Stable and simple theories
(Lecture Notes)

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September 5, 2007

Contents

Preface

These are lecture notes from a seminar imparted the academic years 2005-06 and 2006-07 at the Department of Logic of the University of Barcelona. They are incomplete, in process of revision and enlargement. In particular, the preliminaries contain only a few statements. Advanced knowledge of Model Theory is throughout assumed. The notes are based on the work of many modeltheorists. The names of John Baldwin, Byunghan Kim, Daniel Lascar, Anand Pillay, Bruno Poizat, Saharon Shelah, Frank Wagner and Martin Ziegler deserve special mention. I thank to the participants in the Seminar, Hans Adler, Silvia Barbina, Javier Moreno, Rodrigo Peláez, Juan Francisco Pons, and Joris Potier for their patience and their remarks. Of course, I am the only responsible for the mistakes or inaccuracies that might appear in the text.

Chapter 1

Preliminaries

T is a complete theory of language L with infinite models and $\mathfrak{C}$ is its monster model. A, B, C are subsets of $\mathfrak{C}$ and a, b, c are sequences of elements of $\mathfrak{C}$. $a \in A$ means that all the elements in the sequence a belong to A. We use x,y for single variables but also for sequences of variables.

The existence of indiscernible sequences is usually established using Ramsey's Theorem. It is convenient to introduce here a more powerful method based on Erdös-Rado Theorem.

Proposition 1.1 *If $\kappa \geq |T|$ is a cardinal number, $\lambda = \beth_{(2^\kappa)^+}$, $|A| \leq \kappa$ and $(a_i : i < \lambda)$ is a sequence of sequences a_i of fixed length $\alpha < \kappa^+$, then there is an A-indiscernible sequence $(b_i : i < \omega)$ such that for each $n < \omega$ there are $i_0 < \ldots < i_n < \lambda$ such that $b_0, \ldots, b_n \equiv_A a_{i_0}, \ldots, a_{i_n}$. In most of the applications α is a natural number and therefore the cardinal number λ depends only on $|T|$ and $|A|$.*

Corollary 1.2 *1. If $(a_i : i < \lambda)$ is indiscernible over A, there is some model $M \supseteq A$ such that $(a_i : i < \lambda)$ is indiscernible over M.*

2. If $(a_i : i < \lambda)$ is indiscernible over A, then it is also indiscernible over $\mathrm{acl}^{\mathrm{eq}}(A)$.

A *canonical parameter* of a definable relation R is an imaginary element c such that for all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(R) = R$ if and only if $f(c) = c$. It is unique up to interdefinability and it can be constructed starting with some $\varphi(x, y)$ such that $\varphi(\mathfrak{C}, a) = R$ by $c = a/E$ where E is the 0-definable equivalence relation given by

$$E(b, d) \Leftrightarrow \varphi(\mathfrak{C}, b) = \varphi(\mathfrak{C}, d)$$

The following result on definability and imaginaries will be useful:

Proposition 1.3 *The following are equivalent for any definable relation R:*

1. *R is definable over any model $M \supseteq A$.*
2. *R has only finitely many A-conjugates.*
3. *R is a union of equivalence classes of some A-definable finite (i.e. with finitely many classes) equivalence relation.*
4. *R is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.

Chapter 2

φ-types, stability and simplicity

Definition 2.1 *Let $\varphi(x,y) \in L$. A φ-*formula *over A is a formula of the form $\varphi(x,a)$ or $\neg\varphi(x,a)$ with $a \in A$. A φ-type over A is a consistent set of φ-formulas over A. A φ-type $p(x)$ over A is complete if for every $a \in A$ either $\varphi(x,a) \in p$ or $\neg\varphi(x,a) \in p$. The set of all complete φ-types over A is $S_\varphi(A)$.*

Definition 2.2 *Let $\varphi(x,y) \in L$. A complete φ-type $p(x)$ over A is definable over B if there is a formula $\psi(y) \in L(B)$ such that for all $a \in A$,*

$$\varphi(x,a) \in p \Leftrightarrow \models \psi(a)$$

If B is not mentioned we understand that $A = B$. A complete type $p(x) \in S(A)$ is definable if all its restrictions $p \restriction \varphi$ are definable.

Lemma 2.3 *Let $p(x) \in S_\varphi(M)$ be definable. Then for each $A \supseteq M$ there is a unique $q \in S_\varphi(A)$ extending p which is definable over M.*

Proof: Let $\psi(y) \in L(M)$ be a definition of p. It is easy to check that $\{\varphi(x,a) : a \in A, \models \psi(a)\} \cup \{\neg\varphi(x,a) : a \in A, \models \neg\psi(a)\}$ is consistent and it is in fact a complete φ-type over A extending p. On the other hand, if $q_1, q_2 \in S_\varphi(A)$ are M-definable extensions of p with definitions $\psi_1(y), \psi_2(y) \in L(M)$, then $M \models \forall y(\psi_1(y) \leftrightarrow \psi_2(y))$, which implies $\mathfrak{C} \models \forall y(\psi_1(y) \leftrightarrow \psi_2(y))$ and therefore $q_1 = q_2$. □

Definition 2.4 *Let λ be an infinite cardinal number. We say that φ is λ-*stable *or* stable in λ *if for any set A,*

$$|A| \leq \lambda \Rightarrow |S_\varphi(A)| \leq \lambda$$

It is said that φ is stable *if it is stable in some λ. Otherwise φ is called* unstable.

Proposition 2.5 *The following conditions are equivalent for $\varphi = \varphi(x,y) \in L$.*

1. *$\varphi(x,y)$ is stable.*
2. *$\Gamma_\varphi(\omega)$ is inconsistent, where for any ordinal α,*

   $$\Gamma_\varphi(\alpha) = \{\varphi(x_\nu, y_{\nu\restriction i})^{\nu(i)} : \nu \in 2^\alpha,\ i < \alpha\}$$

   and where $\varphi^0 = \varphi$ and $\varphi^1 = \neg\varphi$.

3. *For any set A, any type $p(x) \in S_\varphi(A)$ is definable.*

4. *$\varphi(x,y)$ is λ-stable for all λ.*

Moreover in 3. one can add that p is definable by a formula of the form

$$\psi(y) = \exists x_1 \dots x_n \exists y_1 \dots y_m \chi(y, x_1, \dots, x_n, y_1, \dots y_m)$$

where χ is a conjunction of formulas of the form $\varphi(x_i, y_j)$, $\neg\varphi(x_i, y_j)$, $\varphi(x_i, y)$ and $\varphi(x_i, y)$-formulas over A.

Proof: *1.* $\Rightarrow$ *2.* Assume $\Gamma_\varphi(\omega)$ is consistent. Let λ be an infinite cardinal number and let μ be the least cardinal number such that $2^\mu > \lambda$. Then $2^{<\mu} \leq \lambda$. Since $\Gamma_\varphi(\mu)$ is also consistent, there is a sequence $(b_\nu : \nu \in 2^{<\mu})$ such that for every $\nu \in 2^\mu$ the set of φ-formulas $p_\nu(x) = \{\varphi(x, b_{\nu\restriction i})^{\nu(i)} : i < \mu\}$ is consistent. Since $p_\nu(x)$ is inconsistent with $p_{\nu'}(x)$ for $\nu \neq \nu'$, it follows that there are $2^\mu > \lambda$ complete φ-types over the set $A = \{b_\nu : \nu \in 2^{<\mu}\}$ but $|A| \leq \lambda$. This shows that $\varphi(x,y)$ is not λ-stable.

2. $\Rightarrow$ *3.* Let $p(x) \in S_\varphi(A)$ and assume $\Gamma_\varphi(\omega)$ is inconsistent. Then also $\Gamma_\varphi(\omega) \cup \bigcup_{\nu \in 2^\omega} p(x_\nu)$ is inconsistent and by compactness there is a least natural number n for which

$$\Gamma_\varphi(n) \cup \bigcup_{\nu \in 2^n} p(x_\nu)$$

is inconsistent. Again by compactness, there is a finite subset $p_0(x) \subseteq p(x)$ such that $\Gamma_\varphi(n) \cup \bigcup_{\nu \in 2^n} p_0(x_\nu)$ is inconsistent. Then $n > 0$ and one can check that for any $a \in A$,

$$\varphi(x,a) \in p \;\Leftrightarrow\; \Gamma_\varphi(n-1) \cup \bigcup_{\nu \in 2^{n-1}} p_0(x_\nu) \cup \{\varphi(x_\nu, a)\} \text{ is consistent}$$

that is

$$\varphi(x,a) \in p \;\Leftrightarrow\; \exists(x_\nu : \nu \in 2^{n-1})\exists(y_\eta : \eta \in 2^{<n-1})(\bigwedge \Gamma_\varphi(n-1) \wedge \bigwedge_{\nu \in 2^{n-1}} p_0(x_\nu) \wedge \varphi(x_\nu, a))$$

which is a definition of p of the form indicated above.

3. $\Rightarrow$ *4.* Since there are at most λ many definitions of the described form over a set A with $|A| \leq \lambda$, there are also λ many complete φ-types over A. This shows that $\varphi(x,y)$ is λ-stable for any λ but uses the hypothesis *3.* with the added information on the form of the definition. Without this information we can only guarantee that it is stable in any $\lambda \geq |T|$. But this is enough since after all we have established that *1.* implies *4.* □

Remark 2.6 *If φ is stable any global φ-type $\mathfrak{p} \in S_\varphi(\mathfrak{C})$ is definable.*

Proof: The proof of *2* $\Rightarrow$ *3* given for Proposition 2.5 works also for $\mathfrak{p} \in S_\varphi(\mathfrak{C})$. □

Definition 2.7 *$\varphi = \varphi(x,y) \in L$ has the order property if for some a_i, b_i $(i < \omega)$ the following holds:*

$$\models \varphi(a_i, b_j) \;\Leftrightarrow\; i < j$$

Remark 2.8 1. *$\varphi(x,y)$ has the order property if and only if for some a_i, b_i $(i < \omega)$, $\models \varphi(a_i, b_j) \;\Leftrightarrow\; i \leq j$.*

2. *$\varphi(x,y)$ has the order property if and only if $\neg\varphi(x,y)$ has the order property.*

3. *In the definition of the order property one can change the index set ω and its order by any infinite linear ordering.*

Lemma 2.9 *Assume $\varphi = \varphi(x, y)$ does not have the order property. If $p \in S_\varphi(A)$ is finitely satisfiable in A (which is always true if A is a model), then p is definable by a positive boolean combination of formulas of the form $\varphi(a, y)$ with $a \in A$.*

Proof: Let $X_1, \dots, X_n$ be a family of subsets of a set A. Consider the relation $R(a, b)$ among elements a, b of A which holds when $\forall i(1 \le i \le n)(a \in X_i \to b \in X_i)$. It is easy to see that a subset $X \subseteq A$ is a positive boolean combination of the sets $X_1, \dots, X_n$ if and only if $b \in X$ whenever $a \in X$ and $R(a, b)$. The reason is that in this situation

$$X(x) \;\Leftrightarrow\; \bigvee_{a \in X} \bigwedge \{X_i(x) : a \in X_i\}$$

We will use this result. Assume p is not definable by a positive boolean combination of formulas of the described form. We inductively define tuples a_i, b_i, c_i $(i \in \omega)$ of elements of A. Suppose a_j, b_j, c_j are defined for $j < i$. By hypothesis $\{a \in A : \varphi(x, a) \in p\}$ is not a positive boolean combination of the sets $X_j = \{a \in A : \models \varphi(c_j, a)\}$ for $j < i$. Then there are $a_i, b_i \in A$ such that $\varphi(x, a_i) \in p$, $\neg\varphi(x, b_i) \in p$ and for all $j < i$, if $\models \varphi(c_j, a_i)$, then $\models \varphi(c_j, b_i)$. Now let c_i be a realization of the finite type $p \restriction \{a_j, b_j : j \le i\}$. The sequences of tuples thus obtained have the property that $\models \varphi(c_j, a_i) \wedge \neg\varphi(c_j, b_i)$ for $i \le j$ but $\models \varphi(c_j, a_i) \to \varphi(c_j, b_i)$ for $j < i$. By Ramsey's Theorem we may assume that always $\models \neg\varphi(c_j, a_i)$ for $j < i$ or always $\models \varphi(c_j, b_i)$ for $j < i$. In the first case we have $i \le j$ if and only if $\models \varphi(c_j, a_i)$. In the second case $i \le j$ if and only if $\models \neg\varphi(c_j, b_i)$. In any case $\varphi(x, y)$ has the order property. □

Proposition 2.10 *$\varphi(x, y)$ is stable if and only if it does not have the order property.*

Proof: If $\varphi(x, y)$ has the order property, then there are $a_i, b_j (i, j \in \mathbb{Q})$ such that for all i, j

$$\models \varphi(a_i, b_j) \;\Leftrightarrow\; i < j$$

Now for each real number r let $p_r(x)$ be the φ-type $\{\varphi(x, b_j) : r < j\} \cup \{\neg\varphi(x, b_j) : r \ge j\}$. Clearly $p_r(x)$ is inconsistent with p_s if $r \ne s$ and thus there are 2^ω many complete φ-types over the countable set $\{b_j : j \in \mathbb{Q}\}$. Hence φ is not stable. For the other direction, assume $\varphi(x, y)$ does not have the order property and let $\lambda \ge |T|$. We use Lemma 2.9 to check that φ is λ-stable. Let A be a set such that $|A| \le \lambda$. We may find a model $M \supseteq A$ such that $|M| \le \lambda$. Since there are at most λ many definitions of φ-types over M and each $p \in S_\varphi(M)$ is definable over M, we conclude that $|S_\varphi(A)| \le |S_\varphi(M)| \le \lambda$. □

Corollary 2.11 *Any boolean combination $\varphi(x, y)$ of stable formulas $\varphi_i(x, y)$ is stable.*

Remark 2.12 *Let $\varphi = \varphi(x, y) \in L$ be stable. By Remark 2.6, any $\mathfrak{p} \in S_\varphi(\mathfrak{C})$ is definable over some set A. If $M \supseteq A$, then $\mathfrak{p}$ has a definition which is a positive boolean combination of formulas of the form $\varphi(b, x)$ with $b \in M$ and which is at the same time equivalent to a formula over A.*

Proof: Since φ does not have the order property, we can apply Lemma 2.9, which gives a positive boolean combination $\psi(x) \in L(M)$ of formulas of the form $\varphi(b, x)$ which defines $\mathfrak{p} \restriction M$. Since there is only one global φ-type extending $\mathfrak{p} \restriction M$ and definable over M, and ψ defines in $\mathfrak{C}$ a φ-type with these properties, it follows that ψ defines $\mathfrak{p}$ and hence ψ is equivalent to a formula over A. □

Definition 2.13 *$\varphi(x, y) \in L$ has the* strict order property *if there are $a_i, (i < \omega)$ such that for all $i < \omega$,*

$$\varphi(\mathfrak{C}, a_i) \subsetneq \varphi(\mathfrak{C}, a_{i+1})$$

Remark 2.14

1. *Clearly, a formula with the strict order property has the order property.*
2. *In the definition of the strict order property one can change the ordered set $(\omega, <)$ for any other infinite linearly ordered set.*
3. *If the formula $\varphi(x, y, a)$ has the strict order property, then also $\varphi(x; y, z)$ has the strict order property.*
4. *There is a formula in T with the strict order property if and only if for some n there is a definable partial order of $\mathfrak{C}^n$ which has infinite chains. In fact is $\varphi(x, y)$ has the strict order property, then*

$$\psi(y_1, y_2) = \forall x(\varphi(x, y_1) \to \varphi(x, y_2)) \wedge \exists x(\varphi(x, y_2) \wedge \neg\varphi(x, y_1))$$

defines such an order.

Definition 2.15 *$\varphi(x, y) \in L$ has the* independence property *if there are sequences $(a_i : i < \omega)$ and $(b_X : X \subseteq \omega)$ such that for all i, X,*

$$\models \varphi(a_i, b_X) \;\Leftrightarrow\; i \in X.$$

Remark 2.16

1. *A formula with the independence property is unstable.*
2. *$\varphi(x, y)$ has the independence property if and only if for each $n < \omega$ there are a_i $(i < n)$ such that for each $X \subseteq n$ the set of formulas*

$$\{\varphi(a_i, x) : i \in X\} \cup \{\neg\varphi(a_i, x) : i \in n \setminus X\}$$

is consistent.

3. *If $\varphi(x, y)$ has the independence property, then also $\varphi^{-1}(y, x) = \varphi(x, y)$ has the independence property.*

Proposition 2.17 *There is a unstable formula in T if and only if there is a formula with the strict order property or there is a formula with the independence property.*

Proof: As already remarked, formulas with the independence property or the strict order property are unstable. Now assume that $\varphi(x, y)$ has the order property but not the independence property. We will see that a certain conjunction $\theta(x, y)$ of $\varphi(x, y)$ with formulas of the form $\varphi(a, y)$ and $\neg\varphi(a, y)$ has the strict order property.

By the order property, there are sequences $(a_i : i \in \mathbb{Q})$ and $(b_i : i \in \mathbb{Q})$ such that $\models \varphi(a_i, b_j)$ iff $i < j$. We may assume that $(a_i : i \in \mathbb{Q})$ is indiscernible. Since φ does not have the independence property, for some $n < \omega$ there is a subset $S \subseteq n$ which is not represented, in the sense that

$$\models \neg\exists y(\bigwedge_{i \in S} \varphi(a_i, y) \wedge \bigwedge_{i \in n \setminus S} \neg\varphi(a_i, y)).$$

S is not an initial segment of n because otherwise some b_j would represent it. But S is obtained as the last step $S = S_m$ of a sequence $S_0, \dots, S_m$ of subsets $S_k \subseteq n$ where S_0 is an initial segment and for each k there is some $m \in S_k$ such that $m + 1 \notin S_k$ and $S_{k+1} = (S_k \smallsetminus \{m\}) \cup \{m+1\}$. Since S_0 is represented but S_m is not, there is some k such that S_k is represented but S_{k+1} is not. Let $U = S_k \cap S_{k+1}$, $V = n \smallsetminus (S_k \cup S_{k+1})$ and let $m \in S_k$ be such that $S_k = U \cup \{m\}$ and $S_{k+1} = U \cup \{m+1\}$. If $\psi(y) = \bigwedge_{i\in U} \varphi(a_i, y) \wedge \bigwedge_{i\in V} \neg\varphi(a_i, y)$, it follows that since S_k is represented,

$$\models \exists y(\psi(y) \wedge \varphi(a_m, y) \wedge \neg\varphi(a_{m+1}, y))$$

but since S_{k+1} is not represented,

$$\models \neg\exists y(\psi(y) \wedge \varphi(a_{m+1}, y) \wedge \neg\varphi(a_m, y)).$$

Hence if $\theta(x, y) = \psi(y) \wedge \varphi(x, y)$ we have that

$$\theta(a_{m+1}, \mathfrak{C}) \subsetneq \theta(a_m, \mathfrak{C}).$$

By indiscernibility, for all rational numbers $m \leq q < q' \leq m + 1$, $\theta(a_{q'}, \mathfrak{C}) \subsetneq \theta(a_q, \mathfrak{C})$ which implies that $\theta(x, y)$ has the strict order property.

$\square$

Definition 2.18 *Let $k \geq 2$ be a natural number. It is said that $\varphi(x, y)$ has the k-*tree property *if there are a_s ($s \in \omega^{<\omega}$) such that*

- *For each $f \in \omega^\omega$, $\{\varphi(x, a_{f\restriction n}) : n < \omega\}$ is consistent.*
- *For each $s \in \omega^{<\omega}$ the set $\{\varphi(x, a_{s^\frown i}) : i < \omega\}$ is k-inconsistent, that is, every subset of k elements is inconsistent.*

The formula φ has the tree property *if it has the k-tree property for some k.*

Proposition 2.19 *If $\varphi(x, y)$ has the strict order property, then $\psi(x; y_1y_2) = \neg\varphi(x, y_1) \wedge \varphi(x, y_2)$ has the 2-tree property.*

Proof: By the strict order property, there is a sequence $(a_p : p < \mathbb{Q})$ such that $\varphi(\mathfrak{C}, a_p) \subsetneq \varphi(\mathfrak{C}, a_q)$ for $p < q \in \mathbb{Q}$. We prove the existence of parameters $b_s = b_s^1 b_s^2$, $(s \in \omega^{<\omega})$ witnessing the 2-tree property of $\psi(x; y_1, y_2)$. The construction is done by induction on the length of s in such a way that for each $s \in \omega^{<\omega}$ there are $p_s < q_s \in \mathbb{Q}$ with $a_{p_s} = b_s^1$ and $a_{q_s} = b_s^2$ and $p_t < p_s < q_s < q_t$ if $t \subsetneq s$. We start with $p_\emptyset = 0$ and $q_\emptyset = 1$. To extend the branch finishing in $s \in \omega$ it is enough to pick two increasing sequences of rational numbers $(p_{s^\frown i} : i < \omega)$ and $(q_{s^\frown i} : i < \omega)$ such that $p_s < p_{s^\frown i} < q_{s^\frown i} < p_{s^\frown i+1} < q_s$. $\square$

Proposition 2.20 *Any formula with the tree property is unstable.*

Proof: Assume $\varphi = \varphi(x, y)$ has the k-tree property. Chose λ such that $\lambda^\omega > 2^\omega$ and $\lambda^\omega > \lambda$. By compactness, there are a_s, $(s \in \lambda^{<\omega})$ such that for each $s \in \lambda^{<\omega}$, $\{\varphi(x, a_{s^\frown i}) : i < \lambda\}$ is k-inconsistent and for each $f \in \lambda^\omega$, $\pi_f(x) = \{\varphi(x, a_{f\restriction n}) : n < \omega\}$ is consistent. Choose for each $f \in \lambda^\omega$ a subset $I_f \subseteq \lambda^\omega$ such that $f \in I_f$ and $p_f(x) = \bigcup_{g\in I_f} \pi_g(x)$ is a maximally consistent union of types π_g. By k-inconsistency I_f is a k-branching tree of height ω and hence $|I_f| \leq 2^\omega$. Since $\lambda^\omega > 2^\omega$, $\{p_f(x) : f \in \lambda^\omega\}$ has cardinality λ^ω. Since it is a set of pairwise incompatible φ-types over a set of parameters $\{a_s : s \in \lambda^{<\omega}\}$ of cardinality λ, we conclude that φ is not λ-stable. $\square$

Definition 2.21 *The theory T is* stable *if all formulas are stable in T, otherwise it is* unstable*. T is* simple *if it does not have formulas with the tree property. It is said that T has* the independence property *if some formula has the independence property in T and it is said that T has* the strict order property *if some formula has the strict order property in T.*

Remark 2.22 *We have seen that*

1. *T is unstable if and only if T has the independence property or it has the strict order property.*
2. *Simple theories do not have the strict order property.*
3. *Stable theories are simple.*

Chapter 3

Δ-types and the local rank $D(\pi, \Delta, k)$

Definition 3.1 *Let $\Delta = \{\varphi_i(x, y_i) : 1 \leq i \leq n\}$ where $\varphi_i(x, y_i) \in L$ for each i. A Δ-*formula *over A is a formula of the form $\varphi_i(x, a)$ or $\neg\varphi_i(x, a)$ with $a \in A$. A Δ-type over A is a consistent set of Δ-formulas over A. A Δ-type $p(x)$ over A is complete if for all $i = 1, \ldots, n$ for every $a \in A$, either $\varphi_i(x, a) \in p$ or $\neg\varphi_i(x, a) \in p$. The set of all complete Δ-types over A is $S_\Delta(A)$. We endow $S_\Delta(A)$ with a compact hausdorff and totally disconnected topology. A basis of clopen sets for it is given by all sets of the form*

$$[\psi] = \{p \in S_\Delta(A) : p \vdash \psi\}$$

for any boolean combination $\psi = \psi(x)$ of Δ-formulas over A.

Definition 3.2 *Let $\Delta = \{\varphi_i(x, y_i) : 1 \leq i \leq n\}$ and let $2 \leq k < \omega$. The D-rank with respect to Δ and k is defined inductively for any set of formulas $\pi = \pi(x)$ by the following clauses:*

1. *$D(\pi, \Delta, k) \geq 0$ if and only if π is consistent.*

2. *$D(\pi, \Delta, k) \geq \alpha + 1$ if and only if there is some i ($1 \leq i \leq n$) and there are a_j, ($j \in \omega$) such that $\{\varphi_i(x, a_j) : j < \omega\}$ is k-inconsistent and for all $j < \omega$, $D(\pi(x) \cup \{\varphi_i(x, a_j)\}, \Delta, k) \geq \alpha$.*

3. *$D(\pi, \Delta, k) \geq \alpha$ if and only if $D(\pi, \Delta, k) \geq \beta$ for all $\beta < \alpha$ if α is a limit ordinal.*

Observe that $\{\alpha : D(\pi, \Delta, k) \geq \alpha\}$ is an initial segment of the ordinals. If $D(\pi, \Delta, k) \geq \alpha$ for all α then we set $D(\pi, \Delta, k) = \infty$; otherwise $D(\pi, \Delta, k)$ is the supremum of all α such that $D(\pi, \Delta, k) \geq \alpha$. In case $\Delta = \{\varphi(x, y)\}$ we use the notation $D(\pi, \varphi, k)$.

Remark 3.3

1. *If $\pi(x) \vdash \pi'(x)$, $\Delta \subseteq \Delta'$ and $k \leq k'$, then $D(\pi(x), \Delta, k) \leq D(\pi'(x), \Delta', k')$*

2. *If $\pi(x)$ and $\pi'(x)$ are equivalent, then $D(\pi(x)), \Delta, k) = D(\pi'(x), \Delta, k)$.*

3. *Given $\pi(x, y)$, a set of formulas over $\emptyset$, given Δ, k, and α, there is some set of formulas $\Phi(y)$ over $\emptyset$, such that for each a, $\models \Phi(a)$ if and only if $D(\pi(x, a), \Delta, k) \geq \alpha$.*

Proof: *2* follows from *1* and to prove *1* one shows by induction on α that

$$D(\pi(x), \Delta, k) \geq \alpha \Rightarrow D(\pi'(x), \Delta', k') \geq \alpha.$$

3 is easily proved by induction on α. $\square$

Lemma 3.4 *Let $\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$ where $\varphi_i(x, y_i) \in L$ for every i. There is a formula*

$$\psi_\Delta = \psi_\Delta(x; y_1, \ldots, y_n, z, z_1, \ldots, z_{2n}) \in L$$

such that

1. *For each A with $|A| \geq 2$ for each Δ-formula $\varphi(x)$ over A there is a tuple $a \in A$ such that $\varphi(x) \equiv \psi_\Delta(x; a)$.*
2. *For each A for each tuple $a \in A$ such that $\psi_\Delta(x; a)$ is consistent, there is a Δ-formula $\varphi(x)$ over A such that $\varphi(x) \equiv \psi_\Delta(x; a)$.*

Proof: Take as $\psi_\Delta(x; y_1, \ldots, y_n, z, z_1, \ldots, z_{2n})$ the following formula:

$$\bigwedge_{i=1}^{n} (z = z_i \rightarrow \varphi_i(x, y_i)) \wedge (z = z_{n+i} \rightarrow \neg\varphi_i(x, y_i)) \wedge \bigvee_{i=1}^{2n} (z = z_i) \wedge \bigwedge_{1 \leq i < j \leq 2n} \neg(z = z_i \wedge z = z_j).$$

Choose $a_0, a_1 \in A$ such $a_0 \neq a_1$. Then for each $a \in A$, $\varphi_i(x, a)$ is equivalent to

$$\psi_\Delta(x; b_1, \ldots, b_n, c, c_1, \ldots, c_{2n})$$

where $b_i = a$ for all $i = 1, \ldots, n$, $c = a_0 = c_i$ and $c_j = a_1$ for $j \neq i$; and $\neg\varphi_i(x, a)$ is equivalent to

$$\psi_\Delta(x; b_1, \ldots, b_n, c, c_1, \ldots, c_{2n})$$

where $b_i = a$ for all $i = 1, \ldots, n$, $c = a_0 = c_{n+i}$ and $c_j = a_1$ for $j \neq n + i$ $\square$

Corollary 3.5 *For each $\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$ where $\varphi_i(x, y_i) \in L$ for every i, there is a formula $\psi_\Delta(x; z) \in L$ such that for each $\pi(x)$, for each k,*

$$D(\pi, \Delta, k) = D(\pi, \psi_\Delta, k)$$

Proof: The formula ψ_Δ is chosen accordingly to Lemma 3.4. By induction on α we see that for each π and k, $D(\pi, \Delta, k) \geq \alpha$ if and only if $D(\pi, \psi_\Delta, k) \geq \alpha$. This is clear for $\alpha = 0$ and follows from the inductive hypothesis for limit α. The case $\alpha + 1$ is easy and only requires to note that Δ is finite and therefore any infinite sequence of Δ-formulas contains an infinite subsequence of instances of a single formula. $\square$

Due to this last result, we will concentrate on the study of $D(\pi, \varphi, k)$ rank. The generalization of the statements to $D(\pi, \Delta, k)$ rank is straightforward.

Definition 3.6 *Let $\varphi(x, y) \in L$ and $2 \leq k < \omega$. The formula $\varphi(x, a)$ k-*divides over A *if there are a_i, $(i < \omega)$ such that $\{\varphi(x, a_i) : i < \omega\}$ is k-inconsistent and $a_i \equiv_A a$ for all $i < \omega$. We say that $\varphi(x, a)$* divides over A *if it k-divides over A for some k. If A is omitted we understand that $A = \emptyset$.*

Proposition 3.7 *Let $(\varphi_i(x, y_i) : i < \alpha)$ be a sequence of L-formulas and let $(k_i : i < \alpha)$ a sequence of natural numbers $k_i \geq 2$. For any partial type $\pi(x)$ over A, those following are equivalent:*

1. *There are b_i, $(i < \alpha)$ such that $\pi(x) \cup \{\varphi_i(x,b_i) : i < \alpha\}$ is consistent and for each $i < \alpha$, $\varphi_i(x,b_i)$ k_i-divides over $A\{b_j : j < i\}$.*

2. *There are a_s, $(s \in \omega^{\leq\alpha})$ such that for each $f \in \omega^\alpha$, $\pi(x) \cup \{\varphi_i(x,a_{f\restriction i+1}) : i < \alpha\}$ is consistent and for each $i < \alpha$, for each $s \in \omega^i$, $\{\varphi_i(x,a_{s^\frown j}) : j < \omega\}$ is k_i-inconsistent.*

Proof: We first prove that *1* implies *2*. Observe that a_s plays no role for s of length 0 or a limit ordinal. We construct a_s for $s \in \omega^i$ by induction in $i \leq \alpha$ with the additional property that $(a_{s\restriction j+1} : j < i) \equiv_A (b_j : j < i)$. Assume $s \in \omega^i$ and a_s has been already obtained. Choose c such that

$$(a_{s\restriction j+1} : j < i)c \equiv_A (b_j : j < i)b_i.$$

Then $\varphi_i(x,c)$ k_i-divides over $A' = A\{a_{s\restriction j+1} : j < i\}$ and therefore we can find $a_{s^\frown l} \equiv_{A'} c$ for all $l < \omega$ such that $\pi(x) \cup \{\varphi_j(x,a_{s\restriction j+1}) : j < i\} \cup \{\varphi_i(x,a_{s^\frown l})\}$ is consistent and $\{\varphi_i(x,a_{s^\frown l}) : l < \omega\}$ is k_i-inconsistent.

For the other direction choose first $\lambda > 2^{|T|+|A|+|\alpha|}$. By compactness there are a_s, $(s \in \lambda^{\leq\alpha})$ such that for each $f \in \lambda^\alpha$, $\pi(x) \cup \{\varphi_i(x,a_{f\restriction i+1}) : i < \alpha\}$ is consistent and for each $i < \alpha$, for each $s \in \lambda^i$, $\{\varphi_i(x,a_{s^\frown l}) : l < \lambda\}$ is k_i-consistent. Observe that by choice of λ, for any $i < \alpha$ for any $s \in \lambda^i$ at least λ of the $a_{s^\frown l}$ have the same type over $A\{a_{s\restriction j+1} : j < i\}$. Hence we can prune the tree obtaining a subtree where this happens for all $a_{s^\frown l}$. Finally choose a branch $f \in \lambda^\alpha$ and put $b_i = a_{f\restriction i+1}$ for all $i < \alpha$. □

Lemma 3.8 *Let $\pi(x)$ be a partial type over A. $D(\pi(x),\Delta,k) \geq \alpha + 1$ if and only if for some $\varphi(x,y) \in \Delta$, for some a, $D(\pi(x) \cup \{\varphi(x,a)\},\Delta,k) \geq \alpha$ and $\varphi(x,a)$ k-divides over A.*

Proof: The direction from right to left is obvious from the definitions of D-rank and of dividing. For the other direction, assume $D(\pi(x),\Delta,k) \geq \alpha + 1$. Let $\lambda > 2^{|T|+|A|}$. From point *3.* in Remark 3.3 and compactness, we see that there are $\varphi(x,y) \in \Delta$ and a_i, $(i < \lambda)$ such that for each $i < \lambda$, $D(\pi(x) \cup \{\varphi(x,a_i)\},\Delta,k) \geq \alpha$ and $\{\varphi(x,a_i) : i < \lambda\}$ is k-inconsistent. By choice of λ, there is an infinite subset $I \subseteq \lambda$ such that $a_i \equiv_A a_j$ for all $i,j \in I$. Then it suffices to take $a = a_i$ with $i \in I$. □

Proposition 3.9 *For any partial type $\pi(x)$ over A, any $\varphi = \varphi(x,y) \in L$ and any k, those following are equivalent:*

1. *$D(\pi(x),\varphi,k) \geq \alpha$*

2. *There is a sequence $(a_i : i < \alpha)$ such that $\pi(x) \cup \{\varphi(x,a_i) : i < \alpha\}$ is consistent and for each $i < \alpha$, $\varphi(x,a_i)$ k-divides over $A\{a_j : j < i\}$.*

Proof: By induction on α. The case $\alpha = 0$ is obvious. For the case α limit we use compactness and the tree characterization given in Proposition 3.7. Let us consider the case case $\alpha+1$. Assume there are a_i, $(i < \alpha+1)$ such that $\pi(x) \cup \{\varphi(x,a_i) : i < \alpha+1\}$ is consistent and for each $i < \alpha + 1$, $\varphi(x,a_i)$ k-divides over $A\{a_j : j < i\}$. By inductive hypothesis $D(\pi(x) \cup \{\varphi(x,a_0)\},\varphi,k) \geq \alpha$ and by Lemma 3.8 we see that $D(\pi(x),\varphi,k) \geq \alpha + 1$. For the other direction, assume now $D(\pi(x),\varphi,k) \geq \alpha + 1$. Again by Lemma 3.8 there is some a_0 such that $\varphi(x,a_0)$ k-divides over A and $D(\pi(x) \cup \{\varphi(x,a_0)\},\varphi,k) \geq \alpha$. By inductive hypothesis there are b_i, $(i < \alpha)$ such that $\pi(x) \cup \{\varphi(x,a_0)\} \cup \{\varphi(x,b_i) : i < \alpha\}$ is consistent and for each $i < \alpha$, $\varphi(x,b_i)$ k-divides over $A \cup \{a_0\} \cup \{b_j : j < i\}$. In case $\alpha < \omega$ we have obtained what we wanted. In case $\alpha \geq \omega$ we use compactness. □

Proposition 3.10 *Fix φ and k.*

1. *If $D(\pi, \varphi, k) \geq \omega$, then $D(\pi, \varphi, k) = \infty$.*
2. *There is a conjunction $\psi(x)$ of formulas from π such that $D(\pi, \varphi, k) = D(\psi, \varphi, k)$.*
3. *$D(\pi(x) \cup \{\psi_1(x) \vee \ldots \vee \psi_n(x)\}, \varphi, k) = \max_{i=1}^{n} D(\pi(x) \cup \{\psi_i(x)\}, \varphi, k)$.*
4. *Any partial type $\pi(x)$ over A can be extended to a complete type $p(x) \in S(A)$ such that $D(\pi, \varphi, k) = D(p, \varphi, k)$.*

Proof: *1.* follows from propositions 3.9 and 3.7 since, by compactness, a tree of length ω can be extended to a similar tree of any height. Similarly for *2* since it is enough to find ψ such that $D(\psi, \varphi, k) \ngeq \alpha + 1$ where $\alpha = D(\pi, \varphi, k)$.

For *3.* we use Proposition 3.9. Assume $\pi(x)$ is over A and $\psi_i(x) = \psi_i(x, b_i)$ where $\psi_i(x, y_i) \in L$. Assume $D(\pi(x) \cup \{\psi_1(x) \vee \ldots \vee \psi_n(x)\}, \varphi, k) \geq \alpha$. There are a_l, $(l < \alpha)$ such that $\pi(x) \cup \{\psi_1(x) \vee \ldots \vee \psi_n(x)\} \cup \{\varphi_l(x, a_l) : l < \alpha\}$ is consistent and for each $l < \alpha$, $\varphi(x, a_l)$ k-divides over $A \cup \{b_1, \ldots, b_n\} \cup \{a_j : j < l\}$. Clearly, for some i, $\pi(x) \cup \{\psi_i(x)\} \cup \{\varphi_l(x, a_l) : l < \alpha\}$ is consistent. Since $\varphi(x, a_l)$ also k-divides over $A \cup \{b_i\} \cup \{a_j : j < l\}$ we conclude that $D(\pi(x) \cup \{\psi_i(x)\}, \varphi, k) \geq \alpha$.

For *4.* use *3.* to guarantee the consistency of

$$\pi(x) \cup \{\neg\psi(x) : \psi(x) \in L(A) \text{ and } D(\pi(x) \cup \{\psi(x)\}, \varphi, k) < D(\pi(x), \varphi, k)\}$$

and take as $p(x)$ any complete type over A extending this consistent set of formulas. $\square$

Proposition 3.11 *1. $\varphi(x, y)$ has the k-tree property if and only if $D(x = x, \varphi, k) = \infty$.*

2. T is simple if and only if $D(x = x, \varphi, k) < \omega$ for all φ and k.

Proof: The first point follows from propositions 3.9 and 3.7 and the second one follows directly from the first. $\square$

Chapter 4

Forking

Definition 4.1 *Let $\pi(x)$ be a set of formulas over B. We say that $\pi(x)$ divides over A if π implies a formula $\varphi(x, a)$ which divides over A. We may always assume that $a \in B$ and that $\varphi(x, a)$ is a conjunction of formulas in $\pi(x)$.*

Remark 4.2

1. *$\varphi(x, a)$ divides over A iff the set $\{\varphi(x, a)\}$ divides over A.*
2. *If $\pi(x)$ is inconsistent, it divides over A.*
3. *A partial type $\pi(x)$ over $\mathrm{acl}(A)$ does not divide over A.*
4. *$\pi(x, a)$ divides over A iff for some infinite A-indiscernible sequence $(a_i : i < \omega)$ with $a_0 = a$, the set of formulas $\bigcup_{i<\omega} \pi(x, a_i)$ is inconsistent.*
5. *$\mathrm{acl}(A) = \{a : \mathrm{tp}(a/Aa)$ does not divide over $A\}$*

Proof: For *2* take as $\varphi(x, y)$ the formula $x \neq x$. For *4* use Ramsey's for the indiscernibility. For *5* consider the formula $x = a$. □

Definition 4.3 *The set of formulas $\pi(x)$ forks over A if for some n there are formulas $\varphi_1(x, a_1), \ldots, \varphi_n(x, a_n)$ such that $\pi(x) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ and every $\varphi_i(x, a_i)$ divides over A. The formula $\varphi(x, a)$ forks over A if the set $\{\varphi(x, a)\}$ forks over A.*

Remark 4.4

1. *If $\pi(x)$ divides over A, then it forks over A.*
2. *If $\pi(x)$ is finitely satisfiable in A, then it does not fork over A.*
3. *$\pi(x)$ forks over A iff a conjunction of formulas in $\pi(x)$ forks over A.*
4. *Let $\pi(x)$ be a partial type over B. If $\pi(x)$ does not fork over A, then it can be extended to a complete type over B which does not fork over A.*

Proof: The first three points follow directly from the definitions. For *4* check the consistency of $\pi(x) \cup \{\neg\varphi(x) : \varphi(x) \in L(B)$ forks over $A\}$ and take as p any complete type over B extending this partial type. □

Lemma 4.5 *Those following are equivalent.*

1. *$\mathrm{tp}(a/Ab)$ does not divide over A.*

2. *For every infinite A-indiscernible sequence I such that $b \in I$, there is $a' \equiv_{Ab} a$ such that I is Aa'-indiscernible.*

3. *For every infinite A-indiscernible sequence I such that $b \in I$, there is $J \equiv_{Ab} I$ such that J is Aa-indiscernible.*

Proof: The equivalence of *2* and *3* follows by conjugation. It is clear that *3* implies *1*. We prove that *1* implies *2*. We may assume that A is empty, that $I = (b_i : i < \omega)$ and that $b = b_0$. Let $p(x, b) = \mathrm{tp}(a/b)$ and let $\Gamma(x, (x_i : i < \omega))$ be a set of formulas expressing that $(x_i : i < \omega)$ is x-indiscernible. It will be enough to prove that $p(x, b) \cup \Gamma(x, (b_i : i < \omega))$ is consistent. By *1* $q(x) = \bigcup_{i<\omega} p(x, b_i)$ is consistent. Let $c \models q$ and let Γ_0 a finite subset of Γ. By Ramsey's Theorem, there is an order preserving $f : \omega \to \omega$ such that $\models \Gamma_0(c, (b_{f(i)} : i < \omega))$. By indiscernibility $(b_i : i < \omega) \equiv (b_{f(i)} : i < \omega)$ and therefore we can find c' such that $c'(b_i : i < \omega) \equiv c(b_{f(i)} : i < \omega)$. Clearly $c' \models q(x) \cup \Gamma_0(x, (b_i : i < \omega))$. □

Proposition 4.6 *If* $\mathrm{tp}(a/B)$ *does not divide over* $A \subseteq B$ *and* $\mathrm{tp}(b/Ba)$ *does not divide over* Aa*, then* $\mathrm{tp}(ab/B)$ *does not divide over* A.

Proof: It is an easy application of point *3* of Lemma 4.5. □

Proposition 4.7 *If* $\varphi(x, a)$ *divides over* A *with respect to* k *and* $\mathrm{tp}(b/Aa)$ *does not divide over* A*, then* $\varphi(x, a)$ *divides over* Ab *with respect to* k.

Proof: Let $I = (a_i : i < \omega)$ be an infinite A-indiscernible sequence such that $a = a_0$ and $\{\varphi(x, a_i) : i < \omega\}$ is k-inconsistent. By Lemma 4.5 there is $J \equiv_{Aa} I$ which is Ab-indiscernible. Then J witnesses that $\varphi(x, a)$ divides over Ab with respect to k. □

Definition 4.8 *A dividing chain for* $\varphi(x, y)$ *is a sequence* $(a_i : i < \alpha)$ *such that* $\{\varphi(x, a_i) : i < \alpha\}$ *is consistent and for every* $i < \alpha$, $\varphi(x, a_i)$ *divides over* $\{a_j : j < i\}$. *If* $\varphi(x, a_i)$ k_i*-divides over* $\{a_j : j < i\}$*, we say that it is a dividing chain with respect to* $(k_i : i < \alpha)$. *We say that* $\varphi(x, y)$ *divides* α *times (with respect to* $(k_i : i < \alpha)$*) if there is a dividing chain of length* α *for* $\varphi(x, y)$ *(with respect to* $(k_i : i < \alpha)$*).*

Remark 4.9
1. $\varphi(x, y)$ *divides* ω *times with respect to* k *iff it has the tree property with respect to* k.

2. *If* $\varphi(x, y)$ *divides* n *times with respect to* k *for every* $n < \omega$*, then it divides* α *times with respect to* k *for every ordinal* α.

3. *If* $\varphi(x, y)$ *divides* ω_1 *times, then for some* $k < \omega$, $\varphi(x, y)$ *divides* ω *times with respect to* k.

