The main topic will be the model theory of NTP₂ theories, but occasionally we will discuss some other topics. If not stated otherwise, \( \perp \) will be forking independence, that is \( \perp = \perp^f \).

1 On limit models

This is a topic from last year. We improve some result on limit models from section 5 of [9] after having read parts of [2]. Let \( p(x) \in S(\emptyset) \). Recall that \( M_p \) is a prime model over a realization of \( p \) and that a \( p \)-limit model is a model \( M \) which is not prime over a realization of \( p \) but it is a union of an elementary chain \( (M_i : i < \omega) \) where each \( M_i \) is prime over a realization of \( p \). Recall also that

\[
\text{SI}_p = \{(a,b) : a,b \models p \text{ and } a \text{ semi-isolates } b\}
\]

Similarly, we define

\[
\text{Ip} = \{(a,b) : a,b \models p \text{ and } a \text{ isolates } b\}
\]

**Proposition 1.1** The following are equivalent in any small theory \( T \):

1. There is a \( p \)-limit model.
2. There are \( a,b \in M_p \) realizing \( p \) and such that \( tp(b/a) \) is nonisolated.
3. \( \text{Ip} \) is not symmetric in \( M_p \).
4. \( \text{Ip} \) is not symmetric.
5. \( \text{SI}_p \) is not symmetric in \( M_p \).

**Proof:**

1 \( \Leftrightarrow \) 2. It is a better version of the proof of \( \Rightarrow \) of Proposition 5.7 of [9]: we have \( M = \bigcup_{i<\omega} M_i \) with \( M_i \) prime over \( a_i \models p \). We claim that for some \( i \), \( tp(a_i/a_0) \) is nonisolated; otherwise every tuple \( a \in M_i \) is isolated over \( a_0 \) for every \( i < \omega \) and then \( M \) is prime over \( a_0 \).

2 \( \Rightarrow \) 3 \( \Rightarrow \) 4 is clear.

4 \( \Rightarrow \) 3. If \( tp(b/a) \) is isolated, it is realized in a prime model over \( a \).

3 \( \Rightarrow \) 5. See Lemma 3.3 in [9].

5 \( \Rightarrow \) 2 is clear.

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2 Weak elimination of hyperimaginaries

Recall from [7] that $T$ has weak elimination of hyperimaginaries (in short, WEH) if every hyperimaginary is interbounded with a sequence of imaginaries. As shown there (Theorem 6.7), in any simple theory with weak elimination of hyperimaginaries, forking and thorn-forking coincide in $T^{eq}$. Moreover (Proposition 6.3), a simple theory $T$ has weak elimination of hyperimaginaries iff $\downarrow^f$ has weak canonical bases in $T^{eq}$. Now we discuss some stuff from [5].

**Definition 2.1** $T$ has dependence witnessed by imaginaries (in short, DWI) if for all hyperimaginaries $a, b$, if $a \not\vdash ^f C b$, then $a \not\vdash ^f C d$ for some imaginary $d \in acl^{eq}(Cb)$. Here $C$ is a set of hyperimaginaries (but we will see that one can assume it is a set of imaginaries) and $acl^{eq}(Cb)$ is the set of all imaginaries that are algebraic over $Cb$. On the other hand, it is enough to obtain a tuple of imaginaries $d$ instead of a single one and, if $T$ is simple, even a set $D \subseteq acl^{eq}(Cb)$ such that $a \not\vdash ^f C D$.

**Proposition 2.2** The following are equivalent to DWI in any simple theory $T$:

1. If $a, b$ are hyperimaginaries and $C$ is a set of imaginaries such that $a \not\vdash ^f C b$, then $a \not\vdash ^f C d$ for some imaginary $d \in acl^{eq}(Cb)$.
2. $T$ has weak elimination of hyperimaginaries.
3. If $a, b$ are hyperimaginaries and $C$ is a set of hyperimaginaries such that $a \not\vdash ^f C b$, then $a \not\vdash ^f C d$ for some imaginary $d \in acl^{eq}(Cb)$.

**Proof:** $1 \Rightarrow 2$. Let $e$ be a hyperimaginary and let $C = acl^{eq}(e)$ (the set of all imaginaries that are algebraic over $e$). Notice that $C = acl^{eq}(C)$. We claim that for every hyperimaginary $a$, $a \not\vdash ^f C e$. Assume not, i.e., $a \not\vdash ^f C e$. By assumption there is some imaginary $d \in acl^{eq}(Ce)$ such that $a \not\vdash ^f C d$. But $acl^{eq}(Ce) = C$, and hence $a \not\vdash ^f C C$, a contradiction. By the claim, $e \not\vdash ^f C e$, which implies $e \in bdd(C)$. Hence $e$ is interbounded with a tuple of imaginaries enumerating $C$.

$2 \Rightarrow 3$. Assume $a, b$ are hyperimaginaries and $C$ is a set of hyperimaginaries such that $a \not\vdash ^f C b$. By WEH, there is some set $B$ of imaginaries such that $bdd(B) = bdd(b)$. Hence $a \not\vdash ^f C B$ and by finite character of forking there is some tuple $d \in B$ such that $a \not\vdash ^f C d$. Clearly, $d \in acl^{eq}(b)$.

**Proposition 2.3** The following are equivalent in any simple theory $T$:

1. $T$ has weak elimination of hyperimaginaries.
2. $T$ has DWI.
3. $\downarrow^f = \downarrow^b$ and $\downarrow^b$ has weak canonical bases in $T^{eq}$.

**Proof:** $1 \Leftrightarrow 2$. By Proposition 2.2.

$1 \Rightarrow 3$. This is Proposition 6.3 and Theorem 6.7 in [7].

$3 \Rightarrow 1$. By Proposition 6.3 in [7].
3 The weak independence theorem

Taken from Ben-Yaacov and Chernikov preprint [4].

Definition 3.1 The weak independence theorem over $A$ is the following statement: if $b \equiv_A b'$, $a \downarrow_A bb'$ and $c \downarrow_A ab$, then there is some $c'$ such that $c'a \equiv_A ca$, $c'b' \equiv_A cb$, and $c' \downarrow_A ab'$.  

Remark 3.2 The independence theorem over $A$ implies the weak independence theorem over $A$.

Proof: Choose first $c_1$ such that $bc \equiv_A b'c_1$. Then $c \equiv_A c_1$, $a \downarrow_A b'$, $c \downarrow_A a$ and $c_1 \downarrow_A b'$. By the independence theorem there is some $c'$ such that $c' \equiv_A a$, $c' \equiv_A b'$ and $c' \downarrow_A ab'$. Clearly, $c'$ satisfies all the requirements.

Recall that $d_A(a, b) \leq n$ iff $\models nc^n_A(a, b)$, that is, iff there are $a_0, \ldots, a_n$ such that $a = a_0, b = a_n$ and for each $i < n$, $\models nc(a_i, a_{i+1})$ (the $a_i$ start and infinite $A$-indiscernible sequence).

Proposition 3.3 Assume forking coincides with dividing over $A$ and the weak independence theorem holds over $A$. Then $T$ is $G$-compact over $A$, and in fact $e \equiv_A d$ iff $d_A(e, d) \leq 3$.

Proof: Note that the hypothesis implies that $A$ is an extension base. Let $e \equiv_A d$. Since $A$ is an extension base, we may fix some global nonforking extension of $\tp(d/A)$. Using this global type we can obtain (as in the case of simple theories) a Morley sequence $(d_i : i < \omega)$ over $A$ starting with $d_0 = d$: first obtain a long $A$-independent sequence using the global type and then use Erdős-Rado to make it $A$-indiscernible. Hence $(d_i : i < \omega)$ is $A$-indiscernible and $d_i \downarrow_A d_{<i}$ for every $i < \omega$. The assumption that forking and dividing over $A$ coincide implies that the pair lemma holds for forking independence: if $a_1 \downarrow_A B$ and $a_2 \downarrow_A c_1$, then $a_1a_2 \downarrow_A B$. Using this and some induction one easily checks that for every $i < \omega$, $d_{\geq 1} \downarrow_A d_{<i}$.

We have then $d_{\geq 1} \downarrow_A d_0$. Without loss of generality, we can assume that $d_{\geq 1} \downarrow_A d_0 d_0 e$ (otherwise we replace $d_{\geq 1}$ by some $d_{\geq 1} \equiv_A d_{\geq 1} e$ having this property). The situation is as follows

- $d_0 \equiv_A e$
- $d_1 \downarrow_A d_0 e$
- $d_{\geq 1} \downarrow_A d_0 d_1$

By the weak independence theorem (with $a = d_1$, $b = d_0$, $b' = e$ and $c = d_{\geq 1}$) there is some $d'_{<1}$ such that $d'_{\geq 1} d_1 \equiv_A d_{\geq 1} d_1$ and $d'_{\geq 1} e \equiv_A d_{\geq 1} d_0$ (we do not need the independence). Notice that the sequences $e + d'_{\geq 1}$ and $d_1 + d'_{\geq 1}$ are $A$-indiscernible. This implies that

$\models nc_A(e, d'_2) \land nc_A(d_1, d'_2) \land nc_A(d_1, d_0)$

Hence $d_A(e, d) \leq 3$. 

3
4 The chain condition

We continue with [4], but changing slightly some things and taking into account some material from [6].

**Definition 4.1** Let $\pi(x)$ be a partial type over $A$. We say that $\pi(x)$ has the chain condition over $A$ with respect to $\pi(x)$ if for every $\varphi(x, y) \in L$, if $\models \text{nc}_A(a_i, a_j)$ and $\pi(x) \cup \{\varphi(x, a_0)\}$ does not fork over $A$, then $\pi(x) \cup \{\varphi(x, a_0) \land \varphi(x, a_1)\}$ does not fork over $A$. If $\pi(x)$ is empty we just say that $T$ has the chain condition over $A$.

**Proposition 4.2** The following are equivalent for any partial type $\pi(x)$ over $A$:

1. $T$ has the chain condition over $A$ with respect to $\pi(x)$.

2. For any cardinal $\kappa \geq |T| + |A| + |x|$, for any family $(\pi_i(x) : i < (2^\kappa)^+)$ of partial types $\pi_i(x)$ of cardinality $\leq \kappa$ extending $\pi(x)$, if $\pi_i(x)$ does not fork over $A$ for every $i < (2^\kappa)^+$, then $\pi_i(x) \cup \pi_j(x)$ does not fork over $A$ for some $i < j < (2^\kappa)^+$.

3. For any cardinal $\kappa \geq |A| + |T| + |x|$, for any family of formulas $(\varphi_i(x, a_i) : i < (2^\kappa)^+)$ with $\varphi_i(x, y) \in L(A)$, if $\pi(x) \cup \{\varphi_i(x, a_i)\}$ does not fork over $A$ for every $i < (2^\kappa)^+$, then $\pi(x) \cup \{\varphi_i(x, a_i) \land \varphi_j(x, a_j)\}$ does not fork over $A$ for some $i < j < (2^\kappa)^+$.

4. For any partial type $\pi(x, y)$ over $A$ extending $\pi(x)$, if $(a_i : i < \omega)$ is $A$-indiscernible and $\pi(x, a_i)$ does not fork over $A$ for every $i < \omega$, then $\bigcup_{i<\omega} \pi(x, a_i)$ does not fork over $A$.

5. If $(a_i : i < \omega) \models \pi(x)$ is $A$-indiscernible $b \models \pi(x)$ and $b \upharpoonright_{A_0} \equiv_{A_0} b$ such that $(a_i : i < \omega)$ is indiscernible over $Ab'$ and $b' \upharpoonright A(a_i : i < \omega)$.

6. If $(a_i : i < \omega) \models \pi(x)$ is $A$-indiscernible $b \models \pi(x)$ and $b \upharpoonright A_0 \equiv_{A_0} (a_i : i < \omega)$ which is indiscernible over $Ab$ and $b \upharpoonright A(a_i' : i < \omega)$.

**Proof:** 1 $\Rightarrow$ 2. We can assume each $\pi_i(x)$ is closed under conjunction. Let $\pi_i(x) = \pi_i(x, a_i) = \{\varphi_{ij}(x, a_i) : j < \kappa\}$ with $\varphi_{ij}(x, y) \in L$. Assume $\pi_i(x, a_i) \cup \pi_j(x, a_j)$ forks over $A$ for every $i < j < (2^\kappa)^+$. Since $\kappa$ is large enough, we can assume that $\varphi_{ij}(x, y) = \varphi_{ji}(x, y)$ for all $i < (2^\kappa)^+$ and $j < \kappa$. By Proposition 3.3 of [6] there are $i < j < (2^\kappa)^+$ such that $\models \text{nc}_A(a_i, a_j)$. We can then choose some $l < \kappa$ such that $\varphi_{ij}(x, a_i) \land \varphi_{ji}(x, a_j)$ forks over $A$, contradicting 1.

2 $\Rightarrow$ 3 is clear.

3 $\Rightarrow$ 4. We can extend the sequence to an $A$-indiscernible sequence $(a_i : i < \lambda)$ where $\kappa = |A| + |T| + |x|$ and $\lambda = (2^\kappa)^+$. By 3 and indiscernibility, for each $i < j < \lambda$, for each $\varphi(x, y) \in \pi(x, y)$, $\varphi(x, a_i) \land \varphi(x, a_j)$ does not fork over $A$. We can inductively generalize this and show that for any $\omega$, for any $i_1 < \ldots < i_n < \lambda$, $\varphi(x, a_{i_1}) \land \ldots \land \varphi(x, a_{i_n})$ does not fork over $A$, which implies that $\bigcup_{i<\lambda} \pi(x, a_i)$ does not fork over $A$. For the induction notice that $(b_i : i < \lambda)$ is $A$-indiscernible if $b_i = a_{i_1(i+1)} \ldots a_{i(i+n)}(n+1)$.

4 $\Rightarrow$ 5. Let $p(x, y) \models S(A)$ be such that $\text{tp}(b/\text{A}_0) = p(x, a_0)$. By 4 $\bigcup_{i<\omega} p(x, a_i)$ does not fork over $A$. Choose $b' \models \bigcup_{i<\omega} p(x, a_i)$ such that $b' \upharpoonright A(a_i : i < \omega)$. We can extend the sequence and then apply Erdős-Rado to extract an $A'$-indiscernible sequence $(a'_i : i < \omega)$ such that $b' \upharpoonright A(a'_i : i < \omega)$. By conjugation over $A$ we obtain a corresponding $b''$ for $(a_i : i < \omega)$.

5 $\Leftrightarrow$ 6. By conjugation over $A$.  

5 ⇒ 1. Let \((a_i : i < \omega)\) be an \(A\)-indiscernible sequence such that \(\varphi(x, a_0)\) does not fork over \(A\). Choose \(b \downarrow A a_0\) such that \(b \models \pi(x)\) and \(\models \varphi(b, a_0)\). By 5 there is some \(b' \equiv_{Aa_0} b\) such that \((a_i : i < \omega)\) is indiscernible over \(Ab'\) and \(b' \downarrow (a_i : i < \omega)\). Since \(b' \models \pi(x)\) and \(\models \varphi(b', a_0) \land \varphi(b', a_1)\), the set \(\pi(x) \cup \{\varphi(x, a_0) \land \varphi(x, a_1)\}\) does not fork over \(A\). □

**Proposition 4.3** If \(T\) has the chain condition over \(A\), then the weak independence theorem over \(A\) holds.

**Proof:** We prove first the case \(\models nc_A(b, b')\). In fact we will see that the general case follows from this particular case without the need of the chain condition.

Assume \(\models nc_A(b, b'), a \downarrow_A bb'\) and \(c \downarrow_A ab\). Since \(a \downarrow Ab'\), by Lemma 10.6 of [8], there is some \(a'\) such that \(\models nc_A(ab, a'b')\). Fix an \(A\)-indiscernible sequence \((a_i b_i : i < \omega)\) with \(a_0, b_0, a b_1 = a, b, a', b'\). Since \(c \downarrow_A a_0 b_0\), by point 5 of Proposition 4.2 there is some \(c' \equiv_{Aa_0b_0} c\) such that \(c' \downarrow_A (a_i b_i : i < \omega)\) and \((a_i b_i : i < \omega)\) is \(Ac'\)-indiscernible. It follows that \(c' \downarrow_A ab', c' b' \equiv_A c'b \equiv_A cb\) and \(c'a \equiv_A ca\).

