Around NTP_2

Enrique Casanovas

July 20, 2013. Last revised April 16, 2014. *

The main topic will be the model theory of NTP_2 theories, but occasionally we will discuss some other topics. If not stated otherwise, $\mathop{\smile\!\!\!\!\!\!|}$ will be forking independence, that is $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$.

1 On limit models

This is a topic from last year. We improve some result on limit models from section 5 of [9] after having read parts of [2]. Let $p(x) \in S(\emptyset)$. Recall that M_p is a prime model over a realization of p and that a p-limit model is a model M which is not prime over a realization of p but it is a union of an elementary chain $(M_i : i < \omega)$ where each M_i is prime over a realization of p. Recall also that

$$\mathrm{SI}_p = \{(a, b) : a, b \models p \text{ and } a \text{ semi-isolates } b\}$$

Similarly, we define

$$\mathrm{I}_p = \{(a, b) : a, b \models p \text{ and } a \text{ isolates } b\}$$

Proposition 1.1 *The following are equivalent in any small theory T:*

1. *There is a p-limit model.*
2. *There are $a, b \in M_p$ realizing p and such that $\mathrm{tp}(b/a)$ is nonisolated.*
3. *I_p is not symmetric in M_p.*
4. *I_p is not symmetric.*
5. *SI_p is not symmetric in M_p.*

Proof: *1* $\Leftrightarrow$ *2*. It is a better version of the proof of $\Rightarrow$ of Proposition 5.7 of [9]: we have $M = \bigcup_{i<\omega} M_i$ with M_i prime over $a_i \models p$. We claim that for some i, $\mathrm{tp}(a_i/a_0)$ is nonisolated; otherwise every tuple $a \in M_i$ is isolated over a_0 for every $i < \omega$ and then M is prime over a_0.

2 $\Rightarrow$ *3* $\Rightarrow$ *4* is clear.

4 $\Rightarrow$ *3*. If $\mathrm{tp}(b/a)$ is isolated, it is realized in a prime model over a.

3 $\Rightarrow$ *5*. See Lemma 3.3 in [9].

5 $\Rightarrow$ *2* is clear. □

*Notes of talks given at the Model Theory Seminar in the academic year 2012-2013. Thanks to all members of the Seminar for their comments: Félix Bou, Esperanza Buitrago, Santiago Cárdenas, Rafel Farré, Joost Joosten, Cory Knapp and Joris Potier.

2 Weak elimination of hyperimaginaries

Recall from [7] that T has *weak elimination of hyperimaginaries* (in short, WEH) if every hyperimaginary is interbounded with a sequence of imaginaries. As shown there (Theorem 6.7), in any simple theory with weak elimination of hyperimaginaries, forking and thorn-forking coincide in T^{eq}. Moreover (Proposition 6.3), a simple theory T has weak elimination of hyperimaginaries iff $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has weak canonical bases in T^{eq}. Now we discuss some stuff from [5].

Definition 2.1 T has *dependence witnessed by imaginaries* (in short, DWI) if for all hyperimaginaries a, b, if $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C b$, then $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C d$ for some imaginary $d \in \mathrm{acl}^{\mathrm{eq}}(Cb)$. Here C is a set of hyperimaginaries (but we will see that one can assume it is a set of imaginaries) and $\mathrm{acl}^{\mathrm{eq}}(Cb)$ is the set of all imaginaries that are algebraic over Cb. On the other hand, it is enough to obtain a tuple of imaginaries d instead of a single one and, if T is simple, even a set $D \subseteq \mathrm{acl}^{\mathrm{eq}}(Cb)$ such that $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C D$.

Proposition 2.2 *The following are equivalent to DWI in any simple theory T:*

1. *If a, b are hyperimaginaries and C is a set of imaginaries such that $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C b$, then $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C d$ for some imaginary $d \in \mathrm{acl}^{\mathrm{eq}}(Cb)$.*
2. *T has weak elimination of hyperimaginaries.*
3. *If a, b are hyperimaginaries and C is a set of hyperimaginaries such that $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C b$, then $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C d$ for some imaginary $d \in \mathrm{acl}^{\mathrm{eq}}(b)$.*

Proof: *1 ⇒ 2.* Let e be a hyperimaginary and let $C = \mathrm{acl}^{\mathrm{eq}}(e)$ (the set of all imaginaries that are algebraic over e). Notice that $C = \mathrm{acl}^{\mathrm{eq}}(C)$. We claim that for every hyperimaginary a, $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C e$. Assume not, i.e., $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C e$. By assumption there is some imaginary $d \in \mathrm{acl}^{\mathrm{eq}}(Ce)$ such that $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C d$. But $\mathrm{acl}^{\mathrm{eq}}(Ce) = C$, and hence $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C C$, a contradiction. By the claim, $e \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C e$, which implies $e \in \mathrm{bdd}(C)$. Hence e is interbounded with a tuple of imaginaries enumerating C.

2 ⇒ 3. Assume a, b are hyperimaginaries and C is a set of hyperimaginaries such that $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C b$. By WEH, there is some set B of imaginaries such that $\mathrm{bdd}(B) = \mathrm{bdd}(b)$. Hence $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B$ and by finite character of forking there is some tuple $d \in B$ such that $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C d$. Clearly, $d \in \mathrm{acl}^{\mathrm{eq}}(b)$. □

Proposition 2.3 *The following are equivalent in any simple theory T:*

1. *T has weak elimination of hyperimaginaries.*
2. *T has DWI.*
3. *$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ has weak canonical bases in T^{eq}.*

Proof: *1 ⇔ 2.* By Proposition 2.2.

1 ⇒ 3. This is Proposition 6.3 and Theorem 6.7 in [7].

3 ⇒ 1. By Proposition 6.3 in [7]. □

3 The weak independence theorem

Taken from Ben-Yaacov and Chernikov preprint [4].

Definition 3.1 The *weak independence theorem over A* is the following statement: if $b \stackrel{\text{Ls}}{\equiv}_A b'$, $a \mathop{\smile\!\!\!\!\!\!\mid}_A bb'$ and $c \mathop{\smile\!\!\!\!\!\!\mid}_A ab$, then there is some c' such that $c'a \equiv_A ca$, $c'b' \equiv_A cb$, and $c' \mathop{\smile\!\!\!\!\!\!\mid}_A ab'$.

Remark 3.2 *The independence theorem over A implies the weak independence theorem over A.*

Proof: Choose first c_1 such that $bc \stackrel{\text{Ls}}{\equiv}_A b'c_1$. Then $c \stackrel{\text{Ls}}{\equiv}_A c_1$, $a \mathop{\smile\!\!\!\!\!\!\mid}_A b'$, $c \mathop{\smile\!\!\!\!\!\!\mid}_A a$ and $c_1 \mathop{\smile\!\!\!\!\!\!\mid}_A b'$. By the independence theorem there is some c' such that $c' \equiv_{Aa} c$, $c' \equiv_{Ab'} c_1$ and $c' \mathop{\smile\!\!\!\!\!\!\mid}_A ab'$. Clearly, c' satisfies all the requirements. □

Recall that $\mathrm{d}_A(a,b) \leq n$ iff $\models \mathrm{nc}^n_A(a,b)$, that is, iff there are $a_0, \ldots, a_n$ such that $a = a_0$, $b = b_n$ and for each $i < n$, $\models \mathrm{nc}(a_i, a_{i+1})$ (a_i, a_{i+1} start and infinite A-indiscernible sequence).

Proposition 3.3 *Assume forking coincides with dividing over A and the weak independence theorem holds over A. Then T is G-compact over A, and in fact $e \stackrel{\text{Ls}}{\equiv}_A d$ iff $\mathrm{d}_A(e,d) \leq 3$.*

Proof: Note that the hypothesis implies that A is an extension base. Let $e \stackrel{\text{Ls}}{\equiv}_A d$. Since A is an extension base, we may fix some global nonforking extension of $\mathrm{tp}(d/A)$. Using this global type we can obtain (as in the case of simple theories) a Morley sequence $(d_i : i < \omega)$ over A starting with $d_0 = d$: first obtain a long A-independent sequence using the global type and then use Erdös-Rado to make it A-indiscernible. Hence $(d_i : i < \omega)$ is A-indiscernible and $d_i \mathop{\smile\!\!\!\!\!\!\mid}_A d_{<i}$ for every $i < \omega$. The assumption that forking and dividing over A coincide implies that the pair lemma holds for forking independence: if $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{Ac_1} B$, then $a_1a_2 \mathop{\smile\!\!\!\!\!\!\mid}_A B$. Using this and some induction one easily checks that for every $i < \omega$, $d_{\geq i} \mathop{\smile\!\!\!\!\!\!\mid}_A d_{<i}$.

We have then $d_{\geq 1} \mathop{\smile\!\!\!\!\!\!\mid}_A d_0$. Without loss of generality, we can assume that $d_{\geq 1} \mathop{\smile\!\!\!\!\!\!\mid}_A d_0 e$ (otherwise we replace $d_{\geq 1}$ by some $d'_{\geq 1} \equiv_{Ad_0} d_{\geq 1}$ having this property). The situation is as follows

- $d_0 \stackrel{\text{Ls}}{\equiv}_A e$
- $d_1 \mathop{\smile\!\!\!\!\!\!\mid}_A d_0 e$
- $d_{>1} \mathop{\smile\!\!\!\!\!\!\mid}_A d_0 d_1$

By the weak independence theorem (with $a = d_1$, $b = d_0$, $b' = e$ and $c = d_{>1}$) there is some $d'_{<i}$ such that $d'_{>1} d_1 \equiv_A d_{>1} d_1$ and $d'_{>1} e \equiv_A d_{>1} d_0$ (we do not need the independence). Notice that the sequences $e + d'_{>1}$ and $d_1 + d'_{>1}$ are A-indiscernible. This implies that

$$\models \mathrm{nc}_A(e, d'_2) \wedge \mathrm{nc}_A(d_1, d'_2) \wedge \mathrm{nc}_A(d_1, d_0)$$

Hence $\mathrm{d}_A(e,d) \leq 3$. □

4 The chain condition

We continue with [4], but changing slightly some things and taking into account some material from [6].

Definition 4.1 Let $\pi(x)$ be a partial type over A. We say that $\pi(x)$ has *the chain condition over A with respect to $\pi(x)$* if for every $\varphi(x,y) \in L$, if $\models \mathrm{nc}_A(a_0,a_1)$ and $\pi(x) \cup \{\varphi(x,a_0)\}$ does not fork over A, then $\pi(x) \cup \{\varphi(x,a_0) \wedge \varphi(x,a_1)\}$ does not fork over A. If $\pi(x)$ is empty we just say that T *has the chain condition over A*.

Proposition 4.2 *The following are equivalent for any partial type $\pi(x)$ over A:*

1. *T has the chain condition over A with respect to $\pi(x)$.*
2. *For any cardinal $\kappa \geq |T| + |A| + |x|$, for any family $(\pi_i(x) : i < (2^\kappa)^+)$ of partial types $\pi_i(x)$ of cardinality $\leq \kappa$ extending $\pi(x)$, if $\pi_i(x)$ does not fork over A for every $i < (2^\kappa)^+$, then $\pi_i(x) \cup \pi_j(x)$ does not fork over A for some $i < j < (2^\kappa)^+$.*
3. *For any cardinal $\kappa \geq |A| + |T| + |x|$, for any family of formulas $(\varphi_i(x,a_i) : i < (2^\kappa)^+)$ with $\varphi_i(x,y) \in L(A)$, if $\pi(x) \cup \{\varphi_i(x,a_i)\}$ does not fork over A for every $i < (2^\kappa)^+$, then $\pi(x) \cup \{(\varphi_i(x,a_i) \wedge \varphi_j(x,a_j))\}$ does not fork over A for some $i < j < (2^\kappa)^+$.*
4. *For any partial type $\pi(x,y)$ over A extending $\pi(x)$, if $(a_i : i < \omega)$ is A-indiscernible and $\pi(x,a_i)$ does not fork over A for every $i < \omega$, then $\bigcup_{i<\omega} \pi(x,a_i)$ does not fork over A.*
5. *If $(a_i : i < \omega)$ is A-indiscernible $b \models \pi(x)$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_A a_0$, then there is some $b' \equiv_{Aa_0} b$ such that $(a_i : i < \omega)$ is indiscernible over Ab' and $b' \mathop{\smile\!\!\!\!\!\!\mid}_A (a_i : i < \omega)$.*
6. *If $(a_i : i < \omega)$ is A-indiscernible $b \models \pi(x)$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_A a_0$, then there is some sequence $(a'_i : i < \omega) \equiv_{Aa_0} (a_i : i < \omega)$ which is indiscernible over Ab and $b \mathop{\smile\!\!\!\!\!\!\mid}_A (a'_i : i < \omega)$.*

Proof: *1* $\Rightarrow$ *2*. We can assume each $\pi_i(x)$ is closed under conjunction. Let $\pi_i(x) = \pi_i(x,a_i) = \{\varphi_{ij}(x,a_i) : j < \kappa\}$ with $\varphi_{ij}(x,y) \in L$. Assume $\pi_i(x,a_i) \cup \pi_j(x,a_j)$ forks over A for every $i < j < (2^\kappa)^+$. Since κ is large enough, we can assume that $\varphi_{ij}(x,y) = \varphi_j(x,y)$ for all $i < (2^\kappa)^+$ and $j < \kappa$. By Proposition 3.3 of [6] there are $i < j < (2^\kappa)^+$ such that $\models \mathrm{nc}_A(a_i,a_j)$. We can then choose some $l < \kappa$ such that $\varphi_l(x,a_i) \wedge \varphi_l(x,a_j)$ forks over A, contradicting *1*.

2 $\Rightarrow$ *3* is clear.

3 $\Rightarrow$ *4*. We can extend the sequence to an A-indiscernible sequence $(a_i : i < \lambda)$ where $\kappa = |A| + |T| + |x|$ and $\lambda = (2^\kappa)^+$. By *3* and indiscernibility, for each $i < j < \lambda$, for each $\varphi(x,y) \in \pi(x,y)$, $\varphi(x,a_i) \wedge \varphi(x,a_j)$ does not fork over A. We can inductively generalize this and show that for any $n < \omega$, for any $i_1 < \ldots < i_n < \lambda$, $\varphi(x,a_{i_1}) \wedge \ldots \wedge \varphi(x,a_{i_n})$ does not fork over A, which implies that $\bigcup_{i<\lambda} \pi(x,a_i)$ does not fork over A. For the induction notice that $(b_i : i < \lambda)$ is A-indiscernible if $b_i = a_{i\cdot(n+1)}, \ldots, a_{(i+1)\cdot(n+1)}$.

4 $\Rightarrow$ *5*. Let $p(x,y) \in S(A)$ be such that $\mathrm{tp}(b/Aa_0) = p(x,a_0)$. By *4* $\bigcup_{i<\omega} p(x,a_i)$ does not fork over A. Choose $b' \models \bigcup_{i<\omega} p(x,a_i)$ such that $b' \mathop{\smile\!\!\!\!\!\!\mid}_A (a_i : i < \omega)$. We can extend the sequence and then apply Erdös-Rado to extract an Ab'-indiscernible sequence $(a'_i : i < \omega) \equiv_A (a_i : i < \omega)$ such that $b' \mathop{\smile\!\!\!\!\!\!\mid}_A (a'_i : i < \omega)$. By conjugation over A we obtain a corresponding b'' for $(a_i : i < \omega)$.

5 $\Leftrightarrow$ *6*. By conjugation over A.

5 ⇒ 1. Let $(a_i : i < \omega)$ be an A-indiscernible sequence such that $\varphi(x, a_0)$ does not fork over A. Choose $b \mathop{\smile\!\!\!\!\!\!\,|}_A a_0$ such that $b \models \pi(x)$ and $\models \varphi(b, a_0)$. By *5* there is some $b' \equiv_{Aa_0} b$ such that $(a_i : i < \omega)$ is indiscernible over Ab' and $b' \mathop{\smile\!\!\!\!\!\!\,|}_A (a_i : i < \omega)$. Since $b' \models \pi(x)$ and $\models \varphi(b', a_0) \wedge \varphi(b', a_1)$, the set $\pi(x) \cup \{\varphi(x, a_0) \wedge \varphi(x, a_1)\}$ does not fork over A. □

Proposition 4.3 *If T has the chain condition over A, then the weak independence theorem over A holds.*

Proof: We prove first the case $\models \mathrm{nc}_A(b, b')$. In fact we will see that the general case follows from this particular case without the need of the chain condition.

Assume $\models \mathrm{nc}_A(b, b')$, $a \mathop{\smile\!\!\!\!\!\!\,|}_A bb'$ and $c \mathop{\smile\!\!\!\!\!\!\,|}_A ab$. Since $a \mathop{\smile\!\!\!\!\!\!\,|}_{Ab} b'$, by Lemma 10.6 of [8], there is some a' such that $\models \mathrm{nc}_A(ab, a'b')$. Fix an A-indiscernible sequence $(a_i b_i : i < \omega)$ with $a_0, b_0, a_1 b_1 = a, b, a', b'$. Since $c \mathop{\smile\!\!\!\!\!\!\,|}_A a_0 b_0$, by point *5* of Proposition 4.2 there is some $c' \equiv_{Aa_0b_0} c$ such that $c' \mathop{\smile\!\!\!\!\!\!\,|}_A (a_i b_i : i < \omega)$ and $(a_i b_i : i < \omega)$ is Ac'-indiscernible. It follows that $c' \mathop{\smile\!\!\!\!\!\!\,|}_A ab'$, $c'b' \equiv_A c'b \equiv_A cb$ and $c'a \equiv_A ca$.