Remark 4.10 *Clearly simplicity is equivalent to the non existence of formulas which divide* ω *times with respect to some fixed* k *and also to the non existence of formulas which divide* ω_1 *times (with respect to possibly varying* k*).*

Proposition 4.11 *The following conditions are equivalent to the simplicity of* T. *Here all the types are assumed to be in finitely many variables.*

1. *For every type* $p(x) \in S(A)$ *there is a* $B \subseteq A$ *such that* $|B| \leq |T|$ *and* p *does not divide over* B.

2. *There is some cardinal κ such that for every type $p(x) \in S(A)$ there is a $B \subseteq A$ such that $|B| \leq \kappa$ and p does not divide over B.*

3. *There is no increasing chain $(p_i(x) : i < |T|^+)$ of types $p_i(x) \in S(A_i)$ such that for every $i < |T|^+$, p_{i+1} divides over A_i.*

4. *For some cardinal κ there is no increasing chain $(p_i(x) : i < \kappa)$ of types $p_i(x) \in S(A_i)$ such that for every $i < \kappa$, p_{i+1} divides over A_i.*

Proof: Simplicity implies *1*, since if $p \in S(A)$ divides over every subset of A of cardinality $\leq |T|$, then we can inductively construct a sequence of formulas $(\varphi_i(x, y_i) : i < |T|^+)$ and a sequence $(a_i : i < |T|^+)$ of parameters $a_i \in A$ such that $\varphi_i(x, a_i) \in p$ and $\varphi_i(x, a_i)$ divides over $\{a_j : j < i\}$. Clearly one formula $\varphi(x, y)$ appears ω_1 times in the sequence and this contradicts simplicity. It is clear that *1* implies *2* and that *3* implies *4*. To show that *1* implies *3*, observe that if the increasing chain $(p_i(x) : i < |T|^+)$ is given and we set $A = \bigcup A_i$ and $p = \bigcup p_i$, then $p(x) \in S(A)$ divides over every subset of A of cardinality $\leq |T|$. The same argument proves *4* from *2*. It remains only to show simplicity from *4*. If T is not simple, then some formula $\varphi(x, y)$ divides κ times. Let $(a_i : i < \kappa)$ be a witness of this. Let a be a realization of $\{\varphi(x, a_i) : i < \kappa\}$, let $A_i = \{a_j : j < i\}$ and let $p_i = \mathrm{tp}(a/A_i)$. The chain $(p_i : i < \kappa)$ contradicts point *4*. □

Lemma 4.12 *Let $\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$, $D(\pi(x) \restriction A, \Delta, k) < \omega$ and $\pi(x) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi(x, a_i)$ divides over A with respect to k. Then $D(\pi(x), \Delta, k) < D(\pi(x) \restriction A, \Delta, k)$.*

Proof: By Proposition 3.10, $D(\pi(x), \Delta, k) \leq D(\pi(x) \restriction A \cup \{\varphi_i(x, a_i)\}, \Delta, k)$ for some i. Let $m = D(\pi(x) \restriction A \cup \{\varphi_i(x, a_i)\}, \Delta, k)$. By Lemma 3.8 $D(\pi(x) \restriction A, \Delta, k) \geq m + 1$. □

Proposition 4.13 *Simplicity is also equivalent to the conditions in Proposition 4.11 if we replace forking for dividing.*

Proof: Point *4* from Proposition 4.11 stated for forking (instead of dividing) implies its original version. The arguments in the proof of Proposition 4.11 showing that *1* implies *2* and *3* and that any of *2* and *3* implies *4* adapt to its version with forking. Moreover it is pretty clear that *3* implies *1* in any version. Hence it will be enough to prove that simple theories verify point *3* in this new version for forking. Assume $(p_i(x) : i < |T|^+)$ is an increasing chain of types $p_i(x) \in S(A_i)$ such that p_{i+1} forks over A_i for all $i < |T|^+$. This means that for all $i < |T|^+$ we can find some $\varphi_1^i(x), \ldots, \varphi_{n_i}^i(x)$ and some numbers $k_{1i}, \ldots, k_{n_i i}$ such that $p_{i+1}(x) \vdash \varphi_1^i(x) \vee \ldots \vee \varphi_{n_i}^i(x)$ and each $\varphi_j^i(x)$ k_{ji}-divides over A_i. Taking the maximum of the numbers, me may assume that they are all equal to some k_i. By counting types, numbers and formulas, we may assume that there are $n, k < \omega$ and some $\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n) \in L$ such that for all $i < |T|^+$, $n = n_i$, $k = k_i$ and there are tuples $a_1^i, \ldots, a_n^i \in A_{i+1}$ for which $\varphi_i(x, a_j^i) = \varphi_j^i(x)$. Let $\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$. By Lemma 4.12 $D(p_i(x), \Delta, k) > D(p_{i+1}(x), \Delta, k)$ for all $i < |T|^+$, which is a contradiction since the rank is finite. □

Corollary 4.14 *Let x be a finite tuple of variables. If T is simple and $p(x) \in S(A)$, then p does not fork over A. Hence, for any $B \supseteq A$ there is a nonforking extension $q(x) \in S(B)$ of p.*

Proof: By Proposition 4.13 and point *4* in Remark 4.4. □

Chapter 5

Independence

Definition 5.1 *We say that A is independent of B over C (written $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$) if for every finite sequence $a \in A$, $\mathrm{tp}(a/BC)$ does not fork over C. In the case $C = \emptyset$ we write $A \mathop{\smile\!\!\!\!\!\!\mid} B$.*

Remark 5.2 *If instead of sets A, B, C we put partially, or everywhere, sequences a, b, c in the independence relation we mean the independence of the enumerated sets. But it is a fact easy to prove that $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $\mathrm{tp}(a/BC)$ does not fork over C for some (any) enumeration a of A.*

Remark 5.3 *The independence relation is invariant under automorphisms and has always the following properties:*

Normality: *$A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ iff $A \mathop{\smile\!\!\!\!\!\!\mid}_C CB$. $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ iff $AC \mathop{\smile\!\!\!\!\!\!\mid}_C B$.*

Finite character: *If $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$ for all finite $a \in A$, $b \in B$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.*

Base monotonicity: *If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $B' \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_{CB'} B$.*

Monotonicity: *If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_C B'$.*

Anti-reflexivity: *If $A \mathop{\smile\!\!\!\!\!\!\mid}_B A$, then $A \subseteq \mathrm{acl}(B)$.*

Proposition 5.4 (Local character) *Let T be simple. For any B, C there is some $A \subseteq B$ such that $|A| \leq |T| + |C|$ and $C \mathop{\smile\!\!\!\!\!\!\mid}_A B$.*

Proof: This is clear for finite C by Proposition 4.13. For the general case, choose first some $A_X \subset B$ such that $|A_X| \leq |T|$ and $X \mathop{\smile\!\!\!\!\!\!\mid}_{A_X} B$ for each finite $X \subseteq C$ and let A be the union of all these sets A_X. Then $|A| \leq |T| + |C|$ and $C \mathop{\smile\!\!\!\!\!\!\mid}_A B$. □

Proposition 5.5 (Closedness) *Let $A \subseteq B$. The set of all complete types $p(x) \in S(B)$ which do not fork over A is a closed set in $S(B)$.*

Proof: Let $\pi(x)$ be the set of all negations $\neg\varphi(x)$ of all formulas $\varphi(x) \in L(B)$ which fork over A. Then $p(x) \in S(B)$ does not fork over A if and only if p extends π. □

Proposition 5.6 (Extension) *Let T be simple and let a be a possibly infinite sequence. If $A \subseteq B$, there is some $a' \equiv_A a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_A B$.*

Proof: If a is a finite tuple this follows easily from Corollary 4.14. The general case follows from the finite case since if $p(x) = \mathrm{tp}(a/A)$ it is enough to prove the consistency of $p(x) \cup \pi(x)$ where $\pi(x)$ is, as in the proof of Proposition 5.5, the set of all $\neg\varphi(x)$ such that $\varphi(x) \in L(B)$ forks over A. □

Remark 5.7 *A type $p(x) \in S(B)$ which does not fork over $A \subseteq B$ has also a global nonforking extension $\mathfrak{p}(x) \in S(\mathfrak{C})$ which does not fork over A. Thus in a simple theory any type has a global nonforking extension.*

Definition 5.8 *Let X be a linearly ordered set. The sequence $(a_i : i \in X)$ is* A-independent *if for every $i \in X$, $a_i \mathop{\smile\!\!\!\!\!\!|}_A \{a_j : j < i\}$. A* Morley sequence over A *is a sequence $(a_i : i \in X)$ which is A-independent and A-indiscernible. It is said to be a Morley sequence in the type $p \in S(A)$ if it is a Morley sequence over A and every a_i realizes p.*

Remark 5.9 *Let X be an infinite linearly ordered set and let $(a_i : i \in X)$ be a Morley sequence in $p(x) \in S(A)$. The sequence is infinite (i.e., $a_i \neq a_j$ for all $i \neq j$) if and only if p is nonalgebraic.*

Lemma 5.10 *If $p(x) \in S(B)$ does not fork over $A \subseteq B$, there is a Morley sequence $(a_i : i < \omega)$ in p which is moreover a Morley sequence over A. Clearly, if p is not algebraic, the sequence is infinite, in the sense that $a_i \neq a_j$ for $i < j < \omega$.*

Proof: Let α be the length of x and let $\kappa = |B| + |T| + |\alpha|$ and $\lambda = \beth_{(2^\kappa)^+}$. Since $p(x)$ does not fork over A, one can construct a sequence $(a_i : i < \lambda)$ of realizations a_i of p such that $a_i \mathop{\smile\!\!\!\!\!\!|}_A B\{a_j : j < i\}$. For this we choose a global extension $\mathfrak{p}$ of p which does not fork over A (see the remark after Proposition 5.6) and we take as a_i a realization of $\mathfrak{p} \restriction B\{a_j : j < i\}$. By Erdös-Rado Theorem (see Proposition 1.1) there is a B-indiscernible sequence $(b_i : i < \omega)$ of realizations of p such that for each $n < \omega$ there are $i_0 < \ldots < i_n < \lambda$ such that

$$b_0, \ldots, b_n \equiv_B a_{i_0}, \ldots, a_{i_n}.$$

Since $(a_i : i < \lambda)$ is A-independent, it follows that $(b_i : i < \omega)$ is also A-independent and hence it is a Morley sequence over A. But $(a_i : i < \lambda)$ is B-independent too and this also transfers to $(b_i : i < \omega)$. Hence it is a Morley sequence in p. □

Remark 5.11 *Let $p(x) \in S(A)$. If there is a Morley sequence $(a_i : i < \omega)$ in p, then for any linearly ordered set X there is a Morley sequence $(b_i : i \in X)$ in p. It is enough to obtain $(b_i : i \in X)$ as an A-indiscernible sequence with the same Ehrenfeucht-Mostowski set as $(a_i : i < \omega)$.*

Lemma 5.12 *Let $(a_i : i \in X)$ be A-independent. If Y, Z are subsets of X such that $Y < Z$, then $\mathrm{tp}((a_i : i \in Z)/A(a_i : i \in Y))$ does not divide over A.*

Proof: It can be assumed that Z is finite. An induction on $|Z|$ using Lemma 4.6 gives easily the result. □

Proposition 5.13 *Let T be simple. The formula $\varphi(x, a)$ divides over A iff for every infinite Morley sequence $(a_i : i < \omega)$ over A in $\mathrm{tp}(a/A)$, $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent.*

Proof: Without loss of generality $A = \emptyset$. Assume that $\varphi(x, a)$ divides over $\emptyset$ but for some infinite Morley sequence the inconsistency fails. Let X be a linearly ordered set isomorphic to the reverse order of the cardinal $|T|^+$. By compactness there is an infinite Morley sequence $a_X = (a_i : i \in X)$ in $\mathrm{tp}(a)$ such that $\{\varphi(x, a_i) : i \in X\}$ is consistent. Let c realize this type. By simplicity there is $Y \subseteq X$ of cardinality at most $|T|$ such that $\mathrm{tp}(c/a_X)$ does not fork over $a_Y = (a_i : i \in Y)$. By choice of the order of X we can find $i \in X$ such that $i < Y$. By Lemma 5.12 $\mathrm{tp}(a_Y/a_i)$ does not divide over $\emptyset$. Since $\varphi(x, a_i)$ divides over $\emptyset$, by Proposition 4.7 it divides over a_Y. But $\mathrm{tp}(c/a_X)$ contains $\varphi(x, a_i)$ and hence it divides (and forks) over a_Y, a contradiction. □

Proposition 5.14 *Let T be simple. A partial type $\pi(x)$ divides over A iff it forks over A.*

Proof: We may assume $\pi(x)$ is a formula $\varphi(x, a)$ and that a is not algebraic over A. Assume $\varphi(x, a)$ does not divide over A but it implies a disjunction $\varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi(x, a_i)$ divides over A. Let $(a^j a_1^j \ldots a_n^j : j < \omega)$ be an infinite Morley sequence in $\mathrm{tp}(aa_1 \ldots a_n/A)$ (a nonalgebraic type). Then $(a^j : j < \omega)$ is an A-indiscernible sequence of realizations of $\mathrm{tp}(a/A)$. By definition of dividing, there exists a realization c of $\{\varphi(x, a^j) : j < \omega\}$. For every $j < \omega$ there is some i such that c realizes some $\varphi_i(x, a_i^j)$. By the pigeonhole principle, there is some i such that for an infinite subset $J \subseteq \omega$, c realizes every $\varphi_i(x, a_i^j)$ with $j \in J$. By indiscernibility, $\{\varphi_i(x, a_i^j) : j < \omega\}$ is consistent and then by Proposition 5.13 $\varphi_i(x, a_i)$ does not divide over A since $(a_i^j : j < \omega)$ is an infinite Morley sequence in $\mathrm{tp}(a_i/A)$. □

Proposition 5.15 (Symmetry) *In a simple theory independence is a symmetric relation, i.e, $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ implies $B \mathop{\smile\!\!\!\!\!\!\;|}_C A$.*

Proof: It is enough to prove that if $\mathrm{tp}(a/Cb)$ does not fork over C, then $\mathrm{tp}(b/Ca)$ does not divide over C. We may assume that $\mathrm{tp}(a/C)$ is not algebraic. By Lemma 5.10 there is an infinite Morley sequence $I = (a_i : i < \omega)$ in $\mathrm{tp}(a/C)$ which is Cb-indiscernible and starts with $a_0 = a$. Let $\varphi(x, y, z)$ be a formula and $c \in C$ such that $\models \varphi(a, b, c)$. We will show that $\varphi(a, y, c)$ does not divide over C. By indiscernibility of I over Cb we know that $\models \varphi(a_i, b, c)$ for all $i < \omega$. Hence $\{\varphi(a_i, y, c) : i < \omega\}$ is consistent. Since $(a_i c : i < \omega)$ is a Morley sequence in $\mathrm{tp}(ac/C)$, by Proposition 5.13 we conclude that $\varphi(a, y, c)$ does not divide over C. □

Proposition 5.16 (Transitivity) *In a simple theory independence is a transitive relation, i.e, whenever $B \subseteq C \subseteq D$, $A \mathop{\smile\!\!\!\!\!\!\;|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_C D$, then $A \mathop{\smile\!\!\!\!\!\!\;|}_B D$.*

Proof: It is a direct consequence of Proposition 5.15, Lemma 4.6 and Proposition 5.14. □

Corollary 5.17 *Let T be simple. If I is an ordered set and $(a_i : i \in I)$ is an A-independent sequence, then $a_i \mathop{\smile\!\!\!\!\!\!\;|}_A \{a_j : j \neq i\}$ for all $i \in I$.*

Proof: By induction on n it is easy to show that for all different $i_1, \ldots, i_{n+1} \in I$, $a_{i_{n+1}} \mathop{\smile\!\!\!\!\!\!\;|}_A a_{i_1}, \ldots, a_{i_n}$. For the inductive case one uses symmetry and Lemma 4.6. □

Proposition 5.18 *Let T be simple, $p(x) \in S(A)$, $A \subseteq B$ and let $\pi(x)$ be a partial type over B. Then $p(x) \cup \pi(x)$ does not fork over A if and only if $D(p, \Delta, k) = D(p \cup \pi, \Delta, k)$ for all Δ, k.*

Proof: The direction from right to left follows from Lemma 4.12. Now assume $p \cup \pi$ is a nonforking extension of p and choose $q(x) \in S(B)$ a type which does not fork over A and extends $p \cup \pi$. We will check that $D(q, \varphi, k) \geq D(p, \varphi, k)$ for all φ, k. From this it will follow that $D(p, \varphi, k) \geq D(p \cup \pi, \varphi, k)$ for all φ, k. We freely use transitivity and symmetry of independence and also the fact that dividing and forking coincide. Let $n = D(p, \varphi, k)$. There is a sequence $(b_i : i < n)$ such that $p(x) \cup \{\varphi(x, b_i) : i < n\}$ is consistent and $\varphi(x, b_i)$ k-divides over $A\{b_j : j < i\}$ for all $i < n$. Let $a \models p(x) \cup \{\varphi(x, b_i) : i < n\}$, let $c \models q$ and let B' be such that $cB \equiv_A aB'$ and $B' \mathop{\smile\!\!\!\!\!\mid}_{Aa} \{b_i : i < n\}$. Then, since $B \mathop{\smile\!\!\!\!\!\mid}_A c$, it follows that $B' \mathop{\smile\!\!\!\!\!\mid}_A a$ and therefore $B' \mathop{\smile\!\!\!\!\!\mid}_A \{b_i : i < n\}$. By Proposition 4.7 $\varphi(x, b_i)$ k-divides over $B'\{b_j : j < i\}$ for all $i < n$. For $q' = \mathrm{tp}(a/B')$ we have then $D(q', \varphi, k) \geq n$. Since q is a conjugate of q', also $D(q, \varphi, k) \geq n$. $\square$

Corollary 5.19 *Let T be simple. Assume $p(x) \in S(A)$ and let $\pi(x, y)$ be a partial type over $\emptyset$. There is a partial type $\Delta(y)$ over A such that for all a, $p(x) \cup \pi(x, a)$ does not fork over A if and only if $\models \Delta(a)$.*

Proof: For any $\varphi = \varphi(x, y) \in L$ and $k < \omega$, let $n_{\varphi,k} = D(p(x), \varphi, k)$. By Proposition 5.18 we know that $p(x) \cup \pi(x, a)$ does not fork over A if and only if for all φ, k, $D(p(x) \cup \pi(x, a), \varphi, k) \geq n_{\varphi,k}$, which can expressed by a partial type over A. $\square$

Corollary 5.20 *Let T be simple and fix $p(x) \in S(A)$.*

1. *For any $n < \omega$ there is a partial type $\Phi(x_1, \ldots, x_n)$ over A such that for any tuple $a_1, \ldots, a_n$ of realizations of p, $\models \Phi(a_1, \ldots, a_n)$ if and only if $(a_1, \ldots, a_n)$ is A-independent.*

2. *For any totally ordered set I there is a partial type $\Phi(x_i : i \in I)$ such that for any sequence $(a_i : i \in I)$, $\models \Phi(a_i : i \in I)$ if and only if $(a_i : i \in I)$ is a Morley sequence in p.*

Proof: *2* follows from *1* and *1* is proved similarly to Corollary 5.19. $\square$

Chapter 6

The local rank $CB_\Delta(\pi)$

Definition 6.1 *Let $\pi(x)$ be a set of formulas over the set A and let*

$$\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$$

be a finite set of formulas $\varphi_i(x, y_i) \in L$. Let m be the length of the tuple of variables x. Since the restriction map $S_m(\mathfrak{C}) \to S_\Delta(\mathfrak{C})$ is closed and the class

$$X_{\pi,\Delta} = \{\mathfrak{p} \in S_\Delta(\mathfrak{C}) : \mathfrak{p}(x) \cup \pi(x) \textit{ is consistent } \}$$

is the image of the closed class $\{\mathfrak{p} \in S_m(\mathfrak{C}) : \pi(x) \subseteq \mathfrak{p}\}$, $X_{\pi,\Delta}$ is closed in $S_\Delta(\mathfrak{C})$. We define the Δ-rank $CB_\Delta(\pi)$ as the Cantor-Bendixson rank of $X_{\pi,\Delta}$ in $S_\Delta(\mathfrak{C})$ and the Δ-multiplicity $Mlt_\Delta(\pi)$ as its Cantor-Bendixson degree.

Lemma 6.2 *If $\pi_1(x) \vdash \pi_2(x)$, then $CB_\Delta(\pi_1) \leq CB_\Delta(\pi_2)$ and in case $CB_\Delta(\pi_1) = CB_\Delta(\pi_2)$, then $Mlt_\Delta(\pi_1) \leq Mlt_\Delta(\pi_2)$.*

Proof: Clear, because if $X_{\pi_i,\Delta} = \{\mathfrak{p} \in S_\varphi(\mathfrak{C}) : \mathfrak{p}$ is consistent with $\pi_i\}$, then $X_{\pi_1,\Delta} \subseteq X_{\pi_2,\Delta}$. □

Remark 6.3 *For each $\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$ where $\varphi_i(x, y_i) \in L$ for every i, there is a formula $\psi_\Delta(x; z) \in L$ such that for each partial type $\pi(x)$, $CB_\Delta(\pi) = CB_{\psi_\Delta}(\pi)$ and $Mlt_\Delta(\pi) = Mlt_{\psi_\Delta}(\pi)$.*

Proof: By Lemma 3.4. □

Proposition 6.4 *Let $\psi(x)$ be a boolean combination of Δ-formulas.*

1. *$CB_\Delta(\psi) \geq 0$ if and only if ψ is consistent.*
2. *$CB_\Delta(\psi) \geq \alpha + 1$ if and only if there is a sequence $(\psi_i(x) : i < \omega)$ of pairwise contradictory boolean combinations $\psi_i(x)$ of Δ-formulas such that $CB_\Delta(\psi(x) \wedge \psi_i(x)) \geq \alpha$ for all $i < \omega$.*
3. *$CB_\Delta(\psi) \geq \alpha$ for limit α if and only if $CB_\Delta(\psi) \geq \beta$ for all $\beta < \alpha$.*
4. *If $CB_\Delta(\psi) = \alpha < \infty$, then $Mlt_\Delta(\psi)$ is the largest $n < \omega$ for which there is a sequence $(\psi_i(x) : i < n)$ of pairwise contradictory boolean combinations $\psi_i(x)$ of Δ-formulas such that $CB_\Delta(\psi(x) \wedge \psi_i(x)) \geq \alpha$ for all $i < n$.*

The formulas ψ_i in 2 and 4 can be chosen as explicitly contradictory conjunctions of Δ-formulas. Moreover in 2 we can fix some $\varphi \in \Delta$ such that each ψ_i is a conjunction of φ-formulas.

Proof: Let $X_{\Delta,\psi}$ be the clopen subset of $S_\Delta(\mathfrak{C})$ of all types $\mathfrak{p} \in S_\Delta(\mathfrak{C})$ such that $\mathfrak{p} \vdash \psi$. $CB_\Delta(\psi)$ is the maximal Cantor-Bendixson rank (in $S_\Delta(\mathfrak{C})$) of the points of $X_{\Delta,\psi}$. Points *1* and *3* are clear. The proof of *4* is similar to the proof of *2*, so we restrict ourselves to *2*. Assume first there are $\psi_i(x)$ $(i < \omega)$ pairwise contradictory boolean combinations of Δ-formulas such that $CB_\Delta(\psi \wedge \psi_i) \geq \alpha$ for each $i < \omega$. For each i choose some $\mathfrak{p}_i \in S_\Delta(\mathfrak{C})$ of Cantor-Bendixson rank and such that $\mathfrak{p}_i \vdash \psi \wedge \psi_i$. Since the ψ_i are pairwise contradictory, all the $\mathfrak{p}_i$ are different. Since $X_{\Delta,\psi}$ contains infinitely many points of rank $\geq \alpha$, it contains some point of rank $\geq \alpha + 1$. Hence $CB_\Delta(\psi) \geq \alpha + 1$.

For the other direction, assume $CB_\Delta(\psi) \geq \alpha + 1$. Then $X_{\Delta,\psi}$ is an open set containing a point of rank $\geq \alpha + 1$. Thus the set Y_0 of points of $X_{\Delta,\psi}$ of rank $\geq \alpha$ is infinite. Clearly, for some Δ-formula θ there are points in Y_0 containing θ and points in Y_0 containing $\neg\theta$ and one of them, say the second one, is infinite. Let then $\theta_0 = \theta$ and let Y_1 be the infinite subset of Y_0 consisting of all point containing $\neg\theta_0$. Now assume that Y_i, ψ_i are defined for all $i \leq n$, that the Y_i build a strictly descending chain of infinite sets, and that Y_{i+1} is the subset of Y_i consisting in all its points containing the Δ-formula $\neg\theta_i$. Again, there is some Δ-formula θ_{n+1} such that some points of Y_{n+1} contain θ_{n+1} and infinitely many points of Y_{n+1} contain $\neg\theta_{n+1}$. For some infinite subset $I \subseteq \omega$ there is a $\varphi \in \Delta$ such that for each $i \in I$, ψ_i is a φ-formula. Without loss of generality, $I = \omega$. We then put $\psi_n = \theta_n \wedge \bigwedge_{i<n} \neg\theta_i$.

□

Proposition 6.5 *Fix Δ and $\pi(x)$.*

1. *There is a boolean combination ψ of Δ-formulas such that $\pi(x) \vdash \psi(x)$, $CB_\Delta(\pi) = CB_\Delta(\psi)$, and $Mlt_\Delta(\pi) = Mlt_\Delta(\psi)$.*

2. *If $\pi(x)$ is over A, it can be extended to a complete type $p(x) \in S(A)$ such that $CB_\Delta(\pi) = CB_\Delta(p)$.*

Proof: *1.* Let $X = \{\mathfrak{p} \in S_\Delta(\mathfrak{C}) : \pi(x) \cup \mathfrak{p}(x)$ is consistent $\}$ be the closed set in $S_\Delta(\mathfrak{C})$ whose Cantor-Bendixson rank determines $CB_\Delta(\pi)$. By general topological reasons, there is a clopen set $U \supseteq X$ of the same Cantor-Bendixson rank and degree. The boolean combination of Δ-formulas $\psi(x)$ such that $U = \{\mathfrak{p} \in S_\Delta(\mathfrak{C}) : \mathfrak{p} \vdash \psi\}$ is the required formula.

2. Take $\mathfrak{p}(x) \in S_\Delta(\mathfrak{C})$ consistent with $\pi(x)$ and of Cantor-Bendixson rank $CB_\Delta(\pi)$ and take any extension $p(x) \in S(A)$ consistent with $\mathfrak{p}(x)$. Clearly $CB_\Delta(p)$ is still the rank of $\mathfrak{p}$.
□

Proposition 6.6 *Those following are equivalent:*

1. *Every $\varphi(x, y) \in \Delta$ is stable.*

2. *$CB_\Delta(x = x) < \omega$*

3. *$CB_\Delta(x = x) < \infty$.*

Proof: Stability of every $\varphi \in \Delta$ means that for each infinite set A, $|S_\Delta(A)| \leq |A|$. It is therefore equivalent to the stability of the formula ψ_Δ given by Lemma 3.4. Hence we may assume that $\Delta = \{\varphi\}$. By Proposition 2.5, stability of φ is equivalent to the inconsistency of the set of formulas $\Gamma_\varphi(\omega)$, where for each ordinal α,

$$\Gamma_\varphi(\alpha) = \{\varphi^{\eta(i)}(x_\eta, y_{\eta\restriction i}) : \eta \in 2^\alpha,\, i < \alpha\}$$

and where $\varphi^0 = \varphi$ and $\varphi^1 = \neg\varphi$. Clearly *2* implies *3*.

1 $\Rightarrow$ *2*. Assume $CB_\varphi(x = x) \geq \omega$. If $\psi(x)$ is a boolean combination of φ-formulas and $CB_\varphi(\psi) \geq n{+}1$ then for some a, $CB_\varphi(\psi(x) \wedge \varphi(x, a)) \geq n$ and $CB_\varphi(\psi \wedge \neg\varphi(x, a)) \geq n$. Since $CB_\varphi(x = x) \geq \omega$ this can be used to construct a binary tree of parameters $(a_s : s \in 2^{<n})$ such that for each $s \in 2^n$ the branch $\{\varphi^{s(i)}(x, a_{s\restriction i}) : i < n\}$ is consistent. This implies that $\Gamma_\varphi(n)$ is consistent. By compactness $\Gamma_\varphi(\omega)$ is consistent and hence φ is unstable.

3 $\Rightarrow$ *1*. Assume φ is unstable but $CB_\varphi(x = x) < \infty$. Hence $\Gamma_\varphi(\omega)$ is consistent and we way find parameters $(a_s : s \in 2^{<\omega})$ such that for each $\eta \in 2^\omega$ the branch $\{\varphi^{\eta(i)}(x, a_{\eta\restriction i}) : i < \omega\}$ is consistent. For any $s \in 2^{<\omega}$, let

$$\psi_s(x) = \bigwedge_{i<n} \varphi^{s(i)}(x, a_{s\restriction i})$$

and choose s for which ψ_s has minimal CB_φ-rank and least Mlt_φ among the formulas with same rank. But $\psi_s(x)$ is equivalent to $(\psi_{s^\frown 0}(x) \vee \psi_{s^\frown 1}(x))$ and the formulas $\psi_{s^\frown 0}$, $\psi_{s^\frown 1}$ are incompatible. So one of them has smaller CB_φ-rank or they have the same rank and one has smaller multiplicity Mlt_φ, a contradiction. $\square$

Remark 6.7 *Let $\varphi = \varphi(x, y) \in L$ be stable, let $\pi(x)$ be a partial type over A, and let $\mathfrak{p} \in S_\varphi(\mathfrak{C})$ be consistent with $\pi(x)$ and of Cantor-Bendixson rank $CB_\varphi(\pi)$. Then $\mathfrak{p}$ is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$. *If $Mlt_\varphi(\pi) = 1$ it is also A-definable.*

Proof: By stability of φ, $\mathfrak{p}$ is definable (see Remark 2.6). All the A-conjugates of $\mathfrak{p}$ have Cantor-Bendixson rank $CB_\varphi(\pi)$ and are consistent with $\pi(x)$; again by stability, its number is bounded by $Mlt_\varphi(\pi) < \omega$. Since $\mathfrak{p}$ has finitely many A-conjugates, by Proposition 1.3 $\mathfrak{p}$ is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable. In case $Mlt_\varphi(\pi) = 1$, $\mathfrak{p}$ is A-invariant and therefore A-definable. $\square$

Lemma 6.8 *Let $\varphi(x, y)$ be stable. Then $\varphi^{-1}(y, x) = \varphi(x, y)$ (changing the role of the variables) is also a stable formula. Let $\mathfrak{p}(x) \in S_\varphi(\mathfrak{C})$ and $\mathfrak{q}(y) \in S_{\varphi^{-1}}(\mathfrak{C})$ and let $d_\mathfrak{p} x\varphi(x, y)$ and $d_\mathfrak{q} y\varphi(x, y)$ be corresponding definitions of $\mathfrak{p}$ and $\mathfrak{q}$ which are boolean combinations of φ^{-1}-formulas and of φ-formulas respectively. Then $\mathfrak{q} \vdash d_\mathfrak{p} x\varphi(x, y)$ if and only if $\mathfrak{p} \vdash d_\mathfrak{q} y\varphi(x, y)$.*

Proof: Let A be a set containing all the parameters of the formulas $d_\mathfrak{p} x\varphi(x, y)$ and $d_\mathfrak{q} y\varphi(x, y)$ defining respectively $\mathfrak{p}$ and $\mathfrak{q}$. Let $(a_n : n \in \omega)$ and $(b_n : n \in \omega)$ be sequences such that $a_n \models \mathfrak{p} \restriction A\{b_i : i < n\}$ and $b_n \models \mathfrak{q} \restriction A\{a_i : i \leq n\}$. In case $d_\mathfrak{p} x\varphi(x, y) \in \mathfrak{q}$ and $d_\mathfrak{q} y\varphi(x, y) \notin \mathfrak{p}$, we would have $\models \varphi(a_m, b_n)$ if and only if $m > n$, and therefore $\varphi(x, y)$ would have the order property. $\square$

Proposition 6.9 *Let φ be stable.*

1. *If $p(x) \in S_\varphi(M)$, then there is only one $\mathfrak{p} \in S_\varphi(\mathfrak{C})$ extending p which is definable over M and hence $Mlt_\varphi(p) = 1$.*

2. *If $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $p(x) \in S(A)$, there is only one $\mathfrak{p} \in S_\varphi(\mathfrak{C})$ consistent with p and definable over A and hence $Mlt_\varphi(p) = 1$.*

Proof: *1.* It is Lemma 2.3 but also a particular case of *2.*

2 Existence follows from Remark 6.7. For the uniqueness, let $\mathfrak{p}_1, \mathfrak{p}_2 \in S_\varphi(\mathfrak{C})$ be two global φ-types consistent with p and A-definable and let $\psi_i(y) \in L(A)$ $(i = 1, 2)$ be corresponding definitions. Fix some model $M \supseteq A$. By Remark 2.12, $\mathfrak{p}_i$ is definable by some $\psi_i'(y) \in L(M)$ which is a positive boolean combination of formulas of the form $\varphi(m, y)$ with $m \in M$, that is, of $\varphi^{-1}(y, x)$-formulas over M (using the notation of Lemma 6.8). Clearly, ψ_i and ψ_i' are equivalent. Let b be a tuple of the same length as y and let us choose (by Remark 6.7) a global type $\mathfrak{q}(y) \in S_{\varphi^{-1}}(\mathfrak{C})$ consistent with $\mathrm{tp}(b/A)$ and definable over A by a formula $\theta(x) \in L(A)$. By Remark 2.12 again, $\mathfrak{q}$ is in fact definable by a positive boolean combination $\theta'(x)$ of φ-formulas over M. Thus $\theta(x)$ is equivalent to $\theta'(x)$. We apply now Lemma 6.8 with $\psi_i'(y) = d_{\mathfrak{p}_i} x\varphi(x, y)$ and $\theta'(x) = d_{\mathfrak{q}} y\varphi(x, y)$ obtaining:

$\varphi(x, b) \in \mathfrak{p}_i$	$\Leftrightarrow$	$\models \psi_i(b)$	because ψ_i defines $\mathfrak{p}_i$
	$\Leftrightarrow$	$\psi_i(y) \in \mathrm{tp}(b/A)$	because $\psi_i(y) \in L(A)$
	$\Leftrightarrow$	$\mathfrak{q} \vdash \psi_i'(y)$	because $\psi_i \equiv \psi_i'$, $\mathfrak{q}(y) \cup \mathrm{tp}(b/A)$ is consistent and ψ_i' is a boolean combination of φ^{-1}-formulas
	$\Leftrightarrow$	$\mathfrak{p}_i \vdash \theta'(x)$	by Lemma 6.8
	$\Leftrightarrow$	$\theta \in p$	because $\theta \equiv \theta'$, $\theta(x) \in L(A)$, $p(x) \cup \mathfrak{p}_i$ is consistent and θ' is a boolean combination of φ-formulas.

This shows that $\mathfrak{p}_1 = \mathfrak{p}_2$. □

Corollary 6.10 *Let $\varphi = \varphi(x, y) \in L$ be stable and let $p(x) \in S(A)$. Every two $\mathfrak{p}(x), \mathfrak{q}(x) \in S_\varphi(\mathfrak{C})$ consistent with $p(x)$ and definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$ *are A-conjugate.*

Proof: Let $\mathfrak{p}, \mathfrak{q}$ be two such types. Let $p_1, q_1 \in S(\mathrm{acl}^{\mathrm{eq}}(A))$ extensions of p such that both $p_1(x) \cup \mathfrak{p}(x)$ and $q_1(x) \cup \mathfrak{q}(x)$ are consistent. Clearly there is some $f \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $p_1^f = q_1$. Then $\mathfrak{q}$ and $\mathfrak{p}^f$ are $\mathrm{acl}^{\mathrm{eq}}(A)$-definable and consistent with q_1. By Proposition 6.9 $\mathfrak{p}^f = \mathfrak{q}$. □

Corollary 6.11 *Let $\varphi = \varphi(x, y) \in L$ be stable, let $p(x) \in S(A)$. For any $\mathfrak{p}(x) \in S_\varphi(\mathfrak{C})$ consistent with p, the following are equivalent:*

1. *$\mathfrak{p}$ is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.
2. *$\mathfrak{p}$ is a point of Cantor-Bendixson rank $CB_\varphi(p)$.*

In case $Mlt_\varphi(p) = 1$ there is only one such $\mathfrak{p} \in X_{p,\varphi}$ and it is in fact A-definable.

Proof: By Remark 6.7 we know that all types in $X_{p,\varphi}$ of rank $CB_\varphi(p)$ are definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Now let $\mathfrak{p}, \mathfrak{q} \in S_\varphi(\mathfrak{C})$ consistent with $p(x)$ be such that $\mathfrak{p}$ is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable and $\mathfrak{q}$ has Cantor-Bendixson rank $CB_\varphi(p)$ in $X_{p,\varphi}$. By Corollary 6.10 they are A-conjugate and therefore $\mathfrak{p}$ has also rank $CB_\varphi(p)$ in $X_{p,\varphi}$. □

Chapter 7

Heirs and coheirs

Definition 7.1 *Let $M \subseteq A$ and $p(x) \in S(A)$. We say that p is a* heir *of $p \restriction M$ or that p heirs from M if for every $\varphi(x,y) \in L(M)$ if $\varphi(x,a) \in p$ for some $a \in A$, then $\varphi(x,m) \in p$ for some $m \in M$. We say that p is a* coheir *of $p \restriction M$ or that p coheirs from M if p is finitely satisfiable in M. The same definitions apply to global types, i.e, to the case $A = \mathfrak{C}$. This definitions make also sense for types in infinitely many variables.*

Remark 7.2 $\mathrm{tp}(a/Mb)$ *heirs from M if and only if* $\mathrm{tp}(b/Ma)$ *coheirs from M.*

Proof: It is just a matter of writing the definitions. $\square$

Lemma 7.3 *1. If $p(x) \in S(M)$, then p heirs and coheirs from M.*

2. If $M \subseteq A$ and $p(x) \in S(A)$ coheirs from M, then for every $B \supseteq A$ there is some $q(x) \in S(B)$ such that $p \subseteq q$ and q coheirs from M.

3. If $M \subseteq A$ and $p(x) \in S(A)$ heirs from M, then for every $B \supseteq A$ there is some $q(x) \in S(B)$ such that $p \subseteq q$ and q heirs from M.

Proof: *1* is clear. For *2* it is enough to check the consistency of the following set of formulas

$$p(x) \cup \{\neg\varphi(x) : \varphi(x) \in L(B) \text{ is not satisfiable in} M\}.$$

3. In this case it suffices to prove that the following set of formulas is consistent

$$p(x) \cup \{\neg\varphi(x,a) : \varphi(x,y) \in L(M),\ a \in A \text{ and there is no } b \in M \text{ such that } \varphi(x,b) \in p \restriction M\}$$

$\square$

Definition 7.4 *Let $p(x) \in S(B)$ and let $A \subseteq B$. We say that p* splits *over A if for some $\varphi(x,y) \in L(A)$ there are $a,b \in B$ such that $a \equiv_A b$, $\varphi(x,a) \in p$ and $\neg\varphi(x,b) \in p$. This applies also to the case $B = \mathfrak{C}$. Note that the same notion is defined if one requires $\varphi(x,y) \in L$. If $\mathfrak{p} \in S(\mathfrak{C})$, then clearly $\mathfrak{p}$ does not split over A if and only if $\mathfrak{p}^f = \mathfrak{p}$ for each $f \in \mathrm{Aut}(\mathfrak{C}/A)$. If moreover $A = M$, a global nonsplitting extension is also called a M-*special *extension.*

Proposition 7.5 *1. The number of global nonsplitting extensions of $p \in S(A)$ is $\leq 2^{2^{|A|+|T|}}$.*

2. *Let $\mathfrak{p}$ be a global nonsplitting extension of $p \in S(A)$. If the sequence $(a_i : i < \alpha)$ is constructed in such a way that for all $i < \alpha$,*

$$a_i \models \mathfrak{p} \restriction A\{a_j : j < i\},$$

then it is A-indiscernible.

Proof: *1.* For each $\varphi(x, y) \in L$, the number of restrictions $\mathfrak{p} \restriction \varphi$ for types $\mathfrak{p} \in S(\mathfrak{C})$ which do not split over A is bounded by the number of sets of types $\mathrm{tp}(a/A)$ of tuples $a \in \mathfrak{C}$ of the length of y. The number of these types is $\leq 2^{|A|+|T|}$ and therefore the number of set of types is $\leq 2^{2^{|A|+|T|}}$.

2. By induction on n we show that for all $i_0 < \ldots < i_n < \alpha$, $a_0, \ldots, a_n \equiv_A a_{i_0}, \ldots, a_{i_n}$. This is clear for $n = 0$ since $\mathrm{tp}(a_{i_0}/A) = p = \mathrm{tp}(a_0/A)$. Consider the case $n + 1$. Let $\varphi(x_0, \ldots, x_{n+1}) \in L(A)$, and let $i_0 < \ldots < i_{n+1} < \alpha$. By inductive hypothesis $a_0, \ldots, a_n \equiv_A a_{i_0}, \ldots, a_{i_n}$. Since $\mathfrak{p}$ does not split over A, $\varphi(a_0, \ldots, a_n, x) \in \mathfrak{p}$ if and only if $\varphi(a_{i_0}, \ldots, a_{i_n}, x) \in \mathfrak{p}$. Since $a_{n+1} \models \mathfrak{p} \restriction Aa_0, \ldots, a_n$ and $a_{i_{n+1}} \models \mathfrak{p} \restriction Aa_{i_0}, \ldots, a_{i_n}$, we conclude that $\models \varphi(a_0, \ldots, a_n, a_{n+1})$ if and only if $\models \varphi(a_{i_0}, \ldots, a_{i_n}, a_{i_{n+1}})$. □

Proposition 7.6 *1. Coheirs are nonsplitting extensions.*

2. *If $\mathfrak{p} \in S(\mathfrak{C})$ does not split over A, then it does not fork over A.*

3. *Coheirs are nonforking extensions.*

4. *If $p(x) \in S(M)$ is definable, then its M-definable extension over $A \supseteq M$ is the only heir of p over A.*

5. *In a simple theory, heirs are nonforking extensions.*

Proof: *1.* Suppose $p(x) \in S(A)$ coheirs from $M \subseteq A$, $a, b \in A$, $a \equiv_M b$, $\varphi(x, y) \in L(M)$ and $\varphi(x, a) \in p$ while $\neg\varphi(x, b) \in p$. Then some $c \in M$ satisfies $\varphi(x, a) \wedge \neg\varphi(x, b)$, which is impossible if $a \equiv_M b$.

2. For a global type forking and dividing is the same. Let $\varphi(x, y) \in L$. If $\varphi(x, a) \in \mathfrak{p}$ and $a_i \equiv_A a$ for each $i < \omega$, then $\varphi(x, a_i) \in \mathfrak{p}$ for each $i < \omega$ and hence $\{\varphi(x, a_i) : i < \omega\}$ is consistent.

3. This can be proved using points *1* and *2* and the extension property of coheirs, but it is also an immediate consequence of the definition of forking as indicated in Remark 4.4.

4. Let $p \in S(M)$ be definable. By the uniqueness of the M-definable extension, we only need to show that heirs are M-definable. Let $q \in S(A)$ be a heir of p, let $\varphi(x, y) \in L$ and $d_p x\varphi(x, y) \in L(M)$ a definition of $p \restriction \varphi$. We show that it is also a definition of $q \restriction \varphi$. If it is not a definition, then for some $a \in A$, $\neg(d_p x\varphi(x, a) \leftrightarrow \varphi(x, a)) \in q$ and therefore for some $a' \in M$, $\neg(d_p x\varphi(x, a') \leftrightarrow \varphi(x, a')) \in p$ contradicting the fact that $d_p x\varphi(x, y)$ defines $p \restriction \varphi$.