Now we prove the general case. Fix \(n\) such that \(d_A(b, b') \leq n\) and fix \(b_0, \ldots, b_n\) such that \(b_0 = b, b_n = b'\) and \(\models nc_A(b_i, b_{i+1})\) for all \(i < n\). Since \(a \downarrow_A b_i b_{i+1}\), we can assume (changing \(a\) and \(c\) if necessary) that \(a \downarrow_A b_0, \ldots, b_n\). Now we check by induction on \(i < n\) that there is some \(c_i\) such that \(c_i b_i \equiv_A c_{i+1} b_{i+1}\), \(c_i a \equiv_A ca\) and \(c_i \downarrow_A ab_i\). This will suffice. We start with \(c_0 = c\). In order to obtain \(c_{i+1}\) from \(c_i\) we apply the particular case proven above: there is some \(c_{i+1}\) such that \(c_{i+1} b_{i+1} \equiv_A c_i b_i, c_{i+1} a \equiv_A c_i a\) and \(c_{i+1} \downarrow_A b_{i+1} a\). Using now the induction hypothesis on \(c_i\) we see that \(c_{i+1} b_{i+1} \equiv_A c b_0\) and \(c_{i+1} a \equiv_A ca\). □

## 5 Indiscernible arrays

References for this section are [4], [10] and [3], but we do some things differently.

We will discuss arrays of the form \(A = (a_{ij} : i < \alpha, j < \beta)\) where \(\kappa, \lambda\) are ordinals (although more generally we could have dealt with linearly ordered sets). We understand that the rows are \((a_{ij} : j < \beta)\) for \(i < \alpha\) and the columns are \((a_{ij} : j < \alpha)\) for \(i < \beta\). Hence our array looks like as follows:

\[
\begin{array}{cccc}
  a_{00} & a_{01} & a_{02} & \cdots \\
  a_{10} & a_{11} & a_{12} & \cdots \\
  a_{20} & a_{21} & a_{22} & \cdots \\
  & & & \\
\end{array}
\]

Each \(a_{ij}\) is a (possibly infinite) tuple and all have the same length. Sometimes we use the notation \(a_i = (a_{ij} : j < \beta)\) for the rows. The sequence of previous rows is \(a_{<i}\). Other notations, like \(a_{>i}\) are self-explanatory.

An \((n, m)\)-subarray of \((a_{ij} : i < \alpha, j < \beta)\) is an array of the form \((a_{ijk} : l = 1, \ldots, n, k = 1, \ldots, m)\) for some \(i_1 < \ldots < i_n < \alpha\) and some \(j_1 < \ldots < j_m < \beta\). In the case \(n = m\) we talk of an \(n\)-subarray.

**Definition 5.1** An array is \(A\)-indiscernible if both the sequence of its rows and the sequence of its columns are \(A\)-indiscernible. Note that \(A\)-indiscernibility of rows imply that each single column is \(A\)-indiscernible and, similarly, \(A\)-indiscernibility of columns imply that each single row is \(A\)-indiscernible.
Remark 5.2 The following are equivalent for any array \( \mathcal{A} = (a_{ij} : i < \alpha, j < \beta) \) and any set \( \mathcal{A} \):

1. \( \mathcal{A} \) is \( \mathcal{A} \)-indiscernible.

2. For all \( n, m < \omega \) all \( (n, m) \)-subarrays of \( \mathcal{A} \) have the same type over \( \mathcal{A} \).

3. For any \( i_1, \ldots, i_n < \alpha \) and \( j_1, \ldots, j_n < \beta \), the type \( \text{tp}(a_{i_1j_1}, \ldots, a_{i_nj_n}/\mathcal{A}) \) depends only on the order (and equality) relations of \( i_1, \ldots, i_n \) and of \( j_1, \ldots, j_n \).

Proof: 1 \( \Rightarrow \) 2. Consider two \( (n, m) \)-subarrays \( a \) and \( b \), the first one determined by \( i_1 < \ldots < i_n < \alpha \) and \( j_1 < \ldots < j_n < \beta \) and the second one by \( i'_1 < \ldots < i'_n < \alpha \) and \( j'_1 < \ldots < j'_n < \beta \). By \( \mathcal{A} \)-indiscernibility of the sequence of columns, \( a \) has same type over \( \mathcal{A} \) as the \( (n, m) \)-subarray \( c \) determined by \( i_1, \ldots, i_n \) and \( j_1', \ldots, j'_n \). Now by \( \mathcal{A} \)-indiscernibility of the sequence of rows, \( c \) has same type over \( \mathcal{A} \) as \( b \).

2 \( \Rightarrow \) 3. If the order of indexes in rows and columns coincide, for some \( n, m \) the corresponding tuples of the array lie in the same positions in some \((n, m)\)-subarrays.

3 \( \Rightarrow \) 1. Clear.

Remark 5.3 Let \( (a_{ij} : i < \alpha, j < \beta) \) be an \( \mathcal{A} \)-indiscernible array. For any two strictly increasing mappings \( f : \alpha \to \beta \) and \( g : \alpha \to \beta \):

\[ (a_{if(i)} : i < \alpha) \equiv_{\mathcal{A}} (a_{ig(i)} : i < \alpha) \]

Proof: By Remark 5.2, since \( (f(i) : i < \alpha) \) and \( (g(i) : i < \alpha) \) are order isomorphic.

Definition 5.4 The rows of an array \( (a_{ij} : i < \alpha, j < \beta) \) are mutually indiscernible over \( \mathcal{A} \) if each row \( a_i = (a_{ij} : j < \beta) \) is indiscernible over \( \mathcal{A} \) and the rest \( a_{i \neq i} = \{a_{ij} : l < \alpha, l \neq i \text{ and } j < \beta \} \) of the array. The array is very indiscernible over \( \mathcal{A} \) if it is \( \mathcal{A} \)-indiscernible and its rows are mutually indiscernible over \( \mathcal{A} \).

Remark 5.5 An array \( (a_{ij} : i < \alpha, j < \beta) \) has rows mutually indiscernible over \( \mathcal{A} \) if and only if for any \( i_1 < \ldots < i_n < \alpha \) and any choice \( b_{i_l} \) and \( b'_{i_l} \) for each \( l = 1, \ldots, n \) of finite increasing sequences with same length of tuples of the row \( i_l \):

\[ b_{i_1}, \ldots, b_{i_n} \equiv_{\mathcal{A}} b'_{i_1}, \ldots, b'_{i_n} \]

Proof: It is clear that this new condition implies mutual indiscernibility of rows. For the other direction, by induction on \( k \leq n \) we check that \( b'_{i_k}, \ldots, b'_{i_{k+1}}, \ldots, b_{i_n} \equiv_{\mathcal{A}} b_{i_k}, \ldots, b_{i_n} \). The case \( k = 1 \) is clear by mutual indiscernibility. For the inductive step note that, by mutual indiscernibility and the inductive hypothesis, \( b'_{i_1}, \ldots, b'_{i_k} b'_{i_{k+1}}, b_{i_{k+2}}, \ldots, b_{i_n} \equiv_{\mathcal{A}} b'_{i_1}, \ldots, b'_{i_k} b_{i_{k+1}}, \ldots, b_{i_n} \).

Remark 5.6 1. If the rows of an array are mutually indiscernible over \( \mathcal{A} \), then the sequence of its columns is \( \mathcal{A} \)-indiscernible.

2. If an array has rows mutually indiscernible over \( \mathcal{A} \) and the sequence of its rows is \( \mathcal{A} \)-indiscernible, then it is very indiscernible over \( \mathcal{A} \).

Proof: By Remark 5.5, since to check indiscernibility of columns it is enough to look at finitely many rows of two given finite sequences of columns.
Remark 5.7 Let $\beta \geq \omega$, assume the columns of $A = (a_{ij} : i < \alpha, j < \beta \cdot \alpha)$ are $A$-indiscernible, and define $b_{ij} = a_{i,\beta+i}$. Then the rows of $B = (b_{ij} : i < \alpha, j < \beta)$ are mutually indiscernible over $A$. Hence, if $A$ is an $A$-indiscernible array, then $B$ is very indiscernible over $A$.

Proof: Note that each $(a_{i,\beta+i} : j < \beta)$ is indiscernible over $A \cup \{a_{i,\beta+k+j} : l < \alpha, (k < i \text{ or } k < l < \alpha) \text{ and } j < \beta\}$. Hence $(b_{ij} : j < \beta)$ is indiscernible over $A \cup \{b_{ij} : j < \beta, l \neq i\}$. By Remark 5.6 $B$ is very indiscernible over $A$.

Proposition 5.8 For any array $A = (a_{ij} : i < \alpha, j < \lambda)$ and any set $A$, if $\kappa \geq |A| + |T| + |a_{ij}| + |\alpha|$ and $\lambda = \sum_{i=0}^{n} \cdot \cdot$, then there is an array $B = (b_{ij} : i < \alpha, j < \omega)$ which has mutually $A$-indiscernible rows and is locally like $A$, in the sense that for every choice in $B$ of finitely many rows and a finite subsequence in each of these rows, there is a corresponding choice of subsequences in the same rows of $A$ having (all together) the same type over $A$.

Proof: By the ordinary method of extracting indiscernible sequences based on Erdös-Rado, there is some array $A' = (a'_{ij} : i < \alpha, j < \omega \cdot \alpha)$ with $A$-indiscernible sequence of columns and such that for every finite subsequence of columns in $A'$ there is a corresponding sequence of columns in $A$ with the same type over $A$. In particular, $A'$ is locally like $A$. The next step is extending the array $A'$ to $(a'_{ij} : i < \alpha, j < \omega \cdot \alpha)$ with $A$-indiscernible sequence of columns. Now by Remark 5.7 if we define $b_{ij} = a'_{i,\omega+i+j}$ for $i < \alpha$ and $j < \omega$, we obtain a new array $B = (b_{ij} : i < \alpha, j < \omega)$ with the required properties. \hfill \Box

6 Array-dividing

Here the main reference is [4]

Definition 6.1 Let $\varphi(x,y) \in L$. We say that $\varphi(x,a)$ array-divides over $A$ if there is some $A$-indiscernible array $(a_{ij} : i, j < \omega)$ with $a = a_0$ and such that $\{\varphi(x,a_{ij}) : i, j < \omega\}$ is inconsistent.

Remark 6.2 If $\varphi(x,a)$ divides over $A$, then it array-divides over $A$.

Proof: Let $(a_{i} : i < \omega)$ be an $A$-indiscernible sequence with $a = a_0$ such that $\{\varphi(x,a_{i}) : i < \omega\}$ is inconsistent. Let $a_{ij} = a_j$. Then the array $(a_{ij} : i, j < \omega)$ witnesses that $\varphi(x,a)$ array-divides over $A$. \hfill \Box

Proposition 6.3 If forking and dividing over $A$ coincide and $T$ has the chain condition over $A$, then dividing and array-dividing over $A$ coincide.

Proof: Assume $\varphi(x,a)$ does not divide (fork) over $A$, let $(a_{i,j} : i, j < \omega)$ be $A$-indiscernible with $a = a_{00}$ and let us check that $\{\varphi(x,a_{ij}) : i, j < \omega\}$ is consistent. Let $n < \omega$. By the chain condition over $A$, $\bigwedge_{i < n} \varphi(x,a_{i0})$ does not fork over $A$. Since the sequence $(b_{ij} : j < \omega)$ with $b_{ij} = (a_{ij} : i < n)$ is $A$-indiscernible, again by the chain condition over $A$, $\bigwedge_{i,j < n} \varphi(x,a_{ij})$ does not fork over $A$ and hence it is consistent. \hfill \Box

Definition 6.4 An $A$-indiscernible sequence $(a_{i} : i < \omega)$ is a universal over $A$ in $\text{tp}(a_0/A)$ if $\{\varphi(x,a_{i}) : i < \omega\}$ is inconsistent for every formula $\varphi(x,y) \in L$ such that $\varphi(x,a_0)$ divides over $A$.

Remark 6.5 1. In a simple theory, every Morley sequence over $A$ is universal over $A$. 2. It is consistent that every $\kappa$-indiscernible sequence over $A$ is universal over $A$. 3. It is consistent that there is a $\kappa$-indiscernible sequence over $A$ which is not universal over $A$.
2. In a NTP\textsubscript{2} theory, every sequence generated over \(A\) by a strict \(A\)-invariant global type is universal over \(A\) (see Proposition 4.4 of [7]); hence (see Proposition 4.11 of [7]) any type over a model \(M\) has universal indiscernible sequences over \(M\).

**Proposition 6.6** If forking and array-dividing over \(A\) coincide and for every type \(p(x)\) over \(A\) there are universal indiscernible sequences over \(A\) in \(p(x)\), then \(T\) has the chain condition over \(A\).

**Proof:** Assume \(\varphi(x, y) \in L\), \(\varphi(x, a_0)\) does not fork over \(A\), \((a_j : j < \omega)\) is \(A\)-indiscernible and \(\varphi(x, a_0) \land \varphi(x, a_1)\) forks over \(A\). Let \(\kappa\) be large enough and extend the sequence to an \(A\)-indiscernible sequence \((a_j : j < \kappa)\). Choose a universal indiscernible sequence \((a_{ij} : i < \omega)\) over \(A\) in the type \(\text{tp}(a_j : j < \kappa/A)\) with \(a_0 = a_j\). By the choice of \(\kappa\) we can extract an \(A\)-indiscernible sequence \((a'_{ij} : i < \omega)\) over the sequence of columns \((a_{ij} : i < \omega)\) as \((a'_{ij} : j < \kappa)\). It follows that \((a'_{ij} : i, j < \omega)\) is an \(A\)-indiscernible array.

By \(A\)-indiscernibility, for every \(j < l < \kappa\), \(\varphi(x, a_{0j}) \land \varphi(x, a_{0l})\) forks over \(A\). Since we have chosen a universal indiscernible sequence, for every \(j < l < \kappa\), \(\{\varphi(x, a_{ij}) : i < \omega\}\) is inconsistent. Hence, for every \(j < l < \omega\), \(\{\varphi(x, a'_{ij}) \land \varphi(x, a'_{il}) : i < \omega\}\) is inconsistent and \(\{\varphi(x, a'_{ij}) : i, j < \omega\}\) is inconsistent too. Since \(a_0 \equiv_A a_{00}\), \(\varphi(x, a_0)\) array-divides (and forks) over \(A\), a contradiction. \(\square\)

**Definition 6.7** A partial type \(\pi(x)\) has TP\textsubscript{2} if for some formula \(\varphi(x, y) \in L\) and some \(k < \omega\) there is some array \((a_{ij} : i, j < \omega)\) such that \(\{\varphi(x, a_{ij}) : j < \omega\}\) is \(k\)-inconsistent for every \(i < \omega\) and \(\pi(x) \cup \{\varphi(x, a_{ij}) : i < \omega\}\) is consistent for every \(f : \omega \rightarrow \omega\). We say that \(\pi\) has NTP\textsubscript{2} if it does not have TP\textsuperscript{2}. A theory \(T\) has TP\textsubscript{2} if the empty partial type has TP\textsubscript{2} in \(T\). Similarly for NTP\textsubscript{2}. As shown in Lemma 7.4, this definition agrees with the one given in [7] (where we required \(k = 2\)).

**Remark 6.8** Nothing changes if in the definition of NTP\textsubscript{2} type one requires that \(\{\varphi(x, a_{ij}) : j < \omega\}\) is \(k\)-inconsistent relatively to \(\pi(x)\), namely, \(p(x) \cup \{\varphi(x, a_{ij}) : i < \omega\}\) is inconsistent for all \(j_1, \ldots, j_k < \omega\). The reason is that in an array witnessing TP\textsubscript{2} of \(\pi(x)\) with this weaker requirement we can assume all rows have the same type over \(A\) and hence adding the same formula \(\theta(x)\) of \(\pi(x)\) to \(\phi(x, y)\) we get \(k\)-inconsistency. If \(\theta(x)\) has parameters in \(A\) we can add the parameters to each tuple in the array.