Now we prove the general case. Fix n such that $\mathrm{d}_A(b, b') \leq n$ and fix $b_0, \ldots, b_n$ such that $b_0 = b$, $b_n = b'$ and $\models \mathrm{nc}_A(b_i, b_{i+1})$ for all $i < n$. Since $a \mathop{\smile\!\!\!\!\!\!\,|}_A b_0 b_n$, we can assume (changing a and c if necessary) that $a \mathop{\smile\!\!\!\!\!\!\,|}_A b_0, \ldots, b_n$. Now we check by induction on $i \leq n$ that there is some c_i such that $c_i b_i \equiv_A cb_0$, $c_i a \equiv_A ca$ and $c_i \mathop{\smile\!\!\!\!\!\!\,|}_A ab_i$. This will suffice. We start with $c_0 = c$. In order to obtain c_{i+1} from c_i we apply the particular case proven above: there is some c_{i+1} such that $c_{i+1} b_{i+1} \equiv_A c_i b_i$, $c_{i+1} a \equiv_A c_i a$ and $c_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_A b_{i+1} a$. Using now the induction hypothesis on c_i we see that $c_{i+1} b_{i+1} \equiv_A cb_0$ and $c_{i+1} a \equiv_A ca$. □

5 Indiscernible arrays

References for this section are [4], [10] and [3], but we do some things differently.

We will discuss arrays of the form $\mathbb{A} = (a_{ij} : i < \alpha, j < \beta)$ where κ, λ are ordinals (although more generally we could have dealt with linearly ordered sets). We understand that the rows are $(a_{ij} : j < \beta)$ for $i < \alpha$ and the columns are $(a_{ij} : j < \alpha)$ for $i < \beta$. Hence our array looks like as follows:

$$\begin{array}{cccc} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

Each a_{ij} is a (possibly infinite) tuple and all have the same length. Sometimes we use the notation $a_i = (a_{ij} : j < \beta)$ for the rows. The sequence of previous rows is $a_{<i}$. Other notations, like $a_{>i}$ are self-explanatory.

An (n, m)-*subarray* of $(a_{ij} : i < \alpha, j < \beta)$ is an array of the form $(a_{i_l j_k} : l = 1, \ldots, n$ and $k = 1, \ldots, m)$ for some $i_1 < \ldots < i_n < \alpha$ and some $j_1 < \ldots < j_m < \beta$. In the case $n = m$ we talk of an n-*subarray.*

Definition 5.1 An array is *A-indiscernible* if both the sequence of its rows and the sequence of its columns are A-indiscernible. Note that A-indiscernibility of rows imply that each single column is A-indiscernible and, similarly, A-indiscernibility of columns imply that each single row is A-indiscernible.

Remark 5.2 *The following are equivalent for any array $\mathbb{A} = (a_{ij} : i < \alpha, j < \beta)$ and any set A:*

1. *$\mathbb{A}$ is A-indiscernible.*
2. *For all $n, m < \omega$ all (n, m)-subarrays of $\mathbb{A}$ have the same type over A.*
3. *For any $i_1, \ldots, i_n < \alpha$ and $j_1, \ldots, j_n < \beta$, the type $\mathrm{tp}(a_{i_1 j_1} \ldots a_{i_n j_n}/A)$ depends only on the order (and equality) relations of $i_1, \ldots, i_n$ and of $j_i, \ldots, j_n$.*

Proof: *1 ⇒ 2.* Consider two (n, m)-subarrays a and b, the first one determined by $i_1 < \ldots < i_n < \alpha$ and $j_1 < \ldots < j_n < \beta$ and the second one by $i'_1 < \ldots < i'_n < \alpha$ and $j'_1 < \ldots < j'_n < \beta$. By A-indiscernibility of the sequence of columns, a has same type over A as the (n, m)-subarray c determined by $i_1, \ldots, i_n$ and $j'_1, \ldots, j'_n$. Now by A-indiscernibility of the sequence of rows, c has same type over A as b.

2 ⇒ 3. If the order of indexes in rows and columns coincide, for some n, m the corresponding tuples of the array lie in the same positions in some (n, m)-subarrays.

3 ⇒ 1. Clear □

Remark 5.3 *Let $(a_{ij} : i < \alpha, j < \beta)$ be an A-indiscernible array. For any two strictly increasing mappings $f : \alpha \to \beta$ and $g : \alpha \to \beta$:*

$$(a_{if(i)} : i < \alpha) \equiv_A (a_{ig(i)} : i < \alpha)$$

Proof: By Remark 5.2, since $(f(i) : i < \alpha)$ and $(g(i) : i < \alpha)$ are order isomorphic. □

Definition 5.4 The rows of an array $(a_{ij} : i < \alpha, j < \beta)$ are *mutually indiscernible over* A if each row $a_i = (a_{ij} : j < \beta)$ is indiscernible over A and the rest $a_{\neq i} = \{a_{lj} : l < \alpha, l \neq i$ and $j < \beta)$ of the array. The array is *very indiscernible over A* if it is A-indiscernible and its rows are mutually indiscernible over A.

Remark 5.5 *An array $(a_{ij} : i < \alpha, j < \beta)$ has rows mutually indiscernible over A if and only if for any $i_1 < \ldots < i_n < \alpha$ and any choice b_{i_l} and b'_{i_l} for each $l = 1, \ldots, n$ of finite increasing sequences with same length of tuples of the row i_l:*

$$b_{i_1}, \ldots, b_{i_n} \equiv_A b'_{i_1}, \ldots, b'_{i_n}$$

Proof: It is clear that this new condition implies mutual indiscernibility of rows. For the other direction, by induction on $k \leq n$ we check that $b'_{i_1}, \ldots, b'_{i_k} b_{i_{k+1}}, \ldots, b_{i_n} \equiv_A b_{i_1}, \ldots, b_{i_n}$. The case $k = 1$ is clear by mutual indiscernibility. For the inductive step note that, by mutual indiscernibility and the induction hypothesis, $b'_{i_1}, \ldots, b'_{i_k} b'_{i_{k+1}}, b_{i_{k+2}} \ldots, b_{i_n} \equiv_A b'_{i_1}, \ldots, b'_{i_k} b_{i_{k+1}}, \ldots, b_{i_n} \equiv_A b_{i_1}, \ldots, b_{i_n}$. □

Remark 5.6 1. *If the rows of an array are mutually indiscernible over A, then the sequence of its columns is A-indiscernible.*

2. *If an array has rows mutually indiscernible over A and the sequence of its rows is A-indiscernible, then it is very indiscernible over A.*

Proof: By Remark 5.5, since to check indiscernibility of columns it is enough to look at finitely many rows of two given finite sequences of columns. □

Remark 5.7 *Let $\beta \geq \omega$, assume the columns of $\mathbb{A} = (a_{ij} : i < \alpha, j < \beta \cdot \alpha)$ are A-indiscernible, and define $b_{ij} = a_{i,\beta \cdot i+j}$. Then the rows of $\mathbb{B} = (b_{ij} : i < \alpha, j < \beta)$ are mutually indiscernible over A. Hence, if $\mathbb{A}$ is an A-indiscernible array, then $\mathbb{B}$ is very indiscernible over A.*

Proof: Note that each $(a_{i,\beta \cdot i+j} : j < \beta)$ is indiscernible over $A \cup \{a_{l,\beta \cdot k+j} : l < \alpha,\ (k < i \text{ or } k < l < \alpha) \text{ and } j < \beta\}$. Hence $(b_{ij} : j < \beta)$ is indiscernible over $A \cup \{b_{lj} : j < \beta, l \neq i\}$. By Remark 5.6 $\mathbb{B}$ is very indiscernible over A. □

Proposition 5.8 *For any array $\mathbb{A} = (a_{ij} : i < \alpha, j < \lambda)$ and any set A, if $\kappa \geq |A| + |T| + |a_{ij}| + |\alpha|$ and $\lambda = \beth_{(2^\kappa)^+}$, then there is an array $\mathbb{B} = (b_{ij} : i < \alpha, j < \omega)$ which has mutually A-indiscernible rows and is locally like $\mathbb{A}$, in the sense that for every choice in $\mathbb{B}$ of finitely many rows and a finite subsequence in each of these rows, there is a corresponding choice of subsequences in the same rows of $\mathbb{A}$ having (all together) the same type over A.*

Proof: By the ordinary method of extracting indiscernible sequences based on Erdös-Rado, there is some array $\mathbb{A}' = (a'_{ij} : i < \alpha, j < \omega)$ with A-indiscernible sequence of columns and such that for every finite subsequence of columns in $\mathbb{A}'$ there is a corresponding sequence of columns in $\mathbb{A}$ with the same type over A. In particular, $\mathbb{A}'$ is locally like $\mathbb{A}$. The next step is extending the array $\mathbb{A}'$ to $(a'_{ij} : i < \alpha, j < \omega \cdot \alpha)$ with A-indiscernible sequence of columns. Now by Remark 5.7 if we define $b_{ij} = a'_{i,\omega \cdot i+j}$ for $i < \alpha$ and $j < \omega$, we obtain a new array $\mathbb{B} = (b_{ij} : i < \alpha, j < \omega)$ with the required properties. □

6 Array-dividing

Here the main reference is [4]

Definition 6.1 Let $\varphi(x, y) \in L$. We say that $\varphi(x, a)$ *array-divides over A* if there is some A-indiscernible array $(a_{ij} : i, j < \omega)$ with $a = a_{00}$ and such that $\{\varphi(x, a_{ij} : i, j < \omega\}$ is inconsistent.

Remark 6.2 *If $\varphi(x, a)$ divides over A, then it array-divides over A.*

Proof: Let $(a_i : i < \omega)$ be an A-indiscernible sequence with $a = a_0$ such that $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent. Let $a_{ij} = a_j$. Then the array $(a_{ij} : i, j < \omega)$ witnesses that $\varphi(x, a)$ array-divides over A. □

Proposition 6.3 *If forking and dividing over A coincide and T has the chain condition over A, then dividing and array-dividing over A coincide.*

Proof: Assume $\varphi(x, a)$ does not divide (fork) over A, let $(a_{ij} : i, j < \omega)$ be A-indiscernible with $a = a_{00}$ and let us check that $\{\varphi(x, a_{ij}) : i, j < \omega\}$ is consistent. Let $n < \omega$. By the chain condition over A, $\bigwedge_{i<n} \varphi(x, a_{i0})$ does not fork over A. Since the sequence $(b_j : j < \omega)$ with $b_j = (a_{ij} : i < n)$ is A-indiscernible, again by the chain condition over A, $\bigwedge_{i,j<n} \varphi(x, a_{ij})$ does not fork over A and hence it is consistent. □

Definition 6.4 An A-indiscernible sequence $(a_i : i < \omega)$ is a *universal over A in* $\mathrm{tp}(a_0/A)$ if $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent for every formula $\varphi(x, y) \in L$ such that $\varphi(x, a_0)$ divides over A.

Remark 6.5 *1. In a simple theory, every Morley sequence over A is universal over A.*

2. *In a* NTP_2 *theory, every sequence generated over* A *by a strict* A*-invariant global type is universal over* A *(see Proposition 4.4 of [7]); hence (see Proposition 4.11 of [7]) any type over a model* M *has universal indiscernible sequences over* M.

Proposition 6.6 *If forking and array-dividing over* A *coincide and for every type* $p(x)$ *over* A *there are universal indiscernible sequences over* A *in* $p(x)$*, then* T *has the chain condition over* A.

Proof: Assume $\varphi(x, y) \in L$, $\varphi(x, a_0)$ does not fork over A , $(a_j : j < \omega)$ is A-indiscernible and $\varphi(x, a_0) \wedge \varphi(x, a_1)$ forks over A. Let κ be large enough and extend the sequence to an A-indiscernible sequence $(a_j : j < \kappa)$. Choose a universal indiscernible sequence $((a_{ij} : j < \kappa) : i < \omega)$ over A in the type $\mathrm{tp}((a_j : j < \kappa)/A)$ with $a_{0j} = a_j$. By the choice of κ we can extract an A-indiscernible sequence $((a'_{ij} : i < \omega) : j < \omega)$ from the sequence of columns $((a_{ij} : i < \omega) : j < \kappa)$. It follows that $(a'_{ij} : i, j < \omega)$ is an A-indiscernible array. By A-indiscernibility, for every $j < l < \kappa$, $\varphi(x, a_{0j}) \wedge \varphi(x, a_{0l})$ forks over A. Since we have chosen a universal indiscernible sequence, for every $j < l < \kappa$, $\{\varphi(x, a_{ij}) \wedge \varphi(x, a_{il}) : i < \omega\}$ is inconsistent. Hence, for every $j < l < \omega$, $\{\varphi(x, a'_{ij}) \wedge \varphi(x, a'_{il}) : i < \omega\}$ is inconsistent and $\{\varphi(x, a'_{ij}) : i, j < \omega\}$ is inconsistent too. Since $a_0 \equiv_A a'_{00}$, $\varphi(x, a_0)$ array-divides (and forks) over A, a contradiction. □

Definition 6.7 A partial type $\pi(x)$ has TP_2 if for some formula $\varphi(x, y) \in L$ and some $k < \omega$ there is some array $(a_{ij} : i, j < \omega)$ such that $\{\varphi(x, a_{ij}) : j < \omega\}$ is k-inconsistent for every $i < \omega$ and $\pi(x) \cup \{\varphi(x, a_{if(i)}) : i < \omega\}$ is consistent for every $f : \omega \to \omega$. We say that π has NTP_2 if it does not have TP_2. A theory T has TP_2 if the empty partial type has TP_2 in T. Similarly for NTP_2. As shown in Lemma 7.4, this definition agrees with the one given in [7] (where we required $k = 2$).

Remark 6.8 *Nothing changes if in the definition of* NTP_2 *type one requires that* $\{\varphi(x, a_{ij}) : j < \omega\}$ *is* k*-inconsistent relatively to* $\pi(x)$*, namely,* $p(x) \cup \{\varphi(x, a_{i,j_1}), \ldots, \varphi(x, a_{i,j_k})\}$ *is inconsistent for all* $j_1 < \ldots < j_k < \omega$. *The reason is that in an array witnessing* TP_2 *of* $\pi(x)$ *with this weaker requirement we can assume all rows have the same type over* A *and hence adding the same formula* $\theta(x)$ *of* $\pi(x)$ *to* $\phi(x, y)$ *we get* k*-inconsistency. If* $\theta(x)$ *has parameters in* A *we can add the parameters to each tuple in the array.*

Lemma 6.9 *Let* $\pi(x)$ *a partial* NTP_2 *type over* A. *If the array* $(a_{ij} : i, j < \omega)$ *is very indiscernible over* A *and the first column* $\{\varphi(x, a_{i0}) : i < \omega\}$ *is consistent with* $\pi(x)$*, then* $\pi(x) \cup \{\varphi(x, a_{ij}) : i, j < \omega\}$ *is consistent.*

Proof: Let $n < \omega$ and let us check that $\pi(x) \cup \{\varphi(x, a_{ij}) : i < n, j < \omega\}$ is consistent. Let $b_{ij} = a_{i\cdot n,j}, \ldots, a_{i\cdot n+(n-1),j}$. Then the array $(b_{ij} : i, j < \omega)$ is also very indiscernible over A. Let $\varphi^n(x; y_0, \ldots, y_{n-1}) = \bigwedge_{i<n} \varphi(x; y_i)$. We need to check that $\pi(x) \cup \{\varphi^n(x, b_{0j}) : j < \omega\}$ is consistent. We know that $\pi(x) \cup \{\varphi^n(x, b_{i0}) : i < \omega\}$ is consistent. By mutual indiscernibility over A (see Remark 5.5), for every $f : \omega \to \omega$, the set $\pi(x) \cup \{\varphi^n(x, b_{i,f(i)}) : i < \omega\}$ is consistent. If $\pi(x) \cup \{\varphi^n(x, b_{0j}) : j < \omega\}$ is inconsistent, then (see Remark 6.8) we may assume that $\{\varphi^n(x, b_{0j}) : j < \omega\}$ is k-inconsistent for some $k < \omega$. Then every $\{\varphi^n(x, b_{ij}) : j < \omega\}$ is k-inconsistent too for every $i < \omega$, and therefore φ^n witnesses that $\pi(x)$ has TP_2. □

Lemma 6.10 *Let* $\pi(x)$ *bea* NTP_2 *type over* A. *If the array* $(a_{ij} : i, j < \omega)$ *is indiscernible over* A *and the diagonal* $\{\varphi(x, a_{ii}) : i < \omega\}$ *is consistent with* $\pi(x)$*, then* $\pi(x) \cup \{\varphi(x, a_{ij}) : i, j < \omega\}$ *is consistent.*

Proof: Assume $\pi(x) \cup \{\varphi(x, a_{ij}) : i, j < \omega\}$ is consistent. We can extend the array to an A-indiscernible array $(a_{ij} : i < \omega, j < \omega \cdot \omega)$. By indiscernibility of columns, $\pi(x) \cup \{\varphi(x, a_{i,\omega \cdot i}) : i < \omega\}$ is consistent. Let $b_{ij} = a_{i,\omega \cdot i+j}$. By Remark 5.7 $(b_{ij} : i, j < \omega)$ is very indiscernible over A. Since $b_{i0} = a_{i,\omega \cdot i}$, by Lemma 6.9 $\pi(x) \cup \{\varphi(x, b_{ij}) : i, j < \omega\}$ is consistent. If we define $c_{ij} = (a_{i,n \cdot j+k} : k < n)$, we get an A-indiscernible array $(c_{ij} : i, j < \omega)$. If $\varphi^n(x; y_0, \ldots, y_{n-1}) = \bigwedge_{i<n} \varphi(x; y_i)$, then the diagonal $\{\varphi(x, c_{ii}) : i < \omega\}$ is consistent with $\pi(x)$. Interchanging rows and columns and applying again this fact, we see that for each $m < \omega$, $\{\bigwedge_{i<m} \varphi^n(x, c_{i+j,i}) : j < \omega\}$ is consistent with $\pi(x)$. In particular, this implies that $\{\varphi(x, a_{ij}) : i < n, j < m\}$ is consistent with $\pi(x)$. Hence $\pi(x) \cup \{\varphi(x, a_{ij}) : i, j < \omega\}$ is consistent. □

Proposition 6.11 *If $\pi(x)$ is a* NTP_2 *type over A, then for any formula $\varphi(x, y) \in L(A)$, for any tuple a, $\pi(x) \cup \{\varphi(x, a)\}$ divides over A iff it array-divides over A.*

Proof: Assume $\pi(x) \cup \{\varphi(x, a)\}$ does not divide over A and let $(a_{ij} : i, j < \omega)$ be an A-indiscernible array with $a = a_{00}$. Since $(a_{ii} : i < \omega)$ is A-indiscernible, $\pi(x) \cup \{\varphi(x, a_{ii}) : i < \omega\}$ is consistent. By Lemma 6.10, $\pi(x) \cup \{\varphi(x, a_{ij}) : i, j < \omega\}$ is consistent. This shows that $\varphi(x, a)$ does not array-divide over A. □

Corollary 6.12 *If T is* NTP_2 *and A is an extension base, then T has the chain condition over A, the weak independence theorem holds over A and T is G-compact over A.*

Proof: By Proposition 4.3, the weak independence theorem follows from the chain condition. Since in a NTP_2 theory forking and dividing coincide over extension bases, by Proposition 3.3 the weak independence theorem implies G-compactness. Hence we only need to check the chain condition over extension bases.