5. Let T be simple and assume $p(x) \in S(A)$ heirs from $M \subseteq A$. Let $a \in A$ be a tuple and let $b \models p$. We want to show that $b \mathop{\smile\!\!\!\!\!\!|}_A a$. By Remark 7.2 $\mathrm{tp}(a/Mb)$ coheirs from M and by point *3* $a \mathop{\smile\!\!\!\!\!\!|}_M b$. The result follows then by symmetry of independence. □

Definition 7.7 *Given $p(x) \in S(M)$, by (M, dp) we refer to the expansion of M to language $L \cup \{R_\varphi : \varphi \in L\}$ where for every $\varphi = \varphi(x, y) \in L$, if $y = y_1, \ldots, y_n$ then R_φ is n-ary and it is interpreted as $\{a \in M : \varphi(x, a) \in p\}$. Let $M \preceq N$ and $p(x) \subseteq q(x) \in S(N)$. We say that q is a* strong heir *of p is $(M, dp) \preceq (N, dq)$. This makes also sense when $N = \mathfrak{C}$.*

Remark 7.8 *1. Strong heirs are heirs.*

2. If $(M, dp) \preceq N'$ and $N = N' \restriction L$, then for some $q(x) \in S(N)$, $p \subseteq q$ and $N' = (N, dq)$.

3. Any strong heir of a nondefinable type is again nondefinable.

Proof: *1* and *2* are easy. For *3*, let $q(x) \in S(N)$ be a strong heir of p, and assume $\varphi(x, y) \in L$, $\psi(y, z) \in L$, $n \in N$ and $\psi(y, n)$ defines $q \restriction \varphi$. Then $(N, dq) \models \exists z \forall y(\psi(y, z) \leftrightarrow R_\varphi(y))$. Since $(M, d) \preceq (N, dq)$, for some $m \in M$, $(M, dp) \models \forall y(\psi(y, m) \leftrightarrow R_\varphi(y))$. Then $\psi(y, m)$ defines $p \restriction \varphi$. $\square$

Proposition 7.9 *If $p(x) \in S(M)$ is not definable, then $p(x)$ has unboundedly many (nondefinable) strong heirs over $\mathfrak{C}$.*

Proof: We show first that $p(x)$ has two strong heirs over some $N \succeq M$. Since p is not definable, (M, dp) is not a definable expansion of M. By Svenonious's Theorem, there is some $N' \succeq (M, dp)$ having some $f \in \mathrm{Aut}(N' \restriction L/M)$ such that $f \notin \mathrm{Aut}(N')$. Let $N = N' \restriction L$. Then for some $q(x) \in S(N)$, $N' = (N, dq)$ and $q^f \neq q$. Clearly q and q^f are two strong heirs of p. Since a strong heir of a nondefinable type is again nondefinable, we can iterate this procedure (taking unions at limits) obtaining for each cardinal κ a family $(p_i(x) : i < \kappa)$ of strong heirs $p_i \in S(M_i)$ of p such that $p_i \cup p_j$ is inconsistent if $i \neq j$. Clearly, each p_i can be extended to a type $\mathfrak{p}_i$ over $\mathfrak{C}$ which is a strong heir of p_i and therefore also of p. $\square$

Definition 7.10 *A* coheir sequence *over A is a sequence $(a_i : i < \alpha)$ such that for some $M \subseteq A$, for all $i < j < \alpha$, $\mathrm{tp}(a_i/A(a_l : l < i)) = \mathrm{tp}(a_j/A(a_l : l < i))$ and $\mathrm{tp}(a_j/A(a_l : l < j))$ coheirs from M.*

Remark 7.11 *1. A coheir sequence over A is a Morley sequence over A.*

2. For any $p(x) \in S(A)$ which coheirs from $M \subseteq A$ there is a coheir sequence $(a_i : i < \alpha)$ over A.

Proof: *1.* Let $p_i = \mathrm{tp}(a_i/A(a_l : l < i))$ and $p_\alpha = \bigcup_{i<\alpha} p_i$. Clearly $a_i \models p_\alpha \restriction A(a_l : l < i)$ and p_α coheirs from M. By point *3* of Proposition 7.6 the sequence is A-independent. By point *1* of Proposition 7.6 and point *2* of Proposition 7.5, it is A-indiscernible.

2. Choose an extension $\mathfrak{p} \in S(\mathfrak{C})$ of p which coheirs from M and choose $a_i \models \mathfrak{p} \restriction A(a_l : l < i)$. $\square$

Proposition 7.12 *Let $\mathfrak{p} \in S(\mathfrak{C})$ be definable. Then $\mathfrak{p}$ does not split over A if and only if it is A-definable.*

Proof: Let $\mathfrak{p}$ be definable and assume it does not split over A. For each $\varphi(x, y) \in L$, $\{a : \varphi(x, a) \in \mathfrak{p}\}$ is definable and A-invariant, and therefore it is definable over A. The other direction is immediate. $\square$

Corollary 7.13 *If $\mathfrak{p} \in S(\mathfrak{C})$ is definable over A, then $\mathfrak{p}$ does not fork over A.*

Proof: Is a consequence of Proposition 7.12 and point *2* of Proposition 7.6. $\square$

Corollary 7.14 *Let $p(x) \in S(M)$ and assume every complete extension of p is definable. Then an extension $q(x)$ of p is a heir of p if and only if it is a coheir of p if and only if it is M-definable.*

Proof: The equivalence of M-definability and heir is given by point *4* in Proposition 7.6. For the rest, by points *2* and *3* of Lemma 7.3 it is enough to check the result in the case of a global extension $\mathfrak{p} \in S(\mathfrak{C})$ of p. Then we can apply Proposition 7.12 and point *1* of Proposition 7.6 to prove that coheirs are heirs. The uniqueness of heirs (point *4* in Proposition 7.6) shows then that also heirs are coheirs. □

Corollary 7.15 *T is stable if and only if heirs are coheirs.*

Proof: If T is stable, Corollary 7.14 establishes that heirs are coheirs. If T is not stable, there is some $p(x) \in S(M)$ not definable. By Proposition 7.9 p has unboundedly many heirs. Since coheirs do no split, by Proposition 7.5 the number of coheirs of p is bounded by $2^{2^{|M|+|T|}}$. Hence some heir is not a coheir. □

Corollary 7.16 *The following are equivalent.*

1. *T is stable*
2. *Each type $p(x) \in S(M)$ has a unique heir over any $A \supseteq M$.*
3. *Each type $p(x) \in S(M)$ has a bounded number of heirs over any $A \supseteq M$.*

Proof: If T is stable, point *4* of Proposition 7.6 shows that p has a unique heir. If T is not stable, there is some $p(x) \in S(M)$ not definable. By Proposition 7.9 p has unboundedly many strong heirs over $\mathfrak{C}$. Clearly, for each $A \supseteq M$ for any strong heir $\mathfrak{p} \in S(\mathfrak{C})$ of p, $\mathfrak{p} \restriction A$ is a heir of p. □

Chapter 8

Stable forking

Proposition 8.1 *Let $\Delta = \{\varphi_i(x, y_i) : i < n\}$ be a set of stable formulas. A type $\mathfrak{p} \in S_\Delta(\mathfrak{C})$ is definable over M if and only if it is finitely satisfiable in M. It fact, if $\mathfrak{p}$ is M-definable and it is consistent with a partial type $\pi(x)$ over M, then $\pi(x) \cup \mathfrak{p}(x)$ is finitely satisfiable in M.*

Proof: We may assume $\Delta = \{\varphi(x, y)\}$. Let $\mathfrak{p}$ be M-definable and let us choose by Remark 2.12 a definition $d_\mathfrak{p} x\varphi(x, y) \in L(M)$, which is a positive boolean combination of formulas of the form $\varphi(b, y)$. Let

$$\varphi(x, a_1), \dots, \varphi(x, a_n), \neg\varphi(x, b_1), \dots, \neg\varphi(x, b_m)$$

be formulas in $\mathfrak{p}$. For $1 \leq i \leq n$ and $1 \leq j \leq m$, let $q_i = \mathrm{tp}_{\varphi^{-1}}(a_i/M)$ and $r_j = \mathrm{tp}_{\varphi^{-1}}(b_j/M)$. Again by Remark 2.12 there are $\mathfrak{q}_i \in S_{\varphi^{-1}}(\mathfrak{C})$ and $\mathfrak{r}_j \in S_{\varphi^{-1}}(\mathfrak{C})$ extending q_i and r_j respectively and having definitions $d_{\mathfrak{q}_i} y\varphi(x, y)$ and $d_{\mathfrak{r}_j} y\varphi(x, y)$ which are positive boolean combinations of formulas of the form $\varphi(x, b)$ with $b \in M$. Then $\models d_\mathfrak{p} x\varphi(x, a_i)$ and $\models \neg d_\mathfrak{p} x\varphi(x, b_j)$ and hence $\mathfrak{q}_i \vdash d_\mathfrak{p} x\varphi(x, y)$ and $\mathfrak{r}_j \vdash \neg d_\mathfrak{p} x\varphi(x, y)$. By Lemma 6.8, $\mathfrak{p} \vdash d_{\mathfrak{q}_i} y\varphi(x, y)$ and $\mathfrak{p} \vdash \neg d_{\mathfrak{r}_j} y\varphi(x, y)$. Since they are formulas over M, for some $c \in M$, $\models d_{\mathfrak{q}_i} y\varphi(c, y)$ and $\models \neg d_{\mathfrak{r}_j} y\varphi(c, y)$ for all i, j. Then $\models \varphi(c, a_i)$ and $\models \neg\varphi(c, b_j)$ for all i, j. Clearly such c can also be found realizing additionally a given finite subset of $\pi(x)$. For the other direction, let us assume $dx\varphi(x, y)$ is a definition of $\mathfrak{p}$ which is not equivalent to a formula over M. Then we can find b, c such that $b \equiv_M c$ and $\models dx\varphi(x, b)$ but $\models \neg dx\varphi(x, c)$. In this case $\varphi(x, b) \in \mathfrak{p}$ and $\neg\varphi(x, c) \in \mathfrak{p}$ but there is no $a \in M$ such that $\models \varphi(a, b) \wedge \neg\varphi(a, c)$. Hence $\mathfrak{p}$ is not finitely satisfiable in M. $\Box$

Proposition 8.2 *Let $\varphi(x, y) \in L$ be stable, let $\mathfrak{p}(x) \in S_\varphi(\mathfrak{C})$ and assume $\mathfrak{p}$ is definable over M and consistent with $\pi(x)$, a partial type over M. For some $q(x) \in S(M)$ extending $\pi(x) \cup \mathfrak{p} \restriction M$ there is a Morley sequence $(c_i : i < \omega)$ in q such that $\mathfrak{p}$ is definable by a positive boolean combination of the formulas $\varphi(c_i, y)$.*

Proof: By Proposition 8.1 $\pi(x) \cup \mathfrak{p}(x)$ is finitely satisfiable in M. It is easy to check the consistency of

$$\pi(x) \cup \mathfrak{p}(x) \cup \{\neg\psi(x) : \psi(x) \in L(\mathfrak{C}) \text{ is not satisfiable in } M\}.$$

Let $\mathfrak{q} \in S(\mathfrak{C})$ be an extension of this set of formulas. Clearly $\mathfrak{q}$ coheirs from M and $\mathfrak{q} \restriction \varphi = \mathfrak{p}$. We claim that for some $n < \omega$ there is a sequence $(c_i : i < n)$ such that $c_i \models \mathfrak{q} \restriction M(c_j : j < i)$

and $\mathfrak{p}$ is definable by a positive boolean combination of the formulas $\varphi(c_i, y)$. Note that if this is the case we can complete the sequence to $(c_i : i < \omega)$, a coheir sequence over M of realizations of $\mathfrak{p} \restriction M$. By Remark 7.11 it is a Morley sequence over M (in $q = \mathfrak{q} \restriction M$). Let us assume that there is no such sequence $(c_i : i < n)$. We proceed as in the proof of Lemma 2.9 obtaining a_i, b_i, c_i such that $\varphi(x, a_i) \in \mathfrak{p}$, $\neg\varphi(x, b_i) \in \mathfrak{p}$, $\models \varphi(c_j, a_i) \to \varphi(c_j, b_i)$ for all $j < i$ and

$$c_i \models \mathfrak{q} \restriction M(a_j : j \leq i)(b_j : j \leq i)(c_j : j < i).$$

As in the proof of Lemma 2.9, this implies that $\varphi(x, y)$ has the order property and is, therefore, unstable. $\square$

Proposition 8.3 *Let $\varphi(x, y) \in L$ be stable. Given A and a, let fix $\mathfrak{q}(y) \in S_{\varphi^{-1}}(\mathfrak{C})$, the only φ^{-1}-type over $\mathfrak{C}$ consistent with $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A)$ and definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Fix a definition $d_{\mathfrak{q}}y\varphi(x, y)$ of $\mathfrak{q}$ which is equivalent to a formula over $\mathrm{acl}^{\mathrm{eq}}(A)$ and it is a positive boolean combination of formulas $\varphi(x, c_i)$ where $(c_i : i < \omega)$ is an indiscernible sequence over $\mathrm{acl}^{\mathrm{eq}}(A)$ of realizations c_i of $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$. Let $\sigma(x)$ be the (finite) disjunction of all A-conjugates of $d_{\mathfrak{q}}y\varphi(x, y)$. For any partial type $\pi(x)$ over A, the following are equivalent.*

1. *$\varphi(x, a) \in \mathfrak{p}$ for some $\mathfrak{p} \in S_{\varphi}(\mathfrak{C})$ definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ and consistent with $\pi(x)$.*
2. *$\pi(x) \cup \{\varphi(x, a)\}$ is finitely satisfiable in every model $M \supseteq A$.*
3. *$\pi(x) \cup \{\varphi(x, a)\}$ does not divide over A.*
4. *Every set of $\mathrm{acl}^{\mathrm{eq}}(A)$-conjugates of $\varphi(x, a)$ is consistent with $\pi(x)$.*
5. *$d_{\mathfrak{q}}y\varphi(x, y)$ is consistent with $\pi(x)$.*
6. *$\sigma(x)$ is consistent with $\pi(x)$.*
7. *Some positive boolean combination of A-conjugates of $\varphi(x, a)$ is equivalent to a formula over A consistent with $\pi(x)$.*

Proof:

1 ⇒ 2 follows directly from Proposition 8.1.

2 ⇒ 3. Let $\psi(x)$ be a finite conjunction of formulas in π and let $(a_i : i < \omega)$ be an A-indiscernible sequence starting with $a = a_0$. By Remark 1.1 $(a_i : i < \omega)$ is indiscernible over some model $M \supseteq A$. There is some $c \in M$ such that $\models \varphi(c, a) \wedge \psi(c)$. By indiscernibility $\models \varphi(c, a_i)$ for every $i < \omega$. Therefore $\{\varphi(x, a_i) \wedge \psi(x) : i < \omega\}$ is consistent and $\varphi(x, a) \wedge \psi(x)$ does not divide over A.

1 ⇒ 4. Any $\mathrm{acl}^{\mathrm{eq}}(A)$-conjugate of $\varphi(x, a)$ is in $\mathfrak{p}$.

3 ⇒ 5. Since the sequence parameters $(c_i : i < \omega)$ build an indiscernible sequence over A and $a \equiv_A c_i$, $\pi(x) \cup \{\varphi(x, c_i) : i < \omega\}$ is consistent. Any positive boolean combination of the formulas $\varphi(x, c_i)$ is therefore consistent with π.

4 ⇒ 5. Same reason since in fact the formulas $\varphi(x, c_i)$ are $\mathrm{acl}^{\mathrm{eq}}(A)$-conjugates of $\varphi(x, a)$.

5 ⇒ 6. Clear by construction of σ.

6 ⇒ 7. $\sigma(x)$ satisfies the requirements in *7*.

7 ⇒ 1. Let $\sigma'(x)$ be a positive boolean combination of A-conjugates of $\varphi(x, a)$ which is equivalent to a formula over A and is consistent with π. By Remark 6.7 there is $\mathfrak{p} \in S_{\varphi}(\mathfrak{C})$ definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ and consistent with $\pi(x) \cup \{\sigma'(x)\}$. Since $\sigma'(x)$ is a disjunction of conjunctions of A-conjugates of $\varphi(x,a)$, some A-conjugate of $\varphi(x,a)$ appears in $\mathfrak{p}$. By conjugation over A, there is also some $\mathfrak{p}' \in S_\varphi(\mathfrak{C})$ definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ and consistent with $\pi(x)$ such that $\varphi(x,a) \in \mathfrak{p}'$. $\square$

Corollary 8.4 *Let $\varphi(x,y) \in L$ be stable and let $\pi(x)$ be a partial φ-type. Those following are equivalent:*

1. *$\pi(x)$ does not fork over A.*
2. *Some $\mathfrak{p} \in S_\varphi(\mathfrak{C})$ extending π is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.
3. *$\pi(x)$ is finitely satisfiable in every $M \supseteq A$.*

Proof: *1* $\Leftrightarrow$ *3* follows from Proposition 8.3. *2* $\Rightarrow$ *3* is a consequence of Lemma 8.1.

1 $\Rightarrow$ *2*. If π does not fork over A it can be extended to some $\mathfrak{p} \in S_\varphi(\mathfrak{C})$ which does not fork over A. By the equivalence *1* $\Leftrightarrow$ *3*, $\mathfrak{p}$ is finitely satisfiable in every $M \supseteq A$ and then, by Lemma 8.1, $\mathfrak{p}$ is definable over every $M \supseteq A$. By Proposition 1.3 $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. $\square$

Corollary 8.5 *Let T be stable, $p(x) \in S(A)$ and $M \subseteq A$ (perhaps $A = \mathfrak{C}$). The following are equivalent:*

1. *$p(x)$ does not fork over M.*
2. *$p(x)$ heirs from M.*
3. *$p(x)$ coheirs from M.*
4. *$p(x)$ is M-definable.*
5. *Some $\mathfrak{p} \in S(\mathfrak{C})$ extending p does not split over M.*

Proof: Equivalence between points *2*, *3*, and *4* follows from Corollary 7.14. Equivalence of *4* and *5* follows from Proposition 7.12. Equivalence of *1* and *3* follows from Corollary 8.4. $\square$

Corollary 8.6 *Let T be stable and let $\mathfrak{p}(x) \in S(\mathfrak{C})$. The following are equivalent:*

1. *$\mathfrak{p}(x)$ does not fork over A.*
2. *$\mathfrak{p}(x)$ coheirs from every $M \supseteq A$.*
3. *$\mathfrak{p}(x)$ heirs from every $M \supseteq A$.*
4. *$\mathfrak{p}(x)$ does not split over any $M \supseteq A$.*
5. *$\mathfrak{p}(x)$ does not fork over any $M \supseteq A$.*
6. *$\mathfrak{p}(x)$ is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.
7. *The Cantor-Bendixson rank of $\mathfrak{p} \restriction \varphi$ is $CB_\varphi(\mathfrak{p} \restriction A)$ for all φ.*
8. *$\mathfrak{p}(x)$ has a bounded orbit in* $\mathrm{Aut}(\mathfrak{C}/A)$ *(in fact of size $\leq 2^{|T|}$).*

Proof: Equivalence between points *2*, *3*, *4*, *5*, and *6* follows from Corollary 8.5 (for *6* observe that $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ if and only if it is definable over every model $M \supseteq A$).

1 $\Leftrightarrow$ *6* follows from Corollary 8.4

6 $\Leftrightarrow$ *7* follows from Corollary 6.11

7 $\Rightarrow$ *8*. The orbit of $\mathfrak{p} \restriction \varphi$ is bounded by $Mlt_\varphi(\mathfrak{p} \restriction A) < \omega$ and hence the orbit of $\mathfrak{p}$ is bounded by $2^{|T|}$.

8 $\Rightarrow$ *6*. Let c_φ be the canonical parameter of the definition of $\mathfrak{p} \restriction \varphi$. Since c_φ has bounded orbit in $\mathrm{Aut}(\mathfrak{C}/A)$, in fact it has finite orbit. Hence $c_\varphi \in \mathrm{acl}^{\mathrm{eq}}(A)$ and $\mathfrak{p} \restriction \varphi$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. □

Corollary 8.7 *Let T be stable, $p(x) \in S(A)$ and $\varphi(x, y) \in L$. The following are equivalent:*

1. *$p(x) \cup \{\varphi(x, a)\}$ does not fork over A.*
2. *$CB_\psi(p \cup \{\varphi(x, a)\}) = CB_\psi(p)$ for all ψ.*
3. *$CB_\varphi(p \cup \{\varphi(x, a)\}) = CB_\varphi(p)$.*

Proof: *1* $\Rightarrow$ *2*. Let $\mathfrak{p} \in S(\mathfrak{C})$ be an extension of $p \cup \{\varphi(x, a)\}$ which does not fork over A. By Corollary 8.6, $CB_\psi(p)$ is the Cantor-Bendixson rank of $\mathfrak{p} \restriction \varphi$. Hence $CB_\psi(p \cup \{\varphi(x, a)\}) \geq CB_\psi(p)$.

2 $\Rightarrow$ *3* is obvious. We prove *3* $\Rightarrow$ *1*. Let $\mathfrak{p} \in S(\mathfrak{C})$ be a nonforking extension of p. By Corollary 8.6, $CB_\varphi(p)$ is the Cantor-Bendixson rank of $\mathfrak{p} \restriction \varphi$. Let $\mathfrak{q} \in S(\mathfrak{C})$ be such $\mathfrak{q} \restriction \varphi$ is consistent with $p \cup \{\varphi(x, a)\}$ and has Cantor-Bendixson rank $CB_\varphi(p \cup \{\varphi(x, a)\})$. By corollaries 6.10 and 6.11 $\mathfrak{p} \restriction \varphi$ and $\mathfrak{q} \restriction \varphi$ are A-conjugate. Since $\varphi(x, a) \in \mathfrak{q} \restriction \varphi$, $p \cup \{\varphi(x, a)\}$ is contained in an A-conjugate of $\mathfrak{p}$, a global type which does not fork over A. Hence $p \cup \{\varphi(x, a)\}$ does not fork over A. □

Corollary 8.8 *Let T be stable, $A \subseteq B$ and $p(x) \in S(B)$. The following are equivalent:*

1. *$p(x)$ does not fork over A.*
2. *$CB_\varphi(p) = CB_\varphi(p \restriction A)$ for all φ.*

Proof: It is an immediate consequence of Corollary 8.7. □

Proposition 8.9 *Let T be simple. If $\varphi(x, y) \in L$ is stable, for every A, a there is some $\sigma(x) \in L(A)$ equivalent to a positive boolean combination of A-conjugates of $\varphi(x, a)$ and such that for every $p(x) \in S(A)$, $\sigma(x) \in p(x)$ if and only if $p(x) \cup \{\varphi(x, a)\}$ does not fork over A.*

Proof: Apply Proposition 8.3 with $p(x) = \pi(x)$. □

Corollary 8.10 (Open mapping theorem) *Let T be stable and let $A \subseteq B$. The set $NF(B, A)$ of all $p(x) \in S(B)$ which do not fork over A is closed in $S(B)$ and the restriction mapping $p \mapsto p \restriction A$ from $NF(B, A)$ onto $S(A)$ is open.*

Proof: The restriction map from $S(\mathfrak{C})$ onto $S(B)$ is continuous and hence closed and the image of $NF(\mathfrak{C}, A)$ is $NF(B, A)$. Hence it is enough to check that $NF(\mathfrak{C}, A)$ is closed. Now,

$$NC(\mathfrak{C}, A) = \bigcap_{M \subseteq A} \{\mathfrak{p} \in S(\mathfrak{C}) : \mathfrak{p} \text{ coheirs from } M\}$$

and for each M, $\{\mathfrak{p} \in S(\mathfrak{C}) : \mathfrak{p}$ coheirs from $M\}$ is closed since it is the closure of $\{\mathrm{tp}(a/\mathfrak{C}) : a \in M\}$. The fact that the restriction map from $NF(B, A)$ onto $S(A)$ is open is an immediate consequence of point *2* of Proposition 8.9. □

Corollary 8.11 *If T is stable, any nonforking extension of a nonisolated type is nonisolated.*

Proof: By Proposition 8.9 or Corollary 8.10. □

Corollary 8.12 *If T is simple, any nonforking extension of a type which is not isolated by stable formulas is neither isolated by stable formulas.*

Proof: By Proposition 8.9. □

Chapter 9

Lascar strong types

Here we will consider relations R and we always mean binary relations among α-sequences of elements of $\mathfrak{C}$ for some ordinal α. Usually α is intended to be a natural number but we do not put restrictions.

Definition 9.1 *A relation R is* bounded *if for some cardinal κ there is no sequence $(a_i : i < \kappa)$ such that $\neg R(a_i, a_j)$ for all $i < j < \kappa$. The relation is* finite *if this bound κ is in fact a natural number. Observe that for definable relations finiteness is equivalent to boundedness. Note also that bounded relations are always reflexive.*

Remark 9.2 *Any intersection of a bounded number of bounded relations is a bounded relation.*

Proof: Let $(R_l : l < \lambda)$ be a sequence of bounded relations. For all $l < \lambda$ let κ_l be a bound for R_l and let $\kappa = \lambda + \sup\{\kappa_l : l < \lambda\}$. Assume that there are $(a_i : i < (2^\kappa)^+)$ such that$\neg R(a_i, a_j)$ for all $i < j < (2^\kappa)^+$, where $R = \bigcap_{l<\lambda} R_l$. By Erdös-Rado ($(2^\kappa)^+ \to (\kappa^+)^2_\kappa$) for some $l < \lambda$ there is a subset $I \subseteq (2^\kappa)^+$ of cardinality κ^+ such that $\neg R_l(a_i, a_j)$ for all $i < j$ in I. This contradicts the choice of κ_l. $\square$

Definition 9.3 *A relation R is A-*invariant *if it is preserved under automorphisms of $\mathfrak{C}$ fixing A pointwise, that is, $R(f(a), f(b))$ whenever $R(a, b)$ and $f \in \mathrm{Aut}(\mathfrak{C}/A)$.*

Lemma 9.4

1. *Every A-invariant relation R is definable by a union of types over A, namely: $R(a, b) \Leftrightarrow ab \models \bigvee_{R(c,d)} \bigwedge \mathrm{tp}(cd/A)$.*
2. *The number of A-invariant relations on sequences of length α is bounded by $2^{2^{|T|+|A|+|\alpha|}}$.*
3. *There is a least A-invariant bounded equivalence relation (among sequences of a fixed length).*

Definition 9.5 *We say that the sequences a, b have the same* Lascar strong type over A *and we write $a \overset{\mathrm{Ls}}{\equiv}_A b$ if a and b are equivalent in the least A-invariant bounded equivalence relation. In case $A = \emptyset$ we omit it.*

Definition 9.6 *Let x, y be finite tuples of variables of the same length. We say that the formula $\theta(x, y)$ is* thick *if it defines a relation which is finite and symmetric. For any set*

A and for any sequences of variables x, y of the same length, the set of all thick formulas over A in (finite subtuples of) the variables x, y will be

$$\mathrm{nc}_A(x, y).$$

In case $A = \emptyset$ we omit it. For every natural number n, $\mathrm{nc}^n_A(x, y)$ is the type

$$\exists y_1 \dots y_{n-1}(\mathrm{nc}_A(x, y_1) \wedge \mathrm{nc}_A(y_1, y_2) \wedge \dots \wedge \mathrm{nc}_A(y_{n-1}, y))$$

Remark 9.7 *1. The conjunction and the disjunction of thick formulas are thick formulas.*

2. Any consequence of a thick formula is a finite formula.

3. If $\varphi(x, y)$ is finite then, $\varphi(x, y) \wedge \varphi(y, x)$ is thick.

Lemma 9.8 *For any $a \neq b$, $\models \mathrm{nc}_A(a, b)$ if and only if a, b start an infinite A-indiscernible sequence.*

Proof: If a, b start an infinite A-indiscernible sequence, then $\models \theta(a, b)$ for any thick formula $\theta(x, y)$ over A. Now assume $\models \mathrm{nc}_A(a, b)$. Let $p(x, y) = \mathrm{tp}(ab/A)$. By Ramsey's Theorem and compactness, to prove that a, b start an infinite A-indiscernible sequence it is enough to check that there is an infinite sequence $(a_i : i < \omega)$ such that $\models p(a_i, a_j)$ for all $i < j < \omega$. For this we have to prove for any $\varphi \in p$, the consistency of $\{\varphi(x_i, x_j) : i < j < \omega\}$. If this set of formulas is inconsistent, then $\neg\varphi(x, y)$ is finite and therefore $(\neg\varphi(x, y) \wedge \neg\varphi(y, x)) \in \mathrm{nc}_A(x, y)$. Hence $\models \neg\varphi(a, b)$, a contradiction. $\square$

Proposition 9.9 *The relation $\overset{\mathrm{Ls}}{\equiv}_A$ of equality of Lascar strong type over A is the transitive closure of the relation of starting an A-indiscernible sequence. Hence it is defined by the infinite disjunction $\bigvee_n \mathrm{nc}^n_A(x, y)$.*

Proof: Since the relation of starting an infinite indiscernible sequence is defined by the type $\mathrm{nc}_A(x, y)$ consisting of finite formulas, it is bounded. Hence its transitive closure E is also bounded. Since E is a bounded A-invariant equivalence relation, $\overset{\mathrm{Ls}}{\equiv}_A \subseteq E$. For the other direction it suffices to show that if a, b start an infinite A-indiscernible sequence then $a \overset{\mathrm{Ls}}{\equiv}_A b$. Let κ be a strict bound for the number of $\overset{\mathrm{Ls}}{\equiv}_A$-classes. Choose an A-indiscernible sequence of length κ starting with a, b . If $a \overset{\mathrm{Ls}}{\not\equiv}_A b$ then by A-invariance $a' \overset{\mathrm{Ls}}{\not\equiv}_A b'$ for any two different a', b' in the sequence, which contradicts the choice of κ. $\square$

Lemma 9.10 *1. If $\models \mathrm{nc}_A(a, b)$, then there is a model $M \supseteq A$ such that $a \equiv_M b$.*

2. If $a \equiv_M b$ for some model $M \supseteq A$, then $\models \mathrm{nc}^2_A(a, b)$.

Proof: *1.* Fix an infinite A-indiscernible sequence I starting with a, b. By Proposition 1.1 I is indiscernible over some model $M \supseteq A$. Then $a \equiv_M b$.

2. Assume that $a \equiv_M b$ for some model $M \supseteq A$. We show that $\models \exists z(\theta(a, z) \wedge \theta(b, z))$ for any thick formula $\theta(x, y)$ over A. Let n be the maximal length of a sequence $a_1, \dots, a_n$ such that $\models \neg\theta(a_i, a_j)$ for all $i < j \leq n$. We can find such $a_1, \dots, a_n$ in M. For some $i, j \leq n$, $\models \theta(a, a_i)$ and $\models \theta(b, a_j)$. Since $a \equiv_M b$ we may take $i = j$. $\square$

Proposition 9.11 *Equality of Lascar strong types over A is the transitive closure of the relation of having the same type over a model containing A.*

Proof: Clear by Proposition 9.9 and Lemma 9.10. □

Definition 9.12 *The group* $\mathrm{Autf}(\mathfrak{C}/A)$ *of* strong automorphisms over A *of the monster model* $\mathfrak{C}$ *is the subgroup of* $\mathrm{Aut}(\mathfrak{C}/A)$ *generated by the automorphisms fixing a small submodel containing* A*:*

$$\mathrm{Autf}(\mathfrak{C}/A) = \langle \bigcup_{M \supseteq A} \mathrm{Aut}(\mathfrak{C}/M) \rangle$$

Corollary 9.13 $a \stackrel{\mathrm{Ls}}{\equiv}_A b$ *if and only if* $f(a) = b$ *for some* $f \in \mathrm{Autf}(\mathfrak{C}/A)$*.*

Proof: It follows from Proposition 9.11. □

Corollary 9.14 *If* $a \stackrel{\mathrm{Ls}}{\equiv}_A b$ *then for any* c *there is some* d *such that* $ac \stackrel{\mathrm{Ls}}{\equiv}_A bd$

Proof: Choose $f \in \mathrm{Autf}(\mathfrak{C}/A)$ such that $f(a) = b$ and put $d = f(c)$. □

Definition 9.15 *Like in the case of* A*-invariance, there is a least type-definable over* A *bounded equivalence relation (among sequences of a given length). We say that the sequences* a, b *have the same* KP-strong type over A *or the same* bounded type over A *and we write* $a \stackrel{\mathrm{bdd}}{\equiv}_A b$ *if* a *and* b *are equivalent in the least type-definable over* A *bounded equivalence relation. We say that* a, b *have the same* strong type over A *and we write* $a \stackrel{\mathrm{s}}{\equiv}_A b$ *if* a *and* b *are equivalent in every* A*-definable finite equivalence relation. As usual, in case* $A = \emptyset$ *we omit it.*

Remark 9.16

1. *If* $a \stackrel{\mathrm{Ls}}{\equiv}_A b$*, then* $a \stackrel{\mathrm{bdd}}{\equiv}_A b$*.*
2. *If* $a \stackrel{\mathrm{bdd}}{\equiv}_A b$*, then* $a \stackrel{\mathrm{s}}{\equiv}_A b$*.*
3. *If* $a \stackrel{\mathrm{s}}{\equiv}_A b$*, then* $a \equiv_A b$*.*

Proof: *1* is clear since every equivalence relation type-definable over A is A-invariant. Similarly for *2* since every A-definable finite equivalence relation is bounded and type-definable over A. For *3* observe that for each $\varphi(x) \in L(A)$, the equivalence relation E defined by $(\varphi(x) \leftrightarrow \varphi(y))$ is A-definable and has only two classes. □

Definition 9.17 *The* strong type *of* a *over* A *is defined by*

$$\mathrm{stp}(a/A) = \mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$$

Lemma 9.18 $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$ *if and only* $a \stackrel{\mathrm{s}}{\equiv}_A b$*.*

Proof: Assume $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$. Let E be a finite A-definable equivalence relation, say defined by $\varphi(x, y, c)$ where $c \in A$ and $\varphi(x, y, z) \in L$. Let $\psi(z) \in \mathrm{tp}(c)$ be the formula expressing that $\varphi(x, y, z)$ defines an equivalence relation in x, y and consider the relation $F(ux; vy)$ defined by

$$F(ux; vy) \;\Leftrightarrow\; (\neg\psi(u) \wedge \neg\psi(v)) \vee (\psi(u) \wedge u = v \wedge \varphi(x, y, u))$$

It is a 0-definable equivalence relation and therefore ac/F and bc/F are imaginary elements. Since $F(cx; cy)$ defines E and E is finite, these imaginaries are algebraic over A, that is, they are elements of $\mathrm{acl}^{\mathrm{eq}}(A)$. This clearly implies $ac/F = bc/F$ and therefore $E(a, b)$.

For the other direction, notice that according to Proposition 1.3 a relation R defined by a formula $\varphi(x) \in \mathrm{acl}^{\mathrm{eq}}(A)$ has finitely many A-conjugates and it is therefore union of classes of a finite A-definable equivalence relation. □

Proposition 9.19 *Let T be stable. If $a \overset{\mathrm{s}}{\equiv}_A b$, $A \subseteq B$, $a \mathop{\smile\!\!\!\!\!\mid}_A B$ and $b \mathop{\smile\!\!\!\!\!\mid}_A B$, then $a \overset{\mathrm{s}}{\equiv}_B b$.*

Proof: Let $p(x) = \mathrm{stp}(a/A) = \mathrm{stp}(b/A)$, let $\mathfrak{p} \in S(\mathfrak{C})$ be a nonforking extension of $\mathrm{stp}(a/B)$ and let $\mathfrak{q} \in S(\mathfrak{C})$ be a nonforking extension of $\mathrm{stp}(b/B)$. Since $a \mathop{\smile\!\!\!\!\!\mid}_A B$, $a \mathop{\smile\!\!\!\!\!\mid}_A \mathrm{acl}^{\mathrm{eq}}(B)$ and therefore $\mathfrak{p}$ does not fork over A. By Corollary 8.6 $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. By the same argument $\mathfrak{q}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ and by Proposition 6.9 $\mathfrak{p} = \mathfrak{q}$. Hence $\mathrm{stp}(a/B) = \mathrm{stp}(b/B)$. □

Corollary 9.20 *If T is stable, $\overset{\mathrm{Ls}}{\equiv}_A = \overset{\mathrm{s}}{\equiv}_A$ for every A.*

Proof: Let $a \overset{\mathrm{s}}{\equiv}_A b$. Choose $M \supseteq A$ such that $M \mathop{\smile\!\!\!\!\!\mid}_A ab$. Then $a \mathop{\smile\!\!\!\!\!\mid}_A M$ and $b \mathop{\smile\!\!\!\!\!\mid}_A M$. By Proposition 9.19 $a \equiv_M b$ and hence by Lemma 9.10 and Proposition 9.9, $a \overset{\mathrm{Ls}}{\equiv}_A b$. □

Theorem 9.21 (Finite equivalence relation theorem) *Let T be stable. Let $A \subseteq B$, $r(x) \in S(A)$ and let $p(x), q(x) \in S(B)$ be two different nonforking extensions of r. Then for some $\varphi(x) \in L(B)$ equivalent to a formula over $\mathrm{acl}^{\mathrm{eq}}(A)$, $\varphi \in p$ while $\neg\varphi \in q$. There is also a finite A-definable equivalence relation E such that*

$$p(x) \cup q(y) \vdash \neg E(x, y)$$

Proof: Let $p'(x) \in S(B \cup \mathrm{acl}^{\mathrm{eq}}(A))$ be an extension of p. If $p'(x) \restriction \mathrm{acl}^{\mathrm{eq}}(A) \cup q(x)$ is consistent then there is some extension $q'(x) \in S(B \cup \mathrm{acl}^{\mathrm{eq}}(A))$ of q such that $p' \restriction \mathrm{acl}^{\mathrm{eq}}(A) = q' \restriction \mathrm{acl}^{\mathrm{eq}}(A)$. But then p' and q' are different nonforking extensions of the same strong type, which contradicts Corollary 9.20. Hence $p'(x) \restriction \mathrm{acl}^{\mathrm{eq}}(A) \cup q(x)$ is inconsistent and there is some $\psi(x) \in p'(x) \restriction \mathrm{acl}^{\mathrm{eq}}(A)$ such that $q(x) \vdash \neg\psi(x)$. Let $\varphi(x)$ be the disjunction of all B-conjugates of ψ. Then $p(x) \vdash \varphi(x)$, $q(x) \vdash \neg\varphi(x)$ and $\varphi(x) \in L(\mathrm{acl}^{\mathrm{eq}}(A))$ is equivalent to a formula over B.

With respect to last assertion, by Proposition 1.3 $\varphi(x)$ defines a union of classes of a finite A-definable equivalence relation E and then clearly $p(x) \cup q(y) \vdash \neg E(x, y)$. □

Chapter 10

The independence theorem

Lemma 10.1 *Let T be simple. If $(a_i : i < \omega + \omega)$ is an infinite A-indiscernible sequence, then $(a_i : \omega \leq i < \omega + \omega)$ is a Morley sequence over $A\{a_i : i < \omega\}$.*

Proof: Let $I = (a_i : i < \omega)$. Clearly $(a_i : \omega \leq i < \omega + \omega)$ is AI-indiscernible. It suffices to show that it is AI-independent. Let X be a finite subset of $\{i : \omega \leq i < \omega + \omega\}$ an let $i < \omega + \omega$ be greater than every element in X. By symmetry it will be enough to check that $a_X \mathop{\smile\!\!\!\!\!\!\;|}_{AI} a_i$, where $a_X = (a_j : j \in X)$. But this is clear since by A-indiscernibility $\mathrm{tp}(a_X/AIa_i)$ is finitely satisfiable in I. $\square$

Proposition 10.2 *Let T be simple and let $\pi(x, y)$ be a set of formulas over $\emptyset$. If $(a_i : i \in I)$ is an A-indiscernible sequence and $\pi(x, a_i)$ does not fork over A for some $i \in I$, then $\bigcup_{i \in I} \pi(x, a_i)$ does not fork over A.*

Proof: For notational convenience, we assume the ordered set I is ω and $\pi(x, a_0)$ does not fork over A. Let us first assume that $(a_i : i < \omega)$ is a Morley sequence over A. Since $\pi(x, a_0)$ does not divide over A, $\bigcup_{i<\omega} \pi(x, a_i)$ is consistent. Let $n < \omega$ and let $\Phi(x, y_0, \ldots, y_{n-1}) = \pi(x, y_1) \cup \ldots \cup \pi(x, y_n)$. We will show that $\Phi(x, a_0, \ldots, a_{n-1})$ does not divide over A. If $b_i = a_{n \cdot i} \ldots a_{n \cdot i + n - 1}$, then $(b_i : i < \omega)$ is an infinite Morley sequence in $\mathrm{tp}(b_0/A)$ and $\bigcup_{i<\omega} \Phi(x, b_i)$ is consistent. By Proposition 5.13, $\Phi(x, b_0)$ does not divide over A.