**Lemma 6.9** Let \(\pi(x)\) a partial NTP\textsubscript{2} type over \(A\). If the array \((a_{ij} : i, j < \omega)\) is very indiscernible over \(A\) and the first column \(\{\varphi(x, a_{i0}) : i < \omega\}\) is consistent with \(\pi(x)\), then \(\pi(x) \cup \{\varphi(x, a_{ij}) : i, j < \omega\}\) is consistent.

**Proof:** Let \(n < \omega\) and let us check that \(\pi(x) \cup \{\varphi(x, a_{ij}) : i < n, j < \omega\}\) is consistent. Let \(b_{ij} = a_{i,n,j}, \ldots, a_{i,n+(n-1),j}\). Then the array \((b_{ij} : i, j < \omega)\) is also very indiscernible over \(A\). Let \(\varphi^n(x, y_0, \ldots, y_{n-1}) = \bigwedge_{i \leq n} \varphi(x, y_i)\). We need to check that \(\pi(x) \cup \{\varphi^n(x, b_{ij}) : j < \omega\}\) is consistent. We know that \(\pi(x) \cup \{\varphi^n(x, b_{ij}) : i < \omega\}\) is consistent. By mutual indiscernibility over \(A\) (see Remark 5.5), for every \(f : \omega \rightarrow \omega\), the set \(\pi(x) \cup \{\varphi^n(x, b_{f(i)}) : i < \omega\}\) is consistent. If \(\pi(x) \cup \{\varphi^n(x, b_{ij}) : j < \omega\}\) is inconsistent, then (see Remark 6.8) we may assume that \(\{\varphi^n(x, b_{ij}) : j < \omega\}\) is \(k\)-inconsistent for some \(k < \omega\). Then every \(\{\varphi^n(x, b_{ij}) : j < \omega\}\) is \(k\)-inconsistent too for every \(i < \omega\), and therefore \(\varphi^n\) witnesses that \(\pi(x)\) has TP\textsubscript{2}. \(\square\)

**Lemma 6.10** Let \(\pi(x)\) be a NTP\textsubscript{2} type over \(A\). If the array \((a_{ij} : i, j < \omega)\) is indiscernible over \(A\) and the diagonal \(\{\varphi(x, a_{ii}) : i < \omega\}\) is consistent with \(\pi(x)\), then \(\pi(x) \cup \{\varphi(x, a_{ij}) : i, j < \omega\}\) is consistent.
Definition 7.1 An inp-pattern of depth \( \kappa \) in a partial type \( \pi(x) \) is an array \( (a_{ij} : i < \kappa, j < \omega) \) together with numbers \( k_i < \omega \) and formulas \( \varphi_i(x,y_i) \in L \) for \( i < \kappa \) such that

1. \( \{\varphi_i(x,a_{ij}) : j < \omega \} \) is \( k_i \)-inconsistent for every \( i < \kappa \).
2. \( \pi(x) \cup \{\varphi_i(x,a_{ij}) : i < \kappa \} \) is consistent for every \( f : \kappa \rightarrow \omega \).

The burden of \( \pi(x) \) is the supremum \( \text{bdn}(\pi) \) of all depths of inp-patterns in \( \pi(x) \). We write \( \text{bdn}(a/A) = \text{bdn}(\text{tp}(a/A)) \).

Remark 7.2 The fact \( \text{bdn}(\pi) = \omega \) means that there are arbitrarily large finite depths of inp-patterns in \( \pi \) and none of depth \( \omega_1 \). This is compatible both with the existence and the
inexistence of patterns of depth \(\omega\). Perhaps it would have been better to define the burden as the first cardinal \(\kappa\) for which there is no inp-pattern of depth \(\kappa\). With this definition one would have \(\text{bdn}(\pi(x)) > \kappa\) iff there is some inp-pattern of depth \(\kappa\) in \(\pi(x)\). But then the difficulty is that with this notion we would have \(\text{bdn}(\pi) = 1\) iff \(\pi\) is algebraic (and there would not exist types of burden zero). This could be repaired by defining the burden as the least \(\kappa\) for which there is no inp-pattern of depth \(\kappa + 1\) (this does not change the burden if it is \(\geq \omega\)).

**Remark 7.3**

1. \(\text{bdn}(\pi(x)) = 0\) iff \(\pi(x)\) is algebraic.

2. If \(\pi(x) \vdash \pi'(x)\), then \(\text{bdn}(\pi(x)) \leq \text{bdn}(\pi'(x))\).

3. \(\text{bdn}(\pi(x)) < \infty\) iff \(\text{bdn}(\pi(x)) < |T|^+\).

**Proof:**

1. Assume \(\pi(x)\) is not algebraic and let \((a_i : i < \omega)\) be a sequence of different realizations of \(\pi\). The inp-pattern of depth 1 defined by the formula \(x = y\), the number \(k = 2\) and the parameters \((a_i : i < \omega)\) shows that \(\text{bdn}(\pi) \geq 1\). Assume now \(\text{bdn}(\pi) \geq 1\), and let \(\varphi(x,y)\) with \(k\) and \((a_i : i < \omega)\) be a pattern of depth 1 in \(\pi(x)\). For each \(i < \omega\) choose a maximal set \(I_i \subseteq \omega\) such that \(\Sigma_i(x) = \pi(x) \cup \{\varphi(x,a_i) : i < \omega\}\) is consistent (any such set has cardinality \(\leq k\)) and choose a realization \(b_i\) of \(\Sigma_i(x)\). If \(j \notin I_i\) then \(\Sigma_j(x) \cup \Sigma_i(x)\) is inconsistent and therefore \(b_i \neq b_j\). Now choose inductively \(i_n \in \omega \setminus \bigcup_{m<n} I_m\). It follows that \(b_{i_0}, b_{i_1}, \ldots\) are different realizations of \(\pi(X)\), which proves that \(\pi(x)\) is not algebraic.

3. In an inp-pattern of depth \(|T|^+\) there is some \(k < \omega\) and some formula \(\varphi(x,y)\) which are used in infinitely many rows. By compactness we can then extend the pattern with \(k\) and \(\varphi\) to any possible depth. \(\square\)

**Lemma 7.4**

If \(\pi\) has \(\text{TP}_2\), then for any set \(A\) there is some very \(A\)-indiscernible array \((a_{ij} : i,j < \omega)\) and some formula \(\varphi(x,y) \in L\) such that \(\{\varphi(x,a_{ij}) : j < \omega\}\) is 2-inconsistent for every \(i < \omega\) and \(\pi(x) \cup \{\varphi(x,a_{ij}) : i < \omega\}\) is consistent for every \(f \in \omega^\omega\).

**Proof:** Start with an array \((a_{ij} : i,j < \omega)\), a natural number \(k\) and a formula \(\varphi(x,y)\) witnessing \(\text{TP}_2\), choose \(\lambda\) large enough and extend the array to \((a_{ij} : i < \lambda, j < \omega)\) with all paths consistent with \(\pi(x)\) and with \(k\)-inconsistent rows. We can extract using Erdős-Rado an array with \(A\)-indiscernible sequence of rows and locally like the previous one. Next, we extend its rows to width \(\lambda\) and we apply Proposition 5.8. After all this, we end up with a very \(A\)-indiscernible array witnessing \(\text{TP}_2\) with \(\varphi(x,y)\) and \(k\). We denote it again by \((a_{ij} : i,j < \omega)\). Now we want to reduce \(k\) to 2. We may assume \(k\) is minimal, that is, no array and formula witness \(\text{TP}_2\) of \(\pi(x)\) with a smaller number. Consider the set \(\{\varphi(x,a_{ij}), \varphi(x,a_{i1}) : i < \omega\}\). If it is consistent with \(\pi(x)\), then (by indiscernibility of the array) the formula \(\psi(x; y_0 y_1) = \varphi(x,y_0) \land \varphi(x,y_1)\) and the array \((a_{12}, a_{21}, a_{2j+1} : i < \omega, j < \omega)\) witness \(\text{TP}_2\) with a smaller number. If it is inconsistent with \(\pi(x)\), we choose \(n < \omega\) such that \(\{\varphi(x,a_{01}), \varphi(x,a_{i1}) : i < n\}\) is inconsistent with \(\pi(x)\). Then the formula \(\psi(x; y_0, \ldots, y_{n-1}) = \bigwedge_{i < n} \varphi(x,y_i)\) together with the array \((a_{01}, \ldots, a_{i+1-1,j} : i < \omega, j < \omega)\) witnesses \(\text{TP}_2\) with \(k = 2\).

**Remark 7.5**

\(T\) is \(\text{NTP}_2\) iff all types have burden \(< \infty\). In fact, if \(T\) has \(\text{TP}_2\), then for every set \(A\) there is a type \(\rho(x) \in S(A)\) with \(\text{bdn}(\rho(x)) = \infty\).

**Proof:** Assume \(T\) has \(\text{TP}_2\) and let \(A\) be any set. Use Lemma 7.4 to witness this with \(\varphi\) and an array \((a_{ij} : i,j < \omega)\) with mutually \(A\)-indiscernible rows. Let \(b \models \{\varphi(x,a_{0j}) : i < \omega\}\) and let \(p(x) = \text{tp}(b/A)\). By Remark 5.5, \((a_{0i} : i < \omega) \equiv_A (a_{1f(i)} : i < \omega)\) and hence \(p(x) \cup \{\varphi(x,a_{1f(i)}) : i < \omega\}\) is consistent for every \(f \in \omega^\omega\). By compactness, \(\text{bdn}(\rho(x)) = \infty\). \(\square\)
Definition 7.6 The rows of an array \((a_{ij} : i < \alpha, j < \beta)\) are almost mutually indiscernible over \(A\) if each row \(a_i = (a_{ij} : j < \beta)\) is indiscernible over \(A_{a_i} = (a_{ij} : l > i)\). Notice that in such arrays, \((a_{i0} : i < \alpha) \equiv_A (a_{i(f(i))} : i < \alpha)\) for every \(f : \alpha \to \beta\).

Lemma 7.7 If the rows of the array \((a_{ij} : i < \alpha, j < \beta)\) are almost mutually indiscernible over \(A\), then there is some array \((a'_{ij} : i < \alpha, j < \beta)\) with rows mutually indiscernible over \(A\) and such that \((a'_{ij} : j < \beta) \equiv_{A_a} (a_{ij} : j < \beta)\) for every \(i < \alpha\).

Proof: As a first step we extend the width of the array to a conveniently large cardinal \(\lambda\) preserving almost mutual indiscernibility. We apply then Proposition 5.8 and compactness to obtain an array \((a'_{ij} : i < \alpha, j < \beta)\) which is locally like our previous array and has mutually \(A\)-indiscernible rows. We claim that \((a'_{i0} : i < \alpha) \equiv_A (a_{i0} : i < \alpha)\). To check this, assume \(i_0 < \ldots < i_k < \alpha\). There are \(j_0, \ldots, j_k < \lambda\) such that \(a'_{i0}, \ldots, a'_{i0} \equiv_A a_{i_0 j_0}, \ldots, a_{i_k j_k}\). But \(a_{i_0 j_0}, \ldots, a_{i_k j_k} \equiv_A a_{i_0 0}, \ldots, a_{i_0 0}\). This proves the claim. Now take \((a''_{ij} : i < \alpha, j < \beta)\) such that \((a'_{ij} : i < \alpha, j < \beta)(a''_{ij} : i < \alpha) \equiv_A (a''_{ij} : i < \alpha, j < \beta)\). The rows of the new array are again mutually \(A\)-indiscernible and moreover \((a''_{ij} : j < \beta)a_{i0} \equiv_A (a_{ij} : j < \beta)a_{i0} \equiv_A (a_{ij} : j < \beta)a_{i0}\) for every \(i < \alpha\).

Lemma 7.8 Let \(\Delta\) be a finite set of formulas of \(L(A)\), let \((a_{ij} : i < n, j < \omega)\) be an array and let \(k < \omega\). For each \(i < n\) there are \(j_0 < \ldots < j_k < \omega\) such that the rows of the array \((a_{ij} : i < n, l < k)\) are \(\Delta\)-indiscernible, meaning that for each \(\varphi(x_1, \ldots, x_r; \alpha)\) \(\in \Delta\), each row is \(\varphi(x_1, \ldots, x_r; a)\)-indiscernible for any tuple \(a\) of elements of the remaining rows.

Proof: By Ramsey's Theorem, there is an infinite subset \(I \subseteq \omega\) such that the sequence of columns of \((a_{ij} : i < n, j \in I)\) is indiscernible with respect to all formulas in \(\Delta\), letting the variables range over elements of any row. By the trick explained in the proof of Remark 5.7 it suffices now to take \(n\) consecutive segments of length \(k+1\) in the increasing enumeration of \(I\), obtaining this way a diagonal of tuples lying in disjoint columns.

Lemma 7.9 Assume the rows of the array \((a_{ij} : i < \alpha, j < \beta)\) are mutually \(A\)-indiscernible and let \(b\) be any tuple. Let \(p_l(x, y) = tp(b_{a_{i0}}/A)\) and suppose \(\bigcup_{i < \alpha, j < \beta} p_l(x, a_{ij})\) is consistent. Then there is some array \((a'_{ij} : i < \alpha, j < \beta)\) with rows mutually indiscernible over \(Ab\) and such that \((a'_{ij} : j < \beta) \equiv_{A_{a_{i0}}} (a_{ij} : j < \beta)\) for all \(i < \alpha\).

Proof: Let \(b' \models \bigcup_{i < \alpha, j < \beta} p_l(x, a_{ij})\). Fix a finite set \(\Delta\) of formulas of \(L(Ab')\), a finite subset \(S \subseteq \alpha\) and some \(k < \omega\). Choose with the Lemma 7.8 a finite array \((a_{i,j} : i \in S, l \leq k)\) with mutually \(\Delta\)-indiscernible rows. By Remark 5.5, this array has the same type over \(A\) as \((a_{i} : i \in S, l \leq k)\). Let \(f \in Aut(C/A)\) send the first array to the second one, and let \(\Delta' = f(\Delta)\). Then \((a_{i,j} : i \in S, l \leq k)\) is \(\Delta'\)-indiscernible, \(\Delta'\) is over \(Af(b')\) and \(f(b')_{a_{i0}} \equiv_A b'_{a_{i0}} \equiv_A b_{a_{i0}}\) for every \(i \in S\). By compactness, the rows of \((a_{ij} : i < \alpha, j < \beta)\) are mutually indiscernible over \(Ab''\) for some \(b''\) such that \(b'' \equiv_{A_{a_{i0}}} b\) for every \(i < \alpha\). By conjugation over \(A\) we obtain the array with the required properties over \(Ab\).

Proposition 7.10 The following are equivalent for any partial type \(\pi(x)\) over \(A\) and any cardinal \(\kappa\):

1. There is no inp-pattern of depth \(\kappa\) in \(\pi(x)\).

2. If \(b \models \pi(x)\) and the rows of the array \((a_{ij} : i < \kappa, j < \omega)\) are almost mutually indiscernible over \(A\), then for some \(i < \kappa\) there is an \(Ab\)-indiscernible sequence \((a'_j : j < \omega) \equiv_{A_{a_{i0}}} (a_{ij} : j < \omega)\).
3. If $b \models \pi(x)$ and the rows of the array $(a_{ij} : i < \kappa, j < \omega)$ are mutually indiscernible over $A$, then for some $i < \kappa$ there is an Ab-indiscernible sequence $(a'_{ij} : j < \omega) \equiv_{Aa_{i0}} (a_{ij} : j < \omega)$.