Consider first the case of $A = M$. As indicated in Remark 6.5, there are universal indiscernible sequences in any type over M. By propositions 6.11 and 6.6, T has the chain condition over M. Consider now the general case of an extension base A. We use point *3* of Proposition 4.2. Let $\kappa \geq |T| + |A| + |x|$ and assume $\varphi_i(x, a_i)$ does not fork over A for every $i < (2^\kappa)^+$. Choose a model $M \supseteq A$ such that $M \mathop{\smile\!\!\!\!\!\!|}_A (a_i : i < (2^\kappa)^+)$. Then no $\varphi_i(x, a_i)$ forks over M and hence $\varphi_i(x, a_i) \wedge \varphi_j(x, a_j)$ does not fork over M for all $i < j < (2^\kappa)^+$. By Lemma 4.8 of [7], $\varphi_i(x, a_i) \wedge \varphi_j(x, a_j)$ does not fork over A. □

7 Burden

Based on [1] and [10].

Definition 7.1 An *inp-pattern* of depth κ in a partial type $\pi(x)$ is an array $(a_{ij} : i < \kappa, j < \omega)$ together with numbers $k_i < \omega$ and formulas $\varphi_i(x, y_i) \in L$ for $i < \kappa$ such that

1. $\{\varphi_i(x, a_{ij}) : j < \omega\}$ is k_i-inconsistent for every $i < \kappa$.
2. $\pi(x) \cup \{\varphi_i(x, a_{if(i)}) : i < \kappa\}$ is consistent for every $f : \kappa \to \omega$.

The *burden* of $\pi(x)$ is the supremum $\mathrm{bdn}(\pi)$ of all depths of inp-patterns in $\pi(x)$. We write $\mathrm{bdn}(a/A) = \mathrm{bdn}(\mathrm{tp}(a/A))$.

Remark 7.2 *The fact* $\mathrm{bdn}(\pi) = \omega$ *means that there are arbitrarily large finite depths of inp-patterns in π and none of depth ω_1. This is compatible both with the existence and the* *inexistence of patterns of depth ω. Perhaps it would have been better to define the burden as the first cardinal κ for which there is no inp-pattern of depth κ. With this definition one would have $\mathrm{bdn}(\pi(x)) > \kappa$ iff there is some inp-pattern of depth κ in $\pi(x)$. But then the difficulty is that with this notion we would have $\mathrm{bdn}(\pi) = 1$ iff π is algebraic (and there would not exist types of burden zero). This could be repaired by defining the burden as the least κ for which there is no inp-pattern of depth $\kappa + 1$ (this does not change the burden if it is $\geq \omega$).*

Remark 7.3 *1.* $\mathrm{bdn}(\pi(x)) = 0$ *iff* $\pi(x)$ *is algebraic.*

2. If $\pi(x) \vdash \pi'(x)$*, then* $\mathrm{bdn}(\pi(x)) \leq \mathrm{bdn}(\pi'(x))$.

3. $\mathrm{bdn}(\pi(x)) < \infty$ *iff* $\mathrm{bdn}(\pi(x)) < |T|^+$

Proof: *1.* Assume $\pi(x)$ is not algebraic and let $(a_i : i < \omega)$ be a sequence of different realizations of π. The inp-pattern of depth 1 defined by the formula $x = y$, the number $k = 2$ and the parameters $(a_i : i < \omega)$ shows that $\mathrm{bdn}(\pi) \geq 1$. Assume now $\mathrm{bdn}(\pi) \geq 1$, and let $\varphi(x, y)$ with k and $(a_i : i < \omega)$ be a inp-pattern of depth 1 in $\pi(x)$. For each $i < \omega$ choose a maximal subset $I_i \subseteq \omega$ such that $\Sigma_i(x) = \pi(x) \cup \{\varphi(x, a_i) : i < \omega\}$ is consistent (any such set has cardinality $\leq k$) and choose a realization b_i of $\Sigma_i(x)$. If $j \notin I_i$ then $\Sigma_i(x) \cup \Sigma_j(x)$ is inconsistent and therefore $b_i \neq b_j$. Now choose inductively $i_n \in \omega \smallsetminus \bigcup_{m<n} I_{i_m}$. It follows that $b_{i_0}, b_{i_1}, \ldots,$ are different realizations of $\pi(X)$, which proves that $\pi(x)$ is not algebraic.

3. In an inp-pattern of depth $|T|^+$ there is some $k < \omega$ and some formula $\varphi(x, y)$ which are used in infinitely many rows. By compactness we can then extend the pattern with k and φ to any possible depth. □

Lemma 7.4 *If π has* TP_2*, then for any set A there is some very A-indiscernible array $(a_{ij} : i, j < \omega)$ and some formula $\varphi(x, y) \in L$ such that $\{\varphi(x, a_{ij}) : j < \omega\}$ is* 2*-inconsistent for every $i < \omega$ and $\pi(x) \cup \{\varphi(x, a_{i,f(i)}) : i < \omega\}$ is consistent for every $f \in \omega^\omega$.*

Proof: Start with an array $(a_{ij} : i, j < \omega)$, a natural number k and a formula $\varphi(x, y)$ witnessing TP_2, choose λ large enough and extend the array to $(a_{ij} : i < \lambda, j < \omega)$ with all paths consistent with $\pi(x)$ and with k-inconsistent rows. We can extract using Erdös-Rado an array with A-indiscernible sequence of rows and locally like the previous one. Next, we extend its rows to width λ and we apply Proposition 5.8. After all this, we end up with a very A-indiscernible array witnessing TP_2 with $\varphi(x, y)$ and k. We denote it again by $(a_{ij} : i, j < \omega)$. Now we want to reduce k to 2. We may assume k is minimal, that is, no array and formula witness TP_2 of $\pi(x)$ with a smaller number. Consider the set $\{\varphi(x, a_{i,0}), \varphi(x, a_{i,1}) : i < \omega\}$. If it is consistent with $\pi(x)$, then (by indiscernibility of the array) the formula $\psi(x; y_0 y_1) = \varphi(x, y_0) \wedge \varphi(x, y_1)$ and the array $(a_{i,2\cdot j} a_{i,2\cdot j+1} : i < \omega, j < \omega)$ witness TP_2 with a smaller number. If it is inconsistent with $\pi(x)$, we choose $n < \omega$ such that $\{\varphi(x, a_{i,0}), \varphi(x, a_{i,1}) : i < n\}$ is inconsistent with $\pi(x)$. Then the formula $\psi(x; y_0, \ldots, y_{n-1}) = \bigwedge_{i<n} \varphi(x, y_i)$ together with the array $(a_{n\cdot i,j}, \ldots, a_{n\cdot(i+1)-1,j} : i < \omega, j < \omega)$ witnesses TP_2 with $k = 2$. □

Remark 7.5 *T is* NTP_2 *iff all types have burden $< \infty$. In fact, if T has* TP_2*, then for every set A there is a type $p(x) \in S(A)$ with* $\mathrm{bdn}(p(x)) = \infty$.

Proof: Assume T has TP_2 and let A be any set. Use Lemma 7.4 to witness this with φ and an array $(a_{ij} : i, j < \omega)$ with mutually A-indiscernible rows. Let $b \models \{\varphi(x, a_{i0}) : i < \omega\}$ and let $p(x) = \mathrm{tp}(b/A)$. By Remark 5.5, $(a_{i0} : i < \omega) \equiv_A (a_{if(i)} : i < \omega)$ and hence $p(x) \cup \{\varphi(x, a_{if(i)}) : i < \omega\}$ is consistent for every $f \in \omega^\omega$. By compactness, $\mathrm{bdn}(p) = \infty$. □

Definition 7.6 The rows of an array $(a_{ij} : i < \alpha, j < \beta)$ are *almost mutually indiscernible* over A if each row $a_i = (a_{ij} : j < \beta)$ is indiscernible over $Aa_{<i}(a_{l0} : l > i)$. Notice that in such arrays, $(a_{i0} : i < \alpha) \equiv_A (a_{if(i)} : i < \alpha)$ for every $f : \alpha \to \beta$.

Lemma 7.7 *If the rows of the array $(a_{ij} : i < \alpha, j < \beta)$ are almost mutually indiscernible over A, then there is some array $(a'_{ij} : i < \alpha, j < \beta)$ with rows mutually indiscernible over A and such that $(a'_{ij} : j < \beta) \equiv_{Aa_{i0}} (a_{ij} : j < \beta)$ for every $i < \alpha$.*

Proof: As a first step we extend the width of the array to a conveniently large cardinal λ preserving almost mutual indiscernibility. We apply then Proposition 5.8 and compactness to obtain an array $(a'_{ij} : i < \alpha, j < \beta)$ which is locally like our previous array and has mutually A-indiscernible rows. We claim that $(a'_{i0} : i < \alpha) \equiv_A (a_{i0} : i < \alpha)$. To check this, assume $i_0 < \ldots < i_k < \alpha$. There are $j_0, \ldots, j_k < \lambda$ such that $a'_{i_00}, \ldots, a'_{i_k0} \equiv_A a_{i_0j_0}, \ldots, a_{i_kj_k}$. But $a_{i_0j_0}, \ldots, a_{i_kj_k} \equiv_A a_{i_00}, \ldots a_{i_k0}$. This proves the claim. Now take $(a''_{ij} : i < \alpha, j < \beta)$ such that $(a'_{ij} : i < \alpha, j < \beta)(a'_{i0} : i < \alpha) \equiv_A (a''_{ij} : i < \alpha, j < \beta)(a_{i0} : i < \alpha)$. The rows of the new array are again mutually A-indiscernible and moreover $(a''_{ij} : j < \beta)a_{i0} \equiv_A (a'_{ij} : j < \beta)a'_{i0} \equiv_A (a_{ij} : j < \beta)a_{i0}$ for every $i < \alpha$. □

Lemma 7.8 *Let Δ be a finite set of formulas of $L(A)$, let $(a_{ij} : i < n, j < \omega)$ be an array and let $k < \omega$. For each $i < n$ there are $j^i_0 < \ldots < j^i_k < \omega$ such that the rows of the array $(a_{i,j^i_l} : i < n, l \leq k)$ are Δ-indiscernible, meaning that for each $\varphi(x_1, \ldots, x_r; y) \in \Delta$, each row is $\varphi(x_1, \ldots, x_r; a)$-indiscernible for any tuple a of elements of the remaining rows.*

Proof: By Ramsey's Theorem, there is an infinite subset $I \subseteq \omega$ such that the sequence of columns of $(a_{ij} : i < n, j \in I)$ is indiscernible with respect to all formulas in Δ, letting the variables range over elements of any row. By the trick explained in the proof of Remark 5.7 it suffices now to take n consecutive segments of length $k+1$ in the increasing enumeration of I, obtaining this way a diagonal of tuples lying in disjoint columns. □

Lemma 7.9 *Assume the rows of the array $(a_{ij} : i < \alpha, j < \beta)$ are mutually A-indiscernible and let b be any tuple. Let $p_i(x, y) = \mathrm{tp}(ba_{i0}/A)$ and suppose $\bigcup_{i<\alpha, j<\beta} p_i(x, a_{ij})$ is consistent. Then there is some array $(a'_{ij} : i < \alpha, j < \beta)$ with rows mutually indiscernible over Ab and such that $(a'_{ij} : j < \beta) \equiv_{Aa_{i0}} (a_{ij} : j < \beta)$ for all $i < \alpha$.*

Proof: Let $b' \models \bigcup_{i<\alpha, j<\beta} p_i(x, a_{ij})$. Fix a finite set Δ of formulas of $L(Ab')$, a finite subset $S \subseteq \alpha$ and some $k < \omega$. Choose with the Lemma 7.8 a finite array $(a_{i,j^i_l} : i \in S, l \leq k)$ with mutually Δ-indiscernible rows. By Remark 5.5, this array has the same type over A as $(a_{il} : i \in S, l \leq k)$. Let $f \in \mathrm{Aut}(\mathfrak{C}/A)$ send the first array to the second one, and let $\Delta' = f(\Delta)$. Then $(a_{il} : i \in S, l \leq k)$ is Δ'-indiscernible, Δ' is over $Af(b')$ and $f(b')a_{i0} \equiv_A b'a_{i,j^i_0} \equiv_A ba_{i0}$ for every $i \in S$. By compactness, the rows of $(a_{ij} : i < \alpha, j < \beta)$ are mutually indiscernible over Ab'' for some b'' such that $b'' \equiv_{Aa_{i0}} b$ for every $i < \alpha$. By conjugation over A we obtain the array with the required properties over Ab. □

Proposition 7.10 *The following are equivalent for any partial type $\pi(x)$ over A and any cardinal κ:*

1. *There is no inp-pattern of depth κ in $\pi(x)$.*
2. *If $b \models \pi(x)$ and the rows of the array $(a_{ij} : i < \kappa, j < \omega)$ are almost mutually indiscernible over A, then for some $i < \kappa$ there is an Ab-indiscernible sequence $(a'_j : j < \omega) \equiv_{Aa_{i0}} (a_{ij} : j < \omega)$.*

3. *If $b \models \pi(x)$ and the rows of the array $(a_{ij} : i < \kappa, j < \omega)$ are mutually indiscernible over A, then for some $i < \kappa$ there is an Ab-indiscernible sequence $(a'_j : j < \omega) \equiv_{Aa_{i0}} (a_{ij} : j < \omega)$.*

Proof: *1 ⇒ 2.* Assume $b \models \pi(x)$ and the rows of $(a_{ij} : i < \kappa, j < \omega)$ are almost mutually indiscernible over A. Let $p_i(x, y) = \mathrm{tp}(ba_{i0}/A)$ and let $q_i(x) = \bigcup_{j<\omega} p_i(x, a_{ij})$. If each $q_i(x)$ is inconsistent, we can find in each case some $\varphi_i(x, y) \in p_i(x, y)$ and some $k_i < \omega$ such that $\{\varphi_i(x, a_{ij}) : j < \omega\}$ is k_i-inconsistent. Notice that $b \models \{\varphi_i(x, a_{i0} : i < \kappa\}$. By almost mutual indiscernibility over A, $(a_{i0} : i < \kappa) \equiv_A (a_{if(i)}) : i < \kappa)$ for each $f : \kappa \to \omega$. Hence $\pi(x) \cup \{\varphi_i(x, a_{if(i)}) : i < \kappa\}$ is consistent for each such f, which shows that we have obtained an inp-pattern of depth κ in $\pi(x)$. If, on the contrary, some $q_i(x)$ is consistent, then we can apply Lemma 7.9 to the array consisting in this single row and we obtain some Ab-indiscernible sequence $(a'_j : j < \omega) \equiv_{Aa_{i0}} (a_{ij} : j < \omega)$.

2 ⇒ 3. Clear.