Now let us consider the general case, where $(a_i : i < \omega)$ is just an A-indiscernible sequence. Choose $J = (b_i : i < \omega)$ such that $(b_i : i < \omega)^\frown(a_i : i < \omega)$ is A-indiscernible. By Lemma 10.1 $(a_i : i < \omega)$ is a Morley sequence over $A \cup J$. Let $p(x, y) \in S(AJ)$ be such that $p(x, a_0)$ extends $\pi(x, a_0)$ and does not fork over A. Then it does not fork over AJ and by the first case, $\bigcup_{i<\omega} p(x, a_i)$ does not fork over AJ. Let $c \models \bigcup_{i<\omega} p(x, a_i)$. Then $c \mathop{\smile\!\!\!\!\!\!\;|}_{AJ} (a_i : i < \omega)$. Since $p(x, a_0)$ does not fork over A, also $c \mathop{\smile\!\!\!\!\!\!\;|}_{A} Ja_0$. Hence $c \mathop{\smile\!\!\!\!\!\!\;|}_{A} J(a_i : i < \omega)$, which shows that $\bigcup_{i<\omega} \pi(x, a_i)$ does not fork over A. $\square$

Lemma 10.3 *Let T be simple. If a, b start an infinite A-indiscernible sequence and $c \mathop{\smile\!\!\!\!\!\!\;|}_{Aa} b$, then for some d, the extended sequences ac, bd start an infinite A-indiscernible sequence also.*

Proof: Assume $A = \emptyset$. Let $c \mathop{\smile\!\!\!\!\!\!\;|}_{a} b$ and assume $I = (a_i : i < \omega)$ is an infinite indiscernible sequence with $a = a_0$ and $b = a_1$. Since $(a_n : n \geq 1)$ is a-indiscernible and $c \mathop{\smile\!\!\!\!\!\!\;|}_{a} b$, by Lemma 4.5 there is an ac-indiscernible sequence $(a'_n : n \geq 1)$ such that $(a_n : n \geq 1) \equiv_{ab} (a'_n : n \geq 1)$. Thus we may assume that $a_n = a'_n$ for all $n \geq 1$. Let $c_0 = c$ and choose for

$n \geq 1$ some c_n such that

$$ca_0 a_1 \ldots \equiv c_n a_n a_{n+1} \ldots$$

Since $(a_n : n \geq 1)$ is ac-indiscernible, $cab \equiv caa_m$. Hence $cab \equiv c_n a_n a_{n+m}$, i.e., in the sequence $(c_n a_n : n < \omega)$ all triangles $c_n a_n a_{n+m}$ have the same type $p(x, y, z) = \mathrm{tp}(cab)$. By Ramsey's Theorem there is an indiscernible sequence $(d_n b_n : n < \omega)$ where all triangles $d_n b_n b_{n+m}$ satisfy $p(x, y, z)$. Clearly we may assume that $c = d_0$, $a = b_0$ and $b = b_1$. Take $d = d_1$. □

Proposition 10.4 *Let T be simple and assume that $\varphi(x, a) \wedge \psi(x, b)$ does not fork over A. If b, b' start an infinite A-indiscernible sequence and $a \mathop{\smile\!\!\!\!\!\!|}_{Ab} b'$, then $\varphi(x, a) \wedge \psi(x, b')$ does not fork over A.*

Proof: Apply Lemma 10.3 finding a' such that $ba, b'a'$ start an infinite A-indiscernible sequence. By Proposition 10.2, $\varphi(x, a) \wedge \psi(x, b) \wedge \varphi(x, a') \wedge \psi(x, b')$ does not fork over A. In particular $\varphi(x, a) \wedge \psi(x, b')$ does not fork over A. □

Corollary 10.5 *Let T be simple and assume that $\varphi(x, a) \wedge \psi(x, b)$ does not fork over A. If $b \overset{\mathrm{Ls}}{\equiv}_A b'$ and $a \mathop{\smile\!\!\!\!\!\!|}_A bb'$, then $\varphi(x, a) \wedge \psi(x, b')$ does not fork over A.*

Proof: Find $b_1, \ldots, b_n$ such that $b = b_1$, $b' = b_n$ and b_i, b_{i+1} start an infinite A-indiscernible sequence. Let a' be such that $a' \equiv_{Abb'} a$ and $a' \mathop{\smile\!\!\!\!\!\!|}_{Abb'} b_1, \ldots b_n$. By Proposition 10.4 we see that $\varphi(x, a') \wedge \psi(x, b_i)$ does not fork over A for all $i \leq n$. Hence $\varphi(x, a) \wedge \psi(x, b')$ does not fork over A. □

Lemma 10.6 *Let T be simple. Let κ be a cardinal number bigger than $|T| + |A|$. If $(a_i : i < \kappa)$ is A-independent and the length of every a_i is smaller than κ, then for any a of length smaller than κ there is some $i < \kappa$ such that $a \mathop{\smile\!\!\!\!\!\!|}_A a_i$.*

Proof: By choice of κ, there is a proper subset $B \subseteq \{a_i : i < \kappa\}$ such that $a \mathop{\smile\!\!\!\!\!\!|}_{AB} \{a_i : i < \kappa\}$. Take $a_i \notin B$. Then $a \mathop{\smile\!\!\!\!\!\!|}_{AB} a_i$ and, by Corollary 5.17, $a_i \mathop{\smile\!\!\!\!\!\!|}_A B$. By symmetry and transitivity, $a \mathop{\smile\!\!\!\!\!\!|}_A a_i$. □

Lemma 10.7 *Let T be simple. For any a, A and $B \supseteq A$ there is a' such that $a' \overset{\mathrm{Ls}}{\equiv}_A a$ and $a' \mathop{\smile\!\!\!\!\!\!|}_A B$.*

Proof: Let κ be a cardinal bigger than $|T| + |B|$ and bigger than the length of a. We may assume that $\mathrm{tp}(a/A)$ is not algebraic. Let $(a_i : i < \kappa)$ be a Morley sequence in $\mathrm{tp}(a/A)$ starting with $a_0 = a$. By Lemma 10.6 there is some $i < \kappa$ such that $B \mathop{\smile\!\!\!\!\!\!|}_A a_i$. Clearly, $a \overset{\mathrm{Ls}}{\equiv}_A a_i$. □

Lemma 10.8 *Let T be simple and $a \overset{\mathrm{Ls}}{\equiv}_A b$. For any c, B there is some d such that $ac \overset{\mathrm{Ls}}{\equiv}_A bd$ and $d \mathop{\smile\!\!\!\!\!\!|}_{Ab} B$.*

Proof: By Corollary 9.14 there is some d' such that $ac \overset{\mathrm{Ls}}{\equiv}_A bd'$ and by Corollary 9.13, there is a strong automorphism $f \in \mathrm{Autf}(\mathfrak{C}/A)$ such that $f(ac) = bd'$. By Lemma 10.7 there is some d such that $d \overset{\mathrm{Ls}}{\equiv}_{Ab} d'$ and $d \mathop{\smile\!\!\!\!\!\!|}_{Ab} B$. Again by Corollary 9.13 there is some $g \in \mathrm{Autf}(\mathfrak{C}/Ab)$ such that $g(d') = d$. It follows that $g \circ f \in \mathrm{Autf}(\mathfrak{C}/A)$ and $g \circ f(ac) = bd$. Hence $ac \overset{\mathrm{Ls}}{\equiv}_A bd$. □

Corollary 10.9 (Independence Theorem) *Let T be simple and $a \mathop{\smile\!\!\!\!\!\!|}_A b$. If there are c, d such that $\models \varphi(c,a)$, $c \mathop{\smile\!\!\!\!\!\!|}_A a$, $\models \psi(d,b)$, $d \mathop{\smile\!\!\!\!\!\!|}_A b$, and $c \overset{\mathrm{Ls}}{\equiv}_A d$, then $\varphi(x,a) \wedge \psi(x,b)$ does not fork over A.*

Proof: Using Lemma 10.8, choose $b' \mathop{\smile\!\!\!\!\!\!|}_{Ac} ab$ such that $cb' \overset{\mathrm{Ls}}{\equiv}_A db$. Then $\models \varphi(c,a) \wedge \psi(c,b')$ and $c \mathop{\smile\!\!\!\!\!\!|}_A ab'$. Therefore $\varphi(x,a) \wedge \psi(x,b')$ does not fork over A. Since $a \mathop{\smile\!\!\!\!\!\!|}_A bb'$ by Corollary 10.5, $\varphi(x,a) \wedge \psi(x,b)$ does not fork over A. $\square$

Corollary 10.10 *Let T be simple.*

1. *Assume A is a common subset of B and C. Assume $B \mathop{\smile\!\!\!\!\!\!|}_A C$, and let $b \mathop{\smile\!\!\!\!\!\!|}_A B$, $c \mathop{\smile\!\!\!\!\!\!|}_A C$, be such that $b \overset{\mathrm{Ls}}{\equiv}_A c$. Then for some $d \mathop{\smile\!\!\!\!\!\!|}_A BC$, $d \equiv_B b$ and $d \equiv_C c$.*
2. *Let $(a_i : i \in I)$ be an A-independent sequence, let $\pi_i(x)$ a partial type over Aa_i which does not fork over A and asume that whenever $(b_i : i \in I)$ is a sequence of realizations $b_i \models \pi_i$ then $b_i \overset{\mathrm{Ls}}{\equiv}_A b_j$ for all $i, j \in I$. Then $\bigcup_{i \in I} \pi_i(x)$ does not fork over A.*
3. *Let $(a_i : i \in I)$ be an M-independent sequence, let $\pi_i(x)$ a partial type over Ma_i which does not fork over M and extends $p(x) \in S(M)$. Then $\bigcup_{i \in I} \pi_i(x)$ does not fork over M.*

Proof: *1* follows from Corollary 10.9. For *2* we may assume $I = \omega$ and then using *1* it is easy to prove by induction that $\pi_0(x) \cup \ldots \cup \pi_n(x)$ does not for over A for all $n < \omega$. *3* follows from *2* since $b_i \equiv_M b_j$ implies $b_i \overset{\mathrm{Ls}}{\equiv}_M b_j$. $\square$

Proposition 10.11 *Let T be simple. If $a \overset{\mathrm{Ls}}{\equiv}_A b$ and $a \mathop{\smile\!\!\!\!\!\!|}_A b$, then a, b start a Morley sequence $(a_i : i < \omega)$ over A.*

Proof: Let $p = \mathrm{tp}(ab/A)$. We prove first that for any cardinal κ there is an infinite A-independent sequence $(a_i : i < \kappa)$ such that $\models p(a_i, a_j)$ for all $i < j < \kappa$. Note that this implies $a_i \overset{\mathrm{Ls}}{\equiv}_A a_j$. The sequence is constructed inductively starting with $a_0 = a$ and $a_1 = b$. We choose as a_α a realization of $\bigcup_{i<\alpha} p(a_i, x)$ such that $a_\alpha \mathop{\smile\!\!\!\!\!\!|}_A (a_i : i < \alpha)$. To do this we need to prove that $\bigcup_{i<\alpha} p(a_i, x)$ does not fork over A. Note that $c \overset{\mathrm{Ls}}{\equiv}_A d$ whenever $c \models p(a_i, x)$ and $d \models p(a_j, x)$. Therefore it is clear that we can apply the generalized version of the Independence Theorem stated in point *2* of Corollary 10.10 to obtain the desired result. Now, once we have this A-independent sequence we still need to make it A-indiscernible. But this can be done easily by Proposition 1.1. $\square$

Proposition 10.12 *If T is simple, then $a \overset{\mathrm{Ls}}{\equiv}_A b$ if and only if there is some c such that a, c start an infinite A-indiscernible sequence and b, c start an infinite indiscernible sequence over A.*

Proof: Assume $a \overset{\mathrm{Ls}}{\equiv}_A b$ and find with Lemma 10.7 some c such that $c \overset{\mathrm{Ls}}{\equiv}_A a$ and $c \mathop{\smile\!\!\!\!\!\!|}_A ab$. By Proposition 10.11 a, c start an infinite Morley sequence over A and b, c start an infinite Morley sequence over A. $\square$

Corollary 10.13 *If T is simple, then the relation $\overset{\mathrm{Ls}}{\equiv}_A$ of equality of Lascar strong types over A is type definable over A by $\exists z(\mathrm{nc}_A(x,z) \wedge \mathrm{nc}_A(y,z))$.*

Proof: Clear, by Proposition 10.12. $\square$

Corollary 10.14 *If T is simple, then $\overset{\mathrm{Ls}}{\equiv}_A = \overset{\mathrm{bdd}}{\equiv}_A$ for every A.*

Proof: By Corollary 10.13, $\overset{\mathrm{Ls}}{\equiv}_A$ is type-definable over A. $\square$

Chapter 11

Canonical bases

T is simple in this chapter.

Definition 11.1 *The* multiplicity *of a type $p(x) \in S(A)$ is the number* $\mathrm{Mlt}(p)$ *of its global nonforking extensions $\mathfrak{p}(x) \in S(\mathfrak{C})$. A* stationary type *is a type of multiplicity* 1*. Thus over any $B \supseteq A$ a stationary type $p(x) \in S(A)$ has exactly one nonforking extension $q(x) \in S(B)$. We use the notation $p|B$ for q.*

Lemma 11.2 *If $p \in S(A)$ is stationary, its global nonforking extension is definable over A.*

Proof: Let $\mathfrak{p}$ be the global nonforking extension of p, and let $\varphi(x, y) \in L$. We will show that $\mathfrak{p} \restriction \varphi$ is A-definable. Let $\Delta_\varphi(y)$ and $\Delta_{\neg\varphi}(y)$ the types over A given by Corollary 5.19 for p and φ and for p and $\neg\varphi$ respectively. By compactness, the conjunction $\psi(y)$ of a finite subset of $\Delta_\varphi(y)$ is inconsistent with $\Delta_{\neg\varphi}(y)$. It is clear that $\psi(y)$ defines $\mathfrak{p} \restriction \varphi$. □

Corollary 11.3 *If types over models are stationary, then T is stable.*

Proof: Lemma 11.2 implies that in this situation every global type is definable. □

Proposition 11.4 1. *If $p \in S(M)$ has bounded multiplicity, then p is stationary.*

2. *If $p \in S(A)$ has bounded multiplicity, then every extension of p over $\mathrm{acl}^{\mathrm{eq}}(A)$ is stationary.*

Proof: *1.* Assume $p \in S(M)$ has two nonforking extensions over $A \supseteq M$, say p_1 and p_2. We will show that no nonforking extension of p is stationary. This implies that p has a unbounded number of nonforking global extensions. Let q be a nonforking extension of p over $B \supseteq M$. To show that q is not stationary we may assume $B \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_M A$. By the Independence Theorem applied to p_1 and q we obtain a type $q_1 \in S(AB)$ extending $q \cup p_1$ which does not fork over M. Similarly, by applying it to p_2 and q we obtain a type $q_2 \in S(AB)$ extending $q \cup p_2$ which does not fork over M. Then q_1, q_2 are two different nonforking extensions of q over AB, which shows q is not stationary.

2 Let $p'(x) \in S(\mathrm{acl}^{\mathrm{eq}}(A))$ be a (nonforking) extension of p and let $M \supseteq A$. Any nonforking extension of p' over M has bounded multiplicity and by point *1* is stationary. We show that p' has only one nonforking extension over M. This will ensure the stationarity of p'. Let $q_1 \in S(M)$ be a nonforking extension of p'. By Lemma 11.2 the global nonforking extension of q_1 is M-definable. Since p' has bounded multiplicity, this global nonforking extension has a bounded number of $\mathrm{acl}^{\mathrm{eq}}(A)$-conjugates and therefore it is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Therefore q_1 is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Now assume $q_2 \in S(\mathrm{acl}^{\mathrm{eq}}(A))$ is another nonforking extension of p'. Again, q_2 is stationary and definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.

Consider the respective definitions $d_1 x\varphi(x,y) \in L(\mathrm{acl}^{\mathrm{eq}}(A))$ and $d_2 x\varphi(x,y) \in L(\mathrm{acl}^{\mathrm{eq}}(A))$ of q_1 and q_2. We show that if $\varphi(x,a) \in q_1$ then $\varphi(x,a) \in q_2$. Let $b_i \models q_i$. Then $b_i \mathop{\smile\!\!\!\!\!\!|}_A M$ and $\models \varphi(b_1,a)$. Let $r(y) = \mathrm{stp}(a/A)$ and let $\Delta(x)$ be the partial type over $\mathrm{acl}^{\mathrm{eq}}(A)$ given by Corollary 5.19 for $r(y)$ and $\varphi^{-1}(y,x) = \varphi(x,y)$. Then $\models \Delta(b_1)$. Since it is a partial type over $\mathrm{acl}^{\mathrm{eq}}(A)$, also $\models \Delta(b_2)$ and therefore there is some a' such that $a' \mathop{\smile\!\!\!\!\!\!|}_A b_2$, $a' \models r(y)$ and $\models \varphi(b_2,a')$. We may find such a' with the additional property that $a' \mathop{\smile\!\!\!\!\!\!|}_{Ab_2} M$. In this case $a' \mathop{\smile\!\!\!\!\!\!|}_M b_2$ and hence by stationarity $\varphi(x,a')$ belongs to the global nonforking extension of q_2, that is, $\models d_2 x\varphi(x,a')$. Since this formula is over $\mathrm{acl}^{\mathrm{eq}}(A)$ and $a \overset{\mathrm{s}}{\equiv}_A a'$ we conclude that $\models d_2 x\varphi(x,a)$, that is, $\varphi(x,a) \in q_2$. □

Remark 11.5 *Let T be stable.*

1. *Any strong type is stationary.*
2. *Any type over a model is stationary.*

Proof: Clear by Proposition 9.19. □

Remark 11.6 *If T is stable, then any two global nonforking extensions of $p(x) \in S(A)$ are A-conjugate.*

Proof: Let $\mathfrak{p}_1, \mathfrak{p}_2 \in S(\mathfrak{C})$ be two nonforking extensions of p and let $p_i = \mathfrak{p}_i \restriction \mathrm{acl}^{\mathrm{eq}}(A)$. As in the proof of Corollary 6.10, there is some $f \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $p_1^f = p_2$. By Remark 11.5, p_2 is stationary. Since $\mathfrak{p}_1^f$ and $\mathfrak{p}_2$ are nonforking extensions of p_2, they coincide. □

Proposition 11.7 *Let T be stable.*

1. $\mathrm{Mlt}(p) \le 2^{|T|}$.
2. *If* $\mathrm{Mlt}(p) \ge \omega$, *then* $\mathrm{Mlt}(p) \ge 2^{\omega}$.

Proof: *1.* Let $p(x) \in S(A)$, assume p has bounded multiplicity and choose some $B \subseteq A$ of cardinality $\le |T|$ such that p does not fork over B. Since every nonforking extension of p is a nonforking extension of $p \restriction B$, it is enough to check that $\mathrm{Mlt}(p \restriction B) \le 2^{|T|}$. Let $M \supseteq B$ be a model of cardinality $\le |T|$. By Remark 11.5 every type over M extending $p \restriction B$ is stationary, $\mathrm{Mlt}(p \restriction B)$ is bounded by the number of extensions of $p \restriction B$ over M and this number is $\le |S(M)| \le 2^{|T|}$.

2. Note that the set of nonforking extensions over $\mathfrak{C}$ of $p(x) \in S(A)$ is a closed set in $S(\mathfrak{C})$ in which (by Remark 11.6) any two points are connected by a homeomorphism induced by an automorphism of $\mathfrak{C}$ over A. Hence in case this set has an isolated point, any other point is isolated and therefore it is finite. In case it does not have isolated points, it is a nonempty perfect set and therefore it contains at least $\ge 2^{\omega}$ points. □

Definition 11.8 *Two stationary $p(x) \in S(A)$, $q(x) \in S(B)$ types are called* parallel *if they have a common nonforking extension. We write then $p \parallel q$. Note that $q = (p|AB) \restriction B$.*

Definition 11.9 *Let $\mathfrak{p} \in S(\mathfrak{C})$ be definable. A subset B of $\mathfrak{C}^{\mathrm{eq}}$ is a* canonical base *of $\mathfrak{p}$ if for every $f \in \mathrm{Aut}(\mathfrak{C})$, $\mathfrak{p}^f = \mathfrak{p}$ if and only if f fixes pointwise B. Clearly, $\mathfrak{p}$ is definable over A if and only if $B \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$.*

Remark 11.10 *1. If $\mathfrak{p}$ is definable and B is a canonical base of $\mathfrak{p}$, then $\mathfrak{p}$ is definable over B.*

2. If B, B' are canonical bases of the definable type $\mathfrak{p}$, then $\mathrm{dcl}^{\mathrm{eq}}(B) = \mathrm{dcl}^{\mathrm{eq}}(B')$.

3. Every definable global type has a canonical base.

Proof: For *3*, choose for every $\varphi(x,y) \in L$ a formula $d_{\mathfrak{p}}x\varphi(x,y)$ defining $\mathfrak{p}(x) \restriction \varphi$ and let $c_\varphi \in \mathfrak{C}^{\mathrm{eq}}$ be the canonical parameter of its definition $d_{\mathfrak{p}}x\varphi(x,y)$. Then $(c_\varphi : \varphi \in L)$ is a canonical base of $\mathfrak{p}$. $\square$

Definition 11.11 *Let T be stable and $p(x) \in S(A)$ be a stationary type. We call B a* canonical base *of p if B is a canonical base of the (definable) global nonforking extension of p. We use the notation $\mathrm{Cb}(p)$ for $\mathrm{dcl}^{\mathrm{eq}}(B)$ where B is a canonical base of p. Finally we define $\mathrm{Cb}(a/A) = \mathrm{Cb}(\mathrm{stp}(a/A))$.*

Remark 11.12 *Let T be stable. B is a canonical base of the stationary type $p \in S(A)$ if and only if for each $f \in \mathrm{Aut}(\mathfrak{C})$: $p \parallel p^f$ if and only if f fixes B pointwise.*

Proposition 11.13 *Let T be stable.*

1. $\mathrm{Cb}(a/A) \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$.

2. If $\mathrm{tp}(a/A)$ is stationary, then $\mathrm{Cb}(a/A) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$.

Proof: If $f \in \mathrm{Aut}(\mathfrak{C}/A)$ and $p(x) \in S(A)$ is stationary, then $p^f = p \parallel p$ and therefore f fixes pointwise $\mathrm{Cb}(p)$. Hence $\mathrm{Cb}(p) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$. For *2* note that if $\mathrm{tp}(a/A)$ is stationary then $\mathrm{Cb}(p) = \mathrm{Cb}(a/A)$. $\square$

Proposition 11.14 *Let T be stable. Let B be a canonical base of $\mathfrak{p}(x) \in S(\mathfrak{C})$. Then $\mathfrak{p}$ does not fork over A if and only if $B \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$. Moreover those following are equivalent:*

1. $\mathfrak{p}$ is definable over A.

2. $B \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$.

3. $\mathfrak{p}$ does not fork over A and $\mathfrak{p} \restriction A$ is stationary.

Proof: If $\mathfrak{p}$ does not fork over A then $p(x) = \mathfrak{p} \restriction \mathrm{acl}^{\mathrm{eq}}A$ is stationary and has B as a canonical base. Hence by Proposition 11.13 $B \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$. On the other hand if $B \subseteq \mathrm{acl}^{\mathrm{eq}}A$ then $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ and hence it does not fork over A.

Equivalence between *1* and *2* is immediate. Now we prove the equivalence with *3*. If $\mathfrak{p}$ is definable over A, then $\mathfrak{p}$ is the only element of its orbit in $\mathrm{Aut}(\mathfrak{C}/A)$ and hence $\mathfrak{p}$ does not fork over A (by Corollary 8.6) and $\mathfrak{p} \restriction A$ is stationary (by Remark 11.6). For the other direction, if $\mathfrak{p}$ does not fork over A and $\mathfrak{p} \restriction A$ is stationary, then clearly $\mathfrak{p}$ is the only element of its orbit in $\mathrm{Aut}(\mathfrak{C}/A)$ and therefore, by definition of canonical base, $B \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$. $\square$

Proposition 11.15 *Let T be stable. If $B \subseteq A$, the following are equivalent.*

1. $a \perp_B A$
2. $\mathrm{Cb}(a/A) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$
3. $\mathrm{Cb}(a/A) = \mathrm{Cb}(a/B)$.

Proof: Equivalence between *1* and *2* follows from Proposition 11.14. Concerning *3*, note that if $a \perp_B A$ then $\mathrm{stp}(a/A)$ and $\mathrm{stp}(a/B)$ have the same global nonforking extension and therefore $\mathrm{Cb}(a/A) = \mathrm{Cb}(a/B)$. On the other hand, if their canonical bases coincide, then $\mathrm{Cb}(a/A) = \mathrm{Cb}(a/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$. □

Lemma 11.16 *Let T be stable. If $a \in \mathrm{dcl}^{\mathrm{eq}}(b)$, then $\mathrm{Cb}(a/A) \subseteq \mathrm{Cb}(b/A)$. Hence, two interdefinable sequences have the same canonical base over any set.*

Proof: Let $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{Cb}(b/A))$. We will show that $\mathrm{stp}(a/A)$ and $\mathrm{stp}(f(a)/f(A))$ are parallel, which easily implies that f fixes pointwise $\mathrm{Cb}(a/A)$. Since $\mathrm{stp}(b/A) \parallel \mathrm{stp}(f(b)/f(A))$, there is some $c \perp_A f(A)$ such that $c \perp_{f(A)} A$, $c \overset{\mathrm{s}}{\equiv}_A b$ and $c \overset{\mathrm{s}}{\equiv}_{f(A)} f(b)$. Let h be a 0-definable mapping such that $h(b) = a$. Then $h(c) \perp_A f(A)$, $h(c) \perp_{f(A)} A$, $h(c) \overset{\mathrm{s}}{\equiv}_A a$ and $h(c) \overset{\mathrm{s}}{\equiv}_{f(A)} f(a)$, and therefore $\mathrm{stp}(a/A) \parallel \mathrm{stp}(f(a)/f(A))$. □

Lemma 11.17 *Let $p(x), q(y) \in S(A)$ and assume one of them is stationary. Let a, a' be realizations of p and let b, b' be realizations of q. If $a \perp_A b$ and $a' \perp_A b'$ then $ab \equiv_A a'b'$. If p and q is stationary, then $\mathrm{tp}(ab/A)$ is also stationary.*

Proof: Without loss of generality, q is stationary. Choose c be such that $ab \equiv_A a'c$. Then $c \equiv_A b'$, $c \perp_A a'$ and $b' \perp_A a'$. Since q is stationary, $c \equiv_{Aa'} b'$. Then $a'b' \equiv_A a'c \equiv_A ab$. With respect to the last assertion, assume $B \supseteq A$, $cd \equiv_A ab$, $c'd' \equiv_A ab$, $cd \perp_A B$, and $c'd' \perp_A B$. Since p is stationary, $c \equiv_B c'$. Similarly, $d \equiv_B d'$. Moreover $c \perp_B d$ and $c' \perp_B d'$. Therefore $cd \equiv_B c'd'$ and we conclude than $\mathrm{tp}(ab/A)$ is stationary. □

Lemma 11.18 *Let $(I, <)$ be a linearly ordered set and for each $i \in I$, let $p_i(x_i) \in S(A)$ be stationary. Let $(a_i : i \in I)$ be a A-independent sequence where $a_i \models p_i$ for all $i \in I$. If $(b_i : i \in I)$ is an A-independent sequence such that $b_i \models p_i$ for all $i \in I$, then $(a_i : i \in I) \equiv_A (b_i : i \in I)$. Moreover $\mathrm{tp}((a_i : i \in I)/A)$ is stationary.*

Proof: We can assume I is finite and then it can be proved easily by induction on $|I|$ using Lemma 11.17. □

Definition 11.19 *Let $p_i(x_i) \in S(A)$ for each $i \in I$ and assume each of the types p_i is stationary. The* product *of the types $(p_i : i \in I)$ is the stationary type $\mathrm{tp}((a_i : i \in I)/A)$ where $(a_i : i \in I)$ is A-independent and $a_i \models p_i$. By Lemma 11.18 it is well defined. We denote it by $\bigotimes_{i \in I} p_i$. In the finite case we use the notation $p_1 \otimes \ldots \otimes p_n$. If all the types p_i are equal to $p(x) \in S(A)$, the notations are p^I and p^n.*

Remark 11.20 *If $(a_i : i < \alpha)$ is an A-independent sequence of realizations of the stationary type $p(x) \in S(A)$, then it is a Morley sequence in p and $\mathrm{tp}((a_i : i < \alpha)/A) = p^\alpha$. Hence, if $(b_i : i < \alpha)$ is another A-independent sequence of realizations of p, then $(a_i : i < \alpha) \equiv_A (b_i : i < \alpha)$.*

Proof: A-indiscernibility of $(a_i : i < \alpha)$ can be justified observing that for each $n < \omega$, for each $i_0 < \ldots < i_n < \alpha$, $\mathrm{tp}(a_0, \ldots, a_n/A) = \mathrm{tp}(a_{i_0}, \ldots, a_{i_n}/A) = p^n$. □

Lemma 11.21 *Let $\varphi(x, y) \in L$ stable, let $\mathfrak{p} \in S(\mathfrak{C})$ and assume $\mathfrak{p} \restriction \varphi$ is M-definable. If c_φ is the canonical parameter of some definition of $\mathfrak{p} \restriction \varphi$ over M, then $c_\varphi \in \mathrm{dcl}^{\mathrm{eq}}(a_i : i < \omega)$ for some Morley sequence $(a_i : i < \omega)$ in $\mathfrak{p} \restriction M$.*

Proof: By Proposition 8.2 $\mathfrak{p} \restriction \varphi$ is definable over some Morley sequence $(a_i : i < \omega)$ in $\mathfrak{p} \restriction M$. □

Proposition 11.22 *If T is stable, then for each Morley sequence $(a_i : i < \omega)$ in $\mathrm{stp}(a/A)$, $\mathrm{Cb}(a/A) \subseteq \mathrm{dcl}^{\mathrm{eq}}(a_i : i < \omega)$.*

Proof: Let $\mathfrak{p}$ be the global nonforking extension of $p(x) = \mathrm{stp}(a/A)$ and fix some $\varphi(x, y) \in L$ and some model $M \supseteq A$. Let c_φ be the canonical parameter of a definition of $\mathfrak{p} \restriction \varphi$. By Lemma 11.21 $c_\varphi \in \mathrm{dcl}^{\mathrm{eq}}(b_i : i < \omega)$ for some Morley sequence $(b_i : i < \omega)$ in $\mathfrak{p} \restriction M$. Note that $(b_i : i < \omega)$ is also a Morley sequence in $\mathrm{stp}(a/A)$. By Remark 11.20 $(a_i : i < \omega) \overset{\mathrm{s}}{\equiv}_A (b_i : i < \omega)$ and therefore there is some $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{acl}^{\mathrm{eq}}(A))$ sending each b_i to a_i. It follows that $c_\varphi \in \mathrm{dcl}^{\mathrm{eq}}(a_i : i < \omega)$. Since $\mathrm{Cb}(a/A)$ is definable over $(c_\varphi : \varphi \in L)$, we conclude that $\mathrm{Cb}(a/A) \subseteq \mathrm{dcl}^{\mathrm{eq}}(a_i : i < \omega)$. □

Chapter 12

More on independence

Notation 12.1 *In this chapter $\mathop{\smile\!\!\!\!\!\!\,|}$ will be an arbitrary ternary invariant relation among sets. We will use $\mathop{\smile\!\!\!\!\!\!\,|}^f$ for the forking-independence relation as defined in 5.1. Sometimes we will say that $\mathop{\smile\!\!\!\!\!\!\,|}$ is invariant just to stress this fact.*

Definition 12.2 *An* independence relation *is a ternary relation $\mathop{\smile\!\!\!\!\!\!\,|}$ among sets satisfying the following axioms:*

1. *Invariance. If $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ and $f \in \mathrm{Aut}(\mathfrak{C})$, then $f(A) \mathop{\smile\!\!\!\!\!\!\,|}_{f(C)} f(B)$.*
2. *Monotonicity. If $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!\,|}_C B'$.*
3. *Right base monotonicity. If $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ and $C \subseteq D \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!\,|}_D B$.*
4. *Right transitivity. If $D \subseteq C \subseteq B$, $B \mathop{\smile\!\!\!\!\!\!\,|}_C A$, and $C \mathop{\smile\!\!\!\!\!\!\,|}_D A$, then $B \mathop{\smile\!\!\!\!\!\!\,|}_D A$.*
5. *Left normality. If $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$, then $AC \mathop{\smile\!\!\!\!\!\!\,|}_C B$.*
6. *Extension. If $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ and $B' \supseteq B$, then $f(A) \mathop{\smile\!\!\!\!\!\!\,|}_C B'$ for some $f \in \mathrm{Aut}(\mathfrak{C}/BC)$.*
7. *Left finite character. If $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_C B$ for all finite $A_0 \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$.*
8. *Weak local character. For every A there is a cardinal number $\kappa(A)$ such that for any B there is some $C \subseteq B$ such that $|C| < \kappa(A)$ and $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$.*

We say that the independence relation $\mathop{\smile\!\!\!\!\!\!\,|}$ is strict *if additionally satisfies*

9. *Anti-reflexivity. If $A \mathop{\smile\!\!\!\!\!\!\,|}_C A$, then $A \subseteq \mathrm{acl}(C)$.*

For a sequence a, $a \mathop{\smile\!\!\!\!\!\!\,|}_C B$ means that $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ where A is the set enumerate by a. Similarly for other notations like $a \mathop{\smile\!\!\!\!\!\!\,|}_C b$, etc.

Remark 12.3 *Note that the property of* right normality

$$\text{if } A \mathop{\smile\!\!\!\!\!\!\,|}_C B \text{ then } A \mathop{\smile\!\!\!\!\!\!\,|}_C BC$$

follows from extension and invariance. Note also that right base monotonicity and weak local character give the so called existence property*:*

$$A \underset{B}{\perp} B.$$

Proposition 12.4 *Assume $\perp$ satisfies the first five axioms of independence and also the extension property. If $a \perp_C B$, then there is a BC-indiscernible sequence $(a_i : i < \omega)$ such that $a_i \equiv_{BC} a$ and $(a_j : j < i) \perp_C a_i$ for all $i < \omega$.*

Proof: Since $a \perp_C B$, by the extension property for any λ we can construct a sequence $(a_i : i < \lambda)$ such that $a_0 = a$, $a_i \equiv_{BC} a_0$, and $a_i \perp_C B(a_j : j < i)$. If we choose λ big enough and we apply Proposition 1.1, we obtain a BC-indiscernible sequence $(a'_i : i < \omega)$ such that for each $n < \omega$ there are $i_0 < \ldots < i_n < \lambda$ such that $a'_0, \ldots, a'_n \equiv_{BC} a_{i_0}, \ldots, a_{i_n}$. By monotonicity and invariance, $a'_i \perp_C (a'_j : j < i)$ for all $i < \omega$. We now claim that for all $n > 0$,

$$(a'_i : 0 < i < n) \underset{C}{\perp} a'_0$$

We prove it by induction on n. It is clear for $n = 1$. By the inductive hypothesis and left normality, $C(a'_i : 0 < i < n) \perp_C a'_0$. By construction of the sequence and right base monotonicity, $a'_n \perp_{C(a'_i : 0<i<n)} a'_0$. By left normality again, $C(a'_i : 0 < i \leq n) \perp_{C(a'_i:0<i<n)} a'_0$. Finally by right transitivity $C(a'_i : 0 < i \leq n) \perp_C a'_0$ and by monotonicity $(a'_i : 0 < i \leq n) \perp_C a'_0$. This finishes the induction.

By compactness, there is a sequence $(a''_i : i < \omega)$ such that for each $n < \omega$, $a''_0, \ldots, a''_n \equiv_{BC} a'_n, \ldots, a'_0$. It is clear that it satisfies the required conditions. □

Proposition 12.5 *Assume $\perp$ satisfies the first five axioms of independence and also the weak local character and the finite character properties. Assume there is a BC-indiscernible sequence $(a_i : i < \omega)$ such that $a_i \equiv_{BC} a$ and $(a_j : j < i) \perp_C a_i$ for all $i < \omega$. Then $B \perp_C a$.*

Proof: Let $\kappa(B)$ be the cardinal given for B by the weak local character property and choose a regular cardinal $\kappa > \kappa(B)$. We can extend our sequence to a BC-indiscernible sequence $(a_i : i < \kappa)$. By finite character and invariance, $(a_j : j < i) \perp_C a_i$ for all $i < \kappa$. By weak local character there is some $D \subseteq C \cup \{a_i : i < \kappa\}$ such that $|D| < \kappa$ and $B \perp_D C(a_i : i < \kappa)$. By regularity of κ, $D \subseteq C \cup \{a_j : j < i\}$ for some $i < \kappa$. By right base monotonicity, $B \perp_{C(a_j:j<i)} C(a_j : j < \kappa)$ and by monotonicity, $B \perp_{C(a_j:j<i)} a_i$. By left normality $BC(a_j : j < i) \perp_{C(a_j:j<i)} a_i$ and also $C(a_j : j < i) \perp_C a_i$. By right transitivity, $BC(a_j : j < i) \perp_C a_i$. By monotonicity $B \perp_C a_i$. Since $a \equiv_{BC} a_i$, by invariance $B \perp_C a$.
□

Corollary 12.6 *Any independence relation is symmetric, that is: if $A \perp_C B$, then $B \perp_C A$.*

Proof: It is an immediate consequence of propositions 12.4 and 12.5. □

Definition 12.7 *For any invariant $\perp$ we define $\perp^*$ as follows: $A \perp^*_C B$ if and only if for all $B' \supseteq B$ there is some $f \in \mathrm{Aut}(\mathfrak{C}/BC)$ such that $f(A) \perp_C B'$.*

Remark 12.8 *For any $\perp$, $\perp^*$ is also invariant and $A \perp^*_C B$ implies $A \perp_C B$.*

Proposition 12.9 *For any monotone $\perp$, $\perp^*$ has the extension property.*

Proof: Let $A \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$ and $B \subseteq B'$. Let a enumerate A and let x be a corresponding sequence of variables. We claim that there is a type $p(x) \in S(CB')$ extending $\mathrm{tp}(a/CB)$ and such that for each cardinal κ there is a κ-saturated model $M \supseteq CB'$ and some $a' \models p$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_C M$. Assume not, and fix for each $p(x) \in S(CB')$ extending $\mathrm{tp}(a/CB)$ a corresponding cardinal κ_p for which there is no κ_p-saturated model $M \supseteq CB'$ with a realization $a' \models p$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_C M$. Let κ be the supremum of all these cardinals κ_p and choose a κ-saturated model $M \supseteq CB'$. Since $a \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$, there is some $a' \equiv_{CB} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_C M$. Then $p(x) = \mathrm{tp}(a'/CB')$ satisfies the requirements of the claim.

Now we use the claim fixing some $p(x) \in S(CB')$ as indicated. Let $a' \models p$. We will show that $a' \mathop{\smile\!\!\!\!\!\!\mid}^*_C B'$. This will establish the extension property for $\mathop{\smile\!\!\!\!\!\!\mid}^*$. Let $B'' \supseteq B'$. We need to show that for some $a'' \equiv_{CB'} a'$ (i.e., some $a'' \models p$), $a'' \mathop{\smile\!\!\!\!\!\!\mid}_C B''$. Let $\kappa = |C \cup B'|^+ + |B''|$ and by the claim choose a κ-saturated $M \supseteq CB'$ and some $a'' \models p$ such that $a'' \mathop{\smile\!\!\!\!\!\!\mid}_C M$. By κ saturation there is an automorphism $f \in \mathrm{Aut}(\mathfrak{C}/CB')$ such that $f(B'') \subseteq M$. By monotonicity $a'' \mathop{\smile\!\!\!\!\!\!\mid}_C f(B'')$. By invariance $f^{-1}(a'') \mathop{\smile\!\!\!\!\!\!\mid}_C B''$. Since $f^{-1}(a'') \models p$ we have finished. □

Remark 12.10 *Each one of the properties of monotonicity, right base monotonicity, right transitivity, left normality, and anti-reflexivity is preserved when passing from* $\mathop{\smile\!\!\!\!\!\!\mid}$ *to* $\mathop{\smile\!\!\!\!\!\!\mid}^*$.

Proposition 12.11 *Assume* $\mathop{\smile\!\!\!\!\!\!\mid}$ *satisfies the first five axioms of independence and also left finite character. If* $\mathop{\smile\!\!\!\!\!\!\mid}^*$ *satisfies weak local character, then* $\mathop{\smile\!\!\!\!\!\!\mid}^*$ *is an independence relation.*

Proof: By Remark 12.10 and Proposition 12.9 we only need to show that $\mathop{\smile\!\!\!\!\!\!\mid}^*$ has left finite character. But first we check that $\mathop{\smile\!\!\!\!\!\!\mid}^*$ is symmetric. Note that $\mathop{\smile\!\!\!\!\!\!\mid}^*$ satisfies the hypotheses of Proposition 12.4 and $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies the hypotheses of Proposition 12.5. Hence $A \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$ implies $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$. Now assume $A \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$ and let us prove that $B \mathop{\smile\!\!\!\!\!\!\mid}^*_C A$. Let $A' \supseteq A$. Since $A' \mathop{\smile\!\!\!\!\!\!\mid}^*_{AC} AC$, by extension there is some $f \in \mathrm{Aut}(\mathfrak{C}/AC)$ such that $f(A') \mathop{\smile\!\!\!\!\!\!\mid}^*_{AC} ACB$. By monotonicity $f(A') \mathop{\smile\!\!\!\!\!\!\mid}^*_{AC} B$. Since $A \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$, by right transitivity and monotonicity of $\mathop{\smile\!\!\!\!\!\!\mid}^*$, $f(A') \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$. Hence $B \mathop{\smile\!\!\!\!\!\!\mid}_C f(A')$ and $f^{-1}(B) \mathop{\smile\!\!\!\!\!\!\mid}_C A'$, which shows that $B \mathop{\smile\!\!\!\!\!\!\mid}^*_C A$.

Assume that for any finite tuple $a \in A$, $a \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$. To prove that $A \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$, consider some $B' \supseteq B$. By existence and extension, there is some $f \in \mathrm{Aut}(\mathfrak{C}/BC)$ such that $f(A) \mathop{\smile\!\!\!\!\!\!\mid}^*_{CB} B'$. Hence $A \mathop{\smile\!\!\!\!\!\!\mid}^*_{CB} f^{-1}(B')$. By symmetry $f^{-1}(B') \mathop{\smile\!\!\!\!\!\!\mid}^*_{BC} A$. For each tuple $a \in A$, we have $a \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^*_{BC} f^{-1}(B')$. By symmetry and right transitivity we obtain then $a \mathop{\smile\!\!\!\!\!\!\mid}^*_C f^{-1}(B')$ for all tuples $a \in A$. Hence $a \mathop{\smile\!\!\!\!\!\!\mid}_C f^{-1}(B')$ for all tuples $a \in A$. By left finite character of $\mathop{\smile\!\!\!\!\!\!\mid}$, $A \mathop{\smile\!\!\!\!\!\!\mid}_C f^{-1}(B')$. By invariance $f(A) \mathop{\smile\!\!\!\!\!\!\mid}_C B'$. □

Proposition 12.12 *Let* $\mathop{\smile\!\!\!\!\!\!\mid}$ *be monotone. Then* $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^*$ *if and only if* $\mathop{\smile\!\!\!\!\!\!\mid}$ *has the extension property.*

Proof: One directions follows from Proposition 12.9. The other direction is clear by definition of $\mathop{\smile\!\!\!\!\!\!\mid}^*$ since $\mathop{\smile\!\!\!\!\!\!\mid}^*$ refines $\mathop{\smile\!\!\!\!\!\!\mid}$. □

Definition 12.13 *It has already mentioned that* $\mathop{\smile\!\!\!\!\!\!\mid}^f$ *is nonforking independence. We define* $\mathop{\smile\!\!\!\!\!\!\mid}^d$ *as nondividing independence. To be precise:*

1. $A \mathop{\smile\!\!\!\!\!\!\mid}^d_C B$ *if and only if for any sequence* $a \in A$, $\mathrm{tp}(a/BC)$ *does not divide over* C.
2. $A \mathop{\smile\!\!\!\!\!\!\mid}^f_C B$ *if and only if for any sequence* $a \in A$, $\mathrm{tp}(a/BC)$ *does not fork over* C.

Proposition 12.14 $(\mathop{\smile\!\!\!\!\!\!|}^{d})^* = \mathop{\smile\!\!\!\!\!\!|}^{f}$.