Proof: 1 $\Rightarrow$ 2. Assume $b \models \pi(x)$ and the rows of $(a_{ij} : i < \kappa, j < \omega)$ are almost mutually indiscernible over $A$. Let $p_i(x, y) = \text{tp}(b|_{a_{i0}}/A)$ and let $q_i(x) = \bigcup_{k \subseteq \omega} p_i(x, a_{i_0})$. If each $q_i(x)$ is inconsistent, we can find in each case some $\varphi_i(x, y) \in p_i(x, y)$ and some $k_i < \omega$ such that $\{\varphi_i(x, a_{i_0}) : j < \omega\}$ is $k_i$-inconsistent. Notice that $b \models \{\varphi_i(x, a_{i0}) : i < \kappa\}$. By almost mutual indiscernibility over $A$, $(a_{i0} : i < \kappa) \equiv_A (a_{i_f(j)}) : i < \kappa$ for each $f : \kappa \to \omega$. Hence $\pi(x) \cup \{\varphi_i(x, a_{i_f(j)}) : i < \kappa\}$ is consistent for each such $f$, which shows that we have obtained an inp-pattern of depth $\kappa$ in $\pi(x)$. If, on the contrary, some $q_i(x)$ is consistent, then we can apply Lemma 7.9 to the array consisting in this single row and we obtain some Ab-indiscernible sequence $(a'_{ij} : j < \omega) \equiv_{Aa_{i0}} (a_{ij} : j < \omega)$.

2 $\Rightarrow$ 3. Clear.

3 $\Rightarrow$ 1. Assume there is an inp-pattern of depth $\kappa$ in $\pi(x)$, with array $(a_{ij} : i < \kappa, j < \omega)$, formulas $(\varphi_i(x, y_i) : i < \kappa)$ and numbers $(k_i : i < \kappa)$. By an application of Proposition 5.8 (after extending the width of the array) we can assume that the rows are mutually $A$-indiscernible. Let $b \models \pi(x) \cup \{\varphi(x, a_{i0}) : i < \kappa\}$. Then $\models \varphi(b, a_{i0})$ and $\{\varphi(x, a_{ij}) : j < \omega\}$ is inconsistent. This means that we can not find some $i < \kappa$ as in point 3. □

Proposition 7.11 If there is an inp-pattern of depth $\kappa_1 \cdot \kappa_2$ in $\text{tp}(b_1b_2/A)$, then either there is an inp-pattern of depth $\kappa_1$ in $\text{tp}(b_1/A)$ or there is an inp-pattern of depth $\kappa_2$ in $\text{tp}(b_2/Ab_1)$.

Proof: We assume that there is no inp-pattern of depth $\kappa_1$ in $\text{tp}(b_1/A)$ and there is no inp-pattern of depth $\kappa_2$ in $\text{tp}(b_2/Ab_1)$, and we will apply Proposition 7.10. Let $(a_{i,j,k} : (i, j) \in \kappa_1 \times \kappa_2, k < \omega)$ be an array with mutually $A$-indiscernible rows. We consider the cartesian product $\kappa_1 \times \kappa_2$ lexicographically ordered and we use the notation $a_{ij} = (a(i,j), k : k < \omega)$. We inductively obtain some sequence $a'_{ij} = (a'_{ij} : j < \omega)$ and some ordinal $\alpha_i < \kappa_2$ for each $i < \kappa_1$ in such a way that

1. $a'_{ij}$ is indiscernible over $b_2D_i$, where $D_i = Aa_{(i,\alpha_i)0}$.
2. $a'_{ij} \equiv_{D_i} a_{(i,\alpha_i)}$.
3. The rows of the array $a'_{ij} \cup a_{(i,\alpha_i)}$ are mutually $A$-indiscernible.

By the assumption (in case $i = 0$) and the inductive hypothesis applied to 3 (cases $i$ successor or limit), the rows $(a'_{ij} : j < \kappa_2)$ are mutually indiscernible over $D_i$. Note that there is no inp-pattern of depth $\kappa_2$ in $\text{tp}(b_2/D_i)$. By Proposition 7.10, for some $\alpha_i < \kappa_2$, there is some $D_i, b_2$-indiscernible sequence $a'_{i} \equiv_{D_i} a_{(i,\alpha_i)}$. Clearly, all conditions of the construction are satisfied. It follows that the rows of $(a'_{ij} : i < \kappa_1)$ are almost mutually indiscernible over $Ab_2$. By Proposition 7.10 there is some $i < \kappa_2$ and some $Ab_1, b_2$-indiscernible sequence $(a_j : j < \omega) \equiv_{Ab_1a_{(i,\alpha_i)}}$. Then $(a_j : j < \omega) \equiv_{Ab_1a_{(i,\alpha_i)}}$ and again by Proposition 7.10 we conclude that there is no inp-pattern of depth $\kappa_1 \cdot \kappa_2$ in $\text{tp}(b_1b_2/A)$.

Remark 7.12 If the burden of a type $\pi$ were defined as the least cardinal $\kappa$ for which there is no inp-pattern of depth $\kappa$ in $\pi$, then Proposition 7.11 would read as follows:

$$\text{bdn}(b_1b_2/A) \leq \text{bdn}(b_1/A) \cdot \text{bdn}(b_2/Ab_1)$$

With our official definition we have for successor cardinal numbers $\kappa_1, \kappa_2 < \omega$:

$$\text{bdn}(b_1/A) < \kappa_1 \& \text{bdn}(b_2/Ab_1) < \kappa_2 \Rightarrow \text{bdn}(b_1b_2/A) < \kappa_1 \cdot \kappa_2$$
Corollary 7.13 If \( T \) has TP\(_2\), then this is witnessed by some formula \( \varphi(x, y) \) where \( x \) is a single variable and the number is \( k = 2 \).

**Proof:** If \( T \) is TP\(_2\), then (by Remark 7.5 and Proposition 7.11) for some single element \( b \), for some set \( A \), \( bdn(b/A) = \infty \). Let \( \varphi(x, y) \) witness it. Then \( x \) is a single variable. As shown in the proof of Lemma 7.4, we can then take \( k = 2 \). \( \square \)

8 \( \text{dp-rank} \)

Based on [1], [10], [14] and [11].

**Definition 8.1** An ict-pattern (meaning independent contradictory types pattern) of depth \( \kappa \) in a partial type \( \pi(x) \) is an array \( (a_{ij} : i < \kappa, j < \omega) \) together with formulas \( \varphi_i(x, y_i) \in \mathcal{L} \) for \( i < \kappa \) such that for every \( f : \kappa \rightarrow \omega \), the set

\[
\Gamma_f(x) = \{ \varphi_i(x, a_{f(i)}): i < \kappa \} \cup \{ \neg \varphi_i(x, a_{ij}): i < \kappa, j \neq f(i) \}
\]

is consistent with \( \pi(x) \). The dependence rank (also called the dp-rank) of \( \pi(x) \) is the supremum \( \text{dprk}(\pi) \) of all depths of ict-patterns in \( \pi(x) \). We write \( \text{dprk}(a/A) = \text{dprk}(\text{tp}(a/A)) \).

**Proposition 8.2**

1. \( \text{bdn}(\pi) \leq \text{dprk}(\pi) \).
2. If \( \pi \) is NIP, then \( \text{bdn}(\pi) = \text{dprk}(\pi) < \infty \); otherwise \( \text{dprk}(\pi) = \infty \).

**Proof:**

1. Let \( \pi(x) \) be a partial type over \( A \). We show that we can obtain an ict-pattern of depth \( \kappa \) in \( \pi(x) \) from an inp-pattern of depth \( \kappa \) in \( \pi(x) \). Let \( (\varphi_i(x, y_i) : i < \kappa) \) and \( (k_i : i < \kappa) \) define an inp-pattern in \( \pi(x) \). We can assume that the rows of the array are mutually indiscernible over \( A \). Choose a realization \( b \) of \( \pi(x) \cup \{ \varphi_i(x, a_{0i}) : i < \kappa \} \). If \( i < \kappa \), there are at most \( k_i < \omega \) indexes \( j < \omega \) such that \( \models \varphi_i(b, a_{ij}) \) and we can delete them of the array. Hence we can assume that \( \models \neg \varphi_i(b, a_{ij}) \) for every \( j \neq 0 \). This means that \( \pi(x) \cup \{ \varphi_i(x, a_{0i}) : i < \kappa \} \cup \{ \neg \varphi_i(x, a_{ij}) : i < \kappa, 0 < j < \omega \} \) is consistent.

By mutual indiscernibility over \( A \), the same happens for any other path \( f : \kappa \rightarrow \omega \). Therefore we have an ict-pattern in \( \pi(x) \).

2. Let \( \pi(x) \) be a partial type over \( A \). If \( \pi(x) \) has IP, witnessed by \( \varphi(x, y) \) and \( (a_{ij} : j < \kappa, i \in \omega) \), \( (b_{ij} : J \subseteq \kappa \times \omega) \) such that \( b_{ij} \models \pi(x) \) and \( \models \varphi(b, a_{ij}) \leftrightarrow (i, j) \in J \), then we can obtain an ict-pattern in \( \pi(x) \) defining \( \varphi_i(x, y_i) = \varphi(x, y) \) for all \( i < \kappa, j < \omega \).

Now assume \( \pi(x) \) is NIP and let us check that \( \text{bdn}(\pi) \geq \text{dprk}(\pi) \). Let \( (\varphi_i(x, y_i) : i < \kappa) \), with \( (a_{ij} : i < \kappa, j < \omega) \) define an ict-pattern in \( \pi(x) \). We can assume that the rows of the array are mutually \( A \)-indiscernible. Let \( \psi_i(x; y_1 y_2) = \varphi_i(x, y_1) \land \neg \varphi_i(x, y_2) \) and let \( b_{ij} = a_{i,2j}a_{i,2j+1} \).

Since \( \pi(x) \) is NIP, \( \pi(x) \cup \{ \psi_i(x; b_{ij})) : j < \omega \} \) is inconsistent. Then for some \( \theta_i(x, z_i) \in \mathcal{L} \) for some tuple \( c_i \in A \), for some \( i < \kappa, \theta_i(x, c_i) \in \pi(x) \) and \( \{ \theta_i(x, c_i) \land \psi_i(x, b_{ij}) : j < \omega \} \) is \( k_i \)-inconsistent. Then the formulas \( \chi_i(x; y_1 y_2, z_i) = \psi_i(x; y_1 y_2) \land \theta_i(x, z_i) \) with parameters \( (b_{ij}c_i : i < \kappa, j < \omega) \) and numbers \( (k_i : i < \kappa) \) define an inp-pattern of depth \( \kappa \) in \( \pi(x) \).

Finally, assume that \( \text{dprk}(\pi) = \infty \) and let us show that \( \pi \) has IP. We may find an ict-pattern of length \( \omega \) in \( \pi(x) \) with constant formula \( \varphi_i(x, y_i) = \varphi(x, y) \). Look at the first column \( (a_{0i} : i < \omega) \). Given \( J \subseteq \omega \) choose \( f : \omega \rightarrow \omega \) be such that \( f(i) = 0 \leftrightarrow i \in J \). Since \( \pi(x) \cup \{ \varphi(x, a_{f(i)}): i < \kappa \} \cup \{ \neg \varphi(x, a_{ij}) : i < \kappa, f(i) \neq j < \omega \} \) is consistent, \( \pi(x) \cup \{ \varphi(x, a_{0i}) : i \in J \} \cup \{ \neg \varphi(x, a_{0i}) : i \in \omega \setminus J \} \) is consistent. Hence \( \varphi(x, y) \) and \( \{ a_{0i} : i < \omega \} \) witness that \( \pi(x) \) has IP. \( \square \)
Remark 8.3  
1. $\text{dprk}(\pi(x)) = 0$ iff $\pi(x)$ is algebraic.

2. If $\pi(x) \vdash \pi'(x)$, then $\text{dprk}(\pi(x)) \leq \text{dprk}(\pi'(x))$.

3. $\text{dprk}(\pi(x)) < \infty$ iff $\text{dprk}(\pi(x)) < |T|^+$

Proof:  
1. If $\pi(x)$ is not algebraic we easily find an ict-pattern of depth 1 in $\pi(x)$ using the formula $x = y$. On the other hand, if there is an ict-pattern of depth 1 in $\pi(x)$, it is clear that $\pi(x)$ has infinitely many realizations.

2 is clear.

3. Similar to the proof of 3 of Remark 7.3. ☐

Proposition 8.4 let $\pi(x)$ be a partial type over $A$. The following are equivalent:

1. There is no ict-pattern of depth $\kappa$ in $\pi(x)$.

2. If $(a_{ij} : i < \kappa, j < \omega)$ has mutually $A$-indiscernible rows and $b \models \pi(x)$, then some row $(a_{ij} : j < \omega)$ is $Ab$-indiscernible.

3. If $(a_{ij} : i < \kappa, j < \omega)$ has mutually $A$-indiscernible rows and $b \models \pi(x)$, then for some $i < \kappa$, $a_{ij} \equiv_{Ab} a_{il}$ for all $j, l < \omega$.

Proof:  
3 $\Rightarrow$ 1. Assume there an ict-pattern of depth $\kappa$ in $\pi(x)$, given by formulas $(\varphi_i(x, y_i) : i < \kappa)$ and some array $(a_{ij} : i < \kappa, j < \omega)$. We may assume that the rows are mutually $A$-indiscernible. Let $b \models \pi(x) \cup \{\varphi_0(x, a_{i0}) : i < \kappa\} \cup \{\neg \varphi_i(x, a_{ij}) : i < \kappa, 0 < j < \omega\}. Then a_{i0} and a_{i1} have different type over $Ab$, contradicting 3.

1 $\Rightarrow$ 3. Assume $(a_{ij} : i < \kappa, j < \omega)$ has mutually $A$-indiscernible rows, $b \models \pi(x)$ and in each row there are elements with different type over $Ab$, say for each $i < \kappa$ there are $\varphi_i(x, y_i) \in L(A)$ and $k, l < \omega$ such that $b \models \varphi_i(b, a_{ik}) \land \neg \varphi_i(b, a_{il})$. Adding the parameters of $\varphi_i$ to each element of the row if necessary, we can assume that $\varphi_i(x, y_i) \in L$. By compactness we can find such an array where the rows have the order type of the integers. Hence it is of the form $(a_{ij} : i < \kappa, j \in \mathbb{Z})$. Let us look at some $i < \kappa$. Either $\{j \in \mathbb{Z} : b \models \varphi_i(b, a_{ij})\}$ is cofinal or $\{j \in \mathbb{Z} : b \models \neg \varphi_i(b, a_{ij})\}$ is cofinal and similarly, either $\{j \in \mathbb{Z} : b \models \varphi_i(b, a_{ij})\}$ or $\{j \in \mathbb{Z} : b \models \neg \varphi_i(b, a_{ij})\}$ is cofinal. Deleting some elements of the row we may assume that either $b \models \varphi_i(b, a_{ik}) \land \neg \varphi_i(b, a_{ij})$ for all $j > 0$ or $b \models \varphi_i(b, a_{ik}) \land \neg \varphi_i(b, a_{ij})$ for all $j > 0$; and similarly we may assume that either $b \models \varphi_i(b, a_{ik}) \land \neg \varphi_i(b, a_{ij})$ for all $j < 0$ or $b \models \varphi_i(b, a_{ik}) \land \neg \varphi_i(b, a_{ij})$ for all $j < 0$. There are four possible cases. The first case is that $b \models \varphi_i(b, a_{ik}) \land \neg \varphi_i(b, a_{ij})$ for all $j \neq 0$. In this case we take $\psi_i(x, y_i) = \varphi_i(x, y_i)$. The second case is similar, with $\varphi$ and $\neg \varphi$ switching its roles. Here we take $\psi_i(x, y_i) = \neg \varphi_i(x, y_i)$. In the third case we have $b \models \varphi_i(b, a_{ij})$ for all $j \leq 0$ and $b \models \varphi_i(b, a_{ij})$ for all $j > 0$. Then we replace the row by $(b_{ij} : j \in \mathbb{Z})$ where $b_i = a_{2i+2i+1}$ and we take $\psi_i(x, y_i) = \varphi_i(x, y_i) \land \neg \varphi_i(x, y_i)$. The fourth case is similar (exchanging $\varphi_i$ and $\neg \varphi_i(x, y_i)$) and has a similar solution. In any case, after modifying the row we obtain a formula $\psi_i$ such that $\pi(x) \cup \{\psi_i(x, a_{i0}) : i < \kappa\} \cup \{\neg \psi_i(x, a_{ij}) : i < \kappa, 0 \neq j \in \mathbb{Z}\}$ is consistent. By Remark 5.5 any other path produces a corresponding set also consistent with $\pi(x)$. Hence (after restricting the row to indexes $\geq 0$) this array and these formulas provide an ict-pattern of depth $\kappa$ in $\pi(x)$.

