equiv_C a_2b_2$.

For the rest it is enough to show that $r_B = \mathfrak{p} \restriction B \otimes_A \mathfrak{q} \restriction B$ for every $B \supseteq A$ complete over A. And this is clear, because $(\mathfrak{q} \restriction B)|_A Ba = \mathfrak{q} \restriction Ba$ for any $a \models \mathfrak{p} \restriction B$. $\square$

More generally, one can define in a similar way the power $\mathfrak{p}^{(\alpha)} = \mathfrak{p}^{(\alpha)}(x_i : i < \alpha)$ for any invariant type $\mathfrak{p}$ and any ordinal α. We set $\mathfrak{p}^{(\alpha+1)}(x_i : i \leq \alpha) = \mathfrak{p}^{(\alpha)}(x_i : i < \alpha) \otimes \mathfrak{p}(x_\alpha)$ and for limit α, $\mathfrak{p}^{(\alpha)}(x_i : i < \alpha) = \bigcup_{\beta<\alpha} \mathfrak{p}^{(\beta)}(x_i : i < \beta)$.

Remark 5.8

1. *If $\mathfrak{q}$ is A-invariant, then $\mathfrak{p} \otimes \mathfrak{q}$ is A-invariant.*
2. *If $\mathfrak{q}, \mathfrak{r}$ are invariant types, then $\mathfrak{p} \otimes (\mathfrak{q} \otimes \mathfrak{r}) = (\mathfrak{p} \otimes \mathfrak{q}) \otimes \mathfrak{r}$.*
3. *If $\mathfrak{p}$ is A-invariant, then $\mathfrak{p}^{(\alpha)}$ is A- invariant for every ordinal α.*
4. *If $\mathfrak{p}$ is A-invariant, then any realization $(a_i : i < \alpha)$ of the power $\mathfrak{p}^{(\alpha)}$ is an A-indiscernible sequence.*

Proof: By Lemma 4.7. $\square$

Definition 5.9 A *Morley sequence* in $\mathfrak{p}$ over A is a sequence $(a_i : i < \alpha)$ such that $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i < \alpha$.

Remark 5.10 *Let $\mathfrak{p}$ be A-invariant. A sequence $(a_i : i < \alpha)$ is a Morley sequence in $\mathfrak{p}$ over A iff it realizes $\mathfrak{p}^{(\alpha)} \restriction A$.*

Proof: Induction on α. $\square$

Lemma 5.11 *$\mathfrak{p}(x)$ is Lascar invariant over A if and only if it is M-invariant for every model $M \supseteq A$.*

Proof: If $A \subseteq M$, then $\mathrm{Aut}(\mathfrak{C}/M) \subseteq \mathrm{Autf}(\mathfrak{C}/A)$. On the other hand, every strong automorphism $f \in \mathrm{Autf}(\mathfrak{C}/A)$ is of the form $f = f_1 \circ \ldots \circ f_n$ where for every i, $f_i \in \mathrm{Aut}(\mathfrak{C}/M_i)$ for some model $M_i \supseteq A$. $\square$

In particular, Lascar invariant types are invariant. Hence product and powers are well-defined.

Remark 5.12 *If $\mathfrak{p}$ is Lascar invariant over A, then $\mathfrak{p}^{(\alpha)}$ is Lascar invariant over A for every ordinal α.*

Proof: By Lemma 5.11 and Remark 5.8. $\square$

6 G-compactness in NIP theories

Lemma 6.1 *Assume the global type $\mathfrak{p}(x)$ is Lascar invariant over A. Any realization $(a_i : i < \omega)$ of $\mathfrak{p}^{(\omega)} \restriction A$ is A-indiscernible.*

Proof: Assume $(a_i : i < \omega) \models \mathfrak{p}^{(\omega)} \restriction A$ and choose a model $M \supseteq A$. Then $\mathfrak{p}$ is M-invariant. Let $(a'_i : i < \omega) \models \mathfrak{p}^{(\omega)} \restriction M$. By Remark 5.8 $(a'_i : i < \omega)$ is M-indiscernible and hence A-indiscernible. Since $(a_i : i < \omega) \equiv_A (a'_i : i < \omega)$, clearly $(a_i : i < \omega)$ is A-indiscernible. $\square$

Theorem 6.2 (Hrushovski-Pillay) *Let $\mathfrak{p}(x)$ be a global type, Lascar invariant over A. If $a \overset{\mathrm{Ls}}{\equiv}_A b$ are realizations of $\mathfrak{p} \restriction A$, then there is some realization $(a_i : i < \omega)$ of $\mathfrak{p}^{(\omega)} \restriction A$ such that the sequences $a, a_0, a_1, \ldots$ and $b, a_0, a_1, \ldots$ are both A-indiscernible. Hence $\mathrm{d}_A(a, b) \leq 2$.*