Proof: By Remark 4.4 we know that $\mathop{\smile\!\!\!\!\!\!|}^{f}$ has the extension property. Since $\mathop{\smile\!\!\!\!\!\!|}^{f}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{d}$, it follows that $\mathop{\smile\!\!\!\!\!\!|}^{f}$ implies $(\mathop{\smile\!\!\!\!\!\!|}^{d})^*$. For the other direction, assume $A(\mathop{\smile\!\!\!\!\!\!|}^{d})^*_C B$ but $A \not\mathop{\smile\!\!\!\!\!\!|}^{f}_C B$. For some tuple $a \in A$, for some formula $\varphi(x,y) \in L$, for some $b \in BC$, $\models \varphi(a,b)$ and $\varphi(x,b)$ forks over C. Then for some $\psi_1(x,y_1),\ldots,\psi_n(x,y_n) \in L$, for some $b_1,\ldots,b_n$, $\models \varphi(x,b) \rightarrow \psi_1(x,b_1) \vee \ldots \vee \psi_n(x,b_n)$ and each $\psi(x,b_i)$ divides over C. Let $B' = Bb_1,\ldots,b_n$. By assumption there is some $a' \equiv_{BC} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^{d}_C B'$. Since $\models \varphi(a',b)$, for some i, $\models \psi_i(a',b_i)$. This implies that $\mathrm{tp}(a'/B')$ divides over C, a contradiction. □

Remark 12.15 $\mathop{\smile\!\!\!\!\!\!|}^{d}$ *has the properties of invariance, monotonicity, right base monotonicity, right transitivity, left normality, finite character and anti-reflexivity. Therefore* $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *satisfies all this properties and moreover it satisfies extension.*

Proof: For right transitivity see Proposition 4.6 and for anti-reflexivity see point *5* in Remark 4.2. The other properties are straightforward. □

Proposition 12.16 *The following are equivalent.*

1. *T is simple*
2. *$\mathop{\smile\!\!\!\!\!\!|}^{f}$ satisfies weak local character.*
3. *$\mathop{\smile\!\!\!\!\!\!|}^{d}$ satisfies weak local character.*
4. *$\mathop{\smile\!\!\!\!\!\!|}^{f}$ is an independence relation.*
5. *$\mathop{\smile\!\!\!\!\!\!|}^{d}$ is an independence relation.*

Proof: We know that simplicity of T implies all the other conditions. It is clear that *4* implies *2* and that *5* implies *3*. It is also clear that *2* implies *3*. We check now that simplicity follows from *3*. Assume T is not simple. Then for some $p(x) \in S(\emptyset)$ for some $\varphi(x,y) \in L$, for some $k < \omega$, $D(p(x),\varphi,k) = \infty$. The cardinal $\kappa(a)$ given by weak local character of $\mathop{\smile\!\!\!\!\!\!|}^{d}$ is clearly the same for any realization of a of p. Let κ be regular and bigger than this cardinal. By Proposition 3.9 there is a sequence $(a_i : i < \kappa)$ such that $p(x) \cup \{\varphi(x,a_i) : i < \kappa\}$ is consistent and for each $i < \kappa$, $\varphi(x,a_i)$ k-divides over $\{a_j : j < i\}$. Let $a \models p(x) \cup \{\varphi(x,a_i) : i < \kappa\}$. By choice of κ, there is some $C \subseteq \{a_i : i < \kappa\}$ such that $|C| < \kappa$ and $\mathrm{tp}(a/\{a_i : i < \kappa\})$ does not divide over C. By regularity of κ, for some $i < \kappa$, $C \subseteq \{a_j : j < i\}$. Then $\mathrm{tp}(a/\{a_j : j \leq i\})$ does not divide over $\{a_j : j < i\}$. But this contradicts the fact that $\models \varphi(a,a_i)$ and that $\varphi(x,a_i)$ divides over $\{a_j : j < i\}$. □

Remark 12.17 *Assume $\mathop{\smile\!\!\!\!\!\!|}$ is invariant and has weak local character. Let α be an ordinal number. There is a cardinal number κ such that for each α-sequence a, for each set B there is some $C \subseteq B$ such that $|C| < \kappa$ and $a \mathop{\smile\!\!\!\!\!\!|}_C B$.*

Proof: Let x be a sequence of variables of length α and let $p(x) \in S(\emptyset)$. By weak local character, for each $a \models p$ there is some cardinal $\kappa(a)$ witnessing the property for a. By invariance $\kappa(a)$ is the same for each $a \models p$. Let us call it κ_p. Now the supremum of all κ_p for $p(x) \in S(\emptyset)$ satisfies the required condition. □

Definition 12.18 *We will be dealing with some arbitrary independence relation $\mathop{\smile\!\!\!\!\!\!|}$ and we would like to use for it the standard terminology developed for nonforking independence $\mathop{\smile\!\!\!\!\!\!|}^f$ in simple theories. By Corollary 12.6 we know that $\mathop{\smile\!\!\!\!\!\!|}$ is symmetric. Therefore $\mathop{\smile\!\!\!\!\!\!|}$ is also left transitive and has right finite character. A* $\mathop{\smile\!\!\!\!\!\!|}$-independent *over C sequence will be a sequence $(a_i : i < \alpha)$ such that $a_i \mathop{\smile\!\!\!\!\!\!|}_C (a_j : j < i)$ for all $i < \alpha$. Such a sequence will be called a* $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence over C *if additionally it is C-indiscernible.*

Let $A \subseteq B$, $p(x) \in S(A)$ and $p(x) \subseteq q(x) \in S(B)$. We say that $q(x)$ is a $\mathop{\smile\!\!\!\!\!\!|}$-free extension *of $p(x)$ if for some $a \models q$, $a \mathop{\smile\!\!\!\!\!\!|}_A B$. In this case we also say that q is $\mathop{\smile\!\!\!\!\!\!|}$-free over A.*

We say that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the Independence Theorem over C, *if whenever $a \equiv_C b$, $C \subseteq A \cap B$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$, $a \mathop{\smile\!\!\!\!\!\!|}_C A$, and $b \mathop{\smile\!\!\!\!\!\!|}_C B$, then there is some $c \mathop{\smile\!\!\!\!\!\!|}_C AB$ such that $c \equiv_A a$ and $c \equiv_B b$. In other terms, if $C \subseteq A \cap B$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$, for any two types $p(x) \in S(A)$ and $q(x) \in S(B)$ which are $\mathop{\smile\!\!\!\!\!\!|}$-free over C and have a common restriction to C, their union can be extended to a complete type over AB which is $\mathop{\smile\!\!\!\!\!\!|}$-free over C.*

Proposition 12.19 *$\mathop{\smile\!\!\!\!\!\!|}^d$ is finer than any independence relation $\mathop{\smile\!\!\!\!\!\!|}$, that is: if $A \mathop{\smile\!\!\!\!\!\!|}^d_C B$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.*

Proof: Assume $a \mathop{\smile\!\!\!\!\!\!|}^d_C b$ but $a \not\mathop{\smile\!\!\!\!\!\!|}_C b$. Let $\kappa(a)$ be the cardinal given for a by the weak local character property and choose a regular $\kappa > \kappa(a)$. We check that there is a $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence $(b_i : i < \kappa)$ over C starting with $b_0 = b$. Since $b \mathop{\smile\!\!\!\!\!\!|}_C C$, there a C-indiscernible sequence $(b_i : i < \kappa)$ starting with $b_0 = b$ which is $\mathop{\smile\!\!\!\!\!\!|}$-independent over C, that is $b_i \mathop{\smile\!\!\!\!\!\!|}_C (b_j : j < i)$ for all $i < \kappa$. Its initial segment $(b_i : i < \omega)$ can be obtained as in Proposition 12.4 (using freely the symmetry of $\mathop{\smile\!\!\!\!\!\!|}$) and for its extension to a sequence of length κ we need only to preserve C-indiscernibility since $\mathop{\smile\!\!\!\!\!\!|}$-independence over C is granted by invariance and finite character. Now let $p(x,y) = \mathrm{tp}(ab/C)$. Since $p(x,b)$ does not divide over C, $\bigcup_{i<\kappa} p(x,b_i)$ is consistent. Let a' be a realization of this union of types. Then $a'b_i \equiv_C ab$ for all $i < \kappa$, which implies that $a' \not\mathop{\smile\!\!\!\!\!\!|}_C b_i$ for all $i < \kappa$. If $a' \mathop{\smile\!\!\!\!\!\!|}_{C(b_j:j<i)} b_i$ then (by transitivity) $a' \mathop{\smile\!\!\!\!\!\!|}_C b_i$, which is not the case. Hence $a' \not\mathop{\smile\!\!\!\!\!\!|}_{C(b_j:j<i)} b_i$ for all $i < \kappa$. But this contradicts the choice of κ since $\kappa(a) = \kappa(a')$ and therefore $a' \mathop{\smile\!\!\!\!\!\!|}_{C(b_j:j<i)} (b_j : j < \kappa)$ for some $i < \kappa$. □

Lemma 12.20 *Let $\mathop{\smile\!\!\!\!\!\!|}$ be an independence relation. Assume $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the Independence Theorem over C. Then for any $p(x,y) \in S(C)$, if $(a_i : i < \alpha)$ is an $\mathop{\smile\!\!\!\!\!\!|}$-independent over C and each $p(x,a_i)$ is a $\mathop{\smile\!\!\!\!\!\!|}$-free extension of its common restriction to C, then $\bigcup_{i<\alpha} p(x,a_i)$ is $\mathop{\smile\!\!\!\!\!\!|}$-free over C.*

Proof: We inductively construct a chain of types $(q_i : i < \alpha)$ such that $q_i(x) \in S(C(a_j : j < i))$ extends $\bigcup_{j<i} p(x,a_j)$ and is $\mathop{\smile\!\!\!\!\!\!|}$-free over C. We begin with $q_0 = p(x,a_0) \upharpoonright C$ and for limit i we put $q_i = \bigcup_{j<i} q_j$ (which is $\mathop{\smile\!\!\!\!\!\!|}$-free by inductive hypothesis and finite character). For the case q_{i+1} we apply the Independence Theorem to $A = C(a_j : j < i)$, $B = Ca_i$, $q_i(x) \in S(A)$, and $p(x,a_i) \in S(B)$ (which are $\mathop{\smile\!\!\!\!\!\!|}$-free extensions of q_0) obtaining a type $q_{i+1}(x) \in S(AB) = S(C(a_j : j < i+1))$ extending $p(x,a_i) \cup q_i(x)$ and $\mathop{\smile\!\!\!\!\!\!|}$-free over C. Since $\bigcup_{i<\alpha} q_i(x)$ is $\mathop{\smile\!\!\!\!\!\!|}$-free over C and contains $\bigcup_{i<\alpha} p(x,a_i)$, also $\bigcup_{i<\alpha} p(x,a_i)$ is $\mathop{\smile\!\!\!\!\!\!|}$-free over C. □

Theorem 12.21 *T is simple if and only if there is an independence relation $\mathop{\smile\!\!\!\!\!\!|}$ in T which satisfies the Independence Theorem over models. Moreover if T is simple and $\mathop{\smile\!\!\!\!\!\!|}$ is as indicated, then $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^d$.*

Proof: If T is simple then clearly $\mathop{\smile\hskip-0.9em^|\ }^d = \mathop{\smile\hskip-0.9em^|\ }^f$ is an independence relation (see Proposition 12.16) and satisfies the Independence Theorem over models (see Corollary 10.10). For the other direction, by Proposition 12.19 we know that $\mathop{\smile\hskip-0.9em^|\ }^d \subseteq \mathop{\smile\hskip-0.9em^|\ }$. We will show now that $\mathop{\smile\hskip-0.9em^|\ } \subseteq \mathop{\smile\hskip-0.9em^|\ }^d$. From this it will follow that $\mathop{\smile\hskip-0.9em^|\ } = \mathop{\smile\hskip-0.9em^|\ }^d$ and hence that $\mathop{\smile\hskip-0.9em^|\ }^d$ has weak local character in T. By Proposition 12.16 T is simple.

Let $a \mathop{\smile\hskip-0.9em^|\ }_C b$. We check that $a \mathop{\smile\hskip-0.9em^|\ }^d_C b$. Let $p(x,y) = \mathrm{tp}(ab/C)$ and let $(b_i : i < \omega)$ be C-indiscernible with $b_0 = b$. We will show that $\bigcup_{i<\omega} p(x,b_i)$ is consistent. Let $\kappa(b)$ be the cardinal number given for b by the weak local character property and choose a regular cardinal $\kappa > \kappa(b)$. Extend the given sequence to a C-indiscernible sequence $(b_i : i \leq \kappa)$. By Corollary 1.2 there is a model $M \supseteq A$ such that $(b_i : i \leq \kappa)$ is M-indiscernible. Starting with $M_0 = M$ it is easy now to construct a chain of models $(M_i : i < \kappa)$ such that $C(b_j : j < i) \subseteq M_i$ and $(b_j : i < j \leq \kappa)$ is M_i-indiscernible. Since $\kappa(b) = \kappa(b_\kappa)$, by choice of κ, $b_k \mathop{\smile\hskip-0.9em^|\ }_{M_i} (M_j : j < \kappa)$ for some $i < \kappa$. Then $b_k \mathop{\smile\hskip-0.9em^|\ }_{M_i} (b_j : i < j < \kappa)$. By invariance and finite character, $(b_j : i < j < \kappa)$ is $\mathop{\smile\hskip-0.9em^|\ }$-independent over M_i and hence it is a $\mathop{\smile\hskip-0.9em^|\ }$-Morley sequence over M_i. By conjugation over C, $(b_i : i < \omega)$ is an $\mathop{\smile\hskip-0.9em^|\ }$-Morley sequence over some model $M \supseteq C$. Let $q(x) \in S(Mb_0)$ a $\mathop{\smile\hskip-0.9em^|\ }$-free extension of $p(x,b_0)$ and choose $p'(x,y) \in S(M)$ such that $q(x) = p'(x,b_0)$. Then $p'(x,b_i) \in S(Mb_i)$ is $\mathop{\smile\hskip-0.9em^|\ }$-free over M (in fact over C). By Lemma 12.20, $\bigcup_{i<\omega} p'(x,b_i)$ is consistent. In particular $\bigcup_{i<\omega} p(x,b_i)$ is consistent. □

Theorem 12.22 *T is stable if and only if there is an independence relation $\mathop{\smile\hskip-0.9em^|\ }$ in T which satisfies one of the two equivalent conditions:*

1. *Types over models are $\mathop{\smile\hskip-0.9em^|\ }$-stationary, that is, for any $p(x) \in S(M)$, for any $B \supseteq M$ there is only one $\mathop{\smile\hskip-0.9em^|\ }$-free extension of p over B.*

2. *Every type has a bounded number of $\mathop{\smile\hskip-0.9em^|\ }$-free extensions, that is, for each sequence of variables x there is a cardinal μ such that for each $p(x) \in S(A)$ for every $B \supseteq A$ there are at most μ $\mathop{\smile\hskip-0.9em^|\ }$-free extensions of p over B.*

Moreover if T is stable and $\mathop{\smile\hskip-0.9em^|\ }$ is as indicated, then $\mathop{\smile\hskip-0.9em^|\ } = \mathop{\smile\hskip-0.9em^|\ }^d$.

Proof: If T is stable, T is simple and $\mathop{\smile\hskip-0.9em^|\ }^d = \mathop{\smile\hskip-0.9em^|\ }^f$ is an independence relation. Moreover (see Remark 11.5 and Proposition 11.7) conditions *1* and *2* hold.

1 implies *2*. Let α be the length of x and let κ be the cardinal given by weak local character according to Remark 12.17. Let $\mu = 2^{|T|+\kappa}$. We want to show that μ is an upper bound for the number of $\mathop{\smile\hskip-0.9em^|\ }$-free extensions of $p(x) \in S(A)$ over any other bigger set. For this we may assume that $|A| \leq \kappa$ because there is some $C \subseteq A$ of cardinality $< \kappa$ such that p is $\mathop{\smile\hskip-0.9em^|\ }$-free over C and then a bound for $p \restriction C$ is also a bound for p. There is a model $M \supseteq A$ of cardinality κ. The number of extensions of p to a complete type over M is bounded by $|S(M)| \leq 2^{|T|+\kappa} = \mu$. Since every type over M is stationary, the number of $\mathop{\smile\hskip-0.9em^|\ }$-free extensions of p over any set is also bounded by μ.

2 implies stability of T and $\mathop{\smile\hskip-0.9em^|\ } = \mathop{\smile\hskip-0.9em^|\ }^d$ (and hence it implies *1*). Fix μ as in *2* and fix an n-tuple of variables x. Choose $\kappa > |T|$ witnessing the weak local character of $\mathop{\smile\hskip-0.9em^|\ }$ for n as in Remark 12.17. Choose $\lambda \geq \mu$ such that $\lambda = \lambda^{<\kappa}$. We show that T is stable in λ. Let $|A| \leq \lambda$. For each $p(x) \in S(A)$ there is some $C \subseteq A$ such that p is $\mathop{\smile\hskip-0.9em^|\ }$-free over C and $|C| < \kappa$. There are $\leq \lambda^{<\kappa} = \lambda$ such subsets $C \subseteq A$, over each such C there are $\leq 2^{|T|+|C|} \leq \lambda^{<\kappa} = \lambda$ types $q(x) \in S(C)$ and for each $q(x) \in S(C)$ there are at most $\mu \leq \lambda$ $\mathop{\smile\hskip-0.9em^|\ }$-free extensions of q over A. The number of types $p(x) \in S(A)$ is therefore bounded by λ.

Thus, T is stable. By Proposition 12.19 we know that $\mathop{\smile\!\!\!\!\!\!|}^d \subseteq \mathop{\smile\!\!\!\!\!\!|}$. To check that $\mathop{\smile\!\!\!\!\!\!|} \subseteq \mathop{\smile\!\!\!\!\!\!|}^d$ assume $p(x) = \mathrm{tp}(a/BC)$ divides over C. Every global extension $\mathfrak{p} \in S(\mathfrak{C})$ of p forks over C and therefore has an unbounded number of C-conjugates. But if p is $\mathop{\smile\!\!\!\!\!\!|}$-free over C then over any bigger set p has an extension which is $\mathop{\smile\!\!\!\!\!\!|}$-free over C and hence the number of its C-conjugates is bounded by μ. Therefore $a \not\mathop{\smile\!\!\!\!\!\!|}_C B$. □

Proposition 12.23 *The following are equivalent.*

1. *T is not simple.*
2. *For some $\varphi(x,y) \in L$ there is an indiscernible sequence $(c_i a_i : i < \omega)$ such that for all $i < \omega$, $\models \varphi(c_i, a_0)$ and $\varphi(x, a_i)$ divides over $\{c_j a_j : j < i\}$.*
3. *For some $\varphi(x,y) \in L$, there are a tuple c and some c-indiscernible sequence $(a_i : i < \omega)$ such that for all $i < \omega$, $\models \varphi(c, a_i)$ and $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$.*
4. *For some $\varphi(x,y) \in L$, there are a tuple c and some c-indiscernible sequence $(a_i : i \leq \omega)$ such that $\models \varphi(c, a_\omega)$ and $\varphi(x, a_\omega)$ divides over $\{a_i : i < \omega\}$.*

Proof: *1 ⇒ 2.* If T is not simple, then (see Proposition 3.9) for some $\varphi(x,y) \in L$, for some $k < \omega$ there is a sequence $(d_i : i < \omega)$ such that $\{\varphi(x, d_i) : i < \omega\}$ is consistent and $\varphi(x, d_i)$ k-divides over $\{d_j : j < i\}$ for each $i < \omega$. By Proposition 1.1 we may assume $(d_i : i < \omega)$ is indiscernible. We now inductively define $(c_i a_i : i < \omega)$ in such a way that $\varphi(x, a_i)$ divides over $\{c_j a_j : j < i\}$ and $\models \varphi(c_i a_0) \wedge \ldots \wedge \varphi(c_i, a_i)$. Indiscernibility can be obtained again by an application of Proposition 1.1. We start the construction with $a_0 = d_0$ choosing then c_0 such that $\models \varphi(c_0, a_0)$. Since $\varphi(x, d_1)$ k-divides over a_0, there is an a_0-indiscernible sequence $(b_i : i < \omega)$ such that $b_i \equiv_{a_0} d_1$ and $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent. By Proposition 1.1 we may assume it is $a_0 c_0$-indiscernible. Set $a_1 = b_0$ and choose c_1 such that $\models \varphi(c_1, a_0) \wedge \varphi(c_1, a_1)$. Then $\varphi(x, a_1)$ k-divides over a_0, c_0. Changing $(d_i : 2 \leq i < \omega)$ by $(d'_i : 2 \leq i < \omega)$ such that $a_0 a_1 (d'_i : 2 \leq i < \omega) \equiv (d_i : i < \omega)$ if necessary we can continue carrying out the construction.

2 ⇒ 3. By indiscernibility we may extend the sequence $(c_i a_i : i < \omega)$ and therefore assume that dividing is always with respect to some fixed $k < \omega$. We can also take $\omega + 1$ as index set, in which case $\models \varphi(c_\omega, a_i)$ for all $i < \omega$. Then put $c = c_\omega$ and note that $(a_i : i < \omega)$ is c–indiscernible and that for all $i < \omega$, $\varphi(x, a_i)$ k-divides over $\{a_j : j < i\}$.

3 ⇒ 4. Extend the sequence $(a_i : i < \omega)$ to a c-indiscernible sequence $(a_i : i \leq \omega)$.

4 ⇒ 1. Assume $\varphi(x, a_\omega)$ k-divides over $\{a_i : i < \omega\}$. By indiscernibility for all $i < \omega$, $\varphi(x, a_i)$ k-divides over $\{a_j : j < i\}$. By c-indiscernibility, $\models \varphi(c, a_i)$ for all $i < \omega$ and therefore $\{\varphi(x, a_i) : i < \omega\}$ is consistent, which contradicts simplicity of T. □

Theorem 12.24 *The following are equivalent*

1. *T is simple.*
2. *$\mathop{\smile\!\!\!\!\!\!|}^d$ is symmetric.*
3. *$\mathop{\smile\!\!\!\!\!\!|}^f$ is symmetric.*
4. *$\mathop{\smile\!\!\!\!\!\!|}^d$ is left transitive.*
5. *$\mathop{\smile\!\!\!\!\!\!|}^f$ is left transitive.*

Proof: By proposition 5.15 and by the fact that in a simple theory $\mathop{\smile\!\!\!\!\!\!|}^{d} = \mathop{\smile\!\!\!\!\!\!|}^{f}$, conditions *2* and *3* follow from *1*.

2 $\Rightarrow$ *4* and *3* $\Rightarrow$ *5*. Since $\mathop{\smile\!\!\!\!\!\!|}^{d}$ and $\mathop{\smile\!\!\!\!\!\!|}^{f}$ are right transitive, it is clear that symmetry implies they are left transitive.

4 $\Rightarrow$ *1* and *5* $\Rightarrow$ *1*. Fix an ordered set of order type $\omega + 2 + \omega^*$, where ω^* is the reverse order of ω, say

$$0 < 1 < \cdots < \omega < \omega + 1 < \cdots < -2 < -1.$$

where $\omega^* = \{-1, -2, \ldots\}$. Assume T is not simple. By Proposition 12.23 and compactness there is some $\varphi(x, y) \in L$ for which there is an indiscernible sequence $(c_i a_i : i \in \omega + 2 + \omega^*)$ such that for each i, $\varphi(x, a_i)$ divides over $\{c_j a_j : j < i\}$ and for all $j \leq i$, $\models \varphi(c_i, a_j)$. Let $I = \{a_i : i \in \omega\}$ and let $J = \{a_i : i \in \omega^*\}$. Since $\models \varphi(c_{\omega+1}, a_\omega)$, $c_{\omega+1} \not\mathop{\smile\!\!\!\!\!\!|}^{d}_{I} J a_\omega$. Since $\mathrm{tp}(c_{\omega+1}/IJ)$ is finitely satisfiable in I, $c_{\omega+1} \mathop{\smile\!\!\!\!\!\!|}^{f}_{I} J$. Since $\mathrm{tp}(c_{\omega+1}/a_\omega IJ)$ is finitely satisfiable in J, $c_{\omega+1} \mathop{\smile\!\!\!\!\!\!|}^{f}_{IJ} a_\omega$. This contradicts left transitivity of $\mathop{\smile\!\!\!\!\!\!|}^{d}$ and $\mathop{\smile\!\!\!\!\!\!|}^{f}$.

$\square$

Chapter 13

Supersimple theories

Definition 13.1 *T is* supersimple *if for all $p \in S(A)$ (in finitely many variables) there is a finite $A_0 \subseteq A$ such that p does not fork over A_0. In other words, for any tuple a, for any set A, there is a finite $A_0 \subseteq A$ such that $a \mathop{\smile\!\!\!\!\mid}_{A_0} A$. By Proposition 4.11 this implies T is simple. T is* superstable *if it is stable and supersimple.*

Definition 13.2 *$\kappa(T)$ is the least cardinal μ such that for each tuple a, for each set A there is some $B \subseteq A$ such that $|B| < \mu$ and $a \mathop{\smile\!\!\!\!\mid}_{B} A$. If there is not such cardinal μ we set $\kappa(T) = \infty$.*

Remark 13.3

1. *T is simple iff $\kappa(T) < \infty$ iff $\kappa(T) \leq |T|^+$.*
2. *T is supersimple iff $\kappa(T) = \omega$.*

Proposition 13.4 *The following are equivalent:*

1. *T is supersimple*
2. *There is no infinite sequence $(\varphi_i(x, a_i) : i < \omega)$ such that $\{\varphi_i(x, a_i) : i < \omega\}$ is consistent and for each $i < \omega$, $\varphi_i(x, a_i)$ divides (forks) over $\{a_j : j < i\}$.*
3. *There is no infinite increasing chain $(p_i(x) : i < \omega)$ of types $p_i(x) \in S(A_i)$ such that each p_{i+1} is a forking (dividing) extension of p_i.*

Proof: Similar to the proof of Proposition 4.11. A forking (dividing) chain of types gives easily a forking (dividing) chain of formulas and conversely. If there are not infinite forking chains of formulas, the theory is simple and therefore forking and dividing coincide. $\square$

Definition 13.5 Lascar ranks *SU and U are ordinal valued (or ∞) and are defined for complete types over sets in finitely many variables. SU is defined by:*

- *$SU(p) \geq 0$.*
- *$SU(p) \geq \alpha + 1$ iff there is a forking extension q of p such that $SU(q) \geq \alpha$.*
- *$SU(p) \geq \alpha$ iff $SU(p) \geq \beta$ for all $\beta < \alpha$ in case α is a limit number.*

As usual, $SU(p) = \infty$ if $SU(p) \geq \alpha$ for all α, and $SU(p) = \alpha$ if $SU(p) \geq \alpha$ but $\mathrm{SU}(p) \not\geq \alpha + 1$. U is defined by the same conditions for 0 and for a limit number β. For a successor ordinal the rule is as follows:

- *For $p(x) \in S(A)$, $U(p) \geq \alpha + 1$ iff for each cardinal number λ there is a set $B \supseteq A$ and there are at least λ many types $q(x) \in S(B)$ extending p and such that $U(q) \geq \alpha$.*

We will use the notation $SU(a/A) = SU(\mathrm{tp}(a/A))$ and $U(a/A) = U(\mathrm{tp}(a/A))$.

Remark 13.6 *SU is a* foundation rank*, the foundation rank of complete types over sets with the relation of being a forking extension. In general, if R is a binary relation, the foundation rank of R is the mapping r assigning to every element of the domain of R an ordinal number (or ∞) according to the following rules:*

1. *$r(a) \geq 0$*
2. *$r(a) \geq \alpha + 1$ iff $r(b) \geq \alpha$ for some b such that aRb.*
3. *$r(a) \geq \alpha$ iff $r(a) \geq \beta$ for all $\beta < \alpha$ if α is a limit number.*

By induction on α (and induction on β in the case $\alpha + 1$) one easily sees that

4. *If $r(a) \geq \alpha$ and $\alpha \geq \beta$ then $r(a) \geq \beta$.*

and therefore if one defines

5. *$r(a) = \infty$ in case $r(a) \geq \alpha$ for all α*
6. *$r(a) = \sup\{\alpha : r(a) \geq \alpha\}$ otherwise,*

it is clear that $r(a) = \alpha$ iff $r(a) \geq \alpha$ and $r(a) \not\geq \alpha + 1$.

Some properties of SU are better understood keeping in mind that it is a foundation rank. The following will be helpful:

7. *If aRb and $r(a) < \infty$, then $r(a) > r(b)$.*
8. *If R is transitive, the rank r is connected: if $r(a) = \alpha < \infty$ and $\beta < \alpha$, then $r(b) = \beta$ for some b such that aRb.*
9. *If there is a sequence $(a_i : i < \omega)$ such that $a = a_0$ and $a_i R a_{i+1}$ for all $i < \omega$, then $r(a) = \infty$.*
10. *If there is an ordinal number α such that for all a, $r(a) \geq \alpha$ implies $r(a) = \infty$ then: if $r(a) = \infty$, then there is a sequence $(a_i : i < \omega)$ such that $a = a_0$ and $a_i R a_{i+1}$ for all $i < \omega$.*

Proof: *7* is clear since by definition if $r(b) \geq \alpha$ and aRb then $r(a) \geq \alpha + 1$. *8* is proven by induction on α using *7*. For *9*, prove that for all i, $r(a_i) \geq \alpha$ for any α by induction on α. For *10* note that the hypothesis implies that if $r(a) = \infty$ then $r(b) = \infty$ for some b such that aRb. □

Remark 13.7 *$SU(p) = 0$ iff p is algebraic iff $U(p) = 0$.*

Proposition 13.8 *Let T be simple and let $p(x) \subseteq q(x)$ be complete types.*

1. *If q is a nonforking extension of p, then $SU(p) = SU(q)$.*
2. *If $SU(p) = SU(q) < \infty$, then q is a nonforking extension of p.*

Proof: *1.* Clearly $SU(p) \geq SU(q)$. We now prove by induction on α that $SU(p) \geq \alpha$ implies $SU(q) \geq \alpha$. Consider the case $SU(p) \geq \alpha + 1$. Let $p(x) \in S(A)$ and $q(x) \in S(B)$. For some $C \supseteq A$ there is a forking extension $p' \in S(C)$ of p such that $SU(p') \geq \alpha$. Changing C if necessary, we may assume that there is some $b \models q$ such that $b \models p'$ and $C \mathop{\smile\!\!\!\!\!\!\;|}_{Ab} B$. Then $b \mathop{\smile\!\!\!\!\!\!\;|}_{C} B$, and hence $q' = \mathrm{tp}(b/CB)$ is a nonforking extension of p'. By inductive hypothesis $SU(q') \geq \alpha$. Since q' is a forking extension of q, $SU(q) \geq \alpha + 1$. Point *2* is clear and corresponds to point *7* of Remark 13.6. $\square$

Proposition 13.9 *If T is stable, then $U = SU$.*

Proof: By Corollary 8.6 in a stable theory a global type $\mathfrak{p} \in S(\mathfrak{C})$ forks over A if and only if it has a bounded orbit in $\mathrm{Aut}(\mathfrak{C}/A)$. By induction on α we prove that $SU(p) \geq \alpha$ iff $U(p) \geq \alpha$. Consider the case $\alpha + 1$. Assume $p \in S(A)$, $SU(p) \geq \alpha + 1$ and $q \in S(B)$ is a forking extension of p with $SU(q) \geq \alpha$. A nonforking extension $\mathfrak{q} \in S(\mathfrak{C})$ of q has unboundedly many A-conjugates. Fix λ and choose a set $C \subseteq B$ such that $\mathfrak{q} \upharpoonright C$ has λ many conjugates over C. By proposition 13.8 and by inductive hypothesis $U(\mathfrak{q} \upharpoonright C) \geq \alpha$ and then all its A-conjugates over C have also U-rank $\geq \alpha$. This means that $U(p) \geq \alpha + 1$. For the other direction, assume $U(p) \geq \alpha + 1$ and choose $\lambda > \mathrm{Mlt}(p)$, the number of nonforking extensions of p. There is a set $B \subseteq A$ over which p has λ extensions of U-rank $\geq \alpha$. By choice of λ, one of them, say $q \in S(B)$ is a forking extension. By inductive hypothesis $SU(q) \geq \alpha$. Then $SU(p) \geq \alpha + 1$. $\square$

Lemma 13.10 *Let T be simple.*

1. *There is some ordinal α such that $SU(p) \geq \alpha$ implies $SU(p) = \infty$*
2. *If $SU(p) = \infty$, there is a forking extension q of p such that $SU(q) = \infty$*

Proof: *1.* Assume for every ordinal α there is a complete type $p_\alpha(x) \in S(A_\alpha)$ such that $\alpha \leq SU(p_\alpha) < \infty$. Since there is a subset $B \subseteq A_\alpha$ such that $|B| \leq |T|$ and p_α does not fork over B, by Lemma 13.8 we may assume that in fact $|A_\alpha| \leq |T|$. For each α there are boundedly many types $p(x) \in S(A_\alpha)$ and therefore there is an ordinal β_α such that $SU(p) \leq \beta_\alpha$ if $p(x) \in S(A_\alpha)$ and $SU(p) < \infty$. Fix an enumeration a_α of A_α. Clearly $\beta_\alpha = \beta_{\alpha'}$ if $\mathrm{tp}(a_\alpha) = \mathrm{tp}(a_{\alpha'})$. This contradicts the fact that there are only boundedly many types $\mathrm{tp}(a_\alpha)$ of such sequences a_α.

2 follows from *1* as shown in points *9*, *10* of Remark 13.6. $\square$

Proposition 13.11 *If T is simple, the following are equivalent for $p \in S(A)$.*

1. *$SU(p) = \infty$*
2. *There is a forking chain of types $(p_n : n < \omega)$ starting with $p = p_0$.*
3. *Some $q \in S(B)$ extending p forks over AB_0 for any finite subset $B_0 \subseteq B$.*

Proof: *2* $\Leftrightarrow$ *3* is like in Proposition 4.11. *1* $\Leftrightarrow$ *2* follows from Lemma 13.10 and points *9*, *10* of 13.6. $\square$

Remark 13.12 *If $p(x) \in S(M)$ is not definable, then $U(p) = \infty$.*

Proof: As explained in the proof of Proposition 7.9 for each cardinal λ there is a model $N \succeq M$ over which there are λ different strong heirs of p. Since all they are again nondefinable, this can be used to show that $U(p) = \infty$. $\square$

Proposition 13.13 *1. T is supersimple if and only if $SU(p) < \infty$ for all p.*

2. T is superstable if and only if $U(p) < \infty$ for all p.

Proof: *1* follows from Proposition 13.11 and Proposition 13.3.

2. If T is superstable, T is stable and by Proposition 13.9 $SU = U$. Since T is also supersimple, by *1* $U(p) < \infty$ for all p. For the other direction, it is enough to show that stability follows from the condition $U(p) < \infty$ for all p. If T is not stable then there is a nondefinable type $p(x) \in S(M)$ over some model M. Then we apply Remark 13.12. $\square$

Remark 13.14 *If $SU(p) = \alpha < \infty$, then for any $\beta < \alpha$ there is some $q \supseteq p$ such that $SU(q) = \beta$.*

Proof: By point *8* of Remark 13.6. $\square$

Notation 13.15 *We will denote by $\alpha \oplus \beta$ the* natural sum *of the ordinals α, β. Every ordinal number α can be written uniquely in* Cantor normal form *as $\alpha = \sum_{i=0}^{k} \omega^{\alpha_i} n_i$ where $\alpha_0 > \ldots > \alpha_k$ are ordinals and $n_0, \ldots, n_k$ are natural numbers > 0. If $\beta = \sum_{i=0}^{j} \omega^{\beta_i} m_i$ is also in Cantor normal form , then $\alpha \oplus \beta = \sum_{i=0}^{l} \omega^{\gamma_i} r_i$ where $\gamma_0 > \ldots > \gamma_l$ enumerates $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_j$ and*

$$r_i = \begin{cases} n_p & \text{if } \gamma_i = \alpha_p \notin \{\beta_0, \ldots, \beta_j\} \\ m_p & \text{if } \gamma_i = \beta_p \notin \{\alpha_0, \ldots, \alpha_k\} \\ n_p + m_q & \text{if } \gamma_i = \alpha_p = \beta_q \end{cases}$$

This sum is the least operation $F : On \times On \to On$ which is strictly increasing in both arguments. Clearly, for natural numbers n, m, $n + m = n \oplus m$.

Theorem 13.16 (Lascar inequalities) *Let T be simple. If $SU(ab/A) < \infty$, then*

$$SU(a/Ab) + SU(b/A) \leq SU(ab/A) \leq SU(a/Ab) \oplus SU(b/A).$$

Proof: It is easy to see by induction on α that if $SU(a/A) \geq \alpha$, then $SU(ab/A) \geq \alpha$. Hence $SU(ab/A) \geq SU(a/A)$. From $SU(ab/A) < \infty$ it follows then $SU(a/A) < \infty$ and $SU(b/A) < \infty$. Then we can freely use Proposition 13.8.

To check the inequality $SU(ab/A) \leq SU(a/Ab) \oplus SU(b/A)$, we prove by induction on α that if $SU(ab/A) \geq \alpha$, then $SU(a/Ab) \oplus SU(b/A) \geq \alpha$. This is clear for $\alpha = 0$ and for limit α. Let us consider the case $\alpha + 1$. Assume $SU(ab/A) \geq \alpha + 1$. For some $B \supseteq A$ we have $SU(ab/B) \geq \alpha$ and $ab \not\downarrow_A B$. Since $ab \not\downarrow_A B$, either $b \not\downarrow_A B$ or $a \not\downarrow_{Ab} B$. Therefore $SU(b/A) > SU(b/B)$ or $SU(a/Ab) > SU(a/Bb)$. By monotonicity of natural addition of ordinal numbers, $SU(a/Ab) \oplus SU(b/A) > SU(a/Bb) \oplus SU(b/B)$. By inductive hypothesis $SU(a/Bb) \oplus SU(b/B) \geq \alpha$. Hence $SU(a/Ab) \oplus SU(b/A) \geq \alpha + 1$.

To check the inequality $SU(a/Ab) + SU(b/A) \leq SU(ab/A)$ we show by induction on α that if $SU(b/A) \geq \alpha$, then $SU(ab/A) \geq SU(a/Ab) + \alpha$. The cases $\alpha = 0$ and α limit are straightforward. For the case $\alpha + 1$, assume $SU(b/A) \geq \alpha + 1$. Then for some $B \supseteq A$,

$SU(b/B) \geq \alpha$ and $b \not\perp_A B$. We may assume that $B \perp_{Ab} a$. By inductive hypothesis $SU(ab/B) \geq SU(a/Bb)+\alpha$. Since $b \not\perp_A B$, also $ab \not\perp_A B$ and then $SU(ab/A) > SU(ab/B)$. Since $a \perp_{Ab} B$ we have $SU(a/Ab) = SU(a/Bb)$. Therefore $SU(ab/A) > SU(ab/B) \geq SU(a/Bb) + \alpha = SU(a/Ab) + \alpha$. We then conclude $SU(ab/A) \geq SU(a/Ab) + \alpha + 1$. □

Corollary 13.17 *If T is simple and $SU(ab/A) < \omega$, then*

$$SU(ab/A) = SU(a/Ab) + SU(b/A)$$

Proof: As remarked above, for natural numbers n, m, $n + m = n \oplus m$. □

Proposition 13.18 *Let T be simple.*

1. *If $a \in \operatorname{acl}(Ab)$, then $SU(ab/A) = SU(b/A)$.*
2. *If $\operatorname{acl}(aA) = \operatorname{acl}(bA)$ then $SU(a/A) = SU(b/A)$.*

Proof: *1.* Clearly $SU(ab/A) \geq S(b/A)$. Moreover it is easy to check by induction on α that $SU(ab/A) \geq \alpha$ implies $SU(b/A) \geq \alpha$. *2* follows from *1.* □

Definition 13.19 *An* abstract rank *is a mapping R assigning an ordinal or ∞ to complete types over sets and satisfying the following conditions:*

1. *If $f \in \operatorname{Aut}(\mathfrak{C})$, then $R(p) = R(p^f)$.*
2. *If $p \subseteq q$, then $R(p) \geq R(q)$.*
3. *If $p \in S(A)$ and $A \subseteq B$, then there is some extension $q \in S(B)$ of p such that $R(p) = R(q)$.*
4. *Let $p \in S(A)$ be such that $R(p) < \infty$. There is a cardinal κ such that for each $B \supseteq A$, p has at most κ extensions $q \in S(B)$ such that $R(p) = R(q)$.*

Remark 13.20 *Let R be an abstract rank. If $p \in S(M)$ is not definable, then $R(p) = \infty$.*

Proof: Choose α minimal for which there is some nondefinable $p \in S(M)$ over some model M with $R(p) = \alpha$. Let κ be the cardinal given by condition *4* in the definition of rank. As shown in the proof of Proposition 7.9 there is a model $N \succeq M$ over which there are κ^+ different strong heirs of p. All are nondefinable and one of them must have rank $< \alpha$, a contradiction. □

Proposition 13.21 *Let R be an abstract rank.*

1. *Let T be stable, $p \subseteq q$, and $R(p) < \infty$. Then $R(p) = R(q)$ iff q is a nonforking extension of p.*
2. *If $R(p) < \infty$ for every complete type p, then T is superstable.*

Proof: *1.* Let $p \in S(A)$, $A \subseteq B$, and $p \subseteq q \in S(B)$. We assume T is stable and $R(p) < \infty$. Fix κ, a bound for the extensions p of rank $R(p)$. We can find a model $M \supseteq A$ such that all nonforking extensions of p over M are A-conjugate in M and such that each forking extension of p over M has more than κ A–conjugates in M. There is an extension $q' \in S(M)$ of q with $R(q) = R(q')$. Now, if q forks over A then also q' forks and therefore q' has more than κ A-conjugates. By definition of rank $R(p) > R(q')$. Now assume

$R(p) > R(q)$ and q does not fork over A. Let $q' \in S(M)$ be a nonforking extension of q and choose $r \in S(M)$, an extension of p of rank $R(p) = R(r)$. As shown before, r does not fork over A. By choice of M, q' and r are A-conjugate. Hence $R(p) = R(r) = R(q') = R(q)$.

2. It suffices to show stability of T since then we can use point *1* to easily verify that T is supersimple. If T is unstable then some type $p \in S(M)$ is nondefinable. By Remark 13.20 $R(p) = \infty$. □

Proposition 13.22 *In a stable theory U is an abstract rank and it is minimal, that is $U(p) \leq R(p)$ for any other abstract rank R.*

Proof: If T is stable, then $U = SU$ and by Proposition 13.8 whenever $p \subseteq q$ and $U(p) < \infty$, q is a nonforking extension of p iff $U(p) = U(q)$. Since in a stable theory a type has only a bounded number of nonforking extensions, the requirements in the definition of abstract rank are fulfilled. Minimality is easily checked showing by induction on α that if R is a rank and $U(p) \geq \alpha$, then $R(p) \geq \alpha$. □

Corollary 13.23 *T is superstable if and only if there is an abstract rank R such that $R(p) < \infty$ for all p.*

Proof: If T is superstable, then U is an abstract rank and $U(p) < \infty$ for all p. The rest follows from Proposition 13.21. □

Proposition 13.24 *Let T be stable and $p(x) \in S(A)$.*

1. *If $U(p) < \infty$, then for any $B \supseteq A$ there are at most $2^{|T|} + |B|$ extensions $q(x) \in S(B)$ of p.*
2. *If $U(p) = \infty$ then for any cardinal $\lambda \geq |T| + |A|$ there is a set $B \supseteq A$ such that $|B| \leq \lambda$ and p has at least λ^ω extensions $q(x) \in S(B)$.*

Proof: If T is stable, then $U = SU$. By Proposition 13.11, if $U(p) < \infty$ then any complete type q over $B \supseteq A$ extending p does not fork over AB_0 for some finite $B_0 \subseteq B$. Since there are only $2^{|T|}$ extensions of p to a complete type $q(x) \in S(AB_0)$ for B_0 finite, and each such type q has at most $2^{|T|}$ nonforking extensions over B, it is easy to check that $2^{|T|} + |B|$ is a correct upper bound for the number of extensions of p over B. On the other hand, if $U(p) = \infty$ by Lemma 13.10 p has a forking extension q of U-rank ∞. Let $\mathfrak{q}$ be a global nonforking extension of q. Then $\mathfrak{q}$ forks over A and therefore it has an unbounded orbit in $\mathrm{Aut}(\mathfrak{C}/A)$. Note that every complete type between p and $\mathfrak{q}$ has U-rank ∞. Fix a set $A_1 \supseteq A$ such that $|A_1| \leq \lambda$ and for which there are different types $r_i(x) \in S(A_1)$ for $i < \lambda$ which can be extended to A-conjugates of $\mathfrak{q}$. Note that $U(r_i) = \infty$. Iterating this procedure we obtain a chain of sets $(A_n : n < \omega)$ of cardinality $|A_n| \leq \lambda$ and a tree of types $(p_s : s \in \lambda^{<\omega})$ such that $p_s \in S(A_n)$ if $s \in \lambda^n$, $p_\emptyset = p$, $p_s \subseteq p_{s'}$ if $s \subseteq s'$, $p_s \neq p_{s'}$ if $s \neq s'$ and $U(p_s) = \infty$. If we put $p_f = \bigcup_{s \subseteq f} p_s$ for $f \in \lambda^\omega$, we obtain a family $(p_f : f \in \lambda^\omega)$ of λ^ω many complete extensions of p over the set $B = \bigcup_{n<\omega} A_n$ of cardinality $|B| \leq \lambda$. □

Theorem 13.25 *The following are equivalent:*

1. *T is superstable.*
2. *For all A, $|S(A)| \leq |A| + 2^{|T|}$.*
3. *For all $\lambda \geq 2^{|T|}$, T is λ-stable.*

4. There is some cardinal μ such that for all $\lambda \geq \mu$, T is λ-stable.

Proof: *1* $\Rightarrow$ *2*. There are only $2^{|T|}$ types over $\emptyset$, and by Proposition 13.24 and Proposition 13.13 each $p(x) \in S(\emptyset)$ has at most $2^{|T|} + |A|$ complete extensions over A.

It is clear that *2* $\Rightarrow$ *3* and that *3* $\Rightarrow$ *4*.

4 $\Rightarrow$ *1*. If T is not superstable then, by Proposition 13.13 there is some $p(x) \in S(A)$ such that $U(p) = \infty$. Choose $\lambda \geq \mu + |T| + |A|$ such that $\lambda^\omega > \lambda$. By Proposition 13.24 there is a set $B \supseteq A$ of cardinality $\leq \lambda$ such that p has at least λ^ω complete extensions over B. Clearly T is not λ-stable. $\square$

Chapter 14

More ranks

Definition 14.1 *D rank is defined for formulas $\varphi(x) \in L(\mathfrak{C})$ as follows:*

1. $D(\varphi(x)) \geq 0$ *iff* $\varphi(x)$ *is consistent.*
2. $D(\varphi(x)) \geq \alpha+1$ *iff for some* $\psi(x,y) \in L$ *for all cardinal numbers* λ *there is an infinite sequence* $(a_i : i < \lambda)$ *such that* $\{\psi(x,a_i) : i < \lambda\}$ *is* k*–inconsistent for some* $k < \omega$ *and for each* $i < \lambda$, $\models \psi(x,a_i) \to \varphi(x)$ *and* $D(\psi(x,a_i)) \geq \alpha$.
3. $D(\varphi(x)) \geq \beta$ *iff* $D(\varphi(x)) \geq \alpha$ *for all* $\alpha < \beta$ *for limit* β.