2 $\iff$ 3. In fact we only need to check 3 $\Rightarrow$ 2. Assume we have a counterexample to 2, given by the array $(a_{ij} : i < \kappa, j < \omega)$ and $b \models \pi(x)$. Consider some row $i < \kappa$. For some $n < \omega$ there are $j_0 < \ldots < j_{n-1} < l_0 < \ldots < l_{n-1} < \omega$ such that $a_{ij_0} \ldots a_{ij_{n-1}} \not\equiv_{Ab} a_{il_0} \ldots a_{il_{n-1}}$. Without loss of generality, $j_k = k$ and $l_k = n + k$. We define a new row $(b_{ij} : j < \omega)$ putting $b_{ij} = a_{i,n,j}, \ldots, a_{i,n,j+n-1}$. We do a corresponding modification in
every other row. That way we obtain an array \((b_{ij} : i < \kappa, j < \omega)\) with mutually \(A\)-indiscernible rows. Note that in each row there are two elements with different type over \(Ab\). Hence we have a counterexample to 3.

9 Simple types in \(\text{NTP}_2\) theories

Based on [10].

**Definition 9.1** Recall from [7] that a partial type \(\pi(x)\) is simple if there are not parameters \((b_f : f \in \omega^\omega), (a_s : s \in \omega^{<\omega})\), formula \(\varphi(x, y) \in L\) and number \(< \omega\) such that

1. \(b_f \models \pi(x)\) for all \(f \in \omega^\omega\).
2. \(b_f \models \varphi(x, a_{f\upharpoonright n})\) for all \(f \in \omega^\omega\) for all \(n < \omega\).
3. \(\{\varphi(x, a_{s\upharpoonright n}) : n < \omega\}\) is \(k\)-inconsistent for all \(s \in \omega^{<\omega}\).

The same notion is defined if we allow \(\varphi(x, y) \in L(A)\) since the additional parameters of \(\varphi\) can be added to the nodes \(a_s\) of the tree.

We say that the partial type \(\Sigma(u)\) is co-simple if there are no such objects \(a_s\) where each \(a_s\) is a tuple of realizations of \(\Sigma(u)\) (this replaces condition 1 on \(\pi\)) and \(\varphi(x, y) \in L(A)\). In this case this does not seem to be equivalent to \(\varphi(x, y) \in L\) since adding the new parameters to the branches \(b_f\) is not enough (\(k\)-inconsistency might be lost). Note that \(y\) is a tuple of the form \(u_1 \ldots u_m\) where each \(u_i\) has the length of \(u\).

**Proposition 9.2** The following are equivalent for any partial type \(\pi(x)\) over \(A\):

1. \(\pi(x)\) is simple.
2. \(D(\pi, \Delta, k) < \omega\) for all \(\Delta, k\).
3. For any completion \(p(x) \in S(B)\) of \(\pi(x)\) there is some \(C \subseteq B\) such that \(|C| \leq |T| + |x|\) and \(p(x)\) does not fork over \(C\).
4. If \(A \subseteq C\), \(a \models \pi\) and \(b \downarrow_C a\), then \(a \downarrow_C b\).

**Proof:** See Proposition 7.3 in [7].

**Remark 9.3**

1. Any extension of a simple type is simple.
2. If \(\pi(x, y)\) is simple, then the type \(\exists y\pi(x, y)\) is simple. Therefore, if \(\pi(x, y)\) is a simple partial type over \(A\), then \(\pi \restriction x = \{\varphi(x) \in L(A) : \pi(x, y) \models \varphi(x)\}\) is simple.
3. \(\text{tp}(a/A)\) and \(\text{tp}(b/Aa)\) are simple if and only if \(\text{tp}(ab/A)\) is simple. More generally, \(\text{tp}(a_i/Aa_{<i})\) is simple for all \(i < \alpha\) if and only if \(\text{tp}((a_i : i < \alpha)/A)\) is simple.
4. If the types \(\pi_i(x_i)\) are simple for all \(i \in I\), then \(\bigcup_{i \in I} \pi_i(x_i)\) is simple.
5. If the types \(\pi_1(x), \pi_2(x)\) are simple, then the type \(\pi_1(x) \lor \pi_2(x)\) is simple.

**Proof:** See remarks 7.2 and 7.5 in [7].

**Remark 9.4** Simple types are \(\text{NTP}_2\).
Proposition 9.5 The following are equivalent for any partial type $\pi(x)$ over $A$:

1. $\pi$ is simple.
2. For any completion $p(x) \in S(B)$ of $\pi(x)$ there is some $C \subseteq B$ such that $|C| \leq |T| + |x|$ and $p(x)$ does not divide (fork) over $C$.
3. If $B \supseteq A$, $(b_i : i < \omega)$ is a Morley sequence in $tp(b/B)$ and $\pi(x) \cup \{\varphi(x, b_i) : i < \omega\}$ divides (forks) over $B$, then $\pi(x) \cup \{\varphi(x, b_i) : i < \omega\}$ is inconsistent.
4. If $B \supseteq A$, $a \models \pi$ and $(b_i : i < \lambda)$ is $B$-independent, where $\lambda = \square(2^\omega)$ and $\kappa \geq |B| + |T| + |a| + |b|$, then $a \downarrow_B b_i$ (or $b \downarrow_B b_i$) for some $i < \lambda$.
5. If $A \subseteq C$, $a \models \pi$ and $b \downarrow_C a$, then $a \downarrow_C b$ (or $a \downarrow_C B$).

Proof: 1 $\Rightarrow$ 2. Non forking implies non dividing. The rest is in Proposition 9.2.

2 $\Rightarrow$ 3. Let $\kappa = |T| + |x|$, let $I$ be of order type $(\kappa^+)^*$ (the reverse ordering of $\kappa^+$) and let $(c_i : i \in I)$ be a $B$-indiscernible sequence with the same Ehrenfeucht-Mostowski type over $B$ as $(b_i : i < \omega)$. Then $(c_i : i \in I)$ is Morley over $B$. If $\pi(x) \cup \{\varphi(x, b_i) : i < \omega\}$ is consistent, then $\pi(x) \cup \{\varphi(x, c_i) : i \in I\}$ is consistent too. Let $a$ be a realization of this set of formulas. By 2, there is some $I_0 \subseteq I$ of cardinality $\leq \kappa$ such that $tp(a/B(c_i : i \in I))$ does not divide (fork) over $B(c_i : i \in I_0)$. Choose $j \in I$ such that $j < I_0$. Then $(c_i : i \in I_0) \downarrow y$ (this is due to a version of Lemma 5.14 of [8] for forking independence which holds because forking independence has left transitivity; see Remark 12.15 of [8]). By Proposition 4.9 of [8] and Lemma 4.8 of [7] $\varphi(x, c_j)$ divides (forks) over $B(c_i : i \in I_0)$, which is a contradiction since $\models \varphi(a, c_j)$.

3 $\Rightarrow$ 4. Assume $a \downarrow_B b_i$ for all $i < \lambda$. By the choice of $\lambda$ we can extract a $Ba$-indiscernible sequence $(b'_i : i < \lambda)$ which is locally like $(b_i : i < \lambda)$ over $Ba$. Then it is a Morley sequence over $B$ and there is some $\varphi(x, y) \in L(B)$ such that $\models \varphi(a, b'_i)$ and $\varphi(x, b'_i)$ divides over $B$ for all $i < \omega$. This contradicts 3. Similar for the forking version.

4 $\Rightarrow$ 5. Since $b \downarrow_C a$, there is a long Morley sequence $(b_i : i < \lambda)$ in $tp(b/Ca)$ which is independent over $C$ and $b = b_0$. By 4, $a \downarrow_C b_i$ for some $i < \lambda$ and by $Ba$-indiscernibility, $a \downarrow_C b_i$. Similar for forking.

5 $\Rightarrow$ 1. Like the proof of the corresponding implication in Proposition 7.3 of [7].

Remark 9.6 Let $\pi(u)$ be a co-simple partial type over $A$. If each $b \in B$ is a tuple of realizations of $\pi(u)$, then any extension $\pi'(u)$ of $\pi(u)$ over $B$ is co-simple.

Proposition 9.7 Let $\pi(u)$ be a partial type over $A$, $\varphi(x, y) \in L(A)$ and $k < \omega$, where $y$ is a tuple $u_1, \ldots, u_m$ and each $u_i$ is a tuple of the length of $u$. The following are equivalent:
1. $\pi(u)$ is not co-simple, witnessed by $\varphi(x,y)$ and $k$.

2. There is a sequence $(a_i : i < \omega)$ where each $a_i$ is a tuple of realizations of $\pi(u)$ and $\varphi(x,a_i)$ divides over $Aa_{<i}$ with respect to $k$ for all $i < \omega$ and $\{\varphi(x,a_i) : i < \omega\}$ is consistent.

3. There is a sequence $(a_i : i < \omega)$ where each $a_i$ is a tuple of realizations of $\pi(u)$ and there is some $b$ such that $|= \varphi(b,a_i)$ for all $i < \omega$, $(a_i : i < \omega)$ is $A$-$b$-indiscernible and $\varphi(x,a_i)$ divides over $Aa_{<i}$ with respect to $k$ for all $i < \omega$.

**Proof:** See the proof of Proposition 12.23 in [8].

**Proposition 9.8** The following are equivalent for any partial type $\pi(u)$ over $A$:

1. $\pi(u)$ is co-simple.

2. If $B$ is a set of realizations of $\pi(u)$ and $q(x) \in S(AB)$, then for some $B' \subseteq B$ of cardinality $\leq |A| + |T| + |u|$, $q(x)$ does not divide (forks) over $AB'$.

3. If $B$ is a set of realizations of $\pi(u)$, $(a_i : i < \omega)$ is a Morley sequence over $AB$, every $a_i$ is a tuple of realizations of $\pi(u)$ and $\varphi(x,a_0)$ divides (forks) over $AB$, then $\{\varphi(x,a_i) : i < \omega\}$ is inconsistent.

4. If $B$ is a set of realizations of $\pi(u)$, and $(a_i : i < \lambda)$ is an $AB$-independent sequence of tuples of realizations of $\pi(u)$, where $\lambda = \beth_2^{\kappa+}$ and $\kappa \geq |AB| + |a_0| + |b|$, then $b \downarrow_{AB} a_i (b \downarrow_{AB} a_i)$ for some $i < \lambda$.

5. If $B$ is a set of realizations of $\pi(u)$, $a$ is a tuple of realizations of $\pi$ and $a \downarrow_{AB} b$, then $b \downarrow_{AB} a (b \downarrow_{AB} a)$.

**Proof:** Like the proof of Proposition 9.5. For $1 \Rightarrow 2$ use also Proposition 9.7.

**Proposition 9.9** Let $\pi(x)$ be a simple partial type over $A$ and assume the chain condition over $A$ holds with respect to $\pi(x)$. Let $(a_i : i < \omega)$ be a Morley sequence in $\pi(x)$ over $A$ and for each $i < \omega$ let $\bar{a}_i$ be an indiscernible sequence over $A$ of length $\omega$ starting with $a_i$. Then we can find $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$ for each $i < \omega$ such that the array $(\bar{b}_i : i < \omega)$ has mutually $A$-indiscernible rows.

**Proof:** By Lemma 7.7 it suffices to find such array with almost mutually $A$-indiscernible rows. We construct the sequences $\bar{b}_i$ by induction on $i$ in such a way that

1. $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$
2. $\bar{b}_i$ is indiscernible over $A\bar{b}_{<i}a_{>i}$
3. $\bar{b}_{<i}a_{>i+1} \downarrow_{A} a_{i+1}$
4. $a_{>i+1} \downarrow_{A} \bar{b}_{<i}$.

Since (by 1) the first element of $\bar{b}_i$ is $a_i$, 2 implies that the rows will be almost mutually $A$-indiscernible. We use the chain condition in the form of point 6 of Proposition 4.2. Note that simplicity of $\pi(x)$ allows us to use symmetry and transitivity of independence when working with tuples composed of realizations of $\pi$. Hence 3 follows from 4 (and the fact that $a_{i+1} \downarrow_{A} a_{>i+1}$, which holds because the sequence is Morley over $A$).
We start noticing that $a_{>0} \downarrow_A a_0$ and then the chain condition gives us some $\bar{b}_0 \equiv_{Aa_0} \pi_0$ which is indiscernible over $Aa_{>0}$ and such that $a_{>0} \downarrow_A \bar{b}_0$ (which is condition 4 for $i = 0$).

Now assume inductively 4 and 3 for $i - 1$, namely, $a_{>1} \downarrow_A \bar{b}_{<1}$ and $\bar{b}_{<1} \downarrow_A a_{>1}$. With the chain condition we choose now $\bar{b}_i \equiv_{Aa_{>1}} \pi_i$ indiscernible over $A\bar{b}_{<1}a_{>1}$ and such that $\bar{b}_{<1}a_{>1} \downarrow_A \bar{b}_i$. Then $a_{>1} \downarrow_A \bar{a}_0 \bar{b}_i$ and this together with the induction hypothesis gives 4:

$$a_{>i} \downarrow_A \bar{b}_{<i}.$$  

\[ \Box \]

**Proposition 9.10** Let $\pi(x)$ be a simple partial type over $A$ and $a \models \pi$. If the chain condition holds over $A$ with respect to $\pi(x)$ and $tp(b/A)$ is NTP$_2$, then $a \downarrow_A b$ implies $b \downarrow_A a$.

**Proof:** Since $a \downarrow_A b$, there is a Morley sequence $(a_i : i < \omega)$ over $Ab$ in $tp(a/Ab)$ which is also Morley over $A$ (see Lemma 5.11 in [8]). We can assume $a = a_0$. Assume $\varphi(x, y) \in L(A)$, $A \models \varphi(b, a)$ and $\varphi(x, a)$ divides over $A$ and let $\pi_0 = (a^0_i : i < \omega)$ be an $A$-indiscernible sequence witnessing it, namely, $\{\varphi(x, a^0_i) : i < \omega\}$ is inconsistent and $a = a^0_0$. Hence $\{\varphi(x, a^0_i) : i < \omega\}$ is $k$-inconsistent for some $k < \omega$. Choose $\pi_i$ such that $\pi_i a_i \equiv_A \pi_0 a_0$. By Proposition 9.9 we may assume that the rows of the array $(\pi_i : i < \omega)$ are mutually $A$-indiscernible. Notice that $b \models \{\varphi(x, a_i) : i < \omega\}$ and hence the first column $\{\varphi(x, a_i) : i < \omega\}$ is consistent with $tp(b/A)$. By Remark 5.5 all paths give similar consistent sets of formulas. Since each row is $k$-consistent, this array shows that $tp(b/A)$ has TP$_2$.  

\[ \Box \]

**Proposition 9.11** Let $\pi(x)$ be a simple partial type over $A$.

1. For any $\varphi(x, y) \in L(A)$, for any tuple $a$ such that $a \downarrow_A A, \pi(x) \cup \{\varphi(x, a)\}$ forks over $A$ iff it divides over $A$.

2. If $A$ is an extension base, the chain condition holds over $A$ with respect to $\pi(x)$.

**Proof:**

1. Assume $\pi(x) \cup \{\varphi(x, a)\}$ forks over $A$. Since $a \downarrow_A A$, there is a Morley sequence $(a_i : i < \omega)$ in $tp(a/A)$. By point 3 of Proposition 9.5 $\pi(x) \cup \{\varphi(x, a_i) : i < \omega\}$ is inconsistent. This implies that $\pi(x) \cup \{\varphi(x, a)\}$ divides over $A$.