3 ⇒ 1. Assume there is an inp-pattern of depth κ in $\pi(x)$, with array $(a_{ij} : i < \kappa, j < \omega)$, formulas $(\varphi_i(x, y_i) : i < \kappa)$ and numbers $(k_i : i < \kappa)$. By an application of Proposition 5.8 (after extending the width of the array) we can assume that the rows are mutually A-indiscernible. Let $b \models \pi(x) \cup \{\varphi_i(x, a_{i0}) : i < \kappa\}$. Then $\models \varphi_i(b, a_{i0})$ and $\{\varphi(x, a_{ij}) : j < \omega\}$ is inconsistent. This means that we can not find some $i < \kappa$ as in point 3. □

Proposition 7.11 *If there is an inp-pattern of depth $\kappa_1 \cdot \kappa_2$ in $\mathrm{tp}(b_1b_2/A)$, then either there is an inp-pattern of depth κ_1 in $\mathrm{tp}(b_1/A)$ or there is an inp-pattern of depth κ_2 in $\mathrm{tp}(b_2/Ab_1)$.*

Proof: We assume that there is no inp-pattern of depth κ_1 in $\mathrm{tp}(b_1/A)$ and there is no inp-pattern of depth κ_2 in $\mathrm{tp}(b_2/Ab_1)$, and we will apply Proposition 7.10. Let $(a_{(i,j),k} : (i, j) \in \kappa_1 \times \kappa_2, k < \omega)$ be an array with mutually A-indiscernible rows. We consider the cartesian product $\kappa_1 \times \kappa_2$ lexicographically ordered and we use the notation $a_{(ij)} = (a_{(i,j),k} : k < \omega)$. We inductively obtain some sequence $a'_i = (a'_{ij} : j < \omega)$ and some ordinal $\alpha_i < \kappa_2$ for each $i < \kappa_1$ in such a way that

1. a'_i is indiscernible over b_2D_i, where $D_i = Aa'_{<i}a_{\geq(i+1,0)}$.
2. $a'_i \equiv_{D_ia_{(i,\alpha_i),0}} a_{(i,\alpha_i)}$.
3. The rows of the array $a'_{\leq i} \cup a_{\geq(i+1,0)}$ are mutually A-indiscernible.

By the assumption (in case $i = 0$) and the inductive hypothesis applied to 3 (cases i successor or limit), the rows $(a_{(i,j)} : j < \kappa_2)$ are mutually indiscernible over D_i. Note that there is no inp-pattern of depth κ_2 in $\mathrm{tp}(b_2/D_i)$. By Proposition 7.10, for some $\alpha_i < \kappa_2$, there is some D_ib_2-indiscernible sequence $a'_i \equiv_{D_ia_{(i,\alpha_i),0}} a_{(i,\alpha_i)}$. Clearly, all conditions of the construction are satisfied. It follows that the rows of $(a'_i : i < \kappa_1)$ are almost mutually indiscernible over Ab_2. By Proposition 7.10 there is some $i < \kappa_2$ and some Ab_1b_2-indiscernible sequence $(a_j : j < \omega) \equiv_{Ab_2a'_{i0}} a'_i$. Then $(a_j : j < \omega) \equiv_{Aa_{(i,\alpha_i)0}} a_{(i,\alpha_i)}$ and again by Proposition 7.10 we conclude that there is no inp-pattern of depth $\kappa_1 \cdot \kappa_2$ in $\mathrm{tp}(b_1b_2/A)$. □

Remark 7.12 *If the burden of a type π were defined as the least cardinal κ for which there is no inp-pattern of depth κ in π, then Proposition 7.11 would read as follows:*

$$\mathrm{bdn}(b_1b_2/A) \leq \mathrm{bdn}(b_1/A) \cdot \mathrm{bdn}(b_2/Ab_1)$$

With our official definition we have for successor cardinal numbers $\kappa_1, \kappa_2 < \omega$:

$$\mathrm{bdn}(b_1/A) < \kappa_1 \ \&\ \mathrm{bdn}(b_2/Ab_1) < \kappa_2 \Rightarrow \mathrm{bdn}(b_1b_2/A) < \kappa_1 \cdot \kappa_2$$

Corollary 7.13 *If T has TP_2, then this is witnessed by some formula $\varphi(x,y)$ where x is a single variable and the number is $k=2$.*

Proof: If T is TP_2, then (by Remark 7.5 and Proposition 7.11) for some single element b, for some set A, $\mathrm{bdn}(b/A)=\infty$. Let $\varphi(x,y)$ witness it. Then x is a single variable. As shown in the proof of Lemma 7.4, we can then take $k=2$. □

8 dp-rank

Based on [1], [10] [14] and [11].

Definition 8.1 An *ict-pattern* (meaning *independent contradictory types pattern*) of depth κ in a partial type $\pi(x)$ is an array $(a_{ij} : i<\kappa, j<\omega)$ together with formulas $\varphi_i(x,y_i)\in L$ for $i<\kappa$ such that for every $f:\kappa\to\omega$, the set

$$\Gamma_f(x)=\{\varphi_i(x,a_{i,f(i)}) : i<\kappa\}\cup\{\neg\varphi_i(x,a_{ij}) : i<\kappa, j\neq f(i)\}$$

is consistent with $\pi(x)$. The *dependence rank* (also called the *dp-rank*) of $\pi(x)$ is the supremum $\mathrm{dprk}(\pi)$ of all depths of ict-patterns in $\pi(x)$. We write $\mathrm{dprk}(a/A)=\mathrm{dprk}(\mathrm{tp}(a/A))$.

Proposition 8.2 *1.* $\mathrm{bdn}(\pi)\leq\mathrm{dprk}(\pi)$.

2. If π is NIP*, then* $\mathrm{bdn}(\pi)=\mathrm{dprk}(\pi)<\infty$*; otherwise* $\mathrm{dprk}(\pi)=\infty$.

Proof: *1.* Let $\pi(x)$ be a partial type over A. We show that we can obtain an ict-pattern of depth κ in $\pi(x)$ from an inp-pattern of depth κ in $\pi(x)$. Let $(\varphi_i(x,y_i) : i<\kappa)$, with $(a_{ij} : i<\kappa)$ and $(k_i : i<\kappa)$ define an inp-pattern in $\pi(x)$. We can assume that the rows of the array are mutually indiscernible over A. Choose a realization b of $\pi(x)\cup\{\varphi_i(x,a_{i0}) : i<\kappa\}$. If $i<\kappa$, there are at most $k_i<\omega$ indexes $j<\omega$ such that $\models\varphi_i(b,a_{ij})$ and we can delete them of the array. Hence we can assume that $\models\neg\varphi_i(b,a_{ij})$ for every $j\neq 0$. This means that $\pi(x)\cup\{\varphi_i(x,a_{i0}) : i<\kappa\}\cup\{\neg\varphi_i(x,a_{ij}) : i<\kappa, 0<j<\omega\}$ is consistent. By mutual indiscernibility over A, the same happens for any other path $f:\kappa\to\omega$. Therefore we have an ict-pattern in $\pi(x)$.

2. Let $\pi(x)$ be a partial type over A. If $\pi(x)$ has IP, witnessed by $\varphi(x,y)$ and $(a_{ij} : j<\kappa, i\in\omega)$, $(b_J : J\subseteq\kappa\times\omega)$ such that $b_J\models\pi(x)$ and $\models\varphi(b_J,a_{ij})\Leftrightarrow(i,j)\in J$, then we can obtain an ict-pattern in $\pi(x)$ defining $\varphi_i(x,y_i)=\varphi(x,y)$ for all $i<\kappa$, $j<\omega$.

Now assume $\pi(x)$ is NIP and let us check that $\mathrm{bdn}(\pi)\geq\mathrm{dprk}(\pi)$. Let $(\varphi_i(x,y_i) : i<\kappa)$, with $(a_{ij} : i<\kappa, j<\omega)$ define an ict-pattern in $\pi(x)$. We can assume that the rows of the array are mutually A-indiscernible. Let $\psi_i(x;y_1y_2)=\varphi_i(x,y_1)\wedge\neg\varphi_i(x,y_2)$ and let $b_{ij}=a_{i,2\cdot j}a_{i,2\cdot j+1}$. Since $\pi(x)$ is NIP, $\pi(x)\cup\{\psi_i(x;b_{ij}) : j<\omega\}$ is inconsistent. Then for some $\theta_i(x,z_i)\in L$ for some tuple $c_i\in A$, for some $k_i<\omega$, $\theta_i(x,c_i)\in\pi(x)$ and $\{\theta_i(x,c_i)\wedge\psi_i(x,b_{ij}) :<\omega\}$ is k_i-inconsistent. Then the formulas $\chi_i(x;y_1y_2,z_i)=\psi_i(x;y_1y_2)\wedge\theta_i(x,z_i)$ with parameters $(b_{ij}c_i : i<\kappa, j<\omega)$ and numbers $(k_i : i<\kappa)$ define a inp-pattern of depth κ in $\pi(x)$.

Finally, assume that $\mathrm{dprk}(\pi)=\infty$ and let us show that π has IP. We may find an ict-pattern of length ω in $\pi(x)$ with constant formula $\varphi_i(x,y_i)=\varphi(x,y)$. Look at the first column $(a_{i0} : i<\omega)$. Given $J\subseteq\omega$ choose $f:\omega\to\omega$ be such that $f(i)=0\Leftrightarrow i\in J$. Since $\pi(x)\cup\{\varphi(x,a_{if(i)}) : i<\kappa\}\cup\{\neg\varphi(x,a_{ij}) : i<\kappa, f(i)\neq j<\omega\}$ is consistent, $\pi(x)\cup\{\varphi(x,a_{i0}) : i\in J\}\cup\{\neg\varphi(x,a_{i0}) : i\in\omega\smallsetminus J\}$ is consistent. Hence $\varphi(x,y)$ and $\{a_{i0} : i<\omega\}$ witness that $\pi(x)$ has IP. □

Remark 8.3 *1.* $\mathrm{dprk}(\pi(x)) = 0$ *iff* $\pi(x)$ *is algebraic.*

2. If $\pi(x) \vdash \pi'(x)$*, then* $\mathrm{dprk}(\pi(x)) \leq \mathrm{dprk}(\pi'(x))$*.*

3. $\mathrm{dprk}(\pi(x)) < \infty$ *iff* $\mathrm{dprk}(\pi(x)) < |T|^+$

Proof: *1.* If $\pi(x)$ is not algebraic we easily find an ict-pattern of depth 1 in $\pi(x)$ using the formula $x = y$. On the other hand, if there is an ict-pattern of depth 1 in $\pi(x)$, it is clear that $\pi(x)$ has infinitely many realizations.

2 is clear.

3. Similar to the proof of *3* of Remark 7.3. □

Proposition 8.4 *let* $\pi(x)$ *be a partial type over* A*. The following are equivalent:*

1. *There is no ict-pattern of depth* κ *in* $\pi(x)$*.*
2. *If* $(a_{ij} : i < \kappa, j < \omega)$ *has mutually* A*-indiscernible rows and* $b \models \pi(x)$*, then some row* $(a_{ij} : j < \omega)$ *is* Ab*-indiscernible.*
3. *If* $(a_{ij} : i < \kappa, j < \omega)$ *has mutually* A*-indiscernible rows and* $b \models \pi(x)$*, then for some* $i < \kappa$*,* $a_{ij} \equiv_{Ab} a_{il}$ *for all* $j, l < \omega$*.*

Proof: *3* $\Rightarrow$ *1.* Assume there an ict-pattern of depth κ in $\pi(x)$, given by formulas $(\varphi_i(x, y_i) : i < \kappa)$ and some array $(a_{ij} : i < \kappa, j < \omega)$. We may assume that the rows are mutually A-indiscernible. Let $b \models \pi(x) \cup \{\varphi_i(x, a_{i,0}) : i < \kappa\} \cup \{\neg\varphi_i(x, a_{ij}) : i < \kappa, 0 < j < \omega\}$. Then a_{i0} and a_{i1} have different type over Ab, contradicting *3*.

1 $\Rightarrow$ *3.* Assume $(a_{ij} : i < \kappa, j < \omega)$ has mutually A-indiscernible rows, $b \models \pi(x)$ and in each row there are elements with different type over Ab, say for each $i < \kappa$ there are $\varphi_i(x, y_i) \in L(A)$ and $k, l < \omega$ such that $\models \varphi_i(b, a_{ik}) \wedge \neg\varphi_i(b, a_{il})$. Adding the parameters of φ_i to each element of the row if necessary, we can assume that $\varphi_i(x, y_i) \in L$. By compactness we can find such an array where the rows have the order type of the integers. Hence it is of the form $(a_{ij} : i < \kappa, j \in \mathbb{Z})$. Let us look at some $i < \kappa$. Either $\{j \in \mathbb{Z} :\models \varphi_i(b, a_{ij})\}$ is cofinal or $\{j \in \mathbb{Z} :\models \neg\varphi_i(b, a_{ij}) : j < \omega\}$ is cofinal and similarly, either $\{j \in \mathbb{Z} :\models \varphi_i(b, a_{ij})\}$ is coinitial or $\{j \in \mathbb{Z} :\models \neg\varphi_i(b, a_{ij}) : j < \omega\}$ is coinitial. Deleting some elements of the row we may assume that either $\models \varphi_i(b, a_{i0}) \wedge \neg\varphi(b, a_{ij})$ for all $j > 0$ or $\models \neg\varphi_i(b, a_{i0}) \wedge \varphi(b, a_{ij})$ for all $j > 0$; and similarly we may assume that either $\models \varphi_i(b, a_{i0}) \wedge \neg\varphi(b, a_{ij})$ for all $j < 0$ or $\models \neg\varphi_i(b, a_{i0}) \wedge \varphi(b, a_{ij})$ for all $j < 0$. There are four possible cases. The first case is that $\models \varphi_i(b, a_{i0}) \wedge \neg\varphi(b, a_{ij})$ for all $j \neq 0$. In this case we take $\psi_i(x, y_i) = \varphi_i(x, y_i)$. The second case is similar, with φ and $\neg\varphi$ switching its roles. Here we take $\psi_i(x, y_i) = \neg\varphi_i(x, y_i)$. In the third case we have $\models \varphi_i(b, a_{ij})$ for all $j \leq 0$ and $\models \varphi_i(b, a_{i,j})$ for all $j > 0$. Then we replace the row by $(b_{ij} : j\mathbb{Z})$ where $b_i = a_{2\cdot i, 2\cdot i+1}$ and we take $\psi_i(x; y_i^1 y_i^2) = \varphi_i(x, y_i^1) \wedge \neg\varphi_i(x, y_i^2)$. The fourth case is similar (exchanging φ_i and $\neg\varphi_i(x, y_i)$) and has a similar solution. In any case, after modifying the row we obtain a formula ψ_i such that $\pi(x) \cup \{\psi_i(x, a_{i0}) : i < \kappa\} \cup \{\neg\psi_i(x, a_{ij}) : i < \kappa, 0 \neq j \in \mathbb{Z}\}$ is consistent. By Remark 5.5 any other path produces a corresponding set also consistent with $\pi(x)$. Hence (after restricting the row to indexes ≥ 0) this array and these formulas provide an ict-pattern of depth κ in $\pi(x)$.

2 $\Leftrightarrow$ *3.* In fact we only need to check *3* $\Rightarrow$ *2*. Assume we have a counterexample to *2*, given by the array $(a_{ij} : i < \kappa, j < \omega)$ and $b \models \pi(x)$. Consider some row $i < \kappa$. For some $n < \omega$ there are $j_0 < \ldots < j_{n-1} < l_0 < \ldots < l_{n-1} < \omega$ such that $a_{ij_0} \ldots a_{ij_{n-1}} \not\equiv_{Ab} a_{il_0} \ldots a_{il_{n-1}}$. Without loss of generality, $j_k = k$ and $l_k = n + k$. We define a new row $(b_{ij} : j < \omega)$ putting $b_{ij} = a_{i,n\cdot j}, \ldots, a_{i,n\cdot j+n-1}$. We do a corresponding modification in every other row. That way we obtain an array $(b_{ij} : i < \kappa, j < \omega)$ with mutually A-indiscernible rows. Note that in each row there are two elements with different type over Ab. Hence we have a counterexample to *3*. □

9 Simple types in NTP_2 theories

Based on [10].

Definition 9.1 Recall from [7] that a partial type $\pi(x)$ is *simple* if there are not parameters $(b_f : f \in \omega^\omega)$, $(a_s : s \in \omega^{<\omega})$, formula $\varphi(x, y) \in L$ and number $< \omega$ such that

1. $b_f \models \pi(x)$ for all $f \in \omega^\omega$.
2. $b_f \models \varphi(x, a_{f\restriction n})$ for all $f \in \omega^\omega$ for all $n < \omega$.
3. $\{\varphi(x, a_{s^\frown n}) : n < \omega\}$ is k-inconsistent for all $s \in \omega^{<\omega}$.

The same notion is defined if we allow $\varphi(x, y) \in L(A)$ since the additional parameters of φ can be added to the nodes a_s of the tree.

We say that the partial type $\Sigma(u)$ is *co-simple* if there are no such objects a_s where each a_s is a tuple of realizations of $\Sigma(u)$ (this replaces condition 1 on π) and $\varphi(x, y) \in L(A)$. In this case this does not seem to be equivalent to $\varphi(x, y) \in L$ since adding the new parameters to the branches b_f is not enough (k-inconsistency might be lost). Note that y is a tuple of the form $u_1 \dots u_m$ where each u_i has the length of u.

Proposition 9.2 *The following are equivalent for any partial type $\pi(x)$ over A:*

1. *$\pi(x)$ is simple.*
2. *$\mathrm{D}(\pi, \Delta, k) < \omega$ for all Δ, k.*
3. *For any completion $p(x) \in S(B)$ of $\pi(x)$ there is some $C \subseteq B$ such that $|C| \leq |T| + |x|$ and $p(x)$ does not fork over C.*
4. *If $A \subseteq C$, $a \models \pi$ and $b \mathop{\smile\!\!\!\!\!|}_C a$, then $a \mathop{\smile\!\!\!\!\!|}_C b$.*

Proof: See Proposition 7.3 in [7]. □

Remark 9.3 1. *Any extension of a simple type is simple.*

2. *If $\pi(x, y)$ is simple, then the type $\exists y \pi(x, y)$ is simple. Therefore, if $\pi(x, y)$ is a simple partial type over A, then $\pi \restriction x = \{\varphi(x) \in L(A) : \pi(x, y) \vdash \varphi(x)\}$ is simple.*
3. *$\mathrm{tp}(a/A)$ and $\mathrm{tp}(b/Aa)$ are simple if and only if $\mathrm{tp}(ab/A)$ is simple. More generally, $\mathrm{tp}(a_i/Aa_{<i})$ is simple for all $i < \alpha$ if and only if $\mathrm{tp}((a_i : i < \alpha)/A)$ is simple.*
4. *If the types $\pi_i(x_i)$ are simple for all $i \in I$, then $\bigcup_{i \in I} \pi_i(x_i)$ is simple.*
5. *If the types $\pi_1(x), \pi_2(x)$ are simple, then the type $\pi_1(x) \vee \pi_2(x)$ is simple.*

Proof: See remarks 7.2 and 7.5 in [7]. □

Remark 9.4 *Simple types are* NTP_2.