The definition is extended to arbitrary sets of formulas $\pi(x)$ *by*

$$D(\pi(x)) = \min\{D(\varphi) : \varphi \textit{ is a finite conjunction of formulas in } \pi(x)\}.$$

Remark 14.2 *If* $\varphi(x) \in L(A)$, *then* $D(\varphi(x)) \geq \alpha+1$ *iff* $\models \psi(x) \to \varphi(x)$ *and* $D(\psi(x)) \geq \alpha$ *for some* $\psi(x) \in L(\mathfrak{C})$ *which divides over* A.

Proposition 14.3

1. *There is an ordinal* α *such that for all* $\varphi(x) \in L(\mathfrak{C})$, *if* $D(\varphi) \geq \alpha$, *then* $D(\varphi) = \infty$.
2. *If* $\varphi(x) \in L(A)$ *and* $D(\varphi(x)) = \infty$, *then* $D(\psi(x)) = \infty$ *for some* $\psi(x)$ *such that* $\models \psi(x) \to \varphi(x)$ *and* $\psi(x)$ *divides over* A.
3. $D(\varphi(x)) = \infty$ *if and only if there is a sequence* $(\varphi_i(x) : i < \omega)$ *of consistent formulas* $\varphi_i(x) \in L(A_i)$ *such that* $\varphi = \varphi_0$, $\models \varphi_{i+1}(x) \to \varphi_i(x)$ *and* $\varphi_{i+1}(x)$ *divides over* $\bigcup_{j \leq i} A_j$.
4. T *is supersimple iff* $D(\varphi) < \infty$ *for all* φ.

Proof: *1* is easy, like in Lemma 13.10, *2* follows from *1*, and *3* follows from *2*. Lastly, *4* follows from *3* and Proposition 13.4. □

Lemma 14.4

1. *If* $\pi_1(x) \vdash \pi_2(x)$, *then* $D(\pi_1) \leq D(\pi_2)$.
2. $D(\pi) = 0$ *if and only if* π *is algebraic.*
3. $D(\varphi \vee \psi) = \max\{D(\varphi), D(\psi)\}$.

4. *If $\pi(x)$ is a partial type over A, there is some $p(x) \in S(A)$ such that $\pi \subseteq p$ and $D(\pi) = D(p)$.*

5. *If $\pi(x)$ is a partial type, there is some finite conjunction $\varphi(x)$ of formulas of $\pi(x)$ such that $D(\pi) = D(\varphi)$.*

Proof: *4* follows from *3*. Concerning *3*, it is clear that $D(\varphi), D(\psi) \leq D(\varphi \vee \psi)$. Then it suffices to show that if $D(\varphi \vee \psi) \geq \alpha$, then $D(\varphi) \geq \alpha$ or $D(\psi) \geq \alpha$, and this can be shown by induction on α. Consider the case $\alpha + 1$. Assume $D(\varphi \vee \psi) \geq \alpha + 1$. For some θ and A, $\models \theta \to (\varphi \vee \psi)$, $(\varphi \vee \psi) \in L(A)$, θ divides over A, and $D(\theta) \geq \alpha$. Note that $\models \theta \leftrightarrow (\theta \wedge \varphi) \vee (\theta \wedge \psi)$ and hence the inductive hypothesis gives $D(\theta \wedge \varphi) \geq \alpha$ or $D(\theta \wedge \psi) \geq \alpha$. We conclude $D(\varphi) \geq \alpha + 1$ or $D(\psi) \geq \alpha + 1$. □

Remark 14.5 *If T is simple, then $SU \leq D$.*

Definition 14.6 *The* continuous rank *RC (also denoted with R^∞) is defined for all sets of formulas (in finitely many variables) as follows:*

1. *$RC(\pi(x)) \geq 0$ iff $\pi(x)$ is consistent.*

2. *$RC(\pi(x)) \geq \alpha + 1$ iff for any conjunction $\varphi(x)$ of formulas in $\pi(x)$ for any cardinal λ there is a sequence $(\pi_i(x) : i < \lambda)$ of partial types $\pi_i(x) \ni \varphi(x)$ such that $CR(\pi_i) \geq \alpha$ and $\pi_i \cup \pi_j$ is inconsistent for all $i < j < \lambda$.*

3. *$RC(\pi(x)) \geq \beta$ iff $RC(\pi(x)) \geq \alpha$ for all $\alpha < \beta$ if β is a limit number.*

For a formula $\varphi(x)$ we set $RC(\varphi) = RC(\{\varphi\})$.

Lemma 14.7 1. *If $\pi(x) \vdash \pi'(x)$, then $RC(\pi) \leq RC(\pi')$.*

2. *$RC(\pi) = 0$ if and only if π is algebraic.*

3. *If $\pi(x)$ is a partial type over A,*

$$RC(\pi) = \min\{RC(\varphi) : \varphi \text{ is a finite conjunction of formulas in } \pi\}$$

and therefore there is a finite conjunction $\varphi(x)$ of formulas in $\pi(x)$ such that $RC(\pi) = RC(\varphi)$.

4. *$RC(\pi \cup \{(\varphi \vee \psi)\}) = \max\{RC(\pi \cup \{\varphi\}), RC(\pi \cup \{\psi\})\}$.*

5. *If $\pi(x)$ is a partial type over A, there is some $p(x) \in S(A)$ such that $\pi \subseteq p$ and $RC(\pi) = RC(p)$.*

Proof: *1.* It is an induction on α: if $RC(\pi) \geq \alpha$, then $RC(\pi') \geq \alpha$. In the case $\alpha + 1$, given φ a conjunction of formulas in π' and given a cardinal λ, we first find ψ, a conjunction of formulas in π such that $\psi \vdash \varphi$, and then we use the hypothesis $RC(\pi) \geq \alpha + 1$ to find a sequence $(\pi_i(x) : i < \lambda)$ of pairwise incompatible types $\pi_i \ni \psi$ with $RC(\pi_i) \geq \alpha$ and then we set $\pi'_i = \pi_i \cup \{\varphi\}$. Since $\pi_i \vdash \pi'_i$, by inductive hypothesis $RC(\pi'_i) \geq \alpha$. Hence $(\pi'_i : i < \lambda)$ witness that $RC(\pi') \geq \alpha + 1$.

For *3*, choose φ, a conjunction of formulas in π of minimal RC-rank, and show by induction on α that $RC(\varphi) \geq \alpha$ implies $RC(\pi) \geq \alpha$.

4. By *1* it is clear that $RC(\pi \cup \{\varphi \vee \psi\}) \geq \max\{RC(\pi \cup \{\varphi\}), RC(\pi \cup \{\psi\})\} \geq \alpha$. Hence we only have to show that if $RC(\pi \cup \{\varphi \vee \psi\}) \geq \alpha$, then $\max\{RC(\pi \cup \{\varphi\}), RC(\pi \cup \{\psi\})\} \geq \alpha$, and this can be done by induction on α. As usual, we consider only the case $\alpha+1$. Assume $RC(\pi \cup \{\varphi\}) \not\geq \alpha+1$ and $RC(\pi \cup \{\psi\}) \not\geq \alpha+1$. Hence we have δ_1, δ_2, conjunctions of formulas in π, and λ_1, λ_2, cardinal numbers, such that there is no sequence $(\pi_i : i < \lambda_1)$ of pairwise incompatible types $\pi_i \ni (\delta_1 \wedge \varphi)$ with $RC(\pi_i) \geq \alpha$ and there is no sequence $(\pi_i : i < \lambda_2)$ of pairwise incompatible types $\pi_i \ni (\delta_2 \wedge \psi)$ with $RC(\pi_i) \geq \alpha$. Let $\delta = (\delta_1 \wedge \delta_2)$ and let $\lambda = \max\{\lambda_1, \lambda_2\}$. There is a sequence $(\pi_i : i < \lambda)$ of pairwise incompatible types $\pi_i \ni (\delta \wedge (\varphi \vee \psi))$ with $RC(\pi_i) \geq \alpha$. Note that $\pi_i \equiv \pi_i \cup \{\delta\} \cup \{\varphi \vee \psi\}$ and then, by *1*, $RC(\pi_i \cup \{\delta\} \cup \{\varphi \vee \psi\}) \geq \alpha$ and by inductive hypothesis either $RC(\pi_i \cup \{\delta\} \cup \{\varphi\}) \geq \alpha$ or $RC(\pi_i \cup \{\delta\} \cup \{\psi\}) \geq \alpha$. Again by *1*, either $RC(\pi_i \cup \{\delta_1\} \cup \{\varphi\}) \geq \alpha$ or $RC(\pi_i \cup \{\delta_2\} \cup \{\psi\}) \geq \alpha$. One of these two possibilities takes place λ times, contradicting the choice of λ_1 and λ_2.

5 follows from *4* as in other similar situations. □

Remark 14.8 *$RC(\pi(x)) \geq \alpha+1$ iff for each $\varphi(x)$, conjunction of formulas of π, for each cardinal λ there is a set A and there is a family $(p_i(x) : i < \lambda)$ of different types $p_i(x) \in S(A)$ such that $RC(p_i) \geq \alpha$ for all $i < \lambda$.*

Proof: By point *5* of Lemma 14.7. □

Proposition 14.9 *If T is stable, then $D = RC$.*

Proof: It is enough to check it for formulas and then it is clear: after Corollary 8.6, for stable T and $\varphi(x) \in L(A)$, $RC(\varphi(x)) \geq \alpha+1$ if and only if there is some $\psi(x)$ such that $\models \psi(x) \to \varphi(x)$, $\psi(x)$ forks over A, and $RC(\varphi) \geq \alpha$. □

Proposition 14.10 *T is superstable if and only if $RC(\varphi) < \infty$ for any φ.*

Proof: One direction follows from Proposition 14.9 and point *4* of Proposition 14.3. For the other direction note that $U(p) \leq RC(p)$ for any complete type p and then apply Proposition 13.13. □

Definition 14.11 *An abstract rank R is a* continuous rank *if for each α, for each A, $\{p(x) \in S(A) : R(p) < \alpha\}$ is an open subset of $S(A)$.*

Proposition 14.12 *If T is stable, RC is the smallest continuous rank in T.*

Proof: By definition and by Lemma 14.7 it is clear that RC always satisfies conditions *1–3* of the definition of abstract rank. For condition *4* we need to assume T is stable. By Proposition 14.9 $RC = D$. If $p(x) \in S(A)$, $RC(p) = \alpha < \infty$, and q is a forking extension of p of the same rank $RC(q) = \alpha$, then q contains a formula $\varphi(x)$ which forks over A. We can assume that $RC(\varphi) = \alpha$ and that φ implies some $\psi(x) \in p$ of rank $RC(\psi) = \alpha$. But then $D(\psi) \geq \alpha+1$, which is a contradiction. Therefore, all extensions q of p with $RC(q) = \alpha$ are nonforking extensions and by stability its number is bounded by the multiplicity of p, which is $\leq 2^{|T|}$. It follows that RC is an abstract rank.

Point *3* of Lemma 14.7 implies that RC is continuous. If R is another continuous rank, then by induction on α one sees that if $RC(p) \geq \alpha$ then $R(p) \geq \alpha$. Consider the case $\alpha+1$. Let $p(x) \in S(A)$ be such that $RC(p) \geq \alpha+1$. We will show that for any $\varphi \in p$ there is some $q \in S(A)$ such that $\varphi \in q$ and $R(q) \geq \alpha+1$. Continuity of R will imply then $R(p) \geq \alpha+1$. Now, by Remark 14.8 for each cardinal λ there is some B such that there are at least λ types $q(x) \in S(B)$ such that $\varphi(x) \in q$ and $RC(q) \geq \alpha$. We may assume that always $A \subseteq B$. Since there are only $2^{|T|+|A|}$ types over A, for some $r(x) \in S(A)$ such that $\varphi \in r$ and for each cardinal λ there is some B such that there are at least λ types $q(x) \in S(B)$ such that $r \subseteq q$ and $RC(q) \geq \alpha$. By inductive hypothesis $R(q) \geq \alpha$ for all such q. By condition *4* in the definition of abstract rank $R(r) \geq \alpha+1$. □

Definition 14.13 *The* Morley rank *of a global type* $\mathfrak{p} \in S_n(\mathfrak{C})$, $RM(\mathfrak{p})$, *is its Cantor-Bendixson rank in the space* $S_n(\mathfrak{C})$. *The Morley rank of a partial type* $\pi(x)$, $RM(\pi)$, *(Where* x *is a* n*-tuple of variables) is the Cantor-Bendixson rank of the closed set* $\{\mathfrak{p} \in S_n(\mathfrak{C}) : \pi \subseteq \pi\}$ *and its* Morley degree, $DM(\pi)$, *is the Cantor-Bendixson degree of this closed set. By compactness,* $DM(\pi)$ *is finite if* $RM(\pi) < \infty$. *It is clear that*

$$RM(\pi) = \max\{RM(\mathfrak{p}) : \pi \subseteq \mathfrak{p}\}$$

For a formula φ *we set* $RM(\varphi) = RM(\{\varphi\}$ *and* $DM(\varphi) = DM(\{\varphi\})$.

Remark 14.14

1. $RM(\varphi(x)) \geq 0$ *iff* $\varphi(x)$ *is consistent*
2. $RM(\varphi(x)) \geq \alpha + 1$ *iff there is a sequence* $(\varphi_i(x) : i < \omega)$ *such that* $\models \varphi_i(x) \to \varphi(x)$, $RM(\varphi_i) \geq \alpha$, *and* $\varphi_i(x) \wedge \varphi_j(x)$ *is inconsistent for all* $i \neq j$.
3. $RM(\varphi) \geq \alpha$ *iff* $RM(\varphi) \geq \beta$ *for all* $\beta < \alpha$ *if* α *is a limit number.*

Proof: These are well-known properties of the Cantor-Bendixson rank of clopen sets in boolean spaces. □

Remark 14.15 $RM(\varphi(x)) \geq \alpha+1$ *iff there for each* $n < \omega$ *there is a sequence* $(\varphi_i(x) : i < n)$ *such that* $\models \varphi_i(x) \to \varphi(x)$, $RM(\varphi_i) \geq \alpha$, *and* $\varphi_i(x) \wedge \varphi_j(x)$ *is inconsistent for all* $i \neq j$. *Hence the degree* $DM(\varphi(x))$ *can be defined (in case* $RM(\varphi) = \alpha < \infty$*) as the maximal* n *for which there is a sequence* $(\varphi_i(x) : i < n)$ *such that* $\models \varphi_i(x) \to \varphi(x)$, $RM(\varphi_i) \geq \alpha$, *and* $\varphi_i(x) \wedge \varphi_j(x)$ *is inconsistent for all* $i \neq j$.

Proof: If for each $n < \omega$ we have such sequence $(\varphi_i(x) : i < n)$, then the number of types $\mathfrak{p}(x) \in S(\mathfrak{C})$ of Cantor-Bendixson rank $\geq \alpha$ containing φ must be infinite. □

Proposition 14.16 *For any partial type* π,

1. $RM(\pi) = \min\{RM(\varphi) : \varphi$ *is a conjunction of formulas in* $\pi\}$
2. $DM(\pi) = \min\{DM(\varphi) : \varphi$ *is a conjunction of formulas in* π *and* $RM(\varphi) = RM(\pi)\}$

Proof: Again this is a well-known property of the Cantor-Bendixson rank and degree of closed sets in boolean spaces. □

Remark 14.17 *Morley rank can be computed in any* ω*-saturated model* M *containing the parameters of the type as the Cantor-Bendixson rank in* $S(M)$ *of the closed set determined by the type.*

Proof: It is enough to check it for formulas and in this case we can use Remark 14.14. The parameters needed in the sequence $(\varphi_i(x) : i < \omega)$ to check that $RM(\varphi(x)) \geq \alpha + 1$ build a countable sequence and its type over the parameters of φ can be realized in M. □

Proposition 14.18 *Morley rank is a continuous rank.*

Proof: All conditions in the definition of an abstract rank are easily seen to be satisfied by Morley rank. The bound for the number of extensions with the same rank of a type $p(x)$ is $DM(p)$. Continuity follows from Proposition 14.16. □

Corollary 14.19 *In a stable theory,* $U \leq RC \leq RM$.

Proof: By propositions 14.18, 13.22, and 14.12. □

Definition 14.20 *T is* totally trascendental *if and only if $RM(\varphi) < \infty$ for all φ.*

Theorem 14.21 *1. If T is λ-stable for some $\lambda < 2^{\omega}$, then T is totally trascendental.*

2. Any totally trascendental theory is λ-stable for all $\lambda \geq |T|$.

Proof: *1.* Assume $RM(\varphi) = \infty$. By standard topological arguments we can build a tree of formulas $(\varphi_s : s \in 2^{<\omega})$ such that $\varphi_\emptyset = \varphi$, $RM(\varphi_s) = \infty$, $\varphi_s \equiv \varphi_{s^\frown 0} \vee \varphi_{s^\frown 1}$ and $\varphi_{s^\frown 0} \wedge \varphi_{s^\frown 1}$ is inconsistent. Every branch $f \in 2^{\omega}$ gives rise to a type $\pi_f = \{\varphi_s : s \subseteq f\}$ and this produces a set of 2^{ω} incompatible partial types over a countable set of parameters, contradicting λ-stability of T.

2. Let $\lambda \geq |T|$ and let $|A| \leq \lambda$. For each $p(x) \in S(A)$ choose some $\varphi_p(x) \in S(A)$ such that $RM(p) = RM(\varphi_p)$ and $DM(p) = DM(\varphi_p)$. Since T is totally trascendental, for any $\psi(x) \in L(A)$, $\psi \in p$ iff $RM(\varphi_p \wedge \psi) = RM(\varphi_p)$ and $DM(\varphi_p \wedge \psi) = DM(\varphi_p)$. It follows that $p \neq q$ implies $\varphi_p \neq \varphi_q$. Hence $|S(A)|$ has as an upper bound the number $|T| + |A|$ of formulas $\varphi(x) \in L(A)$. □

Corollary 14.22 *Totally trascendental theories, and in particular ω-stable theories, are superstable.*

Proof: By theorems 14.21 and 13.25. □

Definition 14.23 *T is* small *if for all n, $|S_n(\emptyset)| \leq \omega$.*

Remark 14.24 *The following are equivalent:*

1. *T is small*
2. *For all n, for all finite A, $|S_n(A)| \leq \omega$.*
3. *For all finite A, $|S_1(A)| \leq \omega$*
4. *T has a saturated countable model.*

Proof: *1* ⇒ *2* can be justified by a standard counting types argument. *2* ⇒ *3* is clear. For *3* ⇒ *4*, the countable saturated model can be constructed as a union $\bigcup_{n\in\omega} A_n$ of countable sets A_n such that each complete 1-type over a finite subset of A_n is realized in A_{n+1}. *4* ⇒ *1* is clear since all $p(x) \in S_n(\emptyset)$ can be realized in the countable saturated model. □

Remark 14.25 *1. ω-categorical theories are small.*

2. ω-stable theories are small.

Proof: Clear. □

Corollary 14.26 *T is ω-stable if and only if T is small, superstable, and every complete type has finite multiplicity.*

Proof: Let T be ω-stable. By Remark 14.25 T is small and by Corollary 14.22 T is superstable. By Theorem 14.21 T is totally trascendental and therefore the multiplicity of a type is its Morley degree. The other direction is just a counting types argument like in the proof of Theorem 13.25. □

Corollary 14.27 *Superstable ω-categorical theories are ω-stable.*

Proof: By Corollary 14.26, since by ω-categoricity for each finite A there are only finitely many complete n-types over A. Moreover by ω-categoricity if A is finite there is a finest finite A-definable equivalence on relation on n-tuples and together with the Finite Equivalence Relation Theorem 9.21 this implies that all multiplicities of complete types over A are finite. □

Definition 14.28 *An abstract rank R is* cantorian *iff any type $p(x) \in S(A)$ has rank $R(p) \geq \alpha + 1$ in case p is an accumulation point of $\{q(x) \in S(A) : R(q) \geq \alpha\}$.*

Proposition 14.29 *Let $p(x) \in S(A)$. Then $RM(p) \geq \alpha + 1$ iff for some $B \supseteq A$ some extension $q(x) \in S(B)$ of p is an accumulation point of $\{r(x) \in S(B) : RM(r) \geq \alpha\}$.*

Proof: Let $RM(p) \geq \alpha+1$ and choose an ω-saturated model $M \supseteq A$ and let $q(x) \in S(M)$ an extension of p of Morley rank $\geq \alpha + 1$. By Remark 14.17 q has Cantor-Bendixson rank $\geq \alpha + 1$ in $S(M)$ and therefore it is an accumulation point of types $r(x) \in S(M)$ of Cantor-Bendixson rank $\geq \alpha$. Again by Remark 14.17, these types r have Morley rank $\geq \alpha$.

For the other direction it is enough to prove that $RM(q) \geq \alpha + 1$, in other words, that RM is cantorian. For this it is enough to show that each $\varphi \in q$ is contained in some $\mathfrak{q} \in S(\mathfrak{C})$ of Cantor-Bendixson rank $\geq \alpha + 1$, that is, φ is contained in infinitely many $\mathfrak{q} \in S(\mathfrak{C})$ of Cantor-Bendixson rank $\geq \alpha$. We know that each such φ is contained in infinitely many types $r(x) \in S(B)$ of Morley rank $\geq \alpha$. But we can choose for each such $r(x) \in S(B)$ an extension $\mathfrak{q}(x) \in S(\mathfrak{C})$ of Cantor-Bendixson rank $\geq \alpha$. □

Proposition 14.30 *RM is the smallest cantorian rank.*

Proof: By propositions 14.18 and 14.29, RM is a cantorian rank. Let R be another cantorian rank. We prove by induction on α that $RM(p) \geq \alpha$ implies $R(p) \geq \alpha$. Consider the case $\alpha + 1$. Assume $p(x) \in S(A)$ and $RM(p) \geq \alpha + 1$. By Proposition 14.29 for some $B \supseteq A$, some $q(x) \in S(B)$ extending p is an accumulation point of $\{r(x) \in S(B) : RM(r) \geq \alpha\}$. By inductive hypothesis, this set is contained in $\{r(x) \in S(B) : R(r) \geq \alpha\}$ and hence q is an accumulation point of this set. Since R is cantorian, $R(q) \geq \alpha + 1$ and therefore $R(p) \geq \alpha + 1$. □

Theorem 14.31 *If T is superstable and ω-categorical, then U is cantorian and therefore $U = RC = RM$.*

Proof: Let $p(x) \in S(A)$ be an accumulation point of $\{q(x) \in S(A) : U(q) \geq \alpha\}$. By corollaries 14.26 and 14.22, every type has finite multiplicity and hence we can find a finite subset $A_0 \subseteq A$ such that p does not fork over A_0 and $p_0 = p \restriction A_0$ has p as its only nonforking extension over A. By ω-categoricity, the type $p_0(x)$ is isolated by some $\varphi_0(x) \in p_0$. By assumption, there is some $q(x) \in S(A)$ such that $\varphi_0(x) \in q$, $U(q) \geq \alpha$ and $p \neq q$. It follows that q forks over A_0. Hence $U(p) = U(p_0) \geq U(q) + 1 \geq \alpha + 1$. The rest follows from Proposition 14.30 and Corollary 14.19. □

Chapter 15

Hyperimaginaries

Definition 15.1 *For any set A, a A-*hiperimaginary *is an equivalence class $[a]_E$ of a sequence a under a type-definable over A equivalence relation E. In order to simplify notation we set $a_E = [a]_E$ and we often identify the equivalence relation E with the partial type over A which defines E. Clearly A-imaginaries are A-hyperimaginaries. A* hiperimaginary *is a $\emptyset$-hyperimaginary. We sometimes use $\mathfrak{C}^{\mathrm{heq}}$ for the class of all hyperimaginaries. If $a = (a_i : i < \alpha)$ for some ordinal α, we say that α is the* length *of the hyperimaginary a_E.* Finitary hyperimaginaries *are hyperimaginaries of finite length.* Countable hyperimaginaries *are hyperimaginaries of countable length.*

Definition 15.2 *An automorphism $f \in \mathrm{Aut}(\mathfrak{C})$ fixes a hyperimaginary a_E if $f(a_E) = a_E$, that is, if $\models E(a, f(a))$. Let A be a class of hyperimaginaries. The* definable closure of A, $\mathrm{dcl}^{\mathrm{heq}}(A)$, *is the class of all hyperimaginaries fixed by all automorphisms fixing pointwise A, that is*

$$\mathrm{dcl}^{\mathrm{heq}}(A) = \{b \in \mathfrak{C}^{\mathrm{heq}} : f(b) = b \text{ for all } f \in \mathrm{Aut}(\mathfrak{C}/A)\}$$

Since a hyperimaginary can have any length, $\mathrm{dcl}^{\mathrm{heq}}(A)$ *is a proper class. As usual, if a is a sequence of hyperimaginaries,* $\mathrm{dcl}^{\mathrm{heq}}(a)$ *is* $\mathrm{dcl}^{\mathrm{heq}}(A)$ *where A is the set enumerated in a. Notice that if A is a set of imaginaries then* $\mathrm{dcl}^{\mathrm{eq}}(A) = \mathrm{dcl}^{\mathrm{heq}}(A) \cap \mathfrak{C}^{\mathrm{eq}}$. *We say that the sequences of hyperimaginaries a,b are* equivalent *if* $\mathrm{dcl}^{\mathrm{heq}}(a) = \mathrm{dcl}^{\mathrm{heq}}(b)$. *This is clearly equivalent to* $\mathrm{Aut}(\mathfrak{C}/a) = \mathrm{Aut}(\mathfrak{C}/b)$. *In this case we write $a \sim b$.*

Lemma 15.3 *Any sequence of hyperimaginaries is equivalent to a hyperimaginary.*

Proof: Let $a = ([a_i]_{E_i} : i \in I)$ be a sequence of hiperimaginaries, where E_i is an equivalence relation among J_i-sequences and $a_i = (a_{(i,j)} : j \in J_i)$. Put $K = \bigcup_{i \in I}\{i\} \times J_i$ and consider the equivalence relation E defined by

$$E((x_{(i,j)} : (i,j) \in K), (y_{(i,j)} : (i,j) \in K)) \leftrightarrow \bigwedge_{i \in I} E_i((x_{(i,j)} : j \in J_i), (y_{(i,j)} : j \in J_i)).$$

Clearly $e = [(a_{(i,j)} : (i,j) \in K)]_E$ is a hyperimaginary and $a \sim e$. $\Box$

Lemma 15.4 *Any hyperimaginary is equivalent to a sequence of countable hyperimaginaries.*

Proof: We can assume that the type $E(x,y)$ defining the equivalence relation E is closed under conjunction and all its formulas are symmetric: $\vdash \varphi(x,y) \rightarrow \varphi(y,x)$ for all $\varphi(x,y) \in E(x,y)$. It will be enough to find for each $\varphi(x,y) \in E(x,y)$ a countable partial type $E_\varphi(x,y) \subseteq E(x,y)$ containing $\varphi(x,y)$ which defines an equivalence relation. Given $\varphi(x,y) \in E(x,y)$ we set $E_\varphi = \{\varphi_n : n \in \omega\}$, where $\varphi_0 = \varphi$ and $\varphi_{n+1}(x,y) \in E(x,y)$ satisfies $\vdash \varphi_{n+1}(x,y) \wedge \varphi_{n+1}(y,z) \rightarrow \varphi_n(x,z)$. Existence of such a φ_{n+1} follows by compactness from the fact that $E(x,y) \cup E(y,z) \vdash \varphi_n(x,z)$. □

Lemma 15.5 *Let $\pi(x)$ be a partial type over A. If E is an equivalence relation on realizations of π and it is type-definable over A, then there exists an equivalence relation F defined for all sequences of the length of x which is type-definable over A and agrees with E in $\pi(\mathfrak{C})$.*

Proof: Set $F(x,y) \Leftrightarrow (\pi(x) \wedge \pi(y) \wedge E(x,y)) \vee x = y$. □

Proposition 15.6 *Let e be a hyperimaginary and let b be a sequence in $\mathfrak{C}$. If $e \in \mathrm{dcl}^{\mathrm{heq}}(b)$ then $e \sim b_E$ for some 0-type-definable equivalence relation E.*

Proof: Let $e = a_F$. Since a_F is type-definable over a and it is b-invariant, it is type-definable over b and there is a partial type $\pi(x,y)$ over $\emptyset$ such that $\pi(x,b)$ defines a_F. Let $p(y) = \mathrm{tp}(b)$. If $b' \models p$ then $\pi(x,b')$ defines an F-class, and hence either defines e or a class disjoint with it. Thus $\exists x(\pi(x,y) \wedge \pi(x,z))$ defines an equivalence relation G in $p(\mathfrak{C})$. By Lemma 15.5 there is an equivalence relation E which is type definable over $\emptyset$ and agrees with G in $p(\mathfrak{C})$. It is easy to see that $e \sim b_E$. □

Corollary 15.7 *If $e \in \mathrm{dcl}^{\mathrm{heq}}(A)$ for some set A of cardinality $\leq \kappa$ then e is equivalent to a hyperimaginary of length $\leq \kappa$.*

Proof: It follows from Proposition 15.6. □

Definition 15.8 *The* algebraic closure *of A, a class of hyperimaginaries, is the class $\mathrm{acl}^{\mathrm{heq}}(A)$ consisting in all hyperimaginaries having finite orbit under the group of all automorphisms fixing pointwise A, that is*

$$\mathrm{acl}^{\mathrm{heq}}(A) = \{b \in \mathfrak{C}^{\mathrm{heq}} : |\{f(b) : f \in \mathrm{Aut}(\mathfrak{C}/A)\}| < \omega\}.$$

As usual, if a enumerates A we put $\mathrm{acl}^{\mathrm{heq}}(a) = \mathrm{acl}^{\mathrm{heq}}(A)$. The bounded closure of A *is the class $\mathrm{bdd}(A)$ consisting in all hyperimaginaries having a bounded orbit under the group of all automorphisms fixing pointwise A, that is*

$$\mathrm{bdd}(A) = \{b \in \mathfrak{C}^{\mathrm{heq}} : |\{f(b) : f \in \mathrm{Aut}(\mathfrak{C}/A)\}| < |\mathfrak{C}|\}.$$

As usual, if a enumerates A we put $\mathrm{acl}^{\mathrm{heq}}(a) = \mathrm{acl}^{\mathrm{heq}}(A)$ and $\mathrm{bdd}(a) = \mathrm{bdd}(A)$. A hyperimaginary b is A-bounded *if $b \in \mathrm{bdd}(A)$.*

Remark 15.9 $\mathfrak{C}^{\mathrm{eq}} \cap \mathrm{bdd}(A) = \mathfrak{C}^{\mathrm{eq}} \cap \mathrm{acl}^{\mathrm{heq}}(A) = \mathrm{acl}^{\mathrm{eq}}(A)$ *for any class of imaginaries A.*

Proof: By compactness, if a hyperimaginary has infinitely many conjugates it has unboundedly many. □

Definition 15.10 *We define now the type $\mathrm{tp}(a_E/b_F)$ of a hyperimaginary a_E over some hyperimaginary b_F. For each formula $\varphi(x,y) \in L$ let $\Phi_{E,F}(x,y)$ be the partial type*

$$\exists x'y'(E(x,x') \wedge F(y,y') \wedge \varphi(x',y')).$$

We define $\mathrm{tp}(a_E/b_F)$ *as the union of all partial types* $\Phi_{E,F}(x,b)$ *where* $\models \varphi(a',b')$ *for some* a'*,* b' *such that* $E(a,a')$ *and* $F(b,b')$*. It is a partial type over* b *but the choice of another representative* b'' *in the* F*-class of* b *gives an equivalent partial type over* b''*. In a similar way we can be define* $\mathrm{tp}(a_E/b)$ *and* $\mathrm{tp}(b/a_E)$ *for any sequence* b *of hyperimaginaries and also* $\mathrm{tp}(a_E/A)$ *for any class* A *of hyperimaginaries. As usual,* $e \equiv_c d$ *means that* $\mathrm{tp}(e/c) = \mathrm{tp}(d/c)$*.*

Proposition 15.11 *The following are equivalent:*

1. $\mathrm{tp}(a_E/c_F) = \mathrm{tp}(b_E/c_F)$
2. $\mathrm{tp}(ac) = \mathrm{tp}(b'c')$ *for some* b'*,* c' *such that* $E(b',b)$ *and* $F(c',c)$*.*
3. $\mathrm{tp}(a'c'') = \mathrm{tp}(b'c')$ *for some* $a', c'', b'c'$ *such that* $E(a',a)$*,* $F(c'',c)$*,* $E(b',b)$ *and* $F(c',c)$*.*
4. *There is some* $f \in \mathrm{Aut}(\mathfrak{C}/c_F)$ *such that* $f(a_E) = b_E$

Proof: *4* $\Rightarrow$ *1*, *2* $\Rightarrow$ *3* and *3* $\Rightarrow$ *4* are clear. For *1* $\Rightarrow$ *2*, notice that if $\mathrm{tp}(a_E/c_F) = \mathrm{tp}(b_E/c_F)$ and $p(x,y) = \mathrm{tp}(a,c)$, then

$$\pi(x,y) = E(x,b) \cup F(y,c) \cup p(x,y)$$

is consistent. If $\models \pi(b',c')$, then $E(b',b)$, $F(c',c)$ and $\mathrm{tp}(ac) = \mathrm{tp}(b'c')$. □

Definition 15.12 *A* complete type over a hyperimaginary e *in the real variables* x *is a type of the form* $p(x) = \mathrm{tp}(a/e)$ *where* $a \in \mathfrak{C}$ *is a sequence of the length of* x*. We use the notation* $p(x) \in S(e)$ *to express this situation. Of course,* $p(x)$ *is a partial type over a representative of* e *but it is complete in the sense that for any* $a, b \models p(x)$ *there is some* $f \in \mathrm{Aut}(\mathfrak{C}/e)$ *such that* $f(a) = b$*.*

Proposition 15.13 *For any hyperimaginary* e*, the relation* $F(x,y) \Leftrightarrow \mathrm{tp}(x/e) = \mathrm{tp}(y/e)$ *is type-definable over any representative of* e*.*

Proof: If $e = a_E$, then $F(x,y) \Leftrightarrow \exists u(E(a,u) \wedge \mathrm{tp}(xa) = \mathrm{tp}(xu))$. □

Proposition 15.14 *For any set of hyperimaginaries* A *there are hyperimaginaries* a, b *such that* $\mathrm{bdd}(A) = \mathrm{dcl}^{\mathrm{heq}}(a)$ *and* $\mathrm{acl}^{\mathrm{heq}}(A) = \mathrm{dcl}^{\mathrm{heq}}(b)$*.*

Proof: By Lemma 15.4 $\mathrm{bdd}(A) = \mathrm{dcl}^{\mathrm{heq}}(B)$ if B is the class of all hyperimaginaries in $\mathrm{bdd}(A)$ of length α for some $\alpha \leq \omega$. For each such $\alpha \leq \omega$ there are at most $2^{|T|}$ many 0-type-definable equivalence relations on α-sequences. For each such equivalence relation E there is an upper bound κ_E for the number of hyperimaginaries e_E in B: there are at most $2^{|T|+|A|}$ possibilities for $p(x) = \mathrm{tp}(e/A)$ and for each such $p(x)$ there are boundedly many $d \models p$ with $d_E \in B$. If κ is the supremum of all these κ_E, it follows that $|B| \leq \kappa + 2^{|T|+|A|}$ and we can choose a sequence c enumerating B. By Lemma 15.3, $c \sim b$ for some hyperimaginary b. Clearly $\mathrm{dcl}^{\mathrm{heq}}(b) = \mathrm{dcl}^{\mathrm{heq}}(B) = \mathrm{bdd}(A)$. The case $\mathrm{acl}^{\mathrm{heq}}(A)$ is similar. □

Lemma 15.15 *For any* A*-hyperimaginary* e*, there is some hyperimaginary* e' *such that* $\mathrm{Aut}(\mathfrak{C}/e') = \{f \in \mathrm{Aut}(\mathfrak{C}/A) : f(e) = e\}$*. If* e *is* A*-bounded,* e' *is* A*-bounded.*

Proof: Let $e = b_E$ where E is a type-definable over A equivalence relation. Let a enumerate A, let $p(x) = \mathrm{tp}(a)$, and let $E = E(x,y;a)$. We define

$$F(xz, yu) \ \Leftrightarrow \ (z = u \wedge p(u) \wedge E(x,y;z)) \vee xz = yu$$

It is a 0-type-definable equivalence relation. It is easy to see that $e' = ba_F$ is as required. □

Proposition 15.16 *$a \stackrel{\text{bdd}}{\equiv}_A b$ if and only if $a \equiv_{\text{bdd}(A)} b$.*

Proof: Consider first the case $A = \emptyset$. By Proposition 15.14 we can assume $\text{bdd}(\emptyset)$ is a single hyperimaginary. The equivalence relation $E(a,b) \Leftrightarrow \text{tp}(a/\text{bdd}(\emptyset)) = \text{tp}(b/\text{bdd}(\emptyset))$ is bounded and by Proposition 15.13 it is type-definable over any representative. Since it is invariant, it is also type-definable over $\emptyset$ and hence $\stackrel{\text{bdd}}{\equiv} \subseteq E$. For the other direction, assume $E(a,b)$. Note that $e = [a]_{\stackrel{\text{bdd}}{\equiv}}$ is a bounded hyperimaginary and thus $e \in \text{bdd}(\emptyset)$. Hence there is some $f \in \text{Aut}(\mathfrak{C}/e)$ such that $f(a) = b$, which implies $a \stackrel{\text{bdd}}{\equiv} b$. The general case can not be obtained by simply applying the case just proven to $T(A)$ since $\text{bdd}(A)$ is the class of all A-bounded hyperimaginaries while $\text{bdd}(\emptyset)$ computed in $T(A)$ is the class of all A-bounded A-hyperimaginaries. But Lemma 15.15 helps to solve this difficulty. □

Lemma 15.17 *For any 0-type-definable equivalence relation E, the following are equivalent:*

1. $a_E \in \text{dcl}^{\text{heq}}(M)$
2. $a_E \in \text{bdd}(M)$
3. *$E(x,a)$ is finitely satisfiable in M.*

Proof: Clearly *1* implies *2*. We will show *2* $\Rightarrow$ *3* and *3* $\Rightarrow$ *1*. Assume first that some formula $\varphi(x,a) \in E(x,a)$ is not satisfiable in M. For each cardinal κ we can build a coheir sequence over M, $(a_i : i < \kappa)$, starting with $a_0 = a$. If $i < j < \kappa$, then $\models \neg\varphi(a_i,a_j)$ since otherwise, by indiscernibility, $\models \varphi(a,a_j)$ and hence $\varphi(x,a)$ would be satisfiable in M. Since κ can be arbitrarily large and the elements of the coheir sequence have the same type over M and have different E-classes, $a_E \notin \text{bdd}(M)$.

For *3* $\Rightarrow$ *1*, assume $E(x,a)$ is finitely satisfiable in M and let $f \in \text{Aut}(\mathfrak{C}/M)$ and $\varphi(x,y) \in E(x,y)$. We will show that $\models \varphi(a,f(a))$. This will imply $E(a,f(a))$ and therefore $f(a_E) = a_E$. We may assume that $E(x,y)$ is closed under conjunction and hence $\vdash \psi(x,z) \wedge \psi(z,y) \rightarrow \varphi(x,y)$ for some symmetric $\psi(x,y) \in E(x,y)$. Since $\psi(x,a)$ is satisfiable in M, there is some $c \in M$ such that $\models \psi(c,a)$. Since $a \equiv_M f(a)$, we have also $\models \psi(c,f(a))$. From this it follows that $\models \varphi(a,f(a))$. □

Proposition 15.18 $\text{bdd}(b) = \bigcap_{b \in \text{dcl}^{\text{heq}}(M)} \text{dcl}^{\text{heq}}(M)$

Proof: If $a \in \text{bdd}(b)$ and $b \in \text{dcl}^{\text{heq}}(M)$ then clearly $a \in \text{bdd}(M)$ and by Lemma 15.17 we conclude $a \in \text{dcl}^{\text{heq}}(M)$. For the other direction, let us choose a model M such that $b \in \text{dcl}^{\text{heq}}(M)$ (for instance, a model containing a representative of b) and let us choose a cardinal $\kappa > 2^{|T|+|M|}$. If $a \notin \text{bdd}(b)$, there is a family $(a_i : i < \kappa)$ of different b-conjugates of a starting with $a_0 = a$. By choice of κ there are $i < j < \kappa$ such that $a_i \equiv_M a_j$. Hence for some $a' \neq a$ we have $a \equiv_M a'$, which implies $a \notin \text{dcl}^{\text{heq}}(M)$. □

Proposition 15.19 *Let $p(x) \in S(A)$.*

1. *The restriction $\stackrel{\text{Ls}}{\equiv}_A\restriction p$ of $\stackrel{\text{Ls}}{\equiv}_A$ to $p(\mathfrak{C})$ is the finest bounded A-invariant equivalence relation on realizations of p.*
2. *The restriction $\stackrel{\text{bdd}}{\equiv}_A\restriction p$ of $\stackrel{\text{bdd}}{\equiv}_A$ to $p(\mathfrak{C})$ is the finest bounded type-definable over A equivalence relation on realizations of p.*

Proof: Since $\overset{\text{Ls}}{\equiv}_A\restriction p$ is bounded and A-invariant, it contains the finest bounded A-invariant equivalence relation E on $p(\mathfrak{C})$. Similarly, $\overset{\text{bdd}}{\equiv}_A\restriction p$ contains the finest bounded type-definable over A equivalence relation F on $p(\mathfrak{C})$. On the other hand,

$$E(x,y) \vee (\neg p(x) \vee \neg p(y))$$

is bounded and A-invariant, and therefore it contains $\overset{\text{Ls}}{\equiv}_A$. The corresponding result with respect to $\overset{\text{bdd}}{\equiv}_A$ and F is more involved. Assume a, b realize $p(x)$ and $a \overset{\text{bdd}}{\equiv}_A b$. We will show that $F(a,b)$. By Proposition 15.16 $a \equiv_{\text{bdd}(A)} b$. Choose with Lemma 15.5 an extension F' of F which is an equivalence relation on all sequences of the length of a, is type-definable over A and agrees with F on p. Then $e = [a]_F = [a]_{F'}$ is an A-bounded A-hyperimaginary. By Lemma 15.15 there is an A-bounded hyperimaginary e' such that $\text{Aut}(\mathfrak{C}/e') = \{f \in \text{Aut}(\mathfrak{C}/A) : f(e) = e\}$. Since $e' \in \text{bdd}(A)$, $a \equiv_{e'} b$, which implies that $f(a) = b$ for some $f \in \text{Aut}(\mathfrak{C}/A)$ such that $f(e) = e$ and therefore implies that $F(a,b)$. □

Proposition 15.20 *If E is a bounded equivalence relation on realizations of $p(x) \in S(A)$ and it is type-definable over A, then there exists a bounded equivalence relation F defined for all sequences of the length of x which is type-definable over A and agrees with E in $p(\mathfrak{C})$.*

Proof: Since $\overset{\text{bdd}}{\equiv}_A\restriction p \subseteq E$, it suffices to set $F(x,y) \Leftrightarrow (p(x) \wedge p(y) \wedge E(x,y)) \vee x \overset{\text{bdd}}{\equiv}_A y$. □

Proposition 15.21 *Any A-bounded A-hyperimaginary is an equivalence class of a bounded type-definable over A equivalence relation.*

Proof: Let E be a type-definable over A equivalence relation and let a_E be an A-bounded A-hyperimaginary. Let $p(x) = \text{tp}(a/A)$ and note that each A-conjugate of a_E is an E-class which is also a union of $\overset{\text{bdd}}{\equiv}_A$-classes. Hence if F is defined by

$$F(x,y) \Leftrightarrow \exists z(p(z) \wedge E(x,z) \wedge E(y,z)) \vee x \overset{\text{bdd}}{\equiv}_A y$$

then F is a bounded equivalence relation which is type-definable over A and $a_E = a_F$. □

Proposition 15.22 *Let e, d be hyperimaginaries such that $e \in \text{bdd}(d)$ and let A be the set of all d-conjugates of e. There is some hyperimaginary c such that* $\text{Aut}(\mathfrak{C}/c) = \{f \in \text{Aut}(\mathfrak{C}) : f(A) = A\}$.