2. By 1 and Proposition 6.11 $\pi(x) \cup \{\varphi(x, a)\}$ forks over $A$ iff it array-divides over $A$. Hence we can easily adapt the proof of Proposition 6.6 (choosing a Morley sequence as universal indiscernible sequence).

\[ \Box \]

**Proposition 9.12** Let $\pi(u)$ be a partial type over $A$. If $\pi(u)$ is not co-simple, then for some model $M \supseteq A$, for some tuple $a$ of realizations of $\pi(u)$, for some $b$, $tp(a/Mb)$ coherits from $M$ but $b \not\downarrow^d_M a$.

**Proof:** Fix some formula $\varphi(x, y)$ witnessing the failure of co-simplicity. Let $T^{sk}$ be some skolemization of $T$. Then $\varphi(x, y)$ witnesses the failure of co-simplicity of $\pi(u)$ in $T^{sk}$. By point 3 of Proposition 9.7, after extending the indiscernible sequence to length $\omega + 1$, in $T^{sk}$ there are a sequence $(a_i : i \leq \omega)$ of tuples $a_i$ of realizations of $\pi$, some $b$ and some number $k < \omega$, such that $\models \varphi(b, a_i)$ for all $i < \omega$, $(a_i : i \leq \omega)$ is $Ab$-indiscernible and $\varphi(x, a_i)$ divides over $Aa_{<i}$ with respect to $k$ for all $i \leq \omega$. Let $M$ be the Skolem hull of $Aa_{<\omega}$. By indiscernibility, $tp(a_{<\omega}/Mb)$ is a coheir of $tp(a_{<\omega}/Mb)$ in $T^{sk}$ and in $T$. By construction $b \not\downarrow^d_A a_{<\omega}$, witnessed in $T^{sk}$ by $\varphi(x, y) \in L$. Since $M$ is definable over $Aa_{<\omega}$, $\varphi(x, y)$ also witnesses in $T^{sk}$ that $b \not\downarrow^d_M a_{<\omega}$. Since $\varphi(x, y) \in L$ we have $b \not\downarrow^d_M a_{<\omega}$ also in $T$.  

\[ \Box \]

**Corollary 9.13** Let $\pi(x)$ be a simple type over the extension base $A$ and let $tp(b/A)$ be NTP$_2$. If $a \models \pi(x)$, then: $a \downarrow_A b$ iff $b \downarrow_A a$.  

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**Proof:** By propositions 9.11, 9.10 and 9.5.

**Corollary 9.14** Let $T$ be NTP$_2$.

1. Simple types are co-simple.

2. If $\pi(x)$ is a simple partial type over $A$ and $a \models \pi(x)$, then for every $b$: $a \downarrow_A b$ iff $b \downarrow_A a$.

3. If $\pi(x)$ is a simple partial type over $A$ and $a \models \pi(x)$, then for every $B, C$: if $a \downarrow_A B$ and $a \downarrow_{AB} C$, then $a \downarrow_A BC$.

**Proof:**

1. By Proposition 9.12 and Proposition 9.10, since in a NTP$_2$ theory the chain condition holds over models.

2. By 1, and propositions 9.2 and 9.8.


**Remark 9.15** We could define the notion of co-NTP$_2$ type similarly to the way co-simple types were defined, requiring that the parameters of the array realize the type (or perhaps are tuples of realizations of the type). Then the proof of Proposition 9.10 would give:

Let $\pi(x)$ be a simple co-NTP$_2$ partial type over $A$ and $a \models \pi$. If the chain condition holds over $A$ with respect to $\pi(x)$, then $a \downarrow_A b$ implies $b \downarrow_A a$.

Moreover parallel to Corollary 9.14 we would have:

1. Simple co-NTP$_2$ types are co-simple.

2. If $\pi(x)$ is a simple co-NTP$_2$ partial type over $A$ and $a \models \pi(x)$, then for every $b$: $a \downarrow_A b$ iff $b \downarrow_A a$.

3. If $\pi(x)$ is a simple co-NTP$_2$ partial type over $A$ and $a \models \pi(x)$, then for every $B, C$: if $a \downarrow_A B$ and $a \downarrow_{AB} C$, then $a \downarrow_A BC$.

The question is then whether simple types are co-NTP$_2$.

## 10 The triangle-free random graph

**Definition 10.1** The theory of the triangle-free random graph is the theory of an irreflexive, symmetric binary relation $R$ without triangles and such that for any disjoint finite sets $A, B$ if no $R$-relations holds between elements of $A$ ( $A$ is an anti-clique), then there is some $c \not\in A \cup B$ such $cRa$ for all $a \in A$ and $\neg cRb$ for all $b \in B$. This can be written as a set of first-order axioms. In this section $T$ will be this theory. We will use here $a, b, c, x$ for elements (points, singletons) and $\bar{a}, \bar{b}, \bar{c}, \bar{x}$ for (possibly infinite) tuples. The following notation will be also convenient: $R_a(A)$ is the set of all $b \in A$ such that $R(a, b)$.

**Remark 10.2** $T$ is complete, is $\omega$-categorical and has quantifier elimination.

**Proof:** We do a back-and-forth between models $M, N$ of $T$. The set of partial isomorphism is the set of all isomorphisms between finite substructures. It is clearly non empty. Assume $f$ is such an isomorphism and let $a \in M$ be a new element. Let $A \cup B$ be the domain of $f$,
where \(a\) is \(R\)-related to all elements in \(A\) and to none in \(B\). Let \(A' = f(A)\) and \(B' = f(B)\). \(A'\) is an anti-clique. By the axioms of \(T\) there is some \(b \in N\) which is \(R\)-related to all elements of \(A'\) and to none of \(B'\) and it is not in the range of \(f\). The mapping \(f \cup \{(a, b)\}\) works.

\[\]
Proposition 10.9 In \( T \downarrow_A = \downarrow^d_A \).

**Proof:** It is enough to check that dividing has the extension property. Let \( \bar{a} = a_1, \ldots, a_n \) and assume \( A \subseteq B \subseteq C \) and \( \bar{a} \downarrow^d_A B \). We want to find some \( \bar{a}' \equiv_B \bar{a} \) such that \( \bar{a}' \downarrow_A C \). We can assume that \( a_i \notin B \) for all \( i = 1, \ldots, n \). Let \( p(x) = tp(\bar{a}/B) \) and let \( \Sigma(x) = p(x) \cup \{ x_i \neq c \land \neg R(x_i, c) : c \in C \setminus B, 1 \leq i \leq n \} \). It is consistent and if \( \bar{a}' \models \Sigma(x) \) then \( \bar{a}' \downarrow_A C \).

Proposition 10.10 Let \( p(x) \in S(A) \) be non algebraic and such that for some \( a \in A \), \( R(x, a) \in p(x) \):

1. If \( B \) is a set of realizations of \( p(x) \), then \( A \downarrow_C B \) if \( AC \cap BC = C \).
2. \( p(x) \) is co-simple.

**Proof:**

2. By 1 and point 4 of Proposition 9.8.

11 Strict nonforking

Based on [12].

Recall from [7] the following definition and facts:

**Definition 11.1**

1. A global type \( p(x) \) is Lascar invariant over \( C \) iff for every \( \phi(x, y) \in L \), for all tuples \( a \equiv_C b \), if \( \phi(x, a) \in p \) then \( \phi(x, b) \in p \).
2. \( a \downarrow^i_c b \) iff for some global type \( p(x) \supseteq tp(a/Cb) \), \( p(x) \) is Lascar invariant over \( C \).
3. \( a \downarrow^{\text{ist}}_A b \) iff for some global type \( p(x) \supseteq tp(a/Cb) \), \( p(x) \) is Lascar invariant over \( C \) and for every \( B \supseteq Cb \), for every \( c \models p(x) \upharpoonright B, B \downarrow_C c \).
4. A sequence \( (a_i : i < \alpha) \) is strictly invariant over \( A \) if for all \( i < \alpha \), \( a_i \downarrow_A a_{<i} \).

**Fact 11.2**

1. \( \downarrow^i \) is a preindependence relation with extension, right-normality, anti-reflexivity and algebraicity properties.
2. \( a \downarrow^i_A b \) implies \( a \downarrow_A b \).
3. If \( a \downarrow^i_A I \) and \( I \) is an \( A \)-indiscernible sequence, then \( I \) is \( Aa \)-indiscernible.
4. \( \downarrow^{\text{ist}} \) is invariant, monotone and has the extension property.

**Definition 11.3** A sequence \( (a_i : i \in I) \) is strictly independent over \( A \) if for all \( (\bar{a}_i : i \in I) \) where each \( \bar{a}_i \) is an \( A \)-indiscernible sequence, there is a mutually \( A \)-indiscernible sequence \( (\bar{b}_i : i \in I) \) such that \( \bar{b}_i \equiv_A a_i \) for all \( i \in I \).

**Remark 11.4** By propositions 9.9 and 9.11, if \( (a_i : i < \alpha) \) is a Morley sequence of realizations of a simple type over an extension base \( A \), then \( (a_i : i < \alpha) \) is strictly independent.

**Lemma 11.5** (Shelah’s lemma) Any strictly invariant sequence is strictly independent.
Proof: By compactness, it is enough to check this for a sequence \( (a_i : i < n) \) where \( n < \omega \). By Lemma 7.7 it is enough to obtain an array with almost mutually indiscernible rows. The proof is by complete induction on \( n \). Assume we have obtained an array \( (b_i : i < n) \) with almost mutually \( A \)-indiscernible rows and such that \( b_i \equiv A_{a_i} a_i \) for all \( i < n \). Since \( a_n \downarrow a_{<n} \), we may assume that \( a_n \downarrow b_{<n} \). Hence \( b_{<n} \downarrow a_n \) and therefore there is some \( b_n \equiv A_{a_n} a_n \) which is \( A_{b_{<n}} \)-indiscernible. If \( i < n \), then \( b_i \) is indiscernible over \( Ab_{<i} a_{<n} \) and \( a_n \downarrow a_{<n} \) \( b_i \) and therefore \( b_i \) is \( A_{b_{<n}} \)-indiscernible.

Lemma 11.6 Let \( T \) be NTP\(_2\) and \( (\varphi_i(x, y_i) : i < |T|^+) \) a sequence of L(A)-formulas. If \( (a_i : i < |T|^+) \) is a strictly A-independent sequence and each \( \varphi_i(x, a_i) \) divides over A, then \( \{\varphi_i(x, a_i) : i < |T|^+\} \) is inconsistent. Proof: Choose for every \( i < |T|^+ \) an A-indiscernible sequence \( \bar{a}_i = (a_{ij} : j < \omega) \) starting with \( a_{00} = a_i \) and such that \( \{\varphi_i(x, a_{ij}) : j < \omega\} \) is inconsistent, hence \( k_i \)-inconsistent for some \( k_i < \omega \). Some formula \( \varphi_i \) and some number \( k_i \) occur jointly \( |T|^+ \) times, so we may assume \( \varphi_i = \varphi(x, y) \) and \( k_i = k \) for all \( i \). Since the sequence is strictly \( A \)-independent, we may assume that the rows of the array \( (a_{ij} : i < |T|^+, j < \omega) \) are mutually \( A \)-indiscernible. If we assume that \( \{\varphi_i(x, a_i) : i < |T|^+\} \) is consistent, then \( \{\varphi(x, a_j(a_i)) : i < |T|^+\} \) is consistent for every \( f : |T|^+ \to \omega \). This shows that \( T \) has TP\(_2\). □

Definition 11.7 An infinite sequence \( (a_i : i \in I) \) is a witness of \( a \) over \( A \) if \( a \equiv A a_i \) for all \( i \) and for every \( \varphi(x, y) \in L(A) \) such that \( \varphi(x, a) \) divides over \( A \), for every infinite countable \( J \subseteq I \), \( \{\varphi(x, a_i) : i \in J\} \) is inconsistent. If the sequence is \( A \)-indiscernible, this is equivalent to \( \{\varphi(x, a_i) : i \in I\} \) being inconsistent and in this case this coincides with the notion of universal indiscernible sequence introduced earlier.

Proposition 11.8 Let \( T \) be NTP\(_2\). Any infinite strictly \( A \)-independent sequence is a witness over \( A \).

Proof: Let \( (a_i : i \in I) \) be strictly independent over \( A \), and assume \( a_i \equiv A a_j \) for all \( i, j \in I \). We can assume \( 0 \in I \). Assume \( \varphi(x, y) \in L(A) \) and \( \varphi(x, a_0) \) divides over \( A \). Choose some \( A \)-indiscernible sequence \( (a_0 : j < \omega) \) starting with \( a_0 = a_00 \) and such that \( \{\varphi(x, a_0) : j < \omega\} \) is inconsistent, hence \( k \)-inconsistent for some \( k < \omega \). For any other \( i \in I \) choose \( \bar{a}_i = (a_{ij} : j < \omega) \equiv A (a_0 : j < \omega) \) with \( a_0 = a_0 \). Clearly \( \{\varphi(x, a_i) : j < \omega\} \) is \( k \)-inconsistent. By strict \( A \)-independence, we may assume the rows of the array \( (a_{ij} : i \in I, j < \omega) \) are mutually indiscernible over \( A \). If \( (a_i : i \in I) \) is not a witness for \( \varphi(x, a) \), then \( \{\varphi(x, a_i) : i \in J\} \) is consistent for some (infinite) countable \( J \subseteq I \). As in the proof of Lemma 11.6, this contradicts NTP\(_2\) of \( T \). □

Corollary 11.9 In a NTP\(_2\) theory, if \( A \) is an extension base for \( \bigcup \downarrow \) then forking over \( A \) coincides with dividing over \( A \).

Proof: Assume \( A \) is an extension base for \( \bigcup \downarrow \), \( \varphi(x, y) \in L \), \( \varphi(x, y) \vdash \varphi_1(x, a_1) \lor \ldots \lor \varphi_n(x, a_n) \) and each \( \varphi_i(x, a_i) \) divides over \( A \). We will show that \( \varphi(x, b) \) divides over \( A \). Let \( \bar{a} = ba_1 \ldots a_n \) and let \( (a_i : i < \omega) \) be a strictly \( A \)-invariant sequence starting with \( a_0 = \bar{a} \). Write \( a_i = b_i a_1 a_2 \ldots a_i \). Since \( (a_i : i < \omega) \) is a strictly \( A \)-invariant sequence, by Proposition 11.8 it is a witness of \( a_j \) over \( A \) and hence \( \{\varphi_j(x, a_j) : i < \omega\} \) is inconsistent. We claim that \( \{\varphi(x, b) : i < \omega\} \) is inconsistent. Assume not and let \( c \) realize this type. For some \( j, c \) realizes infinitely \( \varphi_j(x, a_j) \), a contradiction. □

The previous result was exposed in Proposition 4.5 of [7] with a different proof. Proposition 4.9 of [7] shows the same for the case that \( A \) is an extension base for \( \bigcup \).
Fact 11.10 1. If $T$ is NTP$_2$, then $A$ is an extension base for $\downarrow$ iff it is an extension base for $\downarrow^\text{ist}$.

2. If $T$ is NIP, then $A$ is an extension base for $\downarrow$ iff it is an extension base for $\downarrow^\text{ist}$.

Proof: See Proposition 4.11 and Corollary 4.12 in [7].

Proposition 11.11 The following are equivalent.

1. $a \downarrow^\text{ist} b$.
2. For every $c$ there is some $c' \equiv_A b c$ such that $a \downarrow_A b c'$ and $b c' \downarrow_A a$.
3. For every $c$ there is some $a' \equiv_A a$ such that $a' \downarrow_A b c$ and $b c \downarrow a'$.
4. If $p(x) = \text{tp}(a/Ab)$, the following is consistent:

$$p(x) \cup \{\varphi(x,c) \iff \varphi(x,c') : c \equiv_A c', \varphi(x,y) \in L(A), c,c' \in \mathfrak{C}) \cup \{\neg \varphi(x,b,c) : \varphi(x,y,z) \in L(A), c \in \mathfrak{C} \text{ and } \varphi(a,y,z) \text{ forks over } A\}$$

Proof: By automorphism, 2 $\iff$ 3.