Proof: Assume $\pi(x)$ is a TP_2 type over A, witnessed by $\varphi(x,y)$, $k<\omega$ and the aray $(a_{ij}: i,j<\omega)$. Put $a'_\emptyset = a_{0,0}$ and $a'_s = a_{n,s(n-1)}$ if $n \geq 1$ and $s \in \omega^n$. Note that $\pi(x) \cup \{\varphi(x, a'_{f\restriction n}) : n<\omega\}$ is consistent for each $f \in \omega^\omega$. Hence $(a'_s : s \in \omega^\omega)$ with k and $\varphi(x,y)$ witness that $\pi(x)$ is not simple. □

Note the parenthesis in next proposition. They mean that each statement can be read relatively to dividing or forking. Equivalence between *1*, *2* and *5* with forking was known for us.

Proposition 9.5 *The following are equivalent for any partial type $\pi(x)$ over A:*

1. *π is simple.*
2. *For any completion $p(x) \in S(B)$ of $\pi(x)$ there is some $C \subseteq B$ such that $|C| \leq |T|+|x|$ and $p(x)$ does not divide (fork) over C.*
3. *If $B \supseteq A$, $(b_i : i<\omega)$ is a Morley sequence in $\mathrm{tp}(b/B)$ and $\pi(x) \cup \{\varphi(x,b)\}$ divides (forks) over B, then $\pi(x) \cup \{\varphi(x,b_i) : i<\omega\}$ is inconsistent.*
4. *If $B \supseteq A$, $a \models \pi$ and $(b_i : i<\lambda)$ is B-independent, where $\lambda = \beth_{(2^\kappa)^+}$ and $\kappa \geq |B|+|T+|a|+|b_i|$, then $a \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}_B b_i$ ($a \mathop{\smile\!\!\!\!\!|}_B b_i$) for some $i<\lambda$.*
5. *If $A \subseteq C$, $a \models \pi$ and $b \mathop{\smile\!\!\!\!\!|}_C a$, then $a \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}_C b$ ($a \mathop{\smile\!\!\!\!\!|}_C b$).*

Proof: *1* ⇒ *2*. Non forking implies non dividing. The rest is in Proposition 9.2.

2 ⇒ *3*. Let $\kappa = |T|+|x|$, let I be of order type $(\kappa^+)^*$ (the reverse ordering of κ^+) and let $(c_i : i \in I)$ be a B-indiscernible sequence with the same Ehrenfeucht-Mostowski type over B as $(b_i : i<\omega)$. Then $(c_i : i \in I)$ is Morley over B. If $\pi(x) \cup \{\varphi(x,b_i) : i<\omega\}$ is consistent, then $\pi(x) \cup \{\varphi(x,c_i) : i \in I\}$ is consistent too. Let a be a realization of this set of formulas. By *2*, there is some $I_0 \subseteq I$ of cardinality $\leq \kappa$ such that $\mathrm{tp}(a/B(c_i :\in I))$ does not divide (fork) over $B(c_i : i \in I_0)$. Choose $j \in I$ such that $j < I_0$. Then $(c_i : i \in I_0) \mathop{\smile\!\!\!\!\!|}_B c_j$ (this is due to a version of Lemma 5.14 of [8] for forking independence which holds because forking independence has left transitivity; see Remark 12.15 of [8]). By Proposition 4.9 of [8] and Lemma 4.8 of [7] $\varphi(x,c_j)$ divides (forks) over $B(c_i : i \in I_0)$, which is a contradiction since $\models \varphi(a,c_j)$.

3 ⇒ *4*. Assume $a \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}_B b_i$ for all $i<\lambda$. By the choice of λ we can extract a Ba-indiscernible sequence $(b'_i : i<\omega)$ which is locally like $(b_i : i<\lambda)$ over Ba. Then it is a Morley sequence over B and there is some $\varphi(x,y) \in L(B)$ such that $\models \varphi(a,b'_i)$ and $\varphi(x,b'_i)$ divides over B for all $i<\omega$. This contradicts *3*. Similar for the forking version.

4 ⇒ *5*. Since $b \mathop{\smile\!\!\!\!\!|}_C a$, there is a long Morley sequence $(b_i : i<\lambda)$ in $\mathrm{tp}(b/Ca)$ which is independent over C and $b=b_0$. By *4*, $a \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}_C b_i$ for some $i<\lambda$ and by Ba-indiscernibility, $a \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}_C b$. Similar for forking.

5 ⇒ *1*. Like the proof of the corresponding implication in Proposition 7.3 of [7]. □

Remark 9.6 *Let $\pi(u)$ be a co-simple partial type over A. If each $b \in B$ is a tuple of realizations of $\pi(u)$, then any extension $\pi'(u)$ of $\pi(u)$ over B is co-simple.*

Proposition 9.7 *Let $\pi(u)$ be a partial type over A, $\varphi(x,y) \in L(A)$ and $k<\omega$,where y is a tuple $u_1,\ldots,u_m$ and each u_i is a tuple of the length of u. The following are equivalent:*

1. *$\pi(u)$ is not co-simple, witnessed by $\varphi(x, y)$ and k.*

2. *There is a sequence $(a_i : i < \omega)$ where each a_i is a tuple of realizations of $\pi(u)$ and $\varphi(x, a_i)$ divides over $Aa_{<i}$ with respect to k for all $i < \omega$ and $\{\varphi(x, a_i) : i < \omega\}$ is consistent.*

3. *There is a sequence $(a_i : i < \omega)$ where each a_i is a tuple of realizations of $\pi(u)$ and there is some b such that $\models \varphi(b, a_i)$ for all $i < \omega$, $(a_i : i < \omega)$ is Ab-indiscernible and $\varphi(x, a_i)$ divides over $Aa_{<i}$ with respect to k for all $i < \omega$.*

Proof: See the proof of Proposition 12.23 in [8]. □

Proposition 9.8 *The following are equivalent for any partial type $\pi(u)$ over A:*

1. *$\pi(u)$ is co-simple.*

2. *If B is a set of realizations of $\pi(u)$ and $q(x) \in S(AB)$, then for some $B' \subseteq B$ of cardinality $\leq |A| + |T| + |u|$, $q(x)$ does not divide (fork) over AB'.*

3. *If B is a set of realizations of $\pi(u)$, $(a_i : i < \omega)$ is a Morley sequence over AB, every a_i is a tuple of realizations of $\pi(u)$ and $\varphi(x, a_0)$ divides (forks) over AB, then $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent.*

4. *If B is a set of realizations of $\pi(u)$, and $(a_i : i < \lambda)$ is an AB-independent sequence of tuples of realizations of $\pi(u)$, where $\lambda = \beth_{2^{\kappa^+}}$ and $\kappa \geq |AB| + |a_i| + |b|$, then $b \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{AB} a_i$ ($b \mathop{\smile\!\!\!\!\!\!|}_{AB} a_i$) for some $i < \lambda$.*

5. *If B is a set of realizations of $\pi(u)$, a is a tuple of realizations of π and $a \mathop{\smile\!\!\!\!\!\!|}_{AB} b$, then $b \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{AB} a$ ($b \mathop{\smile\!\!\!\!\!\!|}_{AB} a$).*

Proof: Like the proof of Proposition 9.5. For *1* $\Rightarrow$ *2* use also Proposition 9.7. □

Proposition 9.9 *Let $\pi(x)$ be a simple partial type over A and assume the chain condition over A holds with respect to $\pi(x)$. Let $(a_i : i < \omega)$ be a Morley sequence in $\pi(x)$ over A and for each $i < \omega$ let $\bar{a}_i$ be an indiscernible sequence over A of length ω starting with a_i. Then we can find $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$ for each $i < \omega$ such that the array $(\bar{b}_i : i < \omega)$ has mutually A-indiscernible rows.*

Proof: By Lemma 7.7 it suffices to find such array with almost mutually A-indiscernible rows. We construct the sequences $\bar{b}_i$ by induction on i in such a way that

1. $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$

2. $\bar{b}_i$ is indiscernible over $A\bar{b}_{<i}a_{>i}$

3. $\bar{b}_{\leq i}a_{>i+1} \mathop{\smile\!\!\!\!\!\!|}_A a_{i+1}$

4. $a_{\geq i+1} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}_{\leq i}$.

Since (by *1*) the first element of $\bar{b}_i$ is a_i, *2* implies that the rows will be almost mutually A-indiscernible. We use the chain condition in the form of point *6* of Proposition 4.2. Note that simplicity of $\pi(x)$ allows us to use symmetry and transitivity of independence when working with tuples composed of realizations of π. Hence *3* follows from *4* (and the fact that $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_A a_{>i+1}$, which holds because the sequence is Morley over A).

We start noticing that $a_{>0} \mathop{\smile\!\!\!\!\!\mid}_A a_0$ and then the chain condition gives us some $\bar{b}_0 \equiv_{Aa_0} \bar{a}_0$ which is indiscernible over $Aa_{>0}$ and such that $a_{>0} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}_0$ (which is condition *4* for $i = 0$). Now assume inductively *4* and *3* for $i-1$, namely, $a_{\geq i} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}_{<i}$ and $\bar{b}_{<i}a_{>i} \mathop{\smile\!\!\!\!\!\mid}_A a_i$. With the chain condition we choose now $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$ indiscernible over $A\bar{b}_{<i}a_{>i}$ and such that $\bar{b}_{<i}a_{>i} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}_i$. Then $a_{>i} \mathop{\smile\!\!\!\!\!\mid}_{A\bar{b}_{<i}} \bar{b}_i$ and this together with the induction hypothesis gives *4*: $a_{>i} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}_{\leq i}$. □

Proposition 9.10 *Let $\pi(x)$ be a simple partial type over A and $a \models \pi$. If the chain condition holds over A with respect to $\pi(x)$ and $\mathrm{tp}(b/A)$ is NTP_2, then $a \mathop{\smile\!\!\!\!\!\mid}_A b$ implies $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_A a$.*

Proof: Since $a \mathop{\smile\!\!\!\!\!\mid}_A b$, there is a Morley sequence $(a_i : i < \omega)$ over Ab in $\mathrm{tp}(a/Ab)$ which is also Morley over A (see Lemma 5.11 in [8]). We can assume $a = a_0$. Assume $\varphi(x, y) \in L(A)$, $\models \varphi(b, a)$ and $\varphi(x, a)$ divides over A and let $\bar{a}_0 = (a^0_i : i < \omega)$ be an A-indiscernible sequence witnessing it, namely, $\{\varphi(x, a^0_i) : i < \omega\}$ is inconsistent and $a = a^0_0$. Hence $\{\varphi(x, a^0_i) : i < \omega\}$ is k-inconsistent for some $k < \omega$. Choose $\bar{a}_i$ such that $\bar{a}_i a_i \equiv_A \bar{a}_0 a_0$. By Proposition 9.9 we may assume that the rows of the array $(\bar{a}_i : i < \omega)$ are mutually A-indiscernible. Notice that $b \models \{\varphi(x, a_i) : i < \omega\}$ and hence the first column $\{\varphi(x, a_i) : i < \omega\}$ is consistent with $\mathrm{tp}(b/A)$. By Remark 5.5 all paths give similar consistent sets of formulas. Since each row is k-inconsistent, this array shows that $\mathrm{tp}(b/A)$ has TP_2. □

Proposition 9.11 *Let $\pi(x)$ be a simple partial type over A.*

1. *For any $\varphi(x, y) \in L(A)$, for any tuple a such that $a \mathop{\smile\!\!\!\!\!\mid}_A A$, $\pi(x) \cup \{\varphi(x, a)\}$ forks over A iff it divides over A.*
2. *If A is an extension base, the chain condition holds over A with respect to $\pi(x)$.*

Proof: *1.* Assume $\pi(x) \cup \{\varphi(x, a)\}$ forks over A . Since $a \mathop{\smile\!\!\!\!\!\mid}_A A$, there is a Morley sequence $(a_i : i < \omega)$ in $\mathrm{tp}(a/A)$. By point *3* of Proposition 9.5 $\pi(x) \cup \{\varphi(x, a_i) : i < \omega\}$ is inconsistent. This implies that $\pi(x) \cup \{\varphi(x, a)\}$ divides over A.

2. By *1* and Proposition 6.11 $\pi(x) \cup \{\varphi(x, a)\}$ forks over A iff it array-divides over A. Hence we can easily adapt the proof of Proposition 6.6 (choosing a Morley sequence as universal indiscernible sequence). □

Proposition 9.12 *Let $\pi(u)$ be a partial type over A. If $\pi(u)$ is not co-simple, then for some model $M \supseteq A$, for some tuple a of realizations of $\pi(u)$, for some b, $\mathrm{tp}(a/Mb)$ coinherits from M but $b \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_M a$.*

Proof: Fix some formula $\varphi(x, y)$ witnessing the failure of co-simplicity. Let T^{sk} be some skolemization of T. Then $\varphi(x, y)$ witnesses the failure of co-simplicity of $\pi(u)$ in T^{sk}. By point *3* of Proposition 9.7, after extending the indiscernible sequence to length $\omega+1$, in T^{sk} there are a sequence $(a_i : i \leq \omega)$ of tuples a_i of realizations of π, some b and some number $k < \omega$, such that $\models \varphi(b, a_i)$ for all $i < \omega$, $(a_i : i \leq \omega)$ is Ab-indiscernible and $\varphi(x, a_i)$ divides over $Aa_{<i}$ with respect to k for all $i \leq \omega$. Let M be the Skolem hull of $Aa_{<\omega}$. By indiscernibility, $\mathrm{tp}(a_\omega/Mb)$ is a coheir of $\mathrm{tp}(a_\omega/Mb)$ (in T^{sk} and in T). By construction $b \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_{Aa_{<\omega}} a_\omega$, witnessed in T^{sk} by $\varphi(x, y) \in L$. Since M is definable over $Aa_{<\omega}$, $\varphi(x, y)$ also witnesses in T^{sk} that $b \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_M a_\omega$. Since $\varphi(x, y) \in L$ we have $b \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_M a_\omega$ also in T. □

Corollary 9.13 *Let $\pi(x)$ be a simple type over the extension base A and let $\mathrm{tp}(b/A)$ be NTP_2. If $a \models \pi(x)$, then: $a \mathop{\smile\!\!\!\!\!\mid}_A b$ iff $b \mathop{\smile\!\!\!\!\!\mid}_A a$.*

Proof: By propositions 9.11, 9.10 and 9.5. □

Corollary 9.14 *Let T be* NTP$_2$.

1. *Simple types are co-simple.*
2. *If $\pi(x)$ is a simple partial type over A and $a \models \pi(x)$, then for every b: $a \mathop{\smile\!\!\!\!\!\!\,|}_A b$ iff $b \mathop{\smile\!\!\!\!\!\!\,|}_A a$.*
3. *If $\pi(x)$ is a simple partial type over A and $a \models \pi(x)$, then for every B, C: if $a \mathop{\smile\!\!\!\!\!\!\,|}_A B$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_{AB} C$, then $a \mathop{\smile\!\!\!\!\!\!\,|}_A BC$.*

Proof: *1.* By Proposition 9.12 and Proposition 9.10, since in a NTP$_2$ theory the chain condition holds over models.

2. By *1*, and propositions 9.2 and 9.8.

3. By *2*. □

Remark 9.15 *We could define the notion of co-*NTP$_2$ *type similarly to the way co-simple types were defined, requiring that the parameters of the array realize the type (or perhaps are tuples of realizations of the type). Then the proof of Proposition 9.10 would give:*

*Let $\pi(x)$ be a simple co-*NTP$_2$ *partial type over A and $a \models \pi$. If the chain condition holds over A with respect to $\pi(x)$, then $a \mathop{\smile\!\!\!\!\!\!\,|}_A b$ implies $b \mathop{\smile\!\!\!\!\!\!\,|}_A^{\mathrm{d}} a$.*

Moreover parallel to Corollary 9.14 we would have:

1. *Simple co-*NTP$_2$ *types are co-simple.*
2. *If $\pi(x)$ is a simple co-*NTP$_2$ *partial type over A and $a \models \pi(x)$, then for every b: $a \mathop{\smile\!\!\!\!\!\!\,|}_A b$ iff $b \mathop{\smile\!\!\!\!\!\!\,|}_A a$.*
3. *If $\pi(x)$ is a simple co-*NTP$_2$ *partial type over A and $a \models \pi(x)$, then for every B, C: if $a \mathop{\smile\!\!\!\!\!\!\,|}_A B$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_{AB} C$, then $a \mathop{\smile\!\!\!\!\!\!\,|}_A BC$.*

*The question is then whether simple types are co-*NTP$_2$.