Proof: Let $e = a_E$ and $d = b_G$. We may assume every $\varphi(x,y) \in E(x,y)$ is symmetric. Since $e \in \text{bdd}(d)$, for any $\varphi(x,y)$ there is a maximal $n_\varphi < \omega$ for which there is a sequence $(a_i : i < n_\varphi)$ such that $de \equiv d[a_i]_E$ for each $i < n_\varphi$ and $\models \neg\varphi(a_i, a_j)$ for all $i < j < n_\varphi$. Fix a witnessing sequence $(a_i^\varphi : i < n_\varphi)$. Let $p(z,x) = \text{tp}(d,e)$, let $r_\varphi(z, x_i)_{i<n_\varphi}$ be the type

$$\bigcup_{i<n_\varphi} p(z, x_i) \cup \{\neg\varphi(x_i, x_j) : i < j < n_\varphi\}$$

and let us define $F(z_1, z_2)$ by

$$\bigwedge_{\varphi(x,y)\in E(x,y)} \exists(x_i : i < n_\varphi)(r_\varphi(z_1, x_i)_{i<n_\varphi} \wedge r_\varphi(z_2, x_i)_{i<n_\varphi})$$

Note that F is independent of the choice of representatives z_1, z_2 in G-classes.

Claim: For any $f \in \text{Aut}(\mathfrak{C})$, $\models F(b, f(b))$ iff $f(A) = A$.

Proof of the claim: Assume first $f(A) = A$. Therefore $f^{-1}(A) = A$. Clearly, $\models r_\varphi(b, a_i^\varphi)_{i<n_\varphi}$. Note that $[a_i^\varphi]_E \in A$ and hence $f^{-1}([a_i^\varphi]_E) \in A$ and $f^{-1}([a_i^\varphi]_E) \equiv_d e$, that is, $\models p(b, f^{-1}(a_i^\varphi))$ for all $i < n_\varphi$. It follows that $r_\varphi(f(b), a_i^\varphi)_{i<n_\varphi}$ and thus $F(b, f(b))$. For the other direction, let $e' = [a']_E \in A$ (which means $de \equiv de'$) and assume $F(b, f(b))$. For each $\varphi \in E(x, y)$ choose some $(c_i^\varphi : i < n_\varphi)$ such that $\models r_\varphi(b, c_i^\varphi)_{i<n_\varphi}$ and $\models r_\varphi(f(b), c_i^\varphi)_{i<n_\varphi}$. Then $\models \neg\varphi(c_i^\varphi, c_j^\varphi)$ for $i < j < n_\varphi$ and $d[c_i^\varphi]_E \equiv f(d)[c_i^\varphi]_E \equiv de \equiv de' \equiv f(d)f(e') = f(d)[f(a')]_E$ for all $i < n_\varphi$. By maximality of n_φ, $\models \varphi(f(a'), c_i^\varphi)$ for some $i < n_\varphi$. Therefore $E(f(a'), y) \cup p(b, y)$ is consistent and thus $de \equiv d[f(a')]_E = df(e')$, that is, $f(e') \in A$. This proves $f(A) \subseteq A$. Since also $F(b, f^{-1}(b))$, we get the equality $f(A) = A$.

The claim is obviously true for any other b' such that $[b']_G = d$. From the claim it follows that F is an equivalence relation on realizations of $\mathrm{tp}(d)$. Hence we may assume F is an equivalence relation on all sequences of the length of b. Clearly the hyperimaginary b_F fulfills the requirements.

$\square$

Chapter 16

Forking for hyperimaginaries

Definition 16.1 *Let A be a set of hyperimaginaries and let I be a set linearly ordered by $<$. The sequence of hyperimaginaries $(e_i : i \in I)$ is* indiscernible over A *or it is* A-indiscernible *if for any $n < \omega$, for any two increasing sequences of indices $i_0 < \ldots < i_n$ and $j_0 < \ldots < j_n$, $\mathrm{tp}(e_{i_0}, \ldots, e_{i_n}/A) = \mathrm{tp}(e_{j_0}, \ldots, e_{j_n}/A)$. In practice we may always assume that A is a single hyperimaginary. Note that the type-definable equivalence relations corresponding to the hyperimaginaries e_i are in fact the same and hence we can write $e_i = [a_i]_E$ for a single E.*

Lemma 16.2 *Let d be a hyperimaginary.*

1. *Let I, J be linearly ordered infinite sets. If $(e_i : i \in I)$ is a d-indiscernible sequence of hyperimaginaries, then there is a d-indiscernible sequence $(c_i : j \in J)$ such that for any $n < \omega$, for any two increasing sequences of indices $i_0 < \ldots < i_n \in I$ and $j_0 < \ldots < j_n \in J$, $e_{i_0}, \ldots, e_{i_n} \equiv_d c_{j_0}, \ldots, c_{j_n}$.*
2. *If $(e_i : i \in I)$ and $(d_i : i \in I)$ are d-indiscernible sequence of hyperimaginaries and $(e_i : i \in I_0) \equiv_d (d_i : i \in I_0)$ for each finite subset $I_0 \subseteq I$, then $f((e_i : i \in I)) = (d_i : i \in I)$ for some $f \in \mathrm{Aut}(\mathfrak{C}/d)$.*

Proof: *1.* By compactness. For *2* note that it follows that $(e_i : i \in I) \equiv_d (d_i : i \in I)$. $\square$

Proposition 16.3 *If $(e_i : i \in I)$ is a sequence of hyperimaginaries indiscernible over the hyperimaginary d, then for some representative $\dot{d}$ of d some sequence $(\dot{e}_i : i \in I)$ of corresponding representatives of $(e_i : i \in I)$ is $\dot{d}$-indiscernible.*

Proof: Fix d', a representative of d. Since the sequence we seek is just a realization of some partial type over d' and representatives of the hyperimaginaries e_i, we may assume $(I, <) = (\omega, <)$. Let κ be an infinite cardinal number larger than the length of d', and larger than the length of every representative of e_i, and let $\lambda = \beth_{(2^\kappa)^+}$. By the Lemma 16.2 we can extend $(e_i : i < \omega)$ to a d-indiscernible sequence $(e_i : i < \lambda)$. Choose corresponding representatives $[a_i]_E = e_i$. By Proposition 1.1 there is a d'-indiscernible sequence $(c_i : i < \omega)$ such that for all $n < \omega$ there are some $i_0 < \ldots < i_n < \lambda$ such that $c_0 \ldots c_n \equiv_{\dot{d}} a_{i_0} \ldots a_{i_n}$. Since $([c_i]_E : i < \omega) \equiv_d (e_i : i < \omega)$, for some representative $\dot{d}$ of d there exists a $\dot{d}$-indiscernible sequence $(\dot{e}_i : i < \omega)$ such that $[\dot{e}_i]_E = e_i$. $\square$

Proposition 16.4 *Let d be a hyperimaginary.*

1. *For any hyperimaginary $e \notin \mathrm{bdd}(d)$, there is a d-indiscernible sequence $(e_i : i < \omega)$ of distinct hyperimaginaries starting with $e_0 = e$.*

2. *If the sequence of hyperimaginaries $(e_i : i \in I)$ is d-indiscernible, then it is also indiscernible over $\mathrm{bdd}(d)$.*

Proof: *1* Let $e = a_E$. Since $e \notin \mathrm{bdd}(d)$, for some $\varphi(x,y) \in E(x,y)$ there are $(a_i : i < \omega)$ such that $a = a_0 \equiv_d a_i$ and $\models \neg\varphi(a_i, a_j)$ for all $i < j < \omega$. By Ramsey's Theorem and compactness we find a d-indiscernible sequence $(b_i : i < \omega)$ such that $\models \neg\varphi(b_i, b_j)$ for all $i < j < \omega$ and $b_0 \equiv_d a$. Then a d-conjugate of $([b_i]_E : i < \omega)$ satisfies the requirements.

2. By Proposition 16.3 some sequence $(\dot{e}_i : i \in I)$ of representatives is indiscernible over some representative $\dot{d}$ of d. By Corollary 1.2 $(\dot{e}_i : i \in I)$ is indiscernible over some model M containing $\dot{d}$. By Proposition 15.18, $\mathrm{bdd}(d) \subseteq \mathrm{dcl}^{\mathrm{heq}}(M)$ and hence $(\dot{e}_i : i \in I)$ and $(e_i : i \in I)$ are indiscernible over $\mathrm{bdd}(d)$. □

Definition 16.5 *The formula $\varphi(x,a)$* divides *over the hyperimaginary e (with respect to k) if there is some e-indiscernible sequence $(a_i : i < \omega)$ with $a_0 = a$ for which $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent (k-inconsistent). The formula $\varphi(x,a)$* forks *over e if there are formulas $\psi_1(x,b_1), \ldots, \psi_n(x,b_n)$ such that $\varphi(x,a) \vdash \psi_1(x,b_1) \vee \ldots \vee \psi_n(x,b_n)$ and each $\psi_i(x,b_i)$ divides over e. The set of formulas $\pi(x)$* divides (forks) *over e if $\pi(x)$ implies some formula which divides (forks) over e. The hyperimaginary a is* independent independence! of hyperimaginaries *of the hyperimaginary b over the hyperimaginary e (written $a \mathop{\smile\!\!\!\!\!\!\mid}_e b$) if* $\mathrm{tp}(a/be)$ *does not fork over e. Other notions like Morley sequences can be defined in a similar way and we will make use of them when necessary.*

Proposition 16.6 *A partial type $\pi(x)$ divides over the hyperimaginary e with respect to k if and only if it divides over some representative of e with respect to k.*

Proof: By Proposition 16.3. □

Remark 16.7 *Let $\pi(x,y)$ be a partial type over $\emptyset$. Then $\pi(x,b)$ divides over the hyperimaginary e if and only if for some e-indiscernible sequence $(b_i : i < \omega)$ with $b = b_0$, $\bigcup_{i<\omega} \pi(x, b_i)$ is inconsistent.*

Lemma 16.8 *For any hyperimaginaries a, b, e:* $\mathrm{tp}(a/be)$ *does not divide over e if and only if for any e-indiscernible sequence $I \ni b$ there is some $a' \equiv_{eb} a$ such that I is ea'-indiscernible.*

Proof: We adapt the proof of Lemma 4.5. From right to left it is easy. For the other direction, assume $\mathrm{tp}(a/be)$ does not divide over e and, to simplify notation, let $I = ([b_i]_E : i < \omega)$ be e-indiscernible with $b = [b_0]_E$. By Proposition 16.3 we may assume that $(b_i : i < \omega)$ is indiscernible over some representative $\dot{e}$ of e. Let $\pi(x, \dot{e}, b_0) = \mathrm{tp}(a/eb)$ and let $\Gamma(x, \dot{e}, b_i)_{i<\omega}$ be the set of formulas expressing that $(b_i : i < \omega)$ is indiscernible over $\dot{e}x$. It is enough to show that $\pi(x, \dot{e}, b_0) \cup \Gamma(x, \dot{e}, b_i)_{i<\omega}$ is consistent. By the previous remark, $\bigcup_{i<\omega} \pi(x, \dot{e}, b_i)$ is consistent and can be realized by some c. Let $\Gamma_0(x, \dot{e}, b_i)_{i<\omega}$ be a finite subset of $\Gamma(x, \dot{e}, b_i)_{i<\omega}$. By Ramsey's Theorem, there is a one-to-one mapping $f : \omega \to \omega$ such that $\models \Gamma_0(c, \dot{e}, f(b_i))_{i<\omega}$. Now take some c' such that $c'(b_i : i < \omega) \equiv_{\dot{e}} c(f(b_i) : i < \omega)$ and note that c' realizes $\Gamma_0(x, \dot{e}, b_i)_{i<\omega}$ and $\pi(x, \dot{e}, b_0)$. □

Proposition 16.9 *For any hyperimaginaries a, b, c, d: if* $\mathrm{tp}(b/cd)$ *does not divide over d and* $\mathrm{tp}(a/cbd)$ *does not divide over bd, then* $\mathrm{tp}(ab/cd)$ *does not divide over d.*

Proof: By Lemma 16.8. □

Proposition 16.10 *1. Let $\pi(x)$ be a partial type over A. If π does not fork over the hyperimaginary e, then some completion $p(x) \in S(A)$ of π does not fork over e.*

2. Let a, b, c be hyperimaginaries such that $a \mathop{\smile\!\!\!\!\!\!\!\;\;|}_b c$. Then for any hyperimaginary d, there is some $a' \equiv_{bc} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\!\;\;|}_b cd$.

Proof: *1.* As in Remark 4.4, the reason is that $\pi(x) \cup \{\neg\varphi(x) : \varphi(x) \in L(A) \text{ forks over } e\}$ is consistent. *2.* Fix representatives $\dot{b}, \dot{c}, \dot{d}$ of b, c, d and put $\operatorname{tp}(a/bc) = \pi(x, \dot{b}, \dot{c})$. By *1* there exists a completion $p(x) \in S(\dot{b}\dot{c}\dot{d})$ of $\pi(x, \dot{b}, \dot{c})$ which does not fork over b. Let E be the equivalence relation of a, let $\dot{a} \models p$ and let $a' = \dot{a}_E$. Then $a' \equiv_{bc} a$ and $\dot{a} \mathop{\smile\!\!\!\!\!\!\!\;\;|}_b \dot{c}\dot{d}$. Since $\operatorname{tp}(\dot{a}/b\dot{c}\dot{d}) \vdash \operatorname{tp}(a'/bcd)$ we also have $a' \mathop{\smile\!\!\!\!\!\!\!\;\;|}_b cd$. □

Proposition 16.11 *Let e be a hyperimaginary.*

1. *If $\pi(x)$ divides over e and $\pi(x)$ is a partial type over A, then $\pi(x)$ divides over $\dot{e}$ for any representative $\dot{e}$ of e such that $\dot{e} \mathop{\smile\!\!\!\!\!\!\!\;\;|}_e A$.*
2. *If T is simple, then $a \mathop{\smile\!\!\!\!\!\!\!\;\;|}_e e$ for any sequence a.*
3. *If T is simple, then a partial type $\pi(x)$ forks over e if and only if $\pi(x)$ forks over some representative of e.*

Proof: *1.* Fix $\varphi(x, y) \in L$ and $b \in A$ such that $\pi(x) \vdash \varphi(x, b)$ and $\varphi(x, b)$ divides over e. Then for some e-indiscernible sequence $(b_i : i < \omega)$ with $b = b_0$, $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent. Since $\dot{e} \mathop{\smile\!\!\!\!\!\!\!\;\;|}_e b$, by Lemma 16.8 there is another $\dot{e}$-indiscernible sequence $(b'_i : i < \omega)$ with $b = b'_0$ and such that $\{\varphi(x, b'_i) : i < \omega\}$ is inconsistent. Then $\varphi(x, b)$ divides over $\dot{e}$.

2. Choose a representative $\dot{e}$ of e. We will check that the partial type $\pi(x, \dot{e}) = \operatorname{tp}(a/e)$ does not fork over e. Assume $\pi(x, \dot{e}) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi_i(x, a_i)$ divides over e with respect to k_i. Let $k = \max\{k_1, \ldots, k_n\}$, let $\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$, and let $m = D(\pi(x, \dot{e}), \Delta, k)$. By Proposition 3.10 there is a completion $p(x) \in S(\dot{e}a_1, \ldots, a_n)$ of $\pi(x, \dot{e})$ with $D(p(x), \Delta, k) = m$. For some i, $\varphi_i(x, a_i) \in p$. Now, $\varphi_i(x, a_i)$ divides over e with respect to k and by Proposition 16.6 it divides over some representative $\ddot{e}$ of e with respect to k. Notice that $\pi(x, \dot{e}) \equiv \pi(x, \ddot{e})$. Then $m = D(\pi(x, \ddot{e}) \cup \{\varphi_i(x, a_i)\}, \Delta, k) \geq D(\pi(x, \ddot{e}), \Delta, k) + 1 = m + 1$, a contradiction.

3. Assume $\pi(x) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi_i(x, a_i)$ divides over e. By *2* and Proposition 16.10 we can choose a representative $\dot{e}$ of e such that $\dot{e} \mathop{\smile\!\!\!\!\!\!\!\;\;|}_e a_1, \ldots, a_n$. By *1* every $\varphi_i(x, a_i)$ divides over $\dot{e}$. Hence $\pi(x)$ forks over $\dot{e}$. □

Corollary 16.12 *If T is simple, a partial type forks over a hyperimaginary e if and only if it divides over e.*

Proof: By Proposition 16.11 and Proposition 5.14. □

Proposition 16.13 *Let T be simple. For any hyperimaginaries a, b, c, d:*

1. *If $ab \mathop{\smile\!\!\!\!\!\!\!\;\;|}_c d$, then $a \mathop{\smile\!\!\!\!\!\!\!\;\;|}_c d$ and $a \mathop{\smile\!\!\!\!\!\!\!\;\;|}_{bc} d$.*
2. *If $a \mathop{\smile\!\!\!\!\!\!\!\;\;|}_b cd$, then $a \mathop{\smile\!\!\!\!\!\!\!\;\;|}_b d$ and $a \mathop{\smile\!\!\!\!\!\!\!\;\;|}_{bc} d$.*

3. $a \mathop{\smile\!\!\!\!\!\!\,|}_b b$.

4. *There is some* $a' \equiv_b a$ *such that* $a' \mathop{\smile\!\!\!\!\!\!\,|}_b c$.

5. *If* $a \mathop{\smile\!\!\!\!\!\!\,|}_c d$ *and* $b \mathop{\smile\!\!\!\!\!\!\,|}_{ac} d$, *then* $ab \mathop{\smile\!\!\!\!\!\!\,|}_c d$.

6. $a \mathop{\smile\!\!\!\!\!\!\,|}_b \mathrm{bdd}(b)$.

7. *If* $d \mathop{\smile\!\!\!\!\!\!\,|}_c ab$, *then* $a \mathop{\smile\!\!\!\!\!\!\,|}_c b$ *iff* $a \mathop{\smile\!\!\!\!\!\!\,|}_{cd} b$

Proof: *1* and *2* follow straightforward from Corollary 16.12 and the definition of dividing. To check that $a \mathop{\smile\!\!\!\!\!\!\,|}_{bc} d$ in *1* it could be convenient to use Remark 16.7 applied to $\mathrm{tp}(a/bcd)$.

3. Write $\mathrm{tp}(a/b) = \pi(x, b_0)$, where b_0 is a representative of b. By Corollary 4.14, $\pi(x, b_0)$ does not fork over b_0. If we choose another representative b_1 of b we get equivalent partial types $\pi(x, b_0)$ y $\pi(x, b_1)$ and hence $\pi(x, b_0)$ does not fork over b_1 either. By Proposición 16.6, $\pi(x, b_0)$ does not fork over b.

4 follows from *3* and Proposition 16.10, while *5* follows from Proposition 16.9 and Corollary 16.12.

6. By *4* there is some $a' \equiv_b a$ such that $a' \mathop{\smile\!\!\!\!\!\!\,|}_b \mathrm{bdd}(b)$. There is some $f \in \mathrm{Aut}(\mathfrak{C}/b)$ such that $f(a') = a$. Since f fixes setwise $\mathrm{bdd}(b)$, if we apply f we obtain $a \mathop{\smile\!\!\!\!\!\!\,|}_b \mathrm{bdd}(b)$.

7. Assume $d \mathop{\smile\!\!\!\!\!\!\,|}_c ab$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_c b$. By *2* $d \mathop{\smile\!\!\!\!\!\!\,|}_{ac} b$ and then by *5* $da \mathop{\smile\!\!\!\!\!\!\,|}_c b$. By *1* $a \mathop{\smile\!\!\!\!\!\!\,|}_{cd} b$. Assume now $d \mathop{\smile\!\!\!\!\!\!\,|}_c ab$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_{cd} b$. By *2* $d \mathop{\smile\!\!\!\!\!\!\,|}_c b$ and by *5*, $ad \mathop{\smile\!\!\!\!\!\!\,|}_c b$. Then by *1* $a \mathop{\smile\!\!\!\!\!\!\,|}_c d$. □

Proposition 16.14 (Symmetry and transitivity) *If T is simple, then independence is symmetric and transitive for hyperimaginaries, that is, for any hyperimaginaries a, b, c, d:*

1. $a \mathop{\smile\!\!\!\!\!\!\,|}_b c$ *if and only if* $c \mathop{\smile\!\!\!\!\!\!\,|}_b a$.

2. *If* $a \mathop{\smile\!\!\!\!\!\!\,|}_b c$ *and* $a \mathop{\smile\!\!\!\!\!\!\,|}_{bc} d$, *then* $a \mathop{\smile\!\!\!\!\!\!\,|}_b cd$.

Proof: *2* is a consequence of *1* and of point *5* of Proposition 16.13. For *1*, we show first that we may assume that a is a sequence of elements in $\mathfrak{C}$. For this we use several times Proposition 16.13. Choose with point *4* some representative a_0 of a such that $a_0 \mathop{\smile\!\!\!\!\!\!\,|}_a bc$. By point *2* $a_0 \mathop{\smile\!\!\!\!\!\!\,|}_{ab} c$. If we assume $a \mathop{\smile\!\!\!\!\!\!\,|}_b c$ by point *5* we get $aa_0 \mathop{\smile\!\!\!\!\!\!\,|}_b c$ and by point *1* $a_0 \mathop{\smile\!\!\!\!\!\!\,|}_b c$. Since $a \in \mathrm{dcl}^{\mathrm{heq}}(a_0)$, it follows that $\mathrm{tp}(c/ba_0) \vdash \mathrm{tp}(c/ba)$. If we were able to prove then that $c \mathop{\smile\!\!\!\!\!\!\,|}_b a_0$ we would conclude that $c \mathop{\smile\!\!\!\!\!\!\,|}_b a$. Thus we assume $a \in \mathfrak{C}$.

As a second step we show now that we can also assume c is a sequence of elements in $\mathfrak{C}$. Choose representatives b_0 of b and c_0 of c and let $\pi(x, b_0, c_0) = \mathrm{tp}(a/bc)$. It is a partial type over b_0c_0 and it does not fork over b since we assume $a \mathop{\smile\!\!\!\!\!\!\,|}_b c$. By Proposition 16.10, some completion $p(x) \in S(b_0c_0)$ of $\pi(x, b_0, c_0)$ does not fork over b. Let $a' \models p$. Then $a' \equiv_{bc} a$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_b b_0c_0$ (note that $\mathrm{tp}(a'/b_0c_0) \vdash \mathrm{tp}(a'/b_0c_0b)$). Since these are sequences in $\mathfrak{C}$, $b_0c_0 \mathop{\smile\!\!\!\!\!\!\,|}_b a'$ and then by Proposition 16.13 $c_0 \mathop{\smile\!\!\!\!\!\!\,|}_b a'$. Let $f \in \mathrm{Aut}(\mathfrak{C}/bc)$ be such that $f(a') = a$. Then $f(c_0) \mathop{\smile\!\!\!\!\!\!\,|}_b a$ and $f(c_0)$ is a representative of c. Since $\mathrm{tp}(f(c_0)/ba) \vdash \mathrm{tp}(c/ba)$, we conclude that $c \mathop{\smile\!\!\!\!\!\!\,|}_b a$.

Thus we must finally consider the case where a and c are sequences of elements of $\mathfrak{C}$. Assume $a \mathop{\smile\!\!\!\!\!\!\,|}_b c$. Choose a representative $\dot{b}$ of b such that $\dot{b} \mathop{\smile\!\!\!\!\!\!\,|}_b ac$. By point *7* of Proposition 16.13, $a \mathop{\smile\!\!\!\!\!\!\,|}_{b\dot{b}} c$. Since $b \in \mathrm{dcl}^{\mathrm{heq}}(\dot{b})$, $\dot{b} \sim b\dot{b}$ and therefore $a \mathop{\smile\!\!\!\!\!\!\,|}_{\dot{b}} c$. By symmetry of independence in $\mathfrak{C}$, $c \mathop{\smile\!\!\!\!\!\!\,|}_{\dot{b}} a$. Again, we get $c \mathop{\smile\!\!\!\!\!\!\,|}_{b\dot{b}} a$ and by point *7* of Proposition 16.13, $c \mathop{\smile\!\!\!\!\!\!\,|}_b a$. □

Proposition 16.15 (Local character) *Let T be simple, let a be a hyperimaginary and let $b = (b_i : i \in I)$ be a sequence of hyperimaginaries. Then $a \mathop{\smile\!\!\!\!\!\!|}_{(b_i:i\in J)} b$ for some $J \subseteq I$ such that $|J| \leq |T|$.*

Proof: We may assume $I = \{i : i < \kappa\}$ for some cardinal κ. Choose inductively representatives b'_i of b_i such that $b'_i \mathop{\smile\!\!\!\!\!\!|}_{b_i} b(b'_j : j < i)$ for all $i < \kappa$. It is then easy to see that for all subsets J of κ, $(b'_i : i \in J) \mathop{\smile\!\!\!\!\!\!|}_{(b_i:i\in J)} b$. We can find a subset $J_0 \subseteq \kappa$ such that $|J_0| \leq |T|$ and $\mathrm{tp}(a/b)$ (represented as a partial type over $(b'_i : i < \kappa)$) does not fork over $(b'_i : i \in J_0)$. By symmetry and transitivity it does not fork over $(b_i : i \in J_0)$ either. $\square$

Corollary 16.16 *Let T be simple. For any hyperimaginaries a, b there is some hyperimaginary e of length $\leq |T|$ such that $e \in \mathrm{dcl}^{\mathrm{heq}}(b)$ and $a \mathop{\smile\!\!\!\!\!\!|}_e b$.*

Proof: By Lemma 15.4 there is a sequence $(b_i : i \in I)$ of countable hyperimaginaries b_i such that $b \sim (b_i : i \in I)$. By Proposition 16.15 there is some $J \subseteq I$ such that $|J| \leq |T|$ and $a \mathop{\smile\!\!\!\!\!\!|}_{(b_i:i\in J)} (b_i : i \in I)$. Then $e = (b_i : i \in J)$ satisfies the requirements. $\square$

Proposition 16.17 (Independence Theorem) *Let T be simple, let a, b, c, d be hyperimaginaries such that $a \mathop{\smile\!\!\!\!\!\!|}_M b$, $c \mathop{\smile\!\!\!\!\!\!|}_M a$, $d \mathop{\smile\!\!\!\!\!\!|}_M b$ and $c \equiv_M d$. Then there is some hyperimaginary $e \mathop{\smile\!\!\!\!\!\!|}_M ab$ such that $e \equiv_{Ma} c$ and $e \equiv_{Mb} d$.*

Proof: We may assume that c and d are sequences of elements of $\mathfrak{C}$ (replace c by a representative $\dot{c}$ such that $\dot{c} \mathop{\smile\!\!\!\!\!\!|}_c Ma$ and then replace d by some representative $\dot{d}$ such that $\dot{c} \equiv_M \dot{d}$ and $\dot{d} \mathop{\smile\!\!\!\!\!\!|}_{Md} b$). Choose representatives a_0, b_0 of a and b such that $a_0 \mathop{\smile\!\!\!\!\!\!|}_M b_0$. Consider $\mathrm{tp}(c/aM)$ $\mathrm{tp}(d/bM)$ as partial types over Ma_0 and Mb_0 respectively. They can be extended to complete types $p(x) \in S(Ma_0)$ and $q(x) \in S(Mb_0)$ which do not fork over M. Note that $p \restriction M = q \restriction M$. By the Independence Theorem for ordinary types there is some $e_0 \models p \cup q$ such that $e_0 \mathop{\smile\!\!\!\!\!\!|}_M a_0b_0$. Then e_0 is a representative of the hyperimaginary e we seek. $\square$

Definition 16.18 *For any hyperimaginary e, the group of* strong automorphisms over *e is the group $\mathrm{Autf}(\mathfrak{C}/e)$ generated by all $\mathrm{Aut}(\mathfrak{C}/M)$ where $e \in \mathrm{dcl}^{\mathrm{heq}}(M)$. The* Lascar strong type *over e of a hyperimaginary a is the orbit $\mathrm{Lstp}(a/e)$ of a under $\mathrm{Autf}(\mathfrak{C}/e)$. Hence $\mathrm{Lstp}(a/e) = \mathrm{Lstp}(b/e)$ if and only if for some $n < \omega$ there are models M_i and hyperimaginaries a_i such that $e \in \mathrm{dcl}^{\mathrm{heq}}(M_i)$ and*

$$a = a_0 \equiv_{M_0} a_1 \equiv_{M_1} a_2 \equiv_{M_3} \ldots \equiv_{M_n} a_{n+1} = b$$

Remark 16.19 *For any hyperimaginaries a, b, e:*

1. *$\mathrm{Lstp}(a/e) = \mathrm{Lstp}(b/e)$ if and only if $\mathrm{Lstp}(a'/e) = \mathrm{Lstp}(b'/e)$ for some representatives a' of a and b' of b.*

2. *If $a \equiv_M b$ and $e \in \mathrm{dcl}^{\mathrm{heq}}(M)$ then for some hyperimaginary c, there are infinite e-indiscernible sequences I, J, such that $a, c \in I$ and $c, b \in J$.*

3. *If $a, b \in I$ for some infinite e-indiscernible sequence I, then $a \equiv_M b$ for some model M such that $e \in \mathrm{dcl}^{\mathrm{heq}}(M)$.*

4. *$\mathrm{Lstp}(a/e) = \mathrm{Lstp}(b/e)$ if and only if for some $n < \omega$ there are infinite e-indiscernible sequences I_i and hyperimaginaries a_i such that $a = a_0$, $a_i, a_{i+1} \in I_i$ and $a_{n+1} = b$.*

Proof: *1* is clear. For *2*, take representatives a', b' of a, b such that $a' \equiv_M b'$. By point *2* of Lemma 9.10, there is some c' and some M-indiscernible sequences I, J such that $a', c' \in I$ and $c', b' \in J$. The corresponding sequences of equivalence classes are as required.

3. Let I be a e-indiscernible sequence such that $a, b \in I$. By Proposition 16.3, there are representatives a', b', e' of a, b, e such that for some e'-indiscernible sequence J, $a', b' \in J$. By point *1* of Lemma 9.10, $a' \equiv_M b'$ for some model $M \ni e'$. Then $e \in \mathrm{dcl}^{\mathrm{heq}}(M)$ and $a \equiv_M b$.

4 is a consequence of points *2* and *3*. $\square$

Definition 16.20 *Le e be a hyperimaginary. A type-definable relation E is* type-definable over e *if it is e-invariant. Note that this is equivalent to say that E is type-definable over any representative of e. An equivalence class of a sequence in a type-definable over e equivalence relation is an* e-hyperimaginary.

Lemma 16.21 *Let e be a hyperimaginary. For any e-hyperimaginary h there is a hyperimaginary h' such that $h' \sim he$, that is $\mathrm{Aut}(\mathfrak{C}/h') = \mathrm{Aut}(\mathfrak{C}/he)$. Moreover h' is e-bounded if h is e-bounded.*

Proof: It is a generalization of Lemma 15.15, with a similar proof. Let $h = b_E$ where E is a type-definable over e equivalence relation. Let $\dot{e}$ be a representative of e, say $\dot{e}_G = e$, let $p(x) = \mathrm{tp}(\dot{e})$, and let $E = E(x, y; \dot{e})$. We define

$$F(xz, yu) \;\Leftrightarrow\; (G(z, u) \wedge p(z) \wedge p(u) \wedge E(x, y; z)) \vee xz = yu$$

It is a 0-type-definable equivalence relation and it is easy to see that $h' = b\dot{e}_F$ is as required. $\square$

Lemma 16.22 *Let T be simple. Let a, b, e be hyperimaginaries such that $a \mathop{\smile\!\!\!\!\!\!\;|}_e b$ and $a \equiv_e b$. For any representative a' of a there is some representative b' of b such that $a' \mathop{\smile\!\!\!\!\!\!\;|}_e b'$ and $a' \equiv_e b'$.*

Proof: Choose as b' some representative of b such that $b' \equiv_e a'$ and $b' \mathop{\smile\!\!\!\!\!\!\;|}_{eb} a'$. $\square$

Lemma 16.23 *Let T be simple. Let a, b, e be hyperimaginaries such that $a \equiv_M b$ for some model M such that $e \in \mathrm{dcl}^{\mathrm{heq}}(M)$. Then $a \equiv_N b$ for some model N such that $e \in \mathrm{dcl}^{\mathrm{heq}}(N)$ and $ab \mathop{\smile\!\!\!\!\!\!\;|}_e N$.*

Proof: We may assume that a, b are sequences of elements of $\mathfrak{C}$. Choose a model $M' \equiv_e M$ such that $M' \mathop{\smile\!\!\!\!\!\!\;|}_e M$ and then choose a model N such that $N \equiv_M M'$ and $\mathrm{tp}(N/Mab)$ is a coheir of $\mathrm{tp}(N/M)$. Then $N \mathop{\smile\!\!\!\!\!\!\;|}_e M$ and by Proposition 7.6 $N \mathop{\smile\!\!\!\!\!\!\;|}_M ab$. By transitivity, $N \mathop{\smile\!\!\!\!\!\!\;|}_e ab$. Since $\mathrm{tp}(N/Mab)$ does not split over M and $a \equiv_M b$, it follows that $a \equiv_N b$. $\square$

Proposition 16.24 *Let T be simple. For any hyperimaginaries a, b, e such that $\mathrm{Lstp}(a/e) = \mathrm{Lstp}(b/e)$ and $a \mathop{\smile\!\!\!\!\!\!\;|}_e b$ there is some model M such that $a \equiv_M b$, $ab \mathop{\smile\!\!\!\!\!\!\;|}_e M$ and $e \in \mathrm{dcl}^{\mathrm{heq}}(M)$. Moreover $a, b \in I$ for some infinite M-indiscernible sequence I.*

Proof: Fix $n < \omega$, fix models M_i for $i \leq n$ and sequences a_i for $i \leq n+1$ such that $e \in \mathrm{dcl}^{\mathrm{heq}}(M_i)$ and

$$a = a_0 \equiv_{M_0} a_1 \equiv_{M_1} a_2 \equiv_{M_3} \ldots \equiv_{M_n} a_{n+1} = b$$

By Lemma 16.23 we may assume that $a_i a_{i+1} \mathop{\smile\!\!\!\!\!\!\;|}_e M_i$. Hence we can also assume that $M_i \mathop{\smile\!\!\!\!\!\!\;|}_e a_0, \ldots, a_{n+1}(M_j : j < i)$. It follows that $a_0, \ldots, a_{n+1} \mathop{\smile\!\!\!\!\!\!\;|}_e M_0, \ldots, M_n$. Note that

$a_0 \mathop{\smile\!\!\!\!\!\!\,|}_{M_0,\dots,M_n} a_{n+1}$. Let $b_0 = a_0$, let $b_{n+1} = a_{n+1}$ and for $1 \leq i \leq n$ choose $b_i \equiv_{M_0,\dots,M_n} a_i$ and such that $b_i \mathop{\smile\!\!\!\!\!\!\,|}_{M_0,\dots,M_n} b_{n+1}(b_j : j < i)$ for any $i \leq n$. It follows that $(b_j : j \leq n+1)$ is independent over $M_0, \dots, M_n$. Note that

$$a = b_0 \equiv_{M_0} b_1 \equiv_{M_1} b_2 \equiv_{M_3} \dots \equiv_{M_n} b_{n+1} = b$$

Since $b_i \mathop{\smile\!\!\!\!\!\!\,|}_e M_0, \dots, M_n$ the sequence $(b_j : j \leq n+1)$ is also independent over e. By changing the models M_i as we did before if necessary we may assume that $b_0, \dots, b_{n+1} \mathop{\smile\!\!\!\!\!\!\,|}_e M_0, \dots, M_n$. Independency of $(b_j : j \leq n+1)$ over $M_0, \dots, M_n$ still holds. Now we proceed by induction on n. The case $n = 0$ follows directly from Lemma 16.23. Using the inductive hypothesis it is enough now to consider the case $n = 1$.

We have $a = b_0 \equiv_{M_0} b_1 \equiv_{M_1} b_2 = b$. Let $f \in \mathrm{Aut}(\mathfrak{C}/M_0)$ such that $f(b_1) = b_0$. Then $M_1 \mathop{\smile\!\!\!\!\!\!\,|}_{M_0} b_1b_2$, $f(M_1) \mathop{\smile\!\!\!\!\!\!\,|}_{M_0} b_0$ and $b_0 \mathop{\smile\!\!\!\!\!\!\,|}_{M_0} b_1b_2$ and by the Independence Theorem (Proposition 16.17) there is a model N such that $N \mathop{\smile\!\!\!\!\!\!\,|}_{M_0} b_0b_1b_2$, $N \equiv_{M_0b_1b_2} M_1$ and $N \equiv_{M_0b_0} f(M_1)$. Then $b_0N \equiv b_0f(M_1) \equiv b_1M_1 \equiv b_2M_1 \equiv b_2N$ and clearly $b_0b_2 \mathop{\smile\!\!\!\!\!\!\,|}_e N$.

The last assertion follows from Proposition 10.11 since $a \mathop{\smile\!\!\!\!\!\!\,|}_M b$ and by Lemma 16.22 we can assume a, b are sequences of elements in $\mathfrak{C}$. □

Proposition 16.25 *If T is simple, then for any hyperimaginaries a, b, e the following are equivalent:*

1. $\mathrm{Lstp}(a/e) = \mathrm{Lstp}(b/e)$
2. *For some hyperimaginary c there are infinite e-indiscernible sequences I, J, such that $a, c \in I$ and $c, b \in J$.*
3. $a \equiv_{\mathrm{bdd}(e)} b$.

Proof: By Remark 16.19 it is clear that *2* implies *1* and that *1* implies *3*. To prove *2* from *1* find c such that $c \mathop{\smile\!\!\!\!\!\!\,|}_e ab$ and $\mathrm{Lstp}(a/e) = \mathrm{Lstp}(c/e)$ and then use Proposition 16.24. To find such c one needs to adapt the proof of Lemma 10.7, but this is straightforward. Finally we show that *1* follows from *3*. We may assume that a, b are sequences of elements in $\mathfrak{C}$. The condition in *2* can be expressed by a partial type $\Phi(x, y, d)$ over any representative d of e. Hence $\Phi(x, y, d)$ defines equality of Lascar strong type over e, a bounded equivalence relation E which is type-definable over e. Now a_E is an e-hyperimaginary and by Lemma 16.21 there is a hyperimaginary c such that $c \sim e, a_E$. Since a_E is e-bounded, $c \in \mathrm{bdd}(e)$. By *3*, there is some $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(e))$ such that $f(a) = b$. Since $f(c) = c$, $b_E = f(a_E) = a_E$ and hence $\mathrm{Lstp}(a/e) = \mathrm{Lstp}(b/e)$. □

Corollary 16.26 (Independence theorem for hyperimaginary Lascar strong types) *Let T be simple, let a, b, c, d, h be hyperimaginaries such that $a \mathop{\smile\!\!\!\!\!\!\,|}_h b$, $c \mathop{\smile\!\!\!\!\!\!\,|}_h a$, $d \mathop{\smile\!\!\!\!\!\!\,|}_h b$ and $c \equiv_{\mathrm{bdd}(h)} d$. Then there is some hyperimaginary $e \mathop{\smile\!\!\!\!\!\!\,|}_h ab$ such that $e \equiv_{\mathrm{bdd}(h)a} c$ and $e \equiv_{\mathrm{bdd}(h)b} d$.*

Proof: By Proposition 16.25, $\mathrm{Lstp}(c/h) = \mathrm{Lstp}(d/h)$ and then by Proposition 16.24 $c \equiv_M d$ for some model M such that $cd \mathop{\smile\!\!\!\!\!\!\,|}_h M$ and $h \in \mathrm{dcl}^{\mathrm{heq}}(M)$. We can assume that $M \mathop{\smile\!\!\!\!\!\!\,|}_{cdh} ab$ and therefore $M \mathop{\smile\!\!\!\!\!\!\,|}_h abcd$. It follows that $a \mathop{\smile\!\!\!\!\!\!\,|}_M b$, $c \mathop{\smile\!\!\!\!\!\!\,|}_M a$ and $d \mathop{\smile\!\!\!\!\!\!\,|}_M b$. By Proposition 16.17 there is some $e \mathop{\smile\!\!\!\!\!\!\,|}_M ab$ such that $e \equiv_{Ma} c$ and $e \mathop{\smile\!\!\!\!\!\!\,|}_{Mb} d$. Clearly $e \mathop{\smile\!\!\!\!\!\!\,|}_h ab$, $e \equiv_{\mathrm{bdd}(h)a} c$ and $e \equiv_{\mathrm{bdd}(h)b} d$. □

Chapter 17

Canonical bases revisited

Definition 17.1 *Let $p(x)$ be a complete type over the hyperimaginary h in the real variables x, that is, $p(x) = \mathrm{tp}(c/h)$ where h is a hyperimaginary and c is a sequence of elements of $\mathfrak{C}$. We say that $p(x)$ is an* amalgamation base *if the independence theorem is true for $p(x)$. That is, if for any hyperimaginaries a, b such that $a \mathop{\smile\!\!\!\!\!\!|}_h b$, for any c, d such that $\mathrm{tp}(c/h) = p(x) = \mathrm{tp}(d/h)$, $c \mathop{\smile\!\!\!\!\!\!|}_h a$, and $d \mathop{\smile\!\!\!\!\!\!|}_h b$, there is some $e \mathop{\smile\!\!\!\!\!\!|}_h ab$ such that $e \equiv_{ha} c$ and $e \equiv_{hb} d$. As shown in Proposition 16.17 and in Corollary 16.26, in a simple theory any type over a model and any type over a hyperimaginary of the form $\mathrm{bdd}(h)$ is an amalgamation base.*

If $p(x)$ is an amalgamation base, the amalgamation class *of p is the class $\mathcal{P}_p$ consisting in all amalgamation bases $q(x)$ such that for some $n < \omega$ there are amalgamation bases $(p_i(x) : i \leq n)$ such that $p = p_0$, $q = p_n$ and for every $i < n$, p_i and p_{i+1} have a common nonforking extension. Note that $\mathcal{P}_p = \mathcal{P}_q$ if $q \in \mathcal{P}_p$.*

Remark 17.2 *1. In a simple theory, any stationary type is an amalgamation base.*

2. In a stable theory, any amalgamation base is a stationary type.

Proof: *1.* Given a, b, c, d, h and a stationary type $p(x)$ over h as in the definition of amalgamation base, any realization e of the unique nonforking extension of p over h, a, b satisfies the requirements.