1 $\Rightarrow$ 2. Fix the global extension $p(x) \supseteq \text{tp}(a/Ab)$ and take $a' \models p \upharpoonright Abc$.

3 $\Rightarrow$ 4. Consider a finite fragment $\pi(x)$ of the partial type and let $c$ be a tuple containing all parameters of $\pi(x)$. The tuple $a'$ given by 3 realizes $\pi(x)$.

4 $\Rightarrow$ 1. Extend the partial type to a complete global type.

Definition 11.12 Strict nonforking is defined in the same way as strict invariance except that we require only the global extension $p(x) \supseteq \text{tp}(a/Cb)$ does not fork over $C$ instead of requiring that is Lascar invariant over $C$. We write $a \downarrow^\text{snf} b$ for this. Note that $\downarrow^\text{snf} = \downarrow^\text{ist}$ if $T$ is NIP.

Remark 11.13 1. $a \downarrow^\text{ist} b$ implies $a \downarrow^\text{snf} b$.

2. $\downarrow^\text{snf}$ has the extension property.

3. There is a characterization of $a \downarrow^\text{snf} b$ as in Proposition 11.11, exchanging $\downarrow^\text{ist}$ with $\downarrow^\text{snf}$ and defining the set of formulas as

$$p(x) \cup \{\neg \varphi(x,b,c) : \varphi(x,y,z) \in L(A), c \in \mathfrak{C} \text{ and } \varphi(x,b,c) \text{ forks over } A\} \cup \{\neg \varphi(x,b,c) : \varphi(x,y,z) \in L(A), c \in \mathfrak{C} \text{ and } \varphi(a,y,z) \text{ forks over } A\}$$

Lemma 11.14 $a \downarrow^\text{snf} b$ iff for each $p(z) \in S(\text{Ab})$ the following is consistent:

$$p(z) \cup \{\neg \varphi(a,b,z) : \varphi(x,y,z) \in L(A) \text{ and } \varphi(x,b,c) \text{ forks over } A \text{ for some (all) } c \models p\} \cup \{\neg \varphi(a,b,z) : \varphi(x,y,z) \in L(A) \text{ and } \varphi(a,y,z) \text{ forks over } A\}$$

Proposition 11.15 1. If $(b_i : i < \alpha)$ is a witness of $b = b_0$ over $A$ and all $b_i$ have the same type over $Aa$, then $a \downarrow_A d b$. 

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2. Assume forking and dividing over $A$ coincide. If $\bar{a} = (a_i : i < \alpha)$ is a witness of $a = a_0 \overline{\downarrow}_A b$, and all $a_i$ have the same type over $Ab$, then $a \downarrow_{\text{snf}} b$.

3. Assume forking and dividing over $A$ coincide. If $\bar{a} = (a_i : i < \alpha)$ is an $A$-indiscernible witness of $a = a_0 \overline{\downarrow}_A b$, and $b \downarrow_{\text{snf}} \bar{a}$, then $a \downarrow_{\text{snf}} b$.

**Proof:** 1. Assume $\models \varphi(a,b)$ and $\varphi(x,b)$ divides over $A$. Then $\models \varphi(a,b)$ for all $i < \alpha$, contradicting inconsistency of an infinite subset of $\{\varphi(x,b_i) : i < \alpha\}$.

2. Let $p(z) \in S(\bar{A}b)$. By Lemma 11.14, we must check the consistency of $p(z) \cup \{\neg \varphi(a,b,z) : \varphi(x,y,z) \in L(A)\text{ and } \varphi(x,b,c)\text{ forks over } A\}$ for some $c \models p$. If it is inconsistent, then $p(z) \models \varphi(a,b,z) \lor \psi(a,b,z)$ where $\varphi(x,y,z)$, $\psi(x,y,z) \in L(A)$ and $\varphi(x,b,c), \psi(a,y,z)$ fork and divide over $A$ and $c \models p$. Since $\bar{a}$ is a witness, $\{\psi(a_i,y,z) : i \in I\}$ is consistent and so is $\{\psi(a_i,b,z) : i \in I\}$ for some infinite countable $I \subseteq \alpha$. To simplify notation, assume $I = \omega$. Since $a_i \equiv_{\bar{A}b} a$, $p(z) \models \varphi(a_i,b,z) \lor \psi(a_i,b,z)$ for some $i < \omega$. Since $\bar{a} \downarrow_{\bar{A}} b$, there is some $c' \models p$ such that $\bar{a} \downarrow_{\bar{A}} b c'$. Then $\models \varphi(a_i,b,c')$ for some $i \leq n$. This shows that $\varphi(x,b,c')$ does not fork over $A$. Hence $\varphi(x,b,c)$ does not fork over $A$, a contradiction.

3. By 2, since $b \downarrow_{\text{snf}} \bar{a}$ implies that $\bar{a}$ is also indiscernible over $A$.

**Proposition 11.16** If $T$ is NTP$_2$ and $A$ is an extension base, then $1 \Rightarrow 2 \Rightarrow 3$. If, moreover, $T$ is NIP they are all equivalent and equivalent to $b \downarrow_{\text{snf}} a$.

1. $a \downarrow_{\text{snf}} b$

2. $(a,b)$ is a strictly $A$-independent sequence.

3. $b \downarrow_{\text{snf}} a$

**Proof:** 1 $\Rightarrow$ 2. By Shelah’s Lemma 11.5.

2 $\Rightarrow$ 3. Since $T$ is NTP$_2$, $A$ is an extension base for $\downarrow_{\text{snf}}$ and we can find strictly $A$-invariant $A$-indiscernible infinite sequences $\bar{a}, \bar{b}$ starting with $a$ and with $b$ respectively. Since $(a,b)$ is strictly $A$-independent, we may assume that $\bar{a}$ is $A\bar{a}$-indiscernible and $\bar{b}$ is $A\bar{a}$-indiscernible. By Proposition 11.8 $\bar{a}$ and $\bar{b}$ are witnesses over $A$. By item 1 of Proposition 11.15, $\bar{b} \downarrow_{\bar{A}} a$. Since forking and dividing over $A$ coincide, $b \downarrow_{\bar{A}} a$. By item 2 of Proposition 11.15, $b \downarrow_{\bar{A}} a$.

If $T$ is NIP then $\downarrow_{\text{snf}} = \downarrow_{\text{snf}}$ and hence $b \downarrow_{\text{snf}} a \Rightarrow b \downarrow_{\text{snf}} a \Rightarrow b \downarrow_{\text{snf}} a \Rightarrow a \downarrow_{\text{snf}} b$. □

### 12 Resilient theories

Based on [4] and [12].

**Definition 12.1** Let $S_{EM}^E(A) = \{\text{tp}(\bar{a}/A) : \bar{a} \text{ is an } A\text{-indiscernible } \omega\text{-sequence }\}$. The length of the tuples $a_i$ in all the sequences $\bar{a}$ is the length of $x$. For $p \in S_{EM}^E(A)$, let $\text{cl}^\text{div}(p) = \{\varphi(x,y) \in L(A) : \text{ for some (all) } (a_i : i < \omega) \models p, \{\varphi(a_i,y) : i < \omega\} \text{ is consistent }\}$. For $p, q \in S_{EM}^E(A)$ we define

1. $p \sim_{\text{div}} q$ iff $\text{cl}^\text{div}(p) = \text{cl}^\text{div}(q)$
2. \( p \leq_{\text{div}} q \iff \cl_{A}^{\text{div}}(p) \supseteq \cl_{A}^{\text{div}}(q) \).

**Definition 12.2** Let \( p(x) \in S_x(M) \), where \( A \subseteq M \). The class \( p \) over \( A \) in the fundamental order is defined as \( \cl_{A}^{\text{fund}}(p) = \{ \varphi(x,y) \in L(A) : \varphi(x,m) \in p \text{ for some tuple } m \in M \} \). For \( p(x) \in S_x(M) \), \( q(x) \in S_x(N) \), where \( A \subseteq M \cap N \) we define the fundamental order over \( A \) by

1. \( p \overset{\text{fund}}{\prec}_{A} q \iff \cl_{A}^{\text{fund}}(p) = \cl_{A}^{\text{fund}}(q) \)
2. \( p \overset{\text{fund}}{\preceq}_{A} q \iff \cl_{A}^{\text{fund}}(p) \supseteq \cl_{A}^{\text{fund}}(q) \).

\( O_{x}(A) \) is the class of all types \( p(x) \) over models \( M \supseteq A \) modulo \( \overset{\text{fund}}{\sim}_{A} \). We endow \( O_{x}(A) \) with the induced partial order \( \leq_{A} \).

**Fact 12.3** Let \( T \) be stable, \( A \subseteq M \cap N \), \( p(x) \in S(M) \) and \( q(x) \in S(N) \). Then \( p \overset{\text{fund}}{\sim}_{A} q \iff (M, dp) \equiv_{A} (N, dq) \), where \( (M, dp) \) is the expansion of \( M \) obtained after adding for each \( \varphi(x; x_1, \ldots, x_n) \in L(A) \) some \( n \)-ary relation symbol \( R_{\varphi} \) for the relation \( \{ (a_1, \ldots, a_n) \in M^n : \varphi(x; a_1, \ldots, a_n) \in p \} \) (and similarly for \( N \) and \( q \)).

**Proposition 12.4** Let \( T \) be stable.

1. \( (S_x^{EM}(A)/\sim_{\text{div}}, \leq_{\text{div}}) \cong (O_{x}(A), \leq_{A}) \).
2. For any \( p,q \in S_x^{EM}(A) : p \sim_{\text{div}} q \iff p = q \).

**Proof:** Without loss of generality, \( A = \emptyset \).

1. Let \( p(x) \in S_x(M) \). Choose a Morley sequence \( \bar{a} = (a_i : i < \omega) \) in \( p(x) \) over \( M \) and let \( P = \text{tp}(\bar{a}) \in S_x^{EM} \). We claim that \( \varphi(x,y) \in \cl_{A}^{\text{fund}}(p) \iff \varphi(a_0, y) \) does not fork over \( M \) iff \( \varphi(x,y) \in \cl_{A}^{\text{div}}(P) \). On the one hand, any Morley sequence in a stable theory is a witness and therefore \( \varphi(a_0, y) \) does not fork over \( M \) iff \( \{ \varphi(a_i, y) : i < \omega \} \) is consistent. On the other hand, it is straightforward that if \( \varphi(x,y) \in \cl_{A}^{\text{fund}}(p) \) then \( \models \varphi(a_0, b) \) for some \( b \in M \) and therefore \( \varphi(a_0, b) \) does not fork over \( M \). Finally, if \( \varphi(a_0, y) \) does not fork over \( M \), then \( \models \varphi(a_0, b) \) for some \( b \downarrow M_{a_0} \). If \( p \) is the global nonforking extension of \( p \) then \( \varphi(x,y) \in \cl_{A}^{\text{fund}}(p) \) and since \( p \) is the heir of \( p, \cl_{A}^{\text{fund}}(p) = \cl_{A}^{\text{fund}}(p) \).

It follows that \( \cl_{A}^{\text{fund}}(p) = \cl_{A}^{\text{div}}(P) \) and hence the mapping \( p \mapsto P \) induces an embedding of \( (O_{x}, \leq) \) into \( (S_x^{EM}/\sim_{\text{div}}, \leq_{\text{div}}) \). We check that it is onto. Let \( \bar{a} = (a_i : i < \omega) \) realize \( P \in S_x^{EM} \) and extend it to an indiscernible sequence \( (a_i : i < \omega + \omega) \). Let \( \bar{a}' = (a_i : i < \omega \leq \omega + \omega) \) and choose a model \( M \) containing \( \bar{a} \) and such that \( M \downarrow_{\text{div}} M_{\bar{a}} \). By Erdős-Rado we can assume that \( \bar{a}' \) is \( M \)-indiscernible. It follows that \( \bar{a}' \) is a Morley sequence over \( M \) in \( p(x) = \text{tp}(a_\omega/M) \). Clearly, \( p \) goes to \( P \) in the embedding.

2. Let \( P, Q \in S_x^{EM} \) and assume \( P \sim_{\text{div}} Q \). Choose \( p(x) \in S(M), q(x) \in S(N) \) such that \( p \mapsto P \) and \( q \mapsto Q \) in the isomorphism. By Fact 12.3 there are global types \( p(x) \) and \( q(x) \) extending \( p(x) \) and \( q(x) \) respectively with \( \cl_{A}^{\text{fund}}(p) = \cl_{A}^{\text{fund}}(p), \cl_{A}^{\text{fund}}(q) = \cl_{A}^{\text{fund}}(q) \) and some isomorphism \( f : (C, dp) \cong (C, dq) \). Notice that \( P^f = Q \). There is a one-to-one correspondence between global types non forking over a model and types (over \( \emptyset \)) of Morley sequences over the model constructed with the global type. This correspondence is preserved by automorphisms. Hence \( P^f = Q \). But \( P^f = P \) since they are types over the empty set. Hence, \( P = Q \). \qed
Definition 12.5 For \( p, q \in S^E_M(A) \) we define: \( p \leq^# q \) iff there is some array \((a_{ij} : i, j < \omega)\) such that each row \((a_{ij} : j < \omega)\) realizes \( p \) and each vertical path \((a_{i(f(i))} : i < \omega)\) realizes \( q \).

Proposition 12.6 Let \( p, q \in S^E_M(A) \).

1. If \( p \leq^{\text{div}} q \), then \( p \leq^# q \).

2. \( T \) is NTP2 iff \( p \leq^# q \), then \( p \leq^{\text{div}} q \).

Proof: Without loss of generality, \( A = \emptyset \).

1. Assume \( p \leq^{\text{div}} q \). We will inductively construct \((a_{ij} : i < n, j < \omega)\) such that each row \( \bar{a}_i = (a_{ij} : j < \omega) \) realizes \( p \) and there is some \( \bar{b} \models q \) such that for each \( i_0, \ldots, i_{n-1} < \omega \), \((a_{0i_0}, \ldots, a_{n-1i_{n-1}})^\sim \bar{b} \) is indiscernible. In the case \( n = 0 \) we have not constructed yet any row and it suffices take as \( \bar{b} \) any realization of \( q \). Now, in the general case, choose some \( \bar{c} \)-sequence \( \bar{c} \) such that \( \bar{b}^\sim \bar{c} \) is indiscernible. We define

\[
\bar{r}(\bar{x}_0, \ldots, \bar{x}_{n-1}, y, \bar{z}) = \bigcup_{i < n} p(\bar{x}_i) \cup q(\bar{z}) \cup \bigcup_{i_0, \ldots, i_{n-1} < \omega} (\bar{x}_{0i_0}, \ldots, \bar{x}_{n-1i_{n-1}}, y) \sim \bar{z}
\]

We know that \( \bigcup_{i < \omega} r(\bar{x}_0, \ldots, \bar{x}_{n-1}, y, \bar{z}) \cup q(\bar{y}) \) is consistent (where \( \bar{y} = (y_i : i < \omega) \)) since \( \bar{a}_0, \ldots, \bar{a}_{n-1} \bar{b}^\sim \bar{c} \) realizes it. Since \( p \leq^{\text{div}} q \), it follows that \( \bigcup_{i < \omega} r(\bar{x}_0, \ldots, \bar{x}_{n-1}, y, \bar{z}) \cup p(\bar{y}) \) is consistent and this is what we need for the inductive construction.

2. From left to right. Assume \( p \leq^# q \) but not \( p \leq^{\text{div}} q \). Let \((a_{ij} : i, j < \omega)\) be an array where all rows satisfy \( p \) and all vertical paths satisfy \( q \). If \( \varphi(x, y) \in \text{cl}^{\text{div}}(q) \setminus \text{cl}^{\text{div}}(p) \), then \( \{\varphi(a_{i(f(i))}, y) : i < \omega\} \) is consistent for each \( f : \omega \to \omega \) and for some \( k < \omega \) \( \{\varphi(a_{ij}, y) : j < \omega\} \) is \( k \)-inconsistent for all \( i < \omega \). This shows that \( T \) is TP2. The other direction is similar: if \( T \) is TP2 there is an indiscernible array \((a_{ij} : i, j < \omega)\), some \( k < \omega \) and some formula \( \varphi(x, y) \in L \) witnessing it. If \( p \) is the type of a row and \( q \) is the type of a column, then \( p \leq^# q \) but not \( p \leq^{\text{div}} q \).