10 The triangle-free random graph

Definition 10.1 The theory of the *triangle-free random graph* is the theory of an irreflexive, symmetric binary relation R without triangles and such that for any disjoint finite sets A, B if no R-relations holds between elements of A (A is an *anti-clique*), then there is some $c \notin A \cup B$ such cRa for all $a \in A$ and $\neg cRb$ for all $b \in B$. This can be written as a set of first-order axioms. In this section T will be this theory. We will use here a, b, c, x for elements (points, singletons) and $\bar{a}, \bar{b}, \bar{c}, \bar{x}$ for (possibly infinite) tuples. The following notation will be also convenient: $R_a(A)$ is the set of all $b \in A$ such that $R(a, b)$.

Remark 10.2 *T is complete, is ω-categorical and has quantifier elimination.*

Proof: We do a back-and-forth between models M, N of T. The set of partial isomorphism is the set of all isomorphisms between finite substructures. It is clearly non empty. Assume f is such an isomorphism and let $a \in M$ be a new element. Let $A \cup B$ be the domain of f, where a is R-related to all elements in A and to none in B. Let $A' = f(A)$ and $B' = f(B)$. A' is an anti-clique. By the axioms of T there is some $b \in N$ which is R-related to all elements of A' and to none of B' and it is not in the range of f. The mapping $f \cup \{(a, b)\}$ works. □

Proposition 10.3 *All non algebraic types $p(x) \in S_1(A)$ are* TP_2.

Proof: For $i, j < \omega$ choose $a_{ij}b_{ij}$ such that $a_{ij}Rb_{ik}$ for all $j \neq k$ and no other R-relation holds between all them and between them and A. This array can easily obtained by induction on i. Now let $\varphi(x; yz) = xRy \wedge xRz$. Note that $\{\varphi(x; a_{ij}b_{ij}) : j < \omega\}$ is 2-inconsistent for all $i < \omega$ and $\{\varphi(x; a_{if(i)}b_{if(i)}) : i < \omega\}$ is consistent with $p(x)$ for every $f : \omega \to \omega$. This witnesses TP_2 of $p(x)$. □

Remark 10.4 *In T,* $\mathrm{acl}(A) = A$.

Proof: Let $a \notin A$. Construct inductively $a_0, a_1, \ldots$ such that $a = a_0$ and $a_{i+1} \notin Aa_0 \ldots a_i$, $R_{a_{i+1}}(A) = R_a(A)$. This shows that $\mathrm{tp}(a/A)$ has infinitely many realizations. □

Remark 10.5 *If $(a_i : i < \omega)$ is indiscernible, then $\neg R(a_i, a_j)$ for all $i < j < \omega$.*

Lemma 10.6 *If $a, b, b' \notin C$, $R(a, b), R(a, b')$, $R_b(C) \cap R_{b'}(C) = \emptyset$ and $b \neq b'$, then $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} bb'$.*

Proof: Notice that $\neg R(b, b')$. Let $p(x, b, b') = \mathrm{tp}(a/Cbb')$, with $p(x, y, z) \in S(C)$. We will find some sequence $(b_i b'_i : i < \omega)$ with $bb' \equiv_C b_i b'_i$ and $\Sigma(x) = \bigcup_{i<\omega} p(x, b_i, b'_i)$ 2-inconsistent. We start with $b_0 b'_0 = bb'$. We want now to choose b_1 such that $R_b(C) = R_{b_1}(C)$, $\neg R(b_0, b_1)$ and $R(b'_0, b_1)$. This is possible since $R_b(C) \cup \{b'_0\}$ is an anticlique. Since $b_0 \equiv_C b_1$, we can now choose b'_1 such that $b_0 b'_0 \equiv_C b_1 b'_1$ and $\neg R(b'_1, b'_0)$. By iteration, we obtain the sequence with the additional property that $\neg R(b_i, b_j), \neg R(b'_i b'_j)$ and $R(b'_i, b_j)$ for all $i < j < \omega$. Since $\Sigma(x)$ contains $R(x, b'_i) \wedge R(x, b_j)$ for all $i < j$, it is 2-inconsistent. □

Lemma 10.7 *Assume $a \notin C$ and $B \cap C = \emptyset$. If for all $b, b' \in B$ such that $R(a, b)$ and $R(a, b')$ it happens that $R_b(C) \cap R_{b'}(C) \neq \emptyset$, then $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$.*

Proof: Let $\bar{b}$ enumerate $R_a(B)$ and let $\bar{b}'$ enumerate $B \smallsetminus R_a(B)$. Let $p(x, \bar{b}, \bar{b}') = \mathrm{tp}(a/BC)$ with $p(x, \bar{y}, \bar{y}') \in S(C)$. Assume $(\bar{b}_i \bar{b}'_i : i < \omega)$ is C-indiscernible, with $\bar{b}\bar{b}' = \bar{b}_0 \bar{b}'_0$ and let us check that $\Sigma(x) = \bigcup_{i<\omega} p(x, \bar{b}_i, \bar{b}'_i)$ is consistent. For this it is enough to check that $R_a(C) \cup \{\bar{b}_i : i < \omega\}$ is an anti-clique. Clearly there is no R-relation between elements of $R_a(C)$ and elements of $\bar{b}_i$. Let $\bar{b}_i = (b^i_j : j \in J)$. By indiscernibility $\neg R(b^i_j, b^k_j)$. If $i < k < \omega$ and $j, l \in J$ are different, then $R_{b^i_j}(C) \cap R_{b^k_l}(C) \neq \emptyset$ and therefore $\neg R(b^i_j, b^k_l)$. □

Proposition 10.8 *$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$ iff $AC \cap BC = C$ and for all $a \in A \smallsetminus C$ for all $b, b' \in B \smallsetminus C$: if $R(a, b)$ and $R(a, b')$ then $R_b(C) \cap R_{b'}(C) \neq \emptyset$.*

Proof: Recall that $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B \Leftrightarrow AC \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} BC \Leftrightarrow A \smallsetminus C \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B \smallsetminus C$.

$\Rightarrow$. If $a \in AC \cap BC$ then $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} a$ and therefore $a \in \mathrm{acl}(C) = C$. The rest follows from Lemma 10.6.

$\Leftarrow$. We can assume $A \smallsetminus C$ to be finite and the proof can be realized by induction on $|A \smallsetminus C|$. Using the fact that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$ together with $\bar{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C\bar{a}} B$ implies $\bar{a}\bar{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$, it is enough to check it for the case of a single element a. And the case $A = \{a\}$ follows from Lemma 10.7. □

Proposition 10.9 *In T* $\mathop{\smile\!\!\!\!\!\mid}_A = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_A$.

Proof: It is enough to check that dividing has the extension property. Let $\bar{a} = a_1, \dots, a_n$ and assume $A \subseteq B \subseteq C$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_A B$. We want to find some $\bar{a}' \equiv_B \bar{a}$ such that $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_A C$. We can assume that $a_i \notin B$ for all $i = 1, \dots, n$. Let $p(\bar{x}) = \mathrm{tp}(\bar{a}/B)$ and let $\Sigma(x) = p(\bar{x}) \cup \{x_i \neq c \wedge \neg R(x_i, c) : c \in C \smallsetminus B,\ 1 \leq i \leq n\}$. It is consistent and if $\bar{a}' \models \Sigma(x)$ then $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_A C$. □

Proposition 10.10 *Let* $p(x) \in S(A)$ *be non algebraic and such that for some* $a \in A$, $R(x, a) \in p(x)$*:*

1. *If B is a set of realizations of* $p(x)$*, then* $A \mathop{\smile\!\!\!\!\!\mid}_C B$ *iff* $AC \cap BC = C$.
2. $p(x)$ *is co-simple.*

Proof: *1.* By Proposition 10.8.

2. By *1* and point *4* of Proposition 9.8. □

11 Strict nonforking

Based on [12].

Recall from [7] the following definition and facts:

Definition 11.1 1. A global type $\mathfrak{p}(x)$ is *Lascar invariant* over C iff for every $\varphi(x, y) \in L$, for all tuples $a \stackrel{\mathrm{Ls}}{\equiv}_C b$, if $\varphi(x, a) \in \mathfrak{p}$ then $\varphi(x, b) \in \mathfrak{p}$.

2. $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}_C b$ iff for some global type $\mathfrak{p}(x) \supseteq \mathrm{tp}(a/Cb)$, $\mathfrak{p}(x)$ is Lascar invariant over C.

3. $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}_C b$ iff for some global type $\mathfrak{p}(x) \supseteq \mathrm{tp}(a/Cb)$, $\mathfrak{p}(x)$ is Lascar invariant over C and for every $B \supseteq Cb$, for every $c \models \mathfrak{p}(x) \upharpoonright B$, $B \mathop{\smile\!\!\!\!\!\mid}_C c$.

4. A sequence $(a_i : i < \alpha)$ is *strictly invariant* over A if for all $i < \alpha$, $a_i \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}_A a_{<i}$.

Fact 11.2 *1.* $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ *is a preindependence relation with extension, right-normality, anti-reflexivity and algebraicity properties.*

2. $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}_A b$ *implies* $a \mathop{\smile\!\!\!\!\!\mid}_A b$.

3. If $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}_A I$ *and I is an A-indiscernible sequence, then I is Aa-indiscernible.*

4. $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}$ *is invariant, monotone and has the extension property.*

Definition 11.3 A sequence $(a_i : i \in I)$ is *strictly independent* over A if for all $(\bar{a}_i : i \in I)$ where each $\bar{a}_i$ is an A-indiscernible sequence, there is a mutually A-indiscernible sequence $(\bar{b}_i : i \in I)$ such that $\bar{b}_i \equiv_A \bar{a}_i$ for all $i \in I$.

Remark 11.4 *By propositions 9.9 and 9.11, if* $(a_i : i < \alpha)$ *is a Morley sequence of realizations of a simple type over an extension base A, then* $(a_i : i < \alpha)$ *is strictly independent.*

Lemma 11.5 (Shelah's lemma) *Any strictly invariant sequence is strictly independent.*

Proof: By compactness, it is enough to check this for a sequence $(\bar{a}_i : i < n)$ where $n < \omega$. By Lemma 7.7 it is enough to obtain an array with almost mutually indiscernible rows. The proof is by complete induction on n. Assume we have obtained an array $(\bar{b}_i : i < n)$ with almost mutually A-indiscernible rows and such that $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$ for all $i < n$. Since $a_n \mathop{\smile\!\!\!\!\!\mid}^{\text{ist}}_A a_{<n}$, we may assume that $a_n \mathop{\smile\!\!\!\!\!\mid}^{\text{ist}}_A \bar{b}_{<n}$. Hence $\bar{b}_{<n} \mathop{\smile\!\!\!\!\!\mid}^{\text{d}}_A a_n$ and therefore there is some $\bar{b}_n \equiv_{Aa_n} \bar{a}_n$ which is $A\bar{b}_{<n}$-indiscernible. If $i < n$, then $\bar{b}_i$ is indiscernible over $A\bar{b}_{<i}a_{<n}$ and $a_n \mathop{\smile\!\!\!\!\!\mid}^{\text{i}}_{A\bar{b}_{<i}a_{<n}} \bar{b}_i$ and therefore $\bar{b}_i$ is $A\bar{b}_i a_{\leq n}$-indiscernible. $\square$

Lemma 11.6 *Let T be* NTP$_2$ *and $(\varphi_i(x, y_i) : i < |T|^+)$ a sequence of $L(A)$-formulas. If $(a_i : i < |T|^+)$ is a strictly A-independent sequence and each $\varphi_i(x, a_i)$ divides over A, then $\{\varphi_i(x, a_i) : i < |T|^+\}$ is inconsistent.*

Proof: Choose for every $i < |T|^+$ an A-indiscernible sequence $\bar{a}_i = (a_{ij} : j < \omega)$ starting with $a_{i0} = a_i$ and such that $\{\varphi_i(x, a_{ij}) : j < \omega\}$ is inconsistent, hence k_i-inconsistent for some $k_i < \omega$. Some formula φ_i and some number k_i occur jointly $|T|^+$ times, so we may assume $\varphi_i = \varphi(x, y)$ and $k_i = k$ for all i. Since the sequence is strictly A-independent, we may assume that the rows of the array $(a_{ij} : i < |T|^+, j < \omega)$ are mutually A-indiscernible. If we assume that $\{\varphi(x, a_i) : i < |T|^+\}$ is consistent, then $\{\varphi(x, a_{if(i)}) : i < |T|^+\}$ is consistent for every $f : |T|^+ \to \omega$. This shows that T has TP$_2$. $\square$

Definition 11.7 An infinite sequence $(a_i : i \in I)$ is a *witness* of a over A if $a \equiv_A a_i$ for all i and for every $\varphi(x, y) \in L(A)$ such that $\varphi(x, a)$ divides over A, for every infinite countable $J \subseteq I$, $\{\varphi(x, a_i) : i \in J\}$ is inconsistent. If the sequence is A-indiscernible, this is equivalent to $\{\varphi(x, a_i) : i \in I)$ being inconsistent and in this case this coincides with the notion of universal indiscernible sequence introduced earlier.

Proposition 11.8 *Let T be* NTP$_2$*. Any infinite strictly A-independent sequence is a witness over A.*

Proof: Let $(a_i : i \in I)$ be strictly independent over A, and assume $a_i \equiv_A a_j$ for all $i, j \in I$. We can assume $0 \in I$. Assume $\varphi(x, y) \in L(A)$ and $\varphi(x, a_0)$ divides over A. Choose some A-indiscernible sequence $(a_{0j} : j < \omega)$ starting with $a_0 = a_{00}$ and such that $\{\varphi(x, a_{0j}) : j < \omega\}$ is inconsistent, hence k-inconsistent for some $k < \omega$. For any other $i \in I$ choose $(a_{ij} : j < \omega) \equiv_A (a_{0j} : j < \omega)$ with $a_{i0} = a_i$. Clearly $\{\varphi(x, a_{ij}) : j < \omega\}$ is k-inconsistent. By strict A-independence, we may assume the rows of the array $(a_{ij} : i \in I, j < \omega)$ are mutually indiscernible over A. If $(a_i : i \in I)$ is not a witness for $\varphi(x, y)$, then $\{\varphi(x, a_{i0}) : i \in J\}$ is consistent for some (infinite) countable $J \subseteq I$. As in the proof of Lemma 11.6, this contradicts NTP$_2$ of T. $\square$

Corollary 11.9 *In a* NTP$_2$ *theory, if A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\text{ist}}$ then forking over A coincides with dividing over A.*

Proof: Assume A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\text{ist}}$, $\varphi(x, y) \in L$, $\varphi(x, b) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ and each $\varphi_i(x, a_i)$ divides over A. We will show that $\varphi(x, b)$ divides over A. Let $\bar{a} = ba_1 \ldots a_n$ and let $(\bar{a}_i : i < \omega)$ be a strictly A-invariant sequence starting with $\bar{a}_0 = \bar{a}$. Write $\bar{a}_i = b^i a_1^i \ldots a_n^i$. Since $(a_j^i : i < \omega)$ is a strictly A-invariant sequence, by Proposition 11.8 it is a witness of a_j over A and hence $\{\varphi_j(x, a_j^i) : i < \omega\}$ is inconsistent. We claim that $\{\varphi(x, b^i) : i < \omega\}$ is inconsistent. Assume not and let c realize this type. For some j, c realizes infinitely $\varphi_j(x, a_j^i)$, a contradiction. $\square$

The previous result was exposed in Proposition 4.5 of [7] with a different proof. Proposition 4.9 of [7] shows the same for the case that A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}$.

Fact 11.10 *1. If T is* NTP$_2$*, then A is an extension base for $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}$ iff it is an extension base for $\mathop{\smile\!\!\!\!\mid}^{\mathrm{ist}}$.*

2. If T is NIP*, then A is an extension base for $\mathop{\smile\!\!\!\!\mid}$ iff it is an extension base for $\mathop{\smile\!\!\!\!\mid}^{\mathrm{ist}}$.*

Proof: See Proposition 4.11 and Corollary 4.12 in [7]. □

Proposition 11.11 *The following are equivalent.*

1. *$a \mathop{\smile\!\!\!\!\mid}^{\mathrm{ist}}_A b$.*
2. *For every c there is some $c' \equiv_{Ab} c$ such that $a \mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}_A bc'$ and $bc' \mathop{\smile\!\!\!\!\mid}_A a$.*
3. *For every c there is some $a' \equiv_{Ab} a$ such that $a' \mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}_A bc$ and $bc \mathop{\smile\!\!\!\!\mid}_A a'$.*
4. *If $p(x) = \mathrm{tp}(a/Ab)$, the following is consistent:*

$$p(x) \cup \{\varphi(x,c) \leftrightarrow \varphi(x,c') : c \overset{\mathrm{Ls}}{\equiv}_A c', \varphi(x,y) \in L(A), c, c' \in \mathfrak{C}\} \cup \{\neg\varphi(x,b,c) :$$

$$\varphi(x,y,z) \in L(A), c \in \mathfrak{C} \text{ and } \varphi(a,y,z) \text{ forks over } A\}$$

Proof: By automorphism, *2* $\Leftrightarrow$ *3*.

1 $\Rightarrow$ *3*. Fix the global extension $\mathfrak{p}(x) \supseteq \mathrm{tp}(a/Ab)$ and take $a' \models \mathfrak{p} \restriction Abc$.

3 $\Rightarrow$ *4*. Consider a finite fragment $\pi(x)$ of the partial type and let c be a tuple containing all parameters of $\pi(x)$. The tuple a' given by *3* realizes $\pi(x)$.