Proof: We can assume $a \models \mathfrak{p} \restriction M$ for some model $M \supseteq A$. The reason is that we can choose a model $N \supseteq A$, some $c \models \mathfrak{p} \restriction N$ and some automorphism $f \in \mathrm{Aut}(\mathfrak{C}/A)$ sending c to a. Then put $M = f(N)$, and let $\mathfrak{q} = \mathfrak{p}^f$ be the A-conjugate of $\mathfrak{p}$ by f. Notice that $a \models \mathfrak{q} \restriction M$. Since f maps a Morley sequence in $\mathfrak{p}$ over N into a Morley sequence in $\mathfrak{q}$ over M, it follows that $\mathfrak{p}^{(\omega)} \restriction A = \mathfrak{q}^{(\omega)} \restriction A$.

Now let $(a_i : i < \omega)$ be a realization of $\mathfrak{p}^{(\omega)} \restriction Mab$. It follows that the sequence $a, a_0, a_1, \ldots$ is a Morley sequence in $\mathfrak{p}$ over M and therefore it is M-indiscernible and A-indiscernible. We claim that for each $n < \omega$, $a \equiv_{Aa_0,\ldots,a_n} b$, which implies that also $b, a_0, a_1, \ldots,$ is A-indiscernible. Let $\varphi(x, y_0, \ldots, y_n) \in L(A)$. Since $\mathfrak{p}^{(\omega)}$ does not Lascar split over A,

$$\models \varphi(a, a_0, \ldots, a_n) \Leftrightarrow \varphi(a, y_0, \ldots, y_n) \in \mathfrak{p}^{(\omega)} \Leftrightarrow \varphi(b, y_0, \ldots, y_n) \in \mathfrak{p}^{(\omega)} \Leftrightarrow \models \varphi(b, a_0, \ldots, a_n).$$

$\square$

Corollary 6.3 *If T is* NIP *and $p(x) \in S(A)$ does not fork over A, then for any realizations a, b of $p(x)$: $a \overset{\mathrm{Ls}}{\equiv}_A b$ if and only if $\mathrm{d}_A(a, b) \leq 2$.*

Proof: If $p(x) \in S(A)$ does not fork over A, it has a global nonforking extension $\mathfrak{p}(x)$. Since T is NIP, $\mathfrak{p}(x)$ is Lascar invariant over A. $\square$

An *extension base* is a set A such that no type over A forks over A. It follows that any NIP theory is G-compact over extension bases.

Lemma 6.4 *Let T be* NIP *and let $\mathfrak{p}_1(x), \mathfrak{p}_2(x)$ be global types, Lascar invariant over A. If there is a realization $I = (a_i : i < \omega)$ of $\mathfrak{p}_1^{(\omega)} \upharpoonright A$, such that $\mathfrak{p}_1 \upharpoonright AI = \mathfrak{p}_2 \upharpoonright AI$, then $\mathfrak{p}_1 = \mathfrak{p}_2$.*

Proof: Notice that $I \models \mathfrak{p}_1^{(\omega)} \upharpoonright M$ for some model $M \supseteq A$. We claim that if $I' = (a_i : i < \alpha)$ is an A-indiscernible sequence extending I, then $I'c$ is also A-indiscernible for any $c \models \mathfrak{p}_1 \upharpoonright AI'$ or $c \models \mathfrak{p}_2 \upharpoonright AI'$. Consider the case $c \models \mathfrak{p}_1 \upharpoonright AI'$. Assume $i_0 < \ldots < i_n < \alpha$, $\psi(x_0, \ldots, x_n, y) \in L(A)$ and $\models \psi(a_{i_0}, \ldots, a_{i_n}, c)$. Then $\psi(a_{i_0}, \ldots, a_{i_n}, y) \in \mathfrak{p}_1$. Since $\mathfrak{p}_1$ does not Lascar-split over A and $a_{i_0} \ldots a_{i_n} \stackrel{\text{Ls}}{\equiv}_A a_0 \ldots a_n$, $\psi(a_0, \ldots, a_n, y) \in \mathfrak{p}_1$. Since $a_{n+1} \models \mathfrak{p}_1 \upharpoonright Ma_0 \ldots a_n$, $\models \psi(a_0, \ldots, a_n, a_{n+1})$. The case $c \models \mathfrak{p}_1 \upharpoonright AI'$ is similar but uses the assumption $\mathfrak{p}_1 \upharpoonright AI = \mathfrak{p}_2 \upharpoonright AI$.