Remark 12.8 \( p \leq^# q \) implies \( p \leq^{+} q \).

Definition 12.7 For \( p, q \in S^E_M(A) \) we define: \( p \leq^{+} q \) iff there are \( \bar{a} = (a_i : i \in \mathbb{Z}) \models q \) and \( \bar{b} = (b_i : i \in \mathbb{Z}) \models p \) such that \( a_0 = b_0 \) and \( \bar{b} \) is indiscernible over \( A(a_i : i \neq 0) \).

Remark 12.9 \( T \) is resilient if there are no indiscernible sequences \( \bar{a} = (a_i : i \in \mathbb{Z}) \) and \( \bar{b} = (b_i : i \in \mathbb{Z}) \) and a formula \( \varphi(x, y) \in L \) such that

1. \( a_0 = b_0 \)
2. \( \bar{b} \) is indiscernible over \( (a_i : i \neq 0) \)
3. \( \{\varphi(x, a_i) : i \in \mathbb{Z}\} \) is consistent.
4. \( \{\varphi(x, b_i) : i \in \mathbb{Z}\} \) is inconsistent.

Remark 12.10 If \( T \) is resilient, \( T(A) \) is resilient.

Proof: Add the extra parameters of the formula to each element in the sequences \( \bar{a} \) and \( \bar{b} \).

Proposition 12.11 The following are equivalent:
1. $T$ is resilient.

2. If $p \leq^+ q$, then $p \leq_{\text{div}} q$.

3. If $\bar{a} = (a_i : i \in \mathbb{Z})$ is indiscernible, $\varphi(x,y) \in L$, and $\varphi(x,a_0)$ divides over $(a_i : i \neq 0)$, then $\{\varphi(x,a_i) : i \in \mathbb{Z}\}$ is inconsistent.

4. There is no array $(a_{ij} : i,j < \omega)$, number $k < \omega$ and formula $\varphi(x,y) \in L$ such that $\{\varphi(x,a_0) : i < \omega\}$ is consistent, $\{\varphi(x,a_{ij}) : j < \omega\}$ is $k$-inconsistent for each $i < \omega$ and each row $a_i = (a_{ij} : j < \omega)$ is indiscernible over $(a_{j0} : j \neq i)$.

5. There is a cardinal $\kappa$ such that for each sequence $(a_i : i < \kappa)$ of finite tuples $a_i$, for each finite tuple $b$ there is some $i < \kappa$ such that $b \nind_{(a_{ij} : j \neq i)} a_i$.

**Proof:** $1 \iff 2$ is clear.

$1 \iff 3$. Assume $\varphi(x,a_0)$ divides over $\bar{a}_{\neq 0} = (a_i : i \neq 0)$ and let $\bar{b} = (b_i : i \in \mathbb{Z})$ be an $\bar{a}_{\neq 0}$-indiscernible sequence with $a_0 = b_0$ witnessing it. This means that $\{\varphi(x,b_i) : i \in \mathbb{Z}\}$ is inconsistent. By $1$, $\{\varphi(x,a_i) : i \in \mathbb{Z}\}$ is inconsistent. This proves $1 \implies 3$. The other direction is immediate.

$1 \iff 4$. Let $\bar{a} = (a_i : i \in \mathbb{Z})$ and $\bar{b} = (b_i : i \in \mathbb{Z})$ and $\varphi(x,y) \in L$ witness that $T$ is not resilient. Put $a_{0j} = b_j$. For $i > 0$ choose an automorphism $f_i$ sending $a_{i-1}, \ldots, a_{i-1}, a_0(a_i : l > i)$ to $(a_i : l \geq 0)$ and put $a_{ij} = f_i(a_{ij})$. Since $\bar{b}$ is indiscernible over $a_{i-1}, \ldots, a_{i-1}(a_i : l > i)$, it follows that $(a_{ij} : j \geq 0)$ is indiscernible over $(a_j : 0 \leq j, j \neq i)$. The array $(a_{ij} : i,j \geq 0)$ with the formula $\varphi(x,y)$ provides a counterexample to $4$. This proves $4 \implies 1$. The other direction is immediate.

$4 \implies 5$. Choose $\kappa$ large enough and assume $(a_i : i < \kappa)$ and $b$ provide a counterexample to $5$ for $\kappa$. Let $i < \kappa$. Since $b \nind_{(a_{ij} : j \neq i)} a_i$, for some formula $\varphi_i(x,y_1, \ldots, y_n) \in L$ and some $j_1 < \ldots < j_n$ different from $i$, $\models \varphi_i(b,a_{j_1}, \ldots, a_{j_n})$ and $\varphi_i(x,a_{j_1}, \ldots, a_{j_n})$ divides over $(a_j : j \neq i)$. By choice of $\kappa$, we can assume that all these formulas are the same $\varphi(x;y_1, \ldots, y_n)$. Put $a_i' = a_i, a_{j_1}, \ldots, a_{j_n}$ and choose a $(a_{ij} : j \neq i)$-indiscernible sequence $(a_{ij}' : j < \omega)$ starting with $a_{i0}' = a_i'$ and witnessing that $\varphi(x,a_i')$ divides over $(a_j : j \neq i)$.

$5 \implies 4$. Let $\kappa$ be as in $5$ and assume (by compactness) that the array $(a_{ij} : i < \kappa, j < \omega)$ and the formula $\varphi(x,y) \in L$ contradict $4$. Choose $b = \{\varphi(x,a_0) : i < \kappa\}$. Let $i < \kappa$. Note that $\{\varphi(x,a_{ij}) : j < \omega\}$ is inconsistent and $(a_{ij} : j < \omega)$ is indiscernible over $(a_{i0} : l \neq i)$. This shows that $\varphi(x,a_{i0})$ divides over $(a_{i0} : l \neq i)$. Since $\models \varphi(b,a_0)$, it follows that $b \nind_{(a_{ij} : j \neq i)} a_i$, in contradiction with $5$.

\[\square\]

**Proposition 12.12** 1. NIP theories are resilient.

2. Simple theories are resilient.

3. Resilient theories are NTP2.

**Proof:** 1. Let $T$ be NIP but not resilient. Witness the nonresilience of $T$ by $\bar{a} = (a_i : i \in \mathbb{Z})$, $\bar{b} = (b_i : i \in \mathbb{Z})$ and some formula $\varphi(x,y)$. By NIP there is maximal $k < \omega$ for which $\{\varphi(x,a_i) : i = 1,3, \ldots, 2k+1\} \cup \{\varphi(x,a_i) : i \in \mathbb{Z} \setminus \{1,3, \ldots, 2k+1\}\}$ is consistent. By indiscernibility $\{\varphi(x,a_i) : i = 2,4, \ldots, 2k+2\} \cup \{\varphi(x,a_i) : i \in \mathbb{Z} \setminus \{2,4, \ldots, 2k+2\}\}$ is also consistent and we can realize it by some $d$. Since $\{\varphi(x,b_i) : i \in \mathbb{Z}\}$ is inconsistent, $\models \neg \varphi(d,b_i)$ for some $i \in \mathbb{Z}$. By indiscernibility of $\bar{b}$ over $(a_j : j \neq 0)$, $\{\neg \varphi(x,a_i) : i = \cdots\}.$
\[ 0, 2 \ldots, 2k + 2 \cup \{ \phi(x, a_i) : i \in \mathbb{Z} \setminus \{0, 2, \ldots, 2k + 2 \} \} \] is consistent. This contradicts the maximality of \( k \).

2. We use item 5 of Proposition 12.11. Let \( \kappa = |T|^+ \) and assume \( (a_i : i < \kappa) \) is a sequence of finite tuples and \( b \) is finite. By simplicity there is some \( I \subseteq \kappa \) of cardinality \( \leq |T| \) such that \( b \downarrow_{(a_i : i \in I)} (a_i : i < \kappa) \). Choose \( i < \kappa \), \( i \not\in I \). Then \( b \downarrow_{(a_i : i \not\in I)} a_i \).

3. Assume \( T \) is TP\(_2\), witnessed by some array \( (a_{ij} : i, j < \omega) \) and some formula \( \phi(x, y) \). As shown in section 5, we may assume that the array is very indiscernible. By item 4 of Proposition 12.11, \( T \) is not resilient.

Proof: Since \( T \) is NTP\(_2\) and \( A \) is an extension base, forking over \( A \) and dividing over \( A \) coincide.

\[ \Rightarrow \text{ Note that we can assume } \bar{a} = (a_i : i \in \mathbb{Q}). \text{ Suppose } \models \phi(a_{j1}, \ldots, a_{jn} ; a_i) \text{ for some } j_1 < \ldots < j_l < i < j_{l+1} < \ldots < j_n \text{ and some formula } \phi(x_1, \ldots, x_n ; x) \in L(A). \text{ Let } I \text{ be the interval } (j_l, j_{l+1}). \text{ By indiscernibility, } \models \phi(a_{j1}, \ldots, a_{jn} ; a_q) \text{ for all } q \in I \text{ and therefore } (\phi(x_1, \ldots, x_n ; a_i) : q \in \mathbb{Q}) \text{ is consistent. Since } \bar{a} \text{ is a witness to } \phi(x_1, \ldots, x_n ; a_i) \text{ does not divide (and does not fork) over } A. \text{ Hence } \bar{a} \not\models a_i. \]

\[ \Leftarrow \text{ We can assume } \bar{a} = (a_i : i \in \mathbb{Z}). \text{ Let } \phi(x, y) \in L(A) \text{ and assume } \phi(x, a_0) \text{ divides over } A. \text{ Since } \bar{a} \models a_0 \Downarrow a_0, \phi(x, a_0) \text{ divides over } A \bar{a} \not\models a_0. \text{ Since } T(A) \text{ is resilient, by item 3 of Proposition 12.11, } (\phi(x, a_i) : i \in \mathbb{Z}) \text{ is inconsistent. Hence } \bar{a} \text{ is a witness over } A. \]

13 Weight

Section finished on April 16, 2014. Based on [1]. For more information on weight, see [13] and [15].

Definition 13.1 The preweight of a type \( p(x) \in S(A) \) is the supremum of all cardinals \( \kappa \) for which there is an \( A \)-independent sequence \( (a_i : i < \kappa) \) such that for some \( a \models p, a \Downarrow A a_i \) for all \( i < \kappa \). The weight of \( p(x) \) is the supremum of preweights of all its nonforking extensions. The preweight of \( p \) is \( \text{pwt}(p) \) and \( \text{wt}(p) \) is its weight. If \( p(x) = \text{tp}(a/A) \) we can also write \( \text{pwt}(p) = \text{pwt}(a/A) \) and \( \text{wt}(p) = \text{wt}(a/A) \).

Lemma 13.2 If \( \pi(x) \) is a partial type over \( A \), then \( \text{bdn}(\pi(x)) = \sup \{ \text{bdn}(p(x)) : \pi(x) \subseteq p(x) \in S(A) \} \).

Proof: Given an array \( (a_{ij} : i < \kappa, j < \omega) \) witnessing that \( \text{bdn}(\pi) \geq \kappa \), we may assume that the rows are mutually indiscernible over \( A \). Hence if we choose some completion \( p(x) \in S(A) \) of \( \pi(x) \) which is consistent with the first vertical path, it is consistent with every path in the array. Consequently, the array witnesses that some completion of \( \pi(x) \) over \( A \) has burden \( \geq \kappa \).

Proposition 13.3 Let \( T \) be simple. Let \( p(x) \in S(A), a \models p, (a_i : i < \kappa) \) \( A \)-independent and assume \( a \Downarrow A a_i \) for all \( i < \kappa \). There is an array \( (a_{ij} : i < \kappa, j < \omega) \) with \( a_i = a_{i0} \) witnessing that \( \text{bdn}(p) \geq \kappa \). Hence, \( \text{bdn}(p) \geq \sup \{ \text{pwt}(q) : p \subseteq q \} \).

Proof: We inductively construct Morley sequences \( I_i = (a_{ij} : j < \omega) \) in \( \text{tp}(a_i/AA \rightarrow \text{i} < \omega) \) and in \( \text{tp}(a_{i}/A) \) (i.e., \( A \)-independent) with \( a_i = a_{i0} \) and such that \( a_{i1} \Downarrow A I_{<i} \). Assume \( I_j \)
has been constructed for all $j < i$. Notice that $a_i \downarrow_A a_{>i} I_{<i}$ and choose $I_i = \{ a_{ij} : j < \omega \}$ (see, for instance, Lemma 5.11 in [8]) as a Morley sequence in $\text{tp}(a_{i}/A_{>i} I_{<i})$ and in $\text{tp}(a_{i}/A)$ starting with $a_i = a_{i0}$. It follows that $I_i \downarrow_A a_{>i} I_{<i}$ and hence $I_i \downarrow_A t_{<i} a_{>i}$. By the inductive hypothesis $I_{<i} \downarrow_A a_{>i}$ and therefore $I_{\leq i} \downarrow_A a_{>i}$.

For each $i < \kappa$, since $a_i \not\downarrow_A a_i$ there is some $\varphi_i(x,y_i) \in L(A)$ and some $k_i < \omega$ such that $\varphi_i(x,a_i)$ divides over $A$ with respect to $k_i$. Adding some parameters of $A$ to each $a_i$ if necessary, we may assume that $\varphi_i(x,y_i) \in L$. Since $I_i$ is a Morley sequence in $\text{tp}(a_{i0}/A)$, $\{ \varphi_i(x,a_{i,j}) : i < \omega \}$ is $k_i$-consistent.

Since each $I_i$ is indiscernible over $A a_{>i} I_{<i}$, the rows of the array $(a_{ij} : i < \kappa, j < \omega)$ are almost mutually indiscernible over $A$ and hence for every $f : \kappa \to \omega$, $p(x) \cup \{ \varphi_i(x,a_{i,f(i)}) : i < \kappa \}$ is consistent.

This shows that $\text{bdn}(p) \geq \text{wpt}(p)$. The rest follows from Lemma 13.2. \hfill \Box

**Proposition 13.4** Let $T$ be simple. Assume that the array $(a_{ij} : i < \kappa, j < \omega)$ witnesses that $\text{bdn}(a/A) \geq \kappa$. Then for some set $C \supseteq A$ such that $a \downarrow_A C$, some sequence $(b_i : i < \kappa)$ witnesses that $\text{wpt}(a/C) \geq \kappa$. Hence, for any $p(x) \in S(A)$, $\text{wt}(p) \geq \text{bdn}(p)$.

**Proof:** We may assume that the rows of the array are mutually indiscernible over $A$ and we may extend the rows to the order type $\omega + \omega^*$ preserving mutual indiscernibility over $A$. Let $C = A \cup \{ a_{ij} : i < \kappa, j \in \omega^* \}$. We may assume that $a \downarrow_A C$. For each $i < \kappa$ there is some $\varphi_i(x,y_i) \in L$ and some $k_i$ such that $\{ \varphi_i(x,a_{i,j}) : j < \omega \}$ is $k_i$-consistent. By indiscernibility over $C$, $\varphi_i(x,a_{i0})$ divides over $C$. Hence $a \downarrow_C a_{i0}$ for every $i < \kappa$. Now we show that $(a_{i0} : i < \kappa)$ is $C$-independent: by choice of the order type $\omega^*$ and mutual indiscernibility over $C$, $\text{tp}(a_{i0}/C a_{j0} : j < i)$ is finitely satisfiable in $C$ and therefore $a_{i0} \downarrow_C (a_{i0} : j < i)$. \hfill \Box

**Corollary 13.5** If $T$ is simple and $p(x) \in S(A)$, then

$$\text{bdn}(p) = \text{wt}(p) = \sup \{ \text{wpt}(q) : p \subseteq q \}.$$  

**Proof:** By propositions 13.3 and 13.4. \hfill \Box

**References**