4 $\Rightarrow$ *1*. Extend the partial type to a complete global type. □

Definition 11.12 *Strict nonforking* is defined in the same way as strict invariance except that we require only the global extension $\mathfrak{p}(x) \supseteq \mathrm{tp}(a/Cb)$ does not fork over C instead of requiring that is Lascar invariant over C. We write $a \mathop{\smile\!\!\!\!\mid}^{\mathrm{snf}}_C b$ for this. Note that $\mathop{\smile\!\!\!\!\mid}^{\mathrm{snf}} = \mathop{\smile\!\!\!\!\mid}^{\mathrm{ist}}$ if T is NIP.

Remark 11.13 *1. $a \mathop{\smile\!\!\!\!\mid}^{\mathrm{ist}}_C b$ implies $a \mathop{\smile\!\!\!\!\mid}^{\mathrm{snf}}_C b$.*

2. $\mathop{\smile\!\!\!\!\mid}^{\mathrm{snf}}$ has the extension property.

3. There is a characterization of $a \mathop{\smile\!\!\!\!\mid}^{\mathrm{snf}}_A b$ as in Proposition 11.11, exchanging $\mathop{\smile\!\!\!\!\mid}^{\mathrm{ist}}$ with $\mathop{\smile\!\!\!\!\mid}^{\mathrm{snf}}$ and defining the set of formulas as

$$p(x) \cup \{\neg\varphi(x,b,c) : \varphi(x,y,z) \in L(A), c \in \mathfrak{C} \text{ and } \varphi(x,b,c) \text{ forks over } A\} \cup \{\neg\varphi(x,b,c) :$$

$$\varphi(x,y,z) \in L(A), c \in \mathfrak{C} \text{ and } \varphi(a,y,z) \text{ forks over } A\}$$

Lemma 11.14 *$a \mathop{\smile\!\!\!\!\mid}^{\mathrm{snf}}_A b$ iff for each $p(z) \in S(Ab)$ the following is consistent:*

$$p(z) \cup \{\neg\varphi(a,b,z) : \varphi(x,y,z) \in L(A) \text{ and } \varphi(x,b,c) \text{ forks over } A \text{ for some (all) } c \models p\} \cup$$

$$\{\neg\varphi(a,b,z) : \varphi(x,y,z) \in L(A) \text{ and } \varphi(a,y,z) \text{ forks over } A\}$$

Proposition 11.15 *1. If $(b_i : i < \alpha)$ is a witness of $b = b_0$ over A and all b_i have the same type over Aa, then $a \mathop{\smile\!\!\!\!\mid}^{\mathrm{d}}_A b$.*

2. *Assume forking and dividing over A coincide. If $\bar{a} = (a_i : i < \alpha)$ is a witness of $a = a_0$ over A, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A b$, and all a_i have the same type over Ab, then $a \mathop{\smile\!\!\!\!\!\mid}^{\text{snf}}_A b$.*

3. *Assume forking and dividing over A coincide. If $\bar{a} = (a_i : i < \alpha)$ is an A-indiscernible witness of $a = a_0$ over A, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A b$, and $b \mathop{\smile\!\!\!\!\!\mid}^{\text{i}}_A \bar{a}$, then $a \mathop{\smile\!\!\!\!\!\mid}^{\text{snf}}_A b$.*

Proof: *1.* Assume $\models \varphi(a,b)$ and $\varphi(x,b)$ divides over A. Then $\models \varphi(a, b_i)$ for all $i < \alpha$, contradicting inconsistency of an infinite subset of $\{\varphi(x, b_i) : i < \alpha\}$.

2. Let $p(z) \in S(Ab)$. By Lemma 11.14, we must check the consistency of $p(z) \cup \{\neg\varphi(a,b,z) : \varphi(x,y,z) \in L(A)$ and $\varphi(x,b,c)$ forks over A for some $c \models p\} \cup \{\neg\varphi(a,b,z) : \varphi(x,y,z) \in L(A)$ and $\varphi(a,y,z)$ forks over $A\}$. If it is inconsistent, then $p(z) \vdash \varphi(a,b,z) \vee \psi(a,b,z)$ where $\varphi(x,y,z), \psi(x,y,z) \in L(A)$ and $\varphi(x,b,c), \psi(a,y,z)$ fork (and divide) over A and $c \models p$. Since $\bar{a}$ is a witness, $\{\psi(a_i,y,z) : i \in I\}$ is inconsistent and so is $\{\psi(a_i,b,z) : i \in I\}$ for some infinite countable $I \subseteq \alpha$. To simplify notation, assume $I = \omega$. Since $a_i \equiv_{Ab} a$, $p(z) \vdash \varphi(a_i,b,z) \vee \psi(a_i,b,z)$. By compactness $p(z) \vdash \bigvee_{i=0}^{n} \varphi(a_i,b,z)$ for some $n < \omega$. Since $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A b$, there is some $c' \models p$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A bc'$. Then $\models \varphi(a_i,b,c')$ for some $i \leq n$. This shows that $\varphi(x,b,c')$ does not fork over A. Hence $\varphi(x,b,c)$ does not fork over A, a contradiction.

3. By *2*, since $b \mathop{\smile\!\!\!\!\!\mid}^{\text{i}}_A \bar{a}$ implies that $\bar{a}$ is also indiscernible over Ab. □

Proposition 11.16 *If T is* NTP$_2$ *and A is an extension base, then $1 \Rightarrow 2 \Rightarrow 3$. If, moreover, T is* NIP *they are all equivalent and equivalent to $b \mathop{\smile\!\!\!\!\!\mid}^{\text{ist}}_A a$.*

1. *$a \mathop{\smile\!\!\!\!\!\mid}^{\text{ist}}_A b$*

2. *(a,b) is a strictly A-independent sequence.*

3. *$b \mathop{\smile\!\!\!\!\!\mid}^{\text{snf}}_A a$*

Proof: *1 ⇒ 2.* By Shelah's Lemma 11.5.

2 ⇒ 3. Since T is NTP$_2$, A is an extension base for $\mathop{\smile\!\!\!\!\!\mid}^{\text{ist}}$ and we can find strictly A-invariant A-indiscernible infinite sequences $\bar{a}, \bar{b}$ starting with a and with b respectively. Since (a,b) is strictly A-independent, we may assume that $\bar{a}$ is $A\bar{b}$-indiscernible and $\bar{b}$ is $A\bar{a}$-indiscernible. By Proposition 11.8 $\bar{a}$ and $\bar{b}$ are witnesses over A. By item 1 of Proposition 11.15, $\bar{b} \mathop{\smile\!\!\!\!\!\mid}^{\text{d}}_A a$. Since forking and dividing over A coincide, $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_A a$. By item 2 of Proposition 11.15, $b \mathop{\smile\!\!\!\!\!\mid}^{\text{snf}}_A a$.

If T is NIP then $\mathop{\smile\!\!\!\!\!\mid}^{\text{ist}} = \mathop{\smile\!\!\!\!\!\mid}^{\text{snf}}$ and hence $b \mathop{\smile\!\!\!\!\!\mid}^{\text{snf}}_A a \Rightarrow b \mathop{\smile\!\!\!\!\!\mid}^{\text{ist}}_A a \Rightarrow b \mathop{\smile\!\!\!\!\!\mid}^{\text{snf}}_A a \Rightarrow a \mathop{\smile\!\!\!\!\!\mid}^{\text{snf}}_A b$. □

12 Resilient theories

Based on [4] and [12].

Definition 12.1 Let $S^{EM}_x(A) = \{\text{tp}(\bar{a}/A) : \bar{a}$ is an A-indiscernible ω-sequence $\}$. The length of the tuples a_i in all the sequences $\bar{a}$ is the length of x. For $p \in S^{EM}(A)$, let $\text{cl}^{\text{div}}(p) = \{\varphi(x,y) \in L(A) :$ for some (all) $(a_i : i < \omega) \models p$, $\{\varphi(a_i,y) : i < \omega\}$ is consistent $\}$. For $p, q \in S^{EM}(A)$ we define

1. $p \sim_{\text{div}} q$ iff $\text{cl}^{\text{div}}(p) = \text{cl}^{\text{div}}(q)$

2. $p \leq_{\text{div}} q$ iff $\text{cl}^{\text{div}}(p) \supseteq \text{cl}^{\text{div}}(q)$.

Definition 12.2 Let $p(x) \in S_x(M)$, where $A \subseteq M$. The class of p over A in the *fundamental order* is defined as $\text{cl}_A^{\text{fund}}(p) = \{\varphi(x,y) \in L(A) : \varphi(x,m) \in p \text{ for some tuple } m \in M\}$. For $p(x) \in S_x(M)$, $q(x) \in S_x(N)$, where $A \subseteq M \cap N$ we define the fundamental order over A by

1. $p \overset{\text{fund}}{\sim}_A q$ iff $\text{cl}_A^{\text{fund}}(p) = \text{cl}_A^{\text{fund}}(q)$
2. $p \overset{\text{fund}}{\leq}_A q$ iff $\text{cl}_A^{\text{fund}}(p) \supseteq \text{cl}_A^{\text{fund}}(q)$.

$O_x(A)$ is the class of all types $p(x)$ over models $M \supseteq A$ modulo $\overset{\text{fund}}{\sim}_A$. We endow $O_x(A)$ with the induced partial order $\overset{\text{fund}}{\leq}_A$.

Fact 12.3 *Let T be stable, $A \subseteq M \cap N$, $p(x) \in S(M)$ and $q(x) \in S(N)$. Then $p \overset{\text{fund}}{\sim}_A q$ iff $(M, dp) \equiv_A (N, dq)$, where (M, dp) is the expansion of M obtained after adding for each $\varphi(x; x_1, \ldots, x_n) \in L(A)$ some n-ary relation symbol R_φ for the relation $\{(a_1, \ldots, a_n) \in M^n : \varphi(x; a_1, \ldots, a_n) \in p\}$ (and similarly for N and q).*

Proposition 12.4 *Let T be stable.*

1. *$(S_x^{EM}(A)/\sim_{\text{div}}, \leq_{\text{div}}) \cong (O_x(A), \overset{\text{fund}}{\leq}_A)$.*
2. *For any $p, q \in S_x^{EM}(A)$: $p \sim_{\text{div}} q$ iff $p = q$.*

Proof: Without loss of generality, $A = \emptyset$.

1. Let $p(x) \in S_x(M)$. Choose a Morley sequence $\bar{a} = (a_i : i < \omega)$ in $p(x)$ over M and let $P = \text{tp}(\bar{a}) \in S_x^{EM}$. We claim that $\varphi(x,y) \in \text{cl}^{\text{fund}}(p)$ iff $\varphi(a_0, y)$ does not fork over M iff $\varphi(x,y) \in \text{cl}^{\text{div}}(P)$. On the one hand, any Morley sequence in a stable theory is a witness and therefore $\varphi(a_0, y)$ does not fork over M iff $\{\varphi(a_i, y) : i < \omega\}$ is consistent. On the other hand, it is straightforward that if $\varphi(x,y) \in \text{cl}^{\text{fund}}(p)$ then $\models \varphi(a_0, b)$ for some $b \in M$ and therefore $\varphi(a_0, y)$ does not fork over M. Finally, if $\varphi(a_0, y)$ does not fork over M, then $\models \varphi(a_0, b)$ for some $b \mathop{\smile\!\!\!\!\!\!|}_M a_0$. If $\mathfrak{p}$ is the global nonforking extension of p then $\varphi(x,y) \in \text{cl}^{\text{fund}}(\mathfrak{p})$ and since $\mathfrak{p}$ is the heir of p, $\text{cl}^{\text{fund}}(\mathfrak{p}) = \text{cl}^{\text{fund}}(p)$.

It follows that $\text{cl}^{\text{fund}}(p) = \text{cl}^{\text{div}}(P)$ and hence the mapping $p \mapsto P$ induces an embedding of $(O_x, \overset{\text{fund}}{\leq})$ into $(S_x^{EM}/\sim_{\text{div}}, \leq_{\text{div}})$. We check that it is onto. Let $\bar{a} = (a_i : i < \omega)$ realize $P \in S_x^{EM}$ and extend it to an indiscernible sequence $(a_i : i < \omega + \omega)$. Let $\bar{a}' = (a_i : \omega \leq i < \omega + \omega)$ and choose a model M containing $\bar{a}$ and such that $M \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} \bar{a}'$. By Erdös-Rado we can assume that $\bar{a}'$ is M-indiscernible. It follows that $\bar{a}'$ is a Morley sequence over M in $p(x) = \text{tp}(a_\omega/M)$. Clearly, p goes to P in the embedding.

2. Let $P, Q \in S_x^{EM}$ and assume $P \sim_{\text{div}} Q$. Choose $p(x) \in S(M)$, $q(x) \in S(N)$ such that $p \mapsto P$ and $q \mapsto Q$ in the isomorphism. By Fact 12.3 there are global types $\mathfrak{p}(x)$ and $\mathfrak{q}(x)$ extending $p(x)$ and $q(x)$ respectively with $\text{cl}^{\text{fund}}(p) = \text{cl}^{\text{fund}}(\mathfrak{p})$, $\text{cl}^{\text{fund}}(q) = \text{cl}^{\text{fund}}(\mathfrak{q})$ and some isomorphism $f : (\mathfrak{C}, d\mathfrak{p}) \cong (\mathfrak{C}, d\mathfrak{q})$. Notice that $\mathfrak{p}^f = \mathfrak{q}$. There is a one-to-one correspondence between global types non forking over a model and types (over $\emptyset$) of Morley sequences over the model constructed with the global type. This correspondence is preserved by automorphisms. Hence $P^f = Q$. But $P^f = P$ since they are types over the empty set. Hence, $P = Q$. $\square$

Definition 12.5 For $p, q \in S_x^{EM}(A)$ we define: $p \leq^{\#} q$ iff there is some array $(a_{ij} : i, j < \omega)$ such that each row $(a_{ij} : j < \omega)$ realizes p and each vertical path $(a_{if(i)} : i < \omega)$ realizes q.

Proposition 12.6 *Let $p, q \in S_x^{EM}(A)$.*

1. *If $p \leq^{\mathrm{div}} q$, then $p \leq^{\#} q$.*
2. *T is NTP_2 iff if $p \leq^{\#} q$, then $p \leq^{\mathrm{div}} q$.*

Proof: Without loss of generality, $A = \emptyset$.

1. Assume $p \leq^{\mathrm{div}} q$. We will inductively construct $(a_{ij} : i < n, j < \omega)$ such that each row $\bar{a}_i = (a_{ij} : j < \omega)$ realizes p and there is some $\bar{b} \models q$ such that for each $i_0, \ldots, i_{n-1} < \omega$, $(a_{0i_0}, \ldots, a_{n-1i_{n-1}})^\frown \bar{b}$ is indiscernible. In the case $n = 0$ we have not constructed yet any row and it suffices take as $\bar{b}$ any realization of q. Now, in the general case, choose some ω-sequence $\bar{c}$ such that $\bar{b}^\frown \bar{c}$ is indiscernible. We define

$$r(\bar{x}_0, \ldots, \bar{x}_{n-1}, y, \bar{z}) = \bigcup_{i<n} p(\bar{x}_i) \cup q(\bar{z}) \cup \bigcup_{i_0,\ldots,i_{n-1}<\omega} (x_{0i_0}, \ldots, x_{n-1i_{n-1}}, y)^\frown \bar{z} \text{ is indiscernible}$$

We know that $\bigcup_{i<\omega} r(\bar{x}_0, \ldots, \bar{x}_{n-1}, y_i, \bar{z}) \cup q(\bar{y})$ is consistent (where $\bar{y} = (y_i : i < \omega)$) since $\bar{a}_0, \ldots, \bar{a}_{n-1}^\frown \bar{b}^\frown \bar{c}$ realizes it. Since $p \leq^{\mathrm{div}} q$, it follows that $\bigcup_{i<\omega} r(\bar{x}_0, \ldots, \bar{x}_{n-1}, y_i, \bar{z}) \cup p(\bar{y})$ is consistent and this is what we need for the inductive construction.

2. From left to right. Assume $p \leq^{\#} q$ but not $p \leq^{\mathrm{div}} q$. Let $(a_{ij} : i, j < \omega)$ be an array where all rows satisfy p and all vertical paths satisfy q. If $\varphi(x, y) \in \mathrm{cl}^{\mathrm{div}}(q) \smallsetminus \mathrm{cl}^{\mathrm{div}}(p)$, then $\{\varphi(a_{if(i)}, y) : i < \omega\}$ is consistent for each $f : \omega \to \omega$ and for some $k < \omega$ $\{\varphi(a_{ij}, y) : j < \omega\}$ is k-inconsistent for all $i < \omega$. This shows that T is TP_2. The other direction is similar: if T is TP_2 there is an indiscernible array $(a_{ij} : i, j < \omega)$, some $k < \omega$ and some formula $\varphi(x, y) \in L$ witnessing it. If p is the type of a row and q is the type of a column, then $p \leq^{\#} q$ but not $p \leq^{\mathrm{div}} q$. □

Definition 12.7 For $p, q \in S_x^{EM}(A)$ we define: $p \leq^{+} q$ iff there are $\bar{a} = (a_i : i \in \mathbb{Z}) \models q$ and $\bar{b} = (b_i : i \in \mathbb{Z}) \models p$ such that $a_0 = b_0$ and $\bar{b}$ is indiscernible over $A(a_i : i \neq 0)$.

Remark 12.8 *$p \leq^{\#} q$ implies $p \leq^{+} q$.*

Definition 12.9 T is *resilient* if there are no indiscernible sequences $\bar{a} = (a_i : i \in \mathbb{Z})$ and $\bar{b} = (b_i : i \in \mathbb{Z})$ and a formula $\varphi(x, y) \in L$ such that

1. $a_0 = b_0$
2. $\bar{b}$ is indiscernible over $(a_i : i \neq 0)$
3. $\{\varphi(x, a_i) : i \in \mathbb{Z}\}$ is consistent.
4. $\{\varphi(x, b_i) : i \in \mathbb{Z}\}$ is inconsistent.