Now assume $\varphi(x, y) \in L$, $\varphi(x, b) \in \mathfrak{p}_1$ and $\neg\varphi(x, b) \in \mathfrak{p}_2$. Construct $(c_i : i < \omega)$ in such a way that $c_{2i} \models \mathfrak{p}_1 \upharpoonright AIbc_{<2i}$ and $c_{2i+1} \models \mathfrak{p}_2 \upharpoonright AIbc_{<2i+1}$. Note that I is A-indiscernible. By the claim $I^\frown(c_i : i < \omega)$ is also A-indiscernible. Since $\models \varphi(a_{2i}, b)$ and $\models \varphi(a_{2i+1}, b)$, $\varphi(x, y)$ has the independence property, a contradiction. $\Box$

Lemma 6.5 *Let T be* NIP*, let $f \in \mathrm{Aut}(\mathfrak{C}/A)$ and let $\mathfrak{p}$ be a global type which is Lascar invariant over A. Assume that for each $n < \omega$, for each $a \models \mathfrak{p}^{(n)} \upharpoonright A$, $a \stackrel{\text{Ls}}{\equiv}_A f(a)$. Then $\mathfrak{p}^f = \mathfrak{p}$.*

Proof: Let $I = (a_i : i < \omega) \models \mathfrak{p}^{(\omega)} \upharpoonright A$. By Lemma 6.4 it will suffice to prove $\mathfrak{p} \upharpoonright AI = \mathfrak{p}^f \upharpoonright AI$. Let $\varphi(x, y_0, \ldots, y_n) \in L(A)$ and assume $\varphi(x, a_0, \ldots, a_n) \in \mathfrak{p}$. Since the tuple $a_0, \ldots, a_n$ realizes $\mathfrak{p}^{(n+1)} \upharpoonright A$, by assumption $a_0, \ldots, a_n \stackrel{\text{Ls}}{\equiv}_A f(a_0), \ldots, f(a_n)$. Since $\mathfrak{p}$ does not Lascar split over A, $\varphi(x, f(a_0), \ldots, f(a_n)) \in \mathfrak{p}$. This shows that $\mathfrak{p} \upharpoonright AI \supseteq \mathfrak{p}^f \upharpoonright AI$. Repeating the argument for f^{-1}, $\mathfrak{p} \upharpoonright AI \subseteq \mathfrak{p}^f \upharpoonright AI$. $\Box$

Proposition 6.6 *Assume T has* NIP*. Let M be ω-saturated over $A \subseteq M$ and $p(x) \in S(M)$. Then p Lascar splits over A if and only if p KP-splits over A.*

Proof: We only need to prove the direction from right to left. If $p(x)$ does not Lascar split over A, then it does not fork over A and therefore it has a global extension $\mathfrak{p}$ which does not fork over A. Hence $\mathfrak{p}$ is Lascar invariant over A. It is enough to check that $\mathfrak{p}$ is $\mathrm{bdd}(A)$-invariant. For this purpose, let $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(A))$ and let us prove that $\mathfrak{p}^f = \mathfrak{p}$ using Lemma 6.5. Let $a \models \mathfrak{p}^{(n)} \upharpoonright A$. Note that $f(a)$ also realizes $\mathfrak{p}^{(n)} \upharpoonright A$ and $a \stackrel{\text{KP}}{\equiv}_A f(a)$. Since $\mathfrak{p}^{(n)}$ does not fork over A, by Corollary 6.3, $a \stackrel{\text{Ls}}{\equiv}_A f(a)$. $\Box$

Remark 6.7 *Using the comments made on Lascar completeness after Remark 4.9 we can easily obtain a sharper version of Proposition 6.6: If T is* NIP *and B is Lascar complete over $A \subseteq B$, then for any $p(x) \in S(B)$, p Lascar splits over A if and only if p KP-splits over A.*

There are non G-compact theories. The first example was due to M. Ziegler and it is presented in [5]. There are also ω-categorical examples (see [11] and [6]). L. Newelski proved in [16] that in any small theory, for finites tuples $\stackrel{\text{Ls}}{\equiv}_A$ and $\stackrel{\text{KP}}{\equiv}_A$ coincide for any finite set A. In [9] A. Conversano and A. Pillay exhibit a natural example of a non G-compact theory. It is a principal homogeneous space on G, a saturated elementary extension of the universal cover of the group $\mathrm{SL}(2, \mathbb{R})$. This is a NIP example.

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