Remark 12.10 *If T is resilient, $T(A)$ is resilient.*

Proof: Add the extra parameters of the formula to each element in the sequences $\bar{a}$ and $\bar{b}$. □

Proposition 12.11 *The following are equivalent:*

1. *T is resilient.*

2. *If $p \leq^{+} q$, then $p \leq_{\mathrm{div}} q$.*

3. *If $\bar{a} = (a_i : i \in \mathbb{Z})$ is indiscernible, $\varphi(x, y) \in L$, and $\varphi(x, a_0)$ divides over $(a_i : i \neq 0)$, then $\{\varphi(x, a_i) : i \in \mathbb{Z}\}$ is inconsistent.*

4. *There is no array $(a_{ij} : i, j < \omega)$, number $k < \omega$ and formula $\varphi(x, y) \in L$ such that $\{\varphi(x, a_{i0}) : i < \omega\}$ is consistent, $\{\varphi(x, a_{ij}) : j < \omega\}$ is k-inconsistent for each $i < \omega$ and each row $\bar{a}_i = (a_{ij} : j < \omega)$ is indiscernible over $(a_{j0} : j \neq i)$.*

5. *There is a cardinal κ such that for each sequence $(a_i : i < \kappa)$ of finite tuples a_i, for each finite tuple b there is some $i < \kappa$ such that $b \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_{(a_j : j \neq i)} a_i$.*

Proof: *1* $\Leftrightarrow$ *2* is clear.

1 $\Leftrightarrow$ *3*. Assume $\varphi(x, a_0)$ divides over $\bar{a}_{\neq 0} = (a_i : i \neq 0)$ and let $\bar{b} = (b_i : i \in \mathbb{Z})$ be an $\bar{a}_{\neq 0}$ -indiscernible sequence with $a_0 = b_0$ witnessing it. This means that $\{\varphi(x, b_i) : i \in \mathbb{Z}\}$ is inconsistent. By *1*, $\{\varphi(x, a_i) : i \in \mathbb{Z}\}$ is inconsistent. This proves *1* $\Rightarrow$ *3*. The other direction is immediate.

1 $\Leftrightarrow$ *4*. Let $\bar{a} = (a_i : i \in \mathbb{Z})$ and $\bar{b} = (b_i : i \in \mathbb{Z})$ and $\varphi(x, y) \in L$ witness that T is not resilient. Put $a_{0j} = b_j$. For $i > 0$ choose an automorphism f_i sending $a_{-i}, \ldots, a_{-1}, a_0(a_l : l > i)$ to $(a_l : l \geq 0)$ and put $a_{ij} = f_j(a_{0j})$. Since $\bar{b}$ is indiscernible over $a_{-i}, \ldots, a_{-1}(a_l : l > i)$, it follows that $(a_{ij} : j \geq 0)$ is indiscernible over $(a_j : 0 \leq j, j \neq i)$. The array $(a_{ij} : i, j \geq 0)$ with the formula $\varphi(x, y)$ provides a counterexample to *4*. This proves *4* $\Rightarrow$ *1*. The other direction is immediate.

4 $\Rightarrow$ *5*. Choose κ large enough and assume $(a_i : i < \kappa)$ and b provide a counterexample to *5* for κ. Let $i < \kappa$. Since $b \not\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_{(a_j : j \neq i)} a_i$, for some formula $\varphi_i(x; y, y_1, \ldots, y_{n_i}) \in L$ and some $j_1 < \ldots < j_{n_i}$ different from i, $\models \varphi_i(b; a_i, a_{j_1}, \ldots, a_{j_{n_i}})$ and $\varphi_i(x; a_i, a_{j_1}, \ldots, a_{j_{n_i}})$ divides over $(a_j : j \neq i)$. By choice of κ, we can assume that all these formulas are the same $\varphi(x; y, y_1, \ldots, y_n)$. Put $a'_i = a_i, a_{j_1}, \ldots, a_{j_n}$ and choose a $(a_j : j \neq i)$-indiscernible sequence $(a'_{ij} : j < \omega)$ starting with $a'_{i0} = a'_i$ and witnessing that $\varphi(x; a'_i)$ divides over $(a_j : j \neq i)$.

5 $\Rightarrow$ *4*. Let κ be as in *5* and assume (by compactness) that the array $(a_{ij} : i < \kappa, j < \omega)$ and the formula $\varphi(x, y) \in L$ contradict *4*. Choose $b \models \{\varphi(x, a_{i0}) : i < \kappa\}$. Let $i < \kappa$. Note that $\{\varphi(x, a_{ij}) : j < \omega\}$ is inconsistent and $(a_{ij} : j < \omega)$ is indiscernible over $\{a_{l0} : l \neq i\}$. This shows that $\varphi(x, a_{i0})$ divides over $\{a_{l0} : l \neq i\}$. Since $\models \varphi(b, a_{i0})$, it follows that $b \not\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_{(a_{l0} : l \neq i)} a_i$, in contradiction with *5*.

□

Proposition 12.12 *1.* NIP *theories are resilient.*

2. Simple theories are resilient.

3. Resilient theories are NTP_2*.*

Proof: *1.* Let T be NIP but not resilient. Witness the nonresilience of T by $\bar{a} = (a_i : i \in \mathbb{Z})$, $\bar{b} = (b_i : i \in \mathbb{Z})$ and some formula $\varphi(x, y)$. By NIP there is maximal $k < \omega$ for which $\{\neg\varphi(x, a_i) : i = 1, 3 \ldots, 2k+1\} \cup \{\varphi(x, a_i) : i \in \mathbb{Z} \setminus \{1, 3, \ldots, 2k+1\}\}$ is consistent. By indiscernibility $\{\neg\varphi(x, a_i) : i = 2, 4 \ldots, 2k+2\} \cup \{\varphi(x, a_i) : i \in \mathbb{Z} \setminus \{2, 4, \ldots, 2k+2\}\}$ is also consistent and we can realize it by some d. Since $\{\varphi(x, b_i) : i \in \mathbb{Z}\}$ is inconsistent, $\models \neg\varphi(d, b_i)$ for some $i \in \mathbb{Z}$. By indiscernibility of $\bar{b}$ over $(a_j : j \neq 0)$, $\{\neg\varphi(x, a_i) : i =$

$0, 2 \ldots, 2k+2\} \cup \{\varphi(x, a_i) : i \in \mathbb{Z} \setminus \{0, 2, \ldots, 2k+2\}\}$ is consistent. This contradicts the maximality of k.

2. We use item 5 of Proposition 12.11. Let $\kappa = |T|^+$ and assume $(a_i : i < \kappa)$ is a sequence of finite tuples and b is finite. By simplicity there is some $I \subseteq \kappa$ of cardinality $\leq |T|$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{(a_i : i \in I)} (a_i : i < \kappa)$. Choose $i < \kappa$, $i \notin I$. Then $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{(a_j : j \neq i)} a_i$.

3. Assume T is TP_2, witnessed by some array $(a_{ij} : i, j < \omega)$ and some formula $\varphi(x, y)$. As shown in section 5, we may assume that the array is very indiscernible. By item 4 of Proposition 12.11, T is not resilient. □

Proposition 12.13 *Let T be resilient, A an extension base, and $\bar{a} = (a_i : i < \omega)$ an A-indiscernible sequence. Then $\bar{a}$ is a witness over A iff $\bar{a}_{\neq i} \mathop{\smile\!\!\!\!\!\!\mid}_A a_i$ for every $i < \omega$.*

Proof: Since T is NTP_2 and A is an extension base, forking over A and dividing over A coincide.

$\Rightarrow$. Note that we can assume $\bar{a} = (a_i : i \in \mathbb{Q})$. Suppose $\models \varphi(a_{j_1}, \ldots, a_{j_n}; a_i)$ for some $j_1 < \ldots < j_l < i < j_{l+1} < \ldots < j_n$ and some formula $\varphi(x_1, \ldots, x_n; x) \in L(A)$. Let I be the interval (j_l, j_{l+1}). By indiscernibility, $\models \varphi(a_{j_1}, \ldots, a_{j_n}; a_q)$ for all $q \in I$ and therefore $\{\varphi(x_1, \ldots, x_n; a_q) : q \in \mathbb{Q}\}$ is consistent. Since $\bar{a}$ is a witness, $\varphi(x_1, \ldots, x_n; a_i)$ does not divide (and does not fork) over A. Hence $\bar{a}_{\neq i} \mathop{\smile\!\!\!\!\!\!\mid}_A a_i$.

$\Leftarrow$. We can assume $\bar{a} = (a_i : i \in \mathbb{Z})$. Let $\varphi(x, y) \in L(A)$ and assume $\varphi(x, a_0)$ divides over A. Since $\bar{a}_{\neq 0} \mathop{\smile\!\!\!\!\!\!\mid}_A a_0$, $\varphi(x, a_0)$ divides over $A\bar{a}_{\neq 0}$. Since $T(A)$ is resilient, by item 3 of Proposition 12.11, $\{\varphi(x, a_i) : i \in \mathbb{Z}\}$ is inconsistent. Hence $\bar{a}$ is a witness over A. □

13 Weight

Section finished on April 16, 2014. Based on [1]. For more information on weight, see [13] and [15].

Definition 13.1 The *preweight* of a type $p(x) \in S(A)$ is the supremum of all cardinals κ for which there is an A-independent sequence $(a_i : i < \kappa)$ such that for some $a \models p$, $a \not\mathop{\smile\!\!\!\!\!\!\mid}_A a_i$ for all $i < \kappa$. The *weight* of $p(x)$ is the supremum of preweights of all its nonforking extensions. The preweight of p is $\mathrm{pwt}(p)$ and $\mathrm{wt}(p)$ is its weight. If $p(x) = \mathrm{tp}(a/A)$ we can also write $\mathrm{pwt}(p) = \mathrm{pwt}(a/A)$ and $\mathrm{wt}(p) = \mathrm{wt}(a/A)$.

Lemma 13.2 *If $\pi(x)$ is a partial type over A, then* $\mathrm{bdn}(\pi(x)) = \sup\{\mathrm{bdn}(p(x)) : \pi(x) \subseteq p(x) \in S(A)\}$.

Proof: Given an array $(a_{ij} : i < \kappa, j < \omega)$ witnessing that $\mathrm{bdn}(\pi) \geq \kappa$, we may assume that the rows are mutually indiscernible over A. Hence if we choose some completion $p(x) \in S(A)$ of $\pi(x)$ which is consistent with the first vertical path, it is consistent with every path in the array. Consequently, the array witnesses that some completion of $\pi(x)$ over A has burden $\geq \kappa$. □

Proposition 13.3 *Let T be simple. Let $p(x) \in S(A)$, $a \models p$, $(a_i : i < \kappa)$ A-independent and assume $a \not\mathop{\smile\!\!\!\!\!\!\mid}_A a_i$ for all $i < \kappa$. There is an array $(a_{ij} : i < \kappa, j < \omega)$ with $a_i = a_{i0}$ witnessing that* $\mathrm{bdn}(p) \geq \kappa$. *Hence,* $\mathrm{bdn}(p) \geq \sup\{\mathrm{pwt}(q) : p \subseteq q\}$.

Proof: We inductively construct Morley sequences $I_i = (a_{ij} : j < \omega)$ in $\mathrm{tp}(a_i/Aa_{>i}I_{<i})$ and in $\mathrm{tp}(a_i/A)$ (i.e., A-independent) with $a_i = a_{i0}$ and such that $a_{\geq i} \mathop{\smile\!\!\!\!\!\!\mid}_A I_{<i}$. Assume I_j

has been constructed for all $j < i$. Notice that $a_i \mathop{\smile\kern-0.8em\mid}_A a_{>i} I_{<i}$ and choose $I_i = (a_{ij} : j < \omega)$ (see, for instance, Lemma 5.11 in [8]) as a Morley sequence in $\operatorname{tp}(a_i/Aa_{>i}I_{<i})$ and in $\operatorname{tp}(a_i/A)$ starting with $a_i = a_{i0}$. It follows that $I_i \mathop{\smile\kern-0.8em\mid}_A a_{>i} I_{<i}$ and hence $I_i \mathop{\smile\kern-0.8em\mid}_{AI_{<i}} a_{>i}$. By the inductive hypothesis $I_{<i} \mathop{\smile\kern-0.8em\mid}_A a_{>i}$ and therefore $I_{\leq i} \mathop{\smile\kern-0.8em\mid}_A a_{>i}$.

For each $i < \kappa$, since $a \not\mathop{\smile\kern-0.8em\mid}_A a_i$ there is some $\varphi_i(x, y_i) \in L(A)$ and some $k_i < \omega$ such that $\varphi_i(x, a_i)$ divides over A with respect to k_i. Adding some parameters of A to each a_i if necessary, we may assume that $\varphi_i(x, y_i) \in L$. Since I_i is a Morley sequence in $\operatorname{tp}(a_{i0}/A)$, $\{\varphi_i(x, a_{i,j}) : j < \omega\}$ is k_i-inconsistent.

Since each I_i is indiscernible over $Aa_{>i}I_{<i}$, the rows of the array $(a_{ij} : i < \kappa, j < \omega)$ are almost mutually indiscernible over A and hence for every $f : \kappa \to \omega$, $p(x) \cup \{\varphi_i(x, a_{i,f(i)}) : i < \kappa\}$ is consistent.

This shows that $\operatorname{bdn}(p) \geq \operatorname{pwt}(p)$. The rest follows from Lemma 13.2. □

Proposition 13.4 *Let T be simple. Assume that the array $(a_{ij} : i < \kappa, j < \omega)$ witnesses that $\operatorname{bdn}(a/A) \geq \kappa$. Then for some set $C \supseteq A$ such that $a \mathop{\smile\kern-0.8em\mid}_A C$, some sequence $(b_i : i < \kappa)$ witnesses that $\operatorname{pwt}(a/C) \geq \kappa$. Hence, for any $p(x) \in S(A)$, $\operatorname{wt}(p) \geq \operatorname{bdn}(p)$.*

Proof: We may assume that the rows of the array are mutually indiscernible over A and we may extend the rows to the order type $\omega + \omega^*$ preserving mutual indiscernibility over A. Let $C = A \cup \{a_{ij} : i < \kappa, j \in \omega^*\}$. We may assume that $a \mathop{\smile\kern-0.8em\mid}_A C$. For each $i < \kappa$ there is some $\varphi_i(x, y_i) \in L$ and some k_i such that $\{\varphi_i(x, a_{i,j}) : j < \omega\}$ is k_i-inconsistent. By indiscernibility over C, $\varphi_i(x, a_{i0})$ divides over C. Hence $a \not\mathop{\smile\kern-0.8em\mid}_C a_{i0}$ for every $i < \kappa$. Now we show that $(a_{i0} : i < \kappa)$ is C-independent: by choice of the order type ω^* and mutual indiscernibility over C, $\operatorname{tp}(a_{i0}/C(a_{j0} : j < i))$ is finitely satisfiable in C and therefore $a_{i0} \mathop{\smile\kern-0.8em\mid}_C (a_{j0} : j < i)$. □

Corollary 13.5 *If T is simple and $p(x) \in S(A)$, then*

$$\operatorname{bdn}(p) = \operatorname{wt}(p) = \sup\{\operatorname{pwt}(q) : p \subseteq q\}.$$

Proof: By propositions 13.3 and 13.4. □

References

[1] H. Adler. Strong theories, burden and weight. Preprint, March 2007.

[2] B. Baizhanov, S. Sudoplatov, and V. Verbovskiy. Conditions for non-symmetric relations of semi-isolation. *Siberian Electronic Mathematical Reports*, 9:161–184, 2012.

[3] I. Ben-Yaacov. Simplicity in compact abstract theories. *Journal of Mathematical Logic*, 3:163–191, 2003.

[4] I. Ben-Yaacov and A. Chernikov. An independence theorem for NTP_2 theories. Preprint. arXiv:1207.0289v1, July 2012.

[5] D. A. Brower. *Aspects of stability in simple theories.* PhD thesis, University of Notre Dame, Indiana, April 2012.

[6] E. Casanovas. Dividing and chain conditions. *Archive for Mathematical Logic*, 42(8):815–819, 2003.

[7] E. Casanovas. More on NIP and related topics. Lecture notes of the Model Theory Seminar, University of Barcelona, September 2011.

[8] E. Casanovas. *Simple theories and hyperimaginaries*, volume 39 of *Lecture Notes in Logic*. Cambridge University Press, 2011.

[9] E. Casanovas. The number of countable models. Lecture notes of the Model Theory Seminar, University of Barcelona, March 2012.

[10] A. Chernikov. Theories without the tree property of the second kind. *Annals of Pure and Applied Logic*, 165:695–723, 2014.

[11] I. Kaplan, A. Onshuus, and A. Usvyatsov. Additivity of the dp-rank. *Transactions of the American Mathematical Society*, 365(11):5783–5804, 2013.

[12] I. Kaplan and A. Usvyatsov. Strict independence in dependent theories. Preprint. ArXiv:1208.4062, August 2012.

[13] A. Pillay. *Geometric Stability Theory*. Oxford University Press, 1996.

[14] P. Simon. On dp-minimal ordered structures. *The Journal of Symbolic Logic*, 76(2):448–460, 2011.

[15] F. O. Wagner. *Simple Theories*, volume 503 of *Mathematics and Its Applications*. Kluwer Academic Publishers, Dordrecht, 2000.