2. Let T be stable, let $p(x)$ a complete type over the hyperimaginary h and let us assume p is an amalgamation base. If p is not stationary, then it has two different nonforking extensions over ah for some hyperimaginary a. We can assume that a enumerates a model M such that $h \in \mathrm{dcl}^{\mathrm{heq}}(M)$. Then p has two different nonforking extensions p_1, p_2 over M. Choose now a model N such that $h \in \mathrm{dcl}^{\mathrm{heq}}(N)$ and $N \mathop{\smile\!\!\!\!\!\!|}_h M$. By stability, p_1 is stationary. Let $q_1(x) \in S(N)$ be the unique type over N which is parallel to p_1. Since p is an amalgamation base, we can amalgamate q_1 and p_2, that is, for some $e \mathop{\smile\!\!\!\!\!\!|}_h MN$, $\mathrm{tp}(e/N) = q_1$ and $\mathrm{tp}(e/M) = p_2$. But $\mathrm{tp}(e/M)$ is parallel to $\mathrm{tp}(e/N)$ and hence p_2, p_1 are parallel and they must be the same type over M. □

Proposition 17.3 *Let T be simple, let A be a set of parameters in $\mathfrak{C}$ and $p(x) \in S(A)$. If p is finitely satisfiable in A, then it is an amalgamation base.*

Proof: We prove that $\mathrm{tp}(a/A) \vdash \mathrm{tp}(a/\mathrm{bdd}(A))$. The rest follows from Corollary 16.26. Assume $a \equiv_A b$. By Proposition 15.16 it is enough to check that $a \overset{\mathrm{bdd}}{\equiv}_A b$. Let E be a type-definable over A bounded equivalence relation and let us prove that $E(a,b)$. Consider some $\varphi(x,y) \in E(x,y)$ and choose some thick formula $\psi(x,y) \in E(x,y)$ such that $\models \psi(x,y) \wedge \psi(y,z) \rightarrow \varphi(x,z)$. Since $A \neq \emptyset$ we can choose a maximal sequence $a_0,\ldots,a_n$ in A such that $\neg\psi(a_i,a_j)$ for all $i < j \leq n$. We claim that $\models \psi(a,a_i)$ for some $i \leq n$. Otherwise $\bigwedge_{i\leq n} \neg\psi(x,a_i) \in \mathrm{tp}(a/A)$ and by finite satisfiability we find some $a' \in A$ realizing this formula, contradicting the maximality of $a_0,\ldots,a_n$. Similarly, $\models \psi(b,a_j)$ for some $j \leq n$. Since $a \equiv_A b$, $i = j$. Then $\models \psi(a,a_i) \wedge \psi(a_i,b)$ and by choice of ψ, $\models \varphi(a,b)$. $\square$

Corollary 17.4 *Let T be simple. If $(a_i : i \leq \omega)$ is indiscernible, then $\mathrm{tp}(a_\omega/(a_i : i < \omega))$ is an amalgamation base.*

Proof: It is a consequence of Proposition 17.3 since $\mathrm{tp}(a_\omega/(a_i : i < \omega))$ is finitely satisfiable in $\{a_i : i < \omega\}$. $\square$

Lemma 17.5 *Let T be simple, let R be a type-definable over $\emptyset$ symmetric and reflexive relation on sequences of a given length. Assume that whenever $R(a,b)$, $R(a,c)$, and $b \mathop{\smile\!\!\!\!\!\!\;|}_a c$, then $R(b,c)$. Then the transitive closure of R is the two step composition $R^2 = R \circ R$. Therefore the transitive closure of R is type-definable over $\emptyset$. In fact, for any sequences a,b the following are equivalent:*

1. *For some $n < \omega$ there are $a_0,\ldots,a_n$ such that $a = a_0$, $b = a_n$ and for each $i < n$, $R(a_i,a_{i+1})$.*
2. *There is some c such that $R(a,c)$, $R(b,c)$, $a \mathop{\smile\!\!\!\!\!\!\;|}_b c$, and $b \mathop{\smile\!\!\!\!\!\!\;|}_a c$.*
3. *There is some c such that $R(a,c)$ and $R(b,c)$.*

Proof: Let us fix an equivalence class C of the transitive closure of R. We will restrict our attention to sequences $a,b \in C$. Choose a complete type $p(x) = \mathrm{tp}(a)$ of some $a \in C$. Let $\kappa = |T| + \mathrm{length}(x)$ and fix an enumeration $((\varphi_i,k_i) : i < \kappa)$ of all pairs consisting in some formula $\varphi(x,y) \in L$ and some natural number $k < \omega$. We inductively define the sequence $(n_i : i < \kappa)$: n_i is the greatest $n < \omega$ for which there are $a,b \in C$ such that $a \models p$, $R(a,b)$, $D(\mathrm{tp}(b/a),\varphi_i,k_i) = n$ and for each $j < i$, $D(\mathrm{tp}(b/a),\varphi_j,k_j) \geq n_j$. By compactness, there are $a,b \in C$ such that $a \models p$ and $D(\mathrm{tp}(b/a),\varphi_i,k_i) = n_i$ for all $i < \kappa$.

Claim 1. If $p'(x) = \mathrm{tp}(a')$ for some other $a' \in C$ and we define a similar sequence $(n'_i : i < \kappa)$ for p', then $n_i = n'_i$ for all $i < \kappa$

Proof of the claim: by induction on the length of an R-path in C connecting a realization of p' with a realization of p. Assume a' is connected to a realization of p by an R-path of length $n+1$, and hence $R(a',a'')$ for some a'' connected to a realization of p by an R-path of length n. By the inductive hypothesis we may choose some b such that $R(a'',b)$ and $D(\mathrm{tp}(b/a''),\varphi_i,k_i) = n_i$ for all $i < \kappa$. We may assume that $b \mathop{\smile\!\!\!\!\!\!\;|}_{a''} a'$, and hence the hypothesis of our Lemma gives $R(a',b)$. Then, for all $i < \kappa$, $n'_i \geq D(\mathrm{tp}(b/a'),\varphi_i,k_i) \geq D(\mathrm{tp}(b/a''a'),\varphi_i,k_i) = D(\mathrm{tp}(b/a''),\varphi_i,k_i) = n_i$. Since the situation is completely symmetric, $n_i \geq n'_i$ for all $i < \kappa$. $\square_{\mathit{claim\ 1}}$

We need only to prove *1* $\Rightarrow$ *2*, and it is enough to do it for sequences in C. This can be done by induction on the length n of an R-path joining a and a'. The starting case is clear, and for the case $n+1$ we will need the following

Claim 2. Assume a_1,a_2,a_3,a_4 are an R-path in C, $R(a_2,b)$ and $D(\mathrm{tp}(b/a_2),\varphi_i,k_i) \geq n_i$ for all $i < \kappa$. If $a_2 \mathop{\smile\!\!\!\!\!\!\;|}_{a_3} a_1$, $a_2 \mathop{\smile\!\!\!\!\!\!\;|}_{a_1} a_3$, $a_2 \mathop{\smile\!\!\!\!\!\!\;|}_{a_1a_3} a_4$, and $b \mathop{\smile\!\!\!\!\!\!\;|}_{a_2} a_1a_3a_4$ then $R(a_j,b)$ and $b \mathop{\smile\!\!\!\!\!\!\;|}_{a_j} a_1a_2a_3a_4$ for all j.

Proof of the claim: by the hypothesis of the Lemma we easily see that $R(a_j, b)$ for $j = 1, 2, 3$. Since $a_2 \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_3} a_4$, we see that $R(a_2, a_4)$ and since moreover $b \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_2} a_4$ we conclude that also $R(a_4, b)$. We show by induction on $i < \kappa$ that $n_i = D(\mathrm{tp}(b/a_j), \varphi_i, k_i) = D(\mathrm{tp}(b/a_1a_2a_3a_4), \varphi_i, k_i)$. By the inductive hypothesis and by construction of the sequence $(n_i : i < \kappa)$ we see that $n_i \geq D(\mathrm{tp}(b/a_j), \varphi_i, k_i)$. But since $b \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_2} a_1a_3a_4$

$$D(\mathrm{tp}(b/a_j), \varphi_i, k_i) \geq D(\mathrm{tp}(b/a_1a_2a_3a_4), \varphi_i, k_i) = D(\mathrm{tp}(b/a_2), \varphi_i, k_i) \geq n_i.$$

Hence $b \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_j} a_1a_2a_3a_4$. $\square_{claim\ 2}$

We continue with the case $n + 1$. Assume there is an R-path of length $n + 1$ joining a_1 and a_4. For some a_3, $R(a_3, a_4)$ and there is an R-path of length n joining a_1 and a_3. By inductive hypothesis, there is some a_2 such that $R(a_1, a_2)$, $R(a_3, a_2)$, $a_2 \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_3} a_1$, and $a_2 \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_1} a_3$. We may clearly assume that $a_2 \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_1a_3} a_4$. If we now choose b such that $R(a_2, b)$, $D(\mathrm{tp}(b/a_2), \varphi_i, k_i) \geq n_i$ for all $i < \kappa$ and $b \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_2} a_1a_3a_4$, by *claim 2* we get $R(a_1, b)$, $R(a_4, b)$, $b \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_1} a_4$, and $b \mathop{\smile\!\!\!\!\!\!\!\;|}_{a_4} a_1$. $\square$

Notation 17.6 *We say $p(x, a)$ is an amalgamation base to mean that a is a sequence of elements of $\mathfrak{C}$, $p(x, y)$ is a complete type over $\emptyset$ implying $q(y) = \mathrm{tp}(a/\emptyset)$ and $p(x, a) \in S(a)$ is an amalgamation base. Note that in a simple theory if $q(x)$ is an arbitrary amalgamation base there is always some amalgamation base $p(x, a)$ (where a enumerates a model) with the same amalgamation class $\mathcal{P}_{p(x,a)} = \mathcal{P}_{q(x)}$.*

Lemma 17.7 *Le T be simple and let $p(x, a)$ be an amalgamation base. If $q(x, b) \in \mathcal{P}_{p(x,a)}$ is another amalgamation base, then for some $n < \omega$ there are $a_0, \ldots, a_n$ realizing $\mathrm{tp}(a)$ such that $a = a_0$, and $p(x, a_i)$, $p(x, a_{i+1})$ have a common nonforking extension for all $i < n$ and also $p(x, a_n)$ and $q(x, b)$ have a common nonforking extension.*

Proof: It is enough to prove that for any amalgamation bases $q(x, c)$, $r(x, d)$, if $p(x, a)$ and $q(x, c)$ have a common nonforking extension and also $q(x, c)$ and $r(x, d)$ have a common nonforking extension, then for some $b \equiv a$, $p(x, a)$ and $p(x, b)$ have a common nonforking extension and also $p(x, b)$ and $r(x, d)$ have a common nonforking extension. To check this, let us choose $b \equiv_c a$ such that $b \mathop{\smile\!\!\!\!\!\!\!\;|}_c ad$. Since $b \equiv_c a$, it is clear that $p(x, b)$ and $q(x, c)$ have a common nonforking extension $s_1(x, b, c)$. Now let $s_2(x, c, d)$ a common nonforking extension of $q(x, c)$ and $r(x, d)$. Since $b \mathop{\smile\!\!\!\!\!\!\!\;|}_c d$ and $q(x, c)$ is an amalgamation base, $s_1(x, b, c)$ and $s_2(x, c, d)$ have a common extension $s(x, b, c, d)$ which does not fork over c. Then $s(x, b, c, d)$ is a common nonforking extension of $r(x, d)$ and $p(x, b)$. We finish the proof showing that also $p(x, a)$ and $p(x, b)$ have a common nonforking extension. On the one hand $s_1(x, b, c)$ is a common nonforking extension of $p(x, b)$ and $q(x, c)$. On the other hand $q(x, c)$ and $p(x, a)$ have a common nonforking extension $s_3(x, a, c)$. Since $a \mathop{\smile\!\!\!\!\!\!\!\;|}_c b$, we see that $s_1(x, b, c)$ y $s_3(x, a, c)$ have a common extension $s'(x, a, b, c)$ which does not fork over c. Clearly it is a common nonforking extension of $p(x, a)$ and $p(x, b)$. $\square$

Definition 17.8 *Let $p(x)$ be an amalgamation base and let $\mathcal{P}_p$ be its amalgamation class. A* canonical base *of p is a hyperimaginary e such that for any $f \in \mathrm{Aut}(\mathfrak{C})$, $f(e) = e$ if and only if f fixes setwise $\mathcal{P}_p$. Clearly, if e' is another canonical base of p then $e \sim e'$. Note that $e \in \mathrm{dcl}^{\mathrm{heq}}(a)$ if $p(x)$ is a type over a. Moreover if $q(x) \in \mathcal{P}_p$, then e is also a canonical base of $q(x)$.*

Theorem 17.9 *In a simple theory, any amalgamation base has a canonical base.*

Proof: As remarked above, it is enough to consider an amalgamation base of the form $p(x,a)$ (that is, where a is a sequence in $\mathfrak{C}$). Consider the binary relation R on realizations of $\mathrm{tp}(a)$ defined by: $R(a_1,a_2)$ iff $p(x,a_1)$ and $p(x,a_2)$ have a common nonforking extension. It is reflexive and symmetric. For each $\varphi(x,y) \in L$, for each $k < \omega$ let $n_{\varphi,k} = D(p(x,a),\varphi,k)$. It is then easy to see that R is type-definable by the partial type (over $\emptyset$) expressing that a_1, a_2 realize $\mathrm{tp}(a)$ and for all $\varphi \in L$, for all $k < \omega$, $D(p(x,a_1) \cup p(x,a_2), \varphi, k) \geq n_{\varphi,k}$.

It is easy to check that R satisfies the other conditions of Lemma 17.5 and therefore its transitive closure E is also type-definable. Note that E is an equivalence relation on realizations of $\mathrm{tp}(a)$ and by Lemma 17.7 $E(a,b)$ holds if and only if $p(x,b) \in \mathcal{P}_{p(x,a)}$. By Lemma 15.5 we can extend E to a 0-type-definable equivalence relation on all sequences of the length of a. Hence we can consider the hyperimaginary $e = a_E$. It is clear that e is a canonical base of $p(x,a)$. $\square$

Lemma 17.10 *Let T be simple, let $p(x,a)$ be an amalgamation base and let e be a canonical base of $p(x,a)$. If $q(x,b) \in \mathcal{P}_{p(x,a)}$ and $a \mathop{\smile\!\!\!\!\!\!|}_e b$ then $p(x,a)$ and $q(x,b)$ have a common nonforking extension.*

Proof: Fix amalgamation bases $p_0(x,a_0),\ldots,p_n(x,a_n)$ such that $p_0(x,a_0) = p(x,a)$, $p_n(x,a_n) = q(x,b)$ and for all $i < n$, $p_i(x,a_i)$ and $p_{i+1}(x,a_{i+1})$ have a common nonforking extension. We may assume that all the sequences a_i are of the same length. We may apply 17.5 to the relation R defined by $R(b_1,b_2)$ iff there are $i,j \leq n$ such that $b_1 \equiv a_i$, $b_2 \equiv a_j$ and $p_i(x,b_1)$, $p_j(x,b_2)$ have a common nonforking extension. Hence there is an amalgamation base $r(x,c)$ such that $p(x,a), r(x,c)$ have a common nonforking extension, $r(x,c), q(x,b)$ have a common nonforking extension, $c \mathop{\smile\!\!\!\!\!\!|}_a b$, and $c \mathop{\smile\!\!\!\!\!\!|}_b a$. Since $a \mathop{\smile\!\!\!\!\!\!|}_e b$ and $e \in \mathrm{dcl}^{\mathrm{heq}}(b) \cap \mathrm{dcl}^{\mathrm{heq}}(c)$ from this it follows that $a \mathop{\smile\!\!\!\!\!\!|}_c b$. By amalgamating these types we conclude that $p(x,a)$ y $q(x,b)$ have a common nonforking extension. $\square$

Notation 17.11 *If $p(x)$ is a complete type over a hyperimaginary a and $e \in \mathrm{dcl}^{\mathrm{heq}}(a)$, then by $p(x) \restriction e$ we refer to the type $\mathrm{tp}(b/e)$ where b is an arbitrary realization of p. Note that if $q(x)$ is another complete type and e is also definable over its domain, then the consistency of $p(x) \cup q(x)$ implies $p \restriction e = q \restriction e$.*

Theorem 17.12 *Let T be simple, let $p(x,a)$ be an amalgamation base, and let e be a canonical base of $p(x,a)$. Then*

1. *Any $q(x) \in \mathcal{P}_{p(x,a)}$ is a nonforking extension of $p(x,a) \restriction e$.*
2. *$p(x,a) \restriction e \in \mathcal{P}_{p(x,a)}$.*
3. *If $q(x,b)$ is an amalgamation base and $p(x,a)$, $q(x,b)$ have a common nonforking extension, then $e \in \mathrm{dcl}^{\mathrm{heq}}(b)$.*
4. *If $p(x,a)$ and $q(x,b) \in S(b)$ have a common nonforking extension, then $e \in \mathrm{bdd}(b)$.*

Proof: *1.* We first show that $p(x,a)$ does not fork over e. Choose $b \equiv_e a$ such that $b \mathop{\smile\!\!\!\!\!\!|}_e a$. Then $p(x,b) \in \mathcal{P}_{p(x,a)}$ and by Lemma 17.10 $p(x,a)$ and $p(x,b)$ have a common nonforking extension. Hence, for some $c \models p(x,a) \cup p(x,b)$, $c \mathop{\smile\!\!\!\!\!\!|}_a b$ and $c \mathop{\smile\!\!\!\!\!\!|}_b a$. Since $e \in \mathrm{dcl}^{\mathrm{heq}}(b)$, by transitivity $c \mathop{\smile\!\!\!\!\!\!|}_e a$ and thus $p(x,a)$ does not fork over e. Clearly this implies that any other $q(x,b) \in \mathcal{P}_{p(x,a)}$ (where b is a sequence in $\mathfrak{C}$) does not fork either over e. But we need also to consider the case of a type $q(x) \in \mathcal{P}_{p(x,a)}$ over a hyperimaginary h. Choose a sequence m enumerating a model such that $h \in \mathrm{dcl}^{\mathrm{heq}}(m)$ and let $r(x,m) \in \mathcal{P}_{p(x,a)}$ be a nonforking extension of $q(x)$. Then $r(x,m)$ does not fork over e and hence $q(x)$ does not fork over e.

2. Let $p_0(x) = p(x,a) \restriction e$ and $\mathcal{P} = \mathcal{P}_{p(x,a)}$. Choose $q(x,m) \in \mathcal{P}$ such that m enumerates a model and $e \in \mathrm{dcl}^{\mathrm{heq}}(m)$. Then $p_0 = q(x,m) \restriction e$. Since $\mathrm{bdd}(e) \subseteq \mathrm{dcl}^{\mathrm{heq}}(m)$ and $q(x,m) \restriction \mathrm{bdd}(e)$ is an amalgamation base, by *1* $q(x,m) \restriction \mathrm{bdd}(e) \in \mathcal{P}$. Therefore some extension p_0' of p_0 over $\mathrm{bdd}(e)$ belongs to $\mathcal{P}$. We will see now that this implies that all extensions of p_0 over $\mathrm{bdd}(e)$ belong to $\mathcal{P}$. Let $p_0''(x) = \mathrm{tp}(c/\mathrm{bdd}(e))$ be any such extension and choose b such that $p_0' = \mathrm{tp}(b/\mathrm{bdd}(e))$. Then $f(b) = c$ for some $f \in \mathrm{Aut}(\mathfrak{C}/e)$. Clearly, f fixes setwise $\mathrm{bdd}(e)$ and $(p_0')^f = p_0''$. Since $f(e) = e$, f fixes setwise $\mathcal{P}$, and thus $p_0'' \in \mathcal{P}$, as we wanted to show.

Next we check that p_0 is an amalgamation base. By *1* from this it will follow that $p_0 \in \mathcal{P}$. Let m, n enumerate models such that $e \in \mathrm{dcl}^{\mathrm{heq}}(m) \cap \mathrm{dcl}^{\mathrm{heq}}(n)$ and $m \mathop{\smile\!\!\!\!\!\mid}_e n$. Assume $q_1(x,m)$, $q_2(x,n)$ are nonforking extensions of p_0. Note that $\mathrm{bdd}(e) \subseteq \mathrm{dcl}^{\mathrm{heq}}(m)$ and hence $q_1(x,m)$ extends $q_1(x,m) \restriction \mathrm{bdd}(e)$, which is an extension of p_0 and therefore an element of $\mathcal{P}$. This implies $q_1(x,m) \in \mathcal{P}$. By similar reasons $q_2(x,n) \in \mathcal{P}$. By Lemma 17.10 $q_1(x,m), q_2(x,n)$ have a common nonforking extension which, by transitivity, does not fork over e.

3 is clear since $q(x,b) \in \mathcal{P}_{p(x,a)}$, and *4* follows from *3* because some extension of $q(x,b)$ over $\mathrm{bdd}(b)$ belongs to $\mathcal{P}_{p(x,a)}$. □

Definition 17.13 *If $p(x)$ is an amalgamation base in a simple theory,* $\mathrm{Cb}(p)$ *is, by definition,* $\mathrm{dcl}^{\mathrm{heq}}(e)$ *where e is a canonical base of p. The* canonical type *of $\mathcal{P}_p$ will be $p_0 = p \restriction e$. Notice that $p_0 \in \mathcal{P}_p$ and that $\mathcal{P}_p$ is precisely the set of all nonforking extensions of p_0. For any a, b we define* $\mathrm{Cb}(a/b) = \mathrm{Cb}(\mathrm{Lstp}(a/b))$. *This notation agrees with the one introduced for canonical bases of stationary types in stable theories.*

Lemma 17.14 *For simple T the following are equivalent:*

1. $a \mathop{\smile\!\!\!\!\!\mid}_b c$
2. $\mathrm{Cb}(a/bc) = \mathrm{Cb}(a/b)$
3. $\mathrm{Cb}(a/bc) \subseteq \mathrm{bdd}(b)$

Proof: *1* ⇒ *2*. If $a \mathop{\smile\!\!\!\!\!\mid}_b c$, then $\mathrm{Lstp}(a/bc)$ and $\mathrm{Lstp}(a/b)$ belong to the same amalgamation class.

2 ⇒ *3*. Since $\mathrm{Cb}(a/b) \subseteq \mathrm{bdd}(b)$.

3 ⇒ *1*. This follows from the fact that $a \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{Cb}(a/bc)} bc$. □

Proposition 17.15 *Let T be simple. If $p(x) \in S(A)$ is an amalgamation base and $(a_i : i < \omega)$ is a Morley sequence in p, then* $\mathrm{Cb}(p) \subseteq \mathrm{dcl}^{\mathrm{heq}}(a_i : i < \omega)$. *Moreover, if T is supersimple, then* $\mathrm{Cb}(p) \subseteq \mathrm{bdd}(a_i : i < n)$ *for some $n < \omega$.*

Proof: Extend the Morley sequence $(a_i : i < \omega)$ to a Morley sequence $(a_i : i \leq \omega)$ in p. By Corollary 17.4 $\mathrm{tp}(a_\omega/(a_i : i < \omega)$ is an amalgamation base. Since $\mathrm{tp}(a_\omega/A(a_i : i < \omega))$ is finitely satisfiable in $\{a_i : i < \omega\}$ by Remark 4.4 $a_\omega \mathop{\smile\!\!\!\!\!\mid}_{(a_i:i<\omega)} A$. Since $a_\omega \mathop{\smile\!\!\!\!\!\mid}_A (a_i : i < \omega)$ and $a_\omega \mathop{\smile\!\!\!\!\!\mid}_{(a_i:i<\omega)} A$, $p = \mathrm{tp}(a_\omega/A)$ and $\mathrm{tp}(a_\omega/(a_i : i < \omega))$ have the same amalgamation class. Therefore by Theorem 17.12 $\mathrm{Cb}(p) = \mathrm{Cb}(\mathrm{tp}(a_\omega/(a_i : i < \omega)) \subseteq \mathrm{dcl}^{\mathrm{heq}}(a_i : i < \omega)$.

Assume now T is supersimple. Choose $n < \omega$ such that $a_\omega \mathop{\smile\!\!\!\!\!\mid}_{(a_i:i<n)} (a_i : i < \omega)$. Then $a_\omega \mathop{\smile\!\!\!\!\!\mid}_{(a_i:i<n)} A$ and therefore $\mathrm{Cb}(p) \subseteq \mathrm{bdd}(a_i : i < n)$. □

Chapter 18

Elimination of hyperimaginaries

Definition 18.1 T eliminates a hyperimaginary e *if there is a sequence of imaginaries* $(e_i : i \in I)$ *such that* $e \sim (e_i : i \in I)$. T eliminates hyperimaginaries *if* T *eliminates every hypermaginary.*

Proposition 18.2 *Let* $e = a_E$ *a hyperimaginary and let* $p(x) = \mathrm{tp}(a)$. *Then* T *eliminates* e *if and only if there is a family* $(E_i : i \in I)$ *of* 0*-definable equivalence relations such that* $E \upharpoonright p = (\bigcap_{i \in I} E_i) \upharpoonright p$. *In fact it suffices to require that the* E_i *are* 0*-definable relations whose restrictions* $E_i \upharpoonright p$ *to* $p(\mathfrak{C})$ *are equivalence relations.*

Proof: It is enough to require that the E_i are equivalence relations on $p(\mathfrak{C})$ since, by compactness, it is always possible to find a formula $\varphi_i(x) \in p$ such that E_i is an equivalence relation on $\varphi_i(\mathfrak{C})$.

Assume $e = a_E$ is a hyperimaginary and choose a family $(E_i : i \in I)$ of 0-definable equivalence relations such that on $p(x) = \mathrm{tp}(a)$ the equivalence relation E agrees with $\bigcap_{i \in I} E_i$. Then $e \sim (e_i : i \in I)$ where $e_i = a_{iE_i}$, if a_i is the subtuple of a corresponding to the variables of E_i. For the other direction, by assumption there is a sequence $(e_i : i \in I)$ of imaginaries e_i such that $e \sim (e_i : i \in I)$. Let $p_i(x, y) = \mathrm{tp}(aa_i)$ for each $i \in I$. Then:

1. $E(x, x') \cup p_i(x, y) \cup p_i(x', y') \vdash E_i(y, y')$
2. $p_i(a, a_i)$
3. $p(x) \vdash \exists y p_i(x, y)$

By compactness we can substitute a single formula $\varphi_i(x, y) \in p_i$ for $p_i(x, y)$ and still have these properties. We then define

$$F_i(y, z) \;\Leftrightarrow\; \exists uv(E_i(u, v) \wedge \varphi_i(y, u) \wedge \varphi_i(z, v)).$$

Clearly F_i is definable over $\emptyset$, $F_i \upharpoonright p$ is an equivalence relation, and $E \upharpoonright p = (\bigcap_{i \in I} F_i) \upharpoonright p$. $\square$

Corollary 18.3 *T eliminates hyperimaginaries if and only if for any* $p(x) \in S(\emptyset)$ *for any* 0*-type-definable equivalence relation on* $p(\mathfrak{C})$ *there is a family* $(E_i : i \in I)$ *of* 0*-definable equivalence relations such that* $E = (\bigcap_{i \in I} E_i) \upharpoonright p$. *In fact it suffices to require that the* E_i *are* 0*-definable relations whose restrictions* $E_i \upharpoonright p$ *to* $p(\mathfrak{C})$ *are equivalence relations.*

Proof: By Proposition 18.2. □

Lemma 18.4 *Let E be an intersection of definable (possibly with parameters) equivalence relations. If E is type-definable over $\emptyset$, then E is an intersection of* 0*-definable equivalence relations.*

Proof: Let $E = \bigcap_{i \in I} E_i$ where every E_i is an equivalence relation, defined by $\varphi_i(x, y, a_i)$, with $\varphi_i(x, y, z) \in L$. Assume $\Sigma(x, y)$ is a type over $\emptyset$ defining E and let $p_i(z) = \operatorname{tp}(a_i)$. Then $\Sigma(x, y) \cup p_i(z) \vdash \varphi_i(x, y, z)$, and thus $\Sigma(x, y) \vdash \forall z\, (\psi_i(z) \rightarrow \varphi_i(x, y, z))$ for some $\psi_i(z) \in p_i$. We can choose it so that $\psi_i(z)$ implies $\varphi_i(x, y, z)$ is an equivalence relation in x, y. Then

$$\forall z\, (\psi_i(z) \rightarrow \varphi_i(x, y, z))$$

defines (over $\emptyset$) an equivalence relation F_i such that $E \subseteq F_i \subseteq E_i$. Hence $E = \bigcap_{i \in I} F_i$. □

Proposition 18.5 *If T eliminates hyperimaginaries, then also $T(A)$ eliminates hyperimaginaries.*

Proof: By Lemma 15.15. □

Lemma 18.6 *If a is a sequence of imaginaries, $e \in \operatorname{dcl}^{\mathrm{heq}}(a)$ is a hyperimaginary and $a \in \operatorname{bdd}(e)$, then $e \sim b$ for some sequence b of imaginaries.*

Proof: Let $a = (a_i : i \in I)$ where every a_i is a imaginary. For each finite $J \subseteq I$ let $a_J = (a_i : i \in J)$. Then $a_J \in \operatorname{acl}^{\mathrm{eq}}(e)$. Consider the finite set $b_J = \{f(a_J) : f \in \operatorname{Aut}(\mathfrak{C}/e)\}$ as a single imaginary. This means that $b_J \in \mathfrak{C}^{\mathrm{eq}}$ and $f(b_J) = b_J$ if and only if f permutes the orbit of a_J in $\operatorname{Aut}(\mathfrak{C}/e)$. Now let $b = (b_J : J \subseteq I \text{ is finite })$. It is clear that $b \in \operatorname{dcl}^{\mathrm{heq}}(e)$. We check now that $e \in \operatorname{dcl}^{\mathrm{heq}}(b)$. Assume f fixes b. Then for each finite J, $a_J \equiv_e f(a_J)$ and therefore $a \equiv_e f(a)$. Hence $f(a)e \equiv ae \equiv f(a)f(e)$. Since $e \in \operatorname{dcl}^{\mathrm{heq}}(a)$, also $f(e) \in \operatorname{dcl}^{\mathrm{heq}}(f(a))$ and hence $e = f(e)$. □

Lemma 18.7 *Let T be simple. If $\operatorname{tp}(a/e)$ is an amalgamation base, then $\operatorname{tp}(a/e) \equiv \operatorname{Lstp}(a/e)$.*

Proof: Assume $a' \equiv_e a$. Then $a \mathop{\smile\!\!\!\!\!\!\mid}_e \operatorname{bdd}(e)$, $a' \mathop{\smile\!\!\!\!\!\!\mid}_e \operatorname{bdd}(e)$, and $\operatorname{bdd}(e) \mathop{\smile\!\!\!\!\!\!\mid}_e \operatorname{bdd}(e)$. By definition of amalgamation base, for some $b \mathop{\smile\!\!\!\!\!\!\mid}_e \operatorname{bdd}(e)$, $b \equiv_{\operatorname{bdd}(e)} a$, and $b \equiv_{\operatorname{bdd}(e)} a'$. Therefore $a \equiv_{\operatorname{bdd}(e)} a'$, that is, $\operatorname{Lstp}(a/e) = \operatorname{Lstp}(a'/e)$. □

Proposition 18.8 *Let T be simple. If $e = a_E$ is a hyperimaginary, then $e \in \operatorname{dcl}^{\mathrm{heq}}(\operatorname{Cb}(a/e))$.*

Proof: We first show that $e \in \operatorname{bdd}(\operatorname{Cb}(a/e))$. By Theorem 17.12, $a \mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{Cb}(a/e)} e$ and since $e \in \operatorname{dcl}^{\mathrm{heq}}(a)$, $e \mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{Cb}(a/e)} e$. Therefore $e \in \operatorname{bdd}(\operatorname{Cb}(a/e))$. Since $\operatorname{tp}(a/\operatorname{Cb}(a/e))$ is an amalgamation base, by Lemma 18.7 $\operatorname{tp}(a/\operatorname{Cb}(a/e)) \equiv \operatorname{tp}(a/\operatorname{bdd}(\operatorname{Cb}(a/e))) \vdash \operatorname{tp}(a/e)$. Let $f \in \operatorname{Aut}(\mathfrak{C}/\operatorname{Cb}(a/e))$. Then $a \equiv_{\operatorname{Cb}(a/e)} f(a)$ and therefore $a \equiv_e f(a)$. It follows that $E(a, f(a))$ and hence $f(e) = e$. □

Proposition 18.9 *Let T be simple. If for each amalgamation base $p(x)$, the canonical base $\operatorname{Cb}(p)$ is equivalent to a sequence of imaginaries, then T eliminates hyperimaginaries.*

Proof: Let $e = a_E$ be an hyperimaginary. By assumption there is a sequence d of imaginaries such that $\operatorname{Cb}(a/e) \sim d$. Then $d \in \operatorname{bdd}(e)$. By Proposition 18.8

$$e \in \operatorname{dcl}^{\mathrm{heq}}(\operatorname{Cb}(a/e)) \subseteq \operatorname{dcl}^{\mathrm{heq}}(d).$$

By Lemma 18.6 $e \sim d'$ for some sequence of imaginaries d'. □

Corollary 18.10 *Stable theories eliminate hyperimaginaries.*

Proof: By Proposition 18.9 and by the fact that canonical bases in stable theories are sequences of imaginaries. □

Proposition 18.11 *Let E be a 0-type-definable equivalence relation and let E^* be the equivalence relation given by*

$$E^*(a,b) \Leftrightarrow E(a',b') \text{ for some } a' \equiv a,\ b' \equiv b$$

Then E is an intersection of 0-definable equivalence relations if and only if E^* is an intersection of 0-definable equivalence relations and for each $p(x) \in S(\emptyset)$, $E \restriction p = E \cap (p(\mathfrak{C}) \times p(\mathfrak{C}))$ is an intersection of 0-definable equivalence relations.

Proof: Note that the two following conditions are equivalent to $E^*(a,b)$:

1. $E(a,c)$ for some $c \equiv b$.
2. $E(a,a')$ and $E(b,b')$ for some $a' \equiv b'$.

If $E(a,b)$ is witnessed by $E(a',b')$ where $a' \equiv a$ and $b' \equiv b$, and we choose c such that $ac \equiv a'b'$ then c witness that *1* holds. For *1* $\Rightarrow$ *2* just take $a' = c$ and $b' = b$. Finally if a', b' are as in *2* and we choose c, d such that $a'ad \equiv b'cb$, then $E(c,b)$, $a \equiv c$ and $b \equiv b$.

Now assume $E = \bigcap_{i\in I} E_i$ for 0-definable equivalence relations E_i. Then obviously for each $p \in S(\emptyset)$, E agrees with $\bigcap_{i\in I} E_i$ on p. Moreover $E^* = \bigcap_{i\in I,\varphi\in L} E_{i\varphi}$ where $E_{i\varphi}(x,y)$ is the equivalence relation defined by

$$\exists z(\varphi(z) \wedge E_i(x,z)) \leftrightarrow \exists z(\varphi(z) \wedge E_i(y,z))$$

For the other direction, assume E^* is an intersection of 0-definable equivalence relations and for each $p(x) \in S(\emptyset)$, $E \restriction p = \bigcap_{i\in I_p} E_{ip} \restriction p$ for a family of 0-definable equivalence relations E_{ip}. We can assume that the type $E(x,y)$ defining the equivalence relation E is made of reflexive and symmetric formulas. Fix some $p(x) \in S(\emptyset)$ and choose some $a \models p$. For each $i \in I_p$ we can find some formula $\sigma_{ip}(x,y) \in E(x,y)$ and some $\psi_{ip}(x) \in p$ such that

$$\sigma_{ip}(x,y) \wedge \psi_{ip}(x) \wedge \psi_{ip}(y) \vdash E_{ip}(x,y)$$

We can also find some $\overline{\sigma}_{ip}(x,y) \in E(x,y)$ such that

$$\overline{\sigma}_{ip}(x,y) \wedge \overline{\sigma}_{ip}(y,z) \wedge \overline{\sigma}_{ip}(z,u) \vdash \sigma_{ip}(x,u)$$

and some 0-definable equivalence relation E^*_{ip} in the family whose intersection is E^* such that

$$E^*_{ip}(x,a) \vdash \exists y(\psi_{ip}(y) \wedge \overline{\sigma}_{ip}(x,y))$$

Consider the relation $F_{ip}(x,y)$ defined by the disjunction of $(\neg E^*_{ip}(x,a) \wedge \neg E^*_{ip}(y,a))$ with

$$(E^*_{ip}(x,a) \wedge E^*_{ip}(y,a) \wedge \exists uv(\psi_{ip}(u) \wedge \psi_{ip}(v) \wedge \overline{\sigma}_{ip}(x,u) \wedge \overline{\sigma}_{ip}(y,v) \wedge E_{ip}(u,v)))$$

Note that the definition is in fact independent of the choice of the realization a of p. It is clearly reflexive and symmetric. It is not difficult to see that it is also transitive. We claim that

$$E = E^* \cap \bigcap_{p\in S(\emptyset), i\in I_p} F_{ip}$$

By Lemma 18.4 this will show that E is an intersection of 0-definable equivalence relations.

Assume $E(b,c)$. Then $E^*(b,c)$. Let $p(x) \in S(\emptyset)$, let $i \in I_p$, and let $a \models p$. We want to check that $F_{ip}(c,d)$. We may assume $E^*_{ip}(b,a) \wedge E^*_{ip}(c,a)$. By choice of E^*_{ip} we know that there are c', d' such that $\models \psi_{ip}(c') \wedge \overline{\sigma}_{ip}(c,c')$ and $\models \psi_{ip}(d') \wedge \overline{\sigma}_{ip}(d,d')$. Then $\models \sigma_{ip}(c',d')$ and therefore $E_{ip}(c',d')$.

For the other direction, assume $E^*(c,d)$ and $F_{ip}(c,d)$ for all p, i. Let $p(x) = \mathrm{tp}(c)$. As remarked above, $E(c',d)$ for some $c' \equiv c$. It is enough to show that $E(c,c')$ and for this we have to check that $E_{ip}(c,c')$ for all $i \in I_p$. Note that $F_{ip}(c,d)$ and $F_{ip}(d,c')$ since we have already shown that $E(x,y)$ implies $F_{ip}(x,y)$. Hence $F_{ip}(c,c')$ and by definition of F_{ip} there are b, b' such that $\models \psi_{ip}(b) \wedge \overline{\sigma}_{ip}(c,b) \wedge \psi_{ip}(b') \wedge \overline{\sigma}_{ip}(c',b') \wedge E_{ip}(b,b')$. Note that $\models \psi_{ip}(c) \wedge \psi_{ip}(b) \wedge \sigma_{ip}(c,b)$ and thus $E_{ip}(c,b)$. Similarly $E_{ip}(c',b')$ and we conclude $E_{ip}(c,c')$. $\square$

Lemma 18.12 *Let T be small and let E be a 0-type-definable equivalence relation on $\mathfrak{C}^n$ such that: if $E(a,b)$, $a \equiv a'$ and $b \equiv b'$, then $E(a',b')$. Then E is an intersection of 0-definable equivalence relations.*

Proof: We claim that whenever $\neg E(a,b)$ then for some $\varphi_{ab} \in L$, $\models \varphi_{ab}(a) \wedge \neg\varphi_{ab}(b)$ and $E(x,y) \wedge \varphi_{ab} \wedge \neg\varphi_{ab}(y)$ is inconsistent. If this is the case we can then express E as an intersection of 0-definable equivalence relations as follows:

$$E(x,y) \Leftrightarrow \bigwedge_{\neg E(a,b)} \varphi_{ab}(x) \leftrightarrow \varphi_{ab}(y)$$

In order to prove this claim, assume $\neg E(a,b)$ and set $p(x) = \mathrm{tp}(a)$, $q(x) = \mathrm{tp}(b)$. We first observe that $E(x,y) \cup p(x) \cup q(y)$ is inconsistent and hence we can choose $\varphi(x) \in p(x)$, $\psi(y) \in q(y)$ such that $E(x,y) \wedge \varphi(x) \wedge \psi(y)$ is inconsistent and $\neg\varphi(x) \wedge \neg\psi(x)$ is of minimal Cantor-Bendixson rank α in the space $S_n(\emptyset)$ and of minimal degree in this rank. If $\neg\varphi(x) \wedge \neg\psi(x)$ is inconsistent we set $\varphi_{ab} = \varphi$ and this choice satisfies the requirements. Otherwise we choose a type $p'(x) \in S_n(\emptyset)$ of rank α containing the formula $\neg\varphi(x) \wedge \neg\psi(x)$ and also a realization $c \models p'$. Now, if there if some $a' \models \varphi$ and some $b' \models \psi$ such that $E(a',c)$ and $E(b',c)$ then $E(x,y) \wedge \varphi(x) \wedge \psi(y)$ turns out to be consistent. Hence we may assume that there is no $a' \models \varphi$ such that $E(a',c)$, that is, $E(x,y) \wedge \varphi(x) \wedge p'(y)$ is inconsistent. Therefore $E(x,y) \wedge \varphi(x) \wedge \psi'(y)$ is inconsistent for some $\psi' \in p'$. Note that either $\neg\varphi(x) \wedge \neg\psi'(x)$ has rank $< \alpha$ or has rank α and smaller degree than $\neg\varphi(x) \wedge \neg\psi(x)$. This contradicts the previous choice of $\varphi(x)$ and $\psi(x)$. $\square$

Theorem 18.13 *Let T be small.*

1. *If E is a 0-type-definable equivalence relation on $\mathfrak{C}^n$, then E is an intersection of 0-definable equivalence relations.*

2. *T eliminates all finitary hyperimaginaries.*

3. *For any finite set A, $\stackrel{\mathrm{bdd}}{\equiv}_A = \stackrel{\mathrm{s}}{\equiv}_A$.*

4. *If T is simple, then $\stackrel{\mathrm{Ls}}{\equiv}_A = \stackrel{\mathrm{s}}{\equiv}_A$ for any A.*

Proof: *1.* We apply Proposition 18.11. It is clear that E^* satisfies the hypothesis of Lemma 18.12 and therefore it is an intersection of 0-definable equivalence relations. Now fixe some $p(x) \in S_n(\emptyset)$ and choose some $c \models p$. We have to show that for some family $(E_i : i \in I)$ of 0-definable equivalence relations, $E \restriction p = \bigcap_{i\in I} E_i \restriction p$. Consider the relation

$$F(x,y) \Leftrightarrow \text{ for some } z,\ cx \equiv zy \text{ and } E(c,z)$$

It is an equivalence relation and it is type-definable over c. Since $T(c)$ is small and in $T(c)$ the relation F satisfies the hypothesis of Lemma 18.12, there is some family $(F_i : i \in I)$ of equivalence relations F_i such that $F = \bigcap_{i\in I} F_i$ and for each $i \in I$ there is some $\varphi_i(x,y,z) \in L$ such that $\varphi_i(x,y,c)$ defines F_i. Now let

$$E_i(x,y) \Leftrightarrow \forall u(\varphi_i(u,x,x) \leftrightarrow \varphi_i(u,y,y))$$

It is clearly an equivalence relation. We check that $E \restriction p = \bigcap_{i\in I} E_i \restriction p$. It suffices to see that for any $a \models p$, $E(a,c)$ iff $E_i(a,c)$ for all $i \in I$. Assume $E(a,c)$, let $i \in I$, let b be arbitrary and choose b' such that $ab \equiv cb'$. Then $F(b,b')$ and therefore $\models \varphi_i(b,b',c)$. Since $\varphi_i(x,y,c)$ defines an equivalence relation, $\models \varphi_i(b,c,c) \leftrightarrow \varphi_i(b',c,c)$. By automorphism, $\models \varphi_i(b,c,c) \leftrightarrow \varphi_i(b,a,a)$ and thus $E_i(a,c)$. For the other direction, assume $E_i(a,c)$ for all $i \in I$. Since $\models \varphi_i(a,a,a)$, we get $\models \varphi_i(a,c,c)$ and hence $F(a,c)$. This clearly implies $E(a,c)$.

2. It follows from *1* and Corollary 18.3.

3. Since $T(A)$ is again small we may assume $A = \emptyset$. It is enough to check the equality for finite sequences and this case follows straightforward from *1* since it implies that on n-tuples $\overset{\text{bdd}}{\equiv}$ is an intersection of finite 0-definable equivalence relations.

4. If T is simple, $T(A)$ is also simple and by Corollary 10.14 $\overset{\text{bdd}}{\equiv}_A = \overset{\text{Ls}}{\equiv}_A$. By Corollary 10.13, $a \overset{\text{Ls}}{\equiv}_A b$ iff $a \overset{\text{Ls}}{\equiv}_{A'} b$ for all finite $A' \subseteq A$. Same for $\overset{\text{s}}{\equiv}_A$. Then we can apply *3*.

□

Example 18.14 *1. (Pillay-Poizat) There is a superstable theory T (of U-rank 1) where we can find a 0-type-definable equivalence relation which is not an intersection of 0-definable equivalence relations.*

2. (Adler) There is an ω-categorical (hence small) theory which does not eliminate hyperimaginaries. It is a theory with the strict order property.

Definition 18.15 *A formula $\varphi(x,y) \in L$ is* low *if there is some $n < \omega$ such that for any indiscernible sequence $(a_i : i < \omega)$, if $\{\varphi(x,a_i) : i < \omega\}$ is inconsistent, then it is n-inconsistent. We say that T is* low *if it is simple and every formula is low in T.*

Definition 18.16 *Let $\varphi(x,y) \in L$. For any set of formulas $\pi(x)$ we define the rank $D(\pi,\varphi)$ as follows*

1. *$D(\pi,\varphi) \geq 0$ iff $\pi(x)$ is consistent.*
2. *$D(\pi,\varphi) \geq \alpha + 1$ iff for some a, $\varphi(x,a)$ divides over the parameters of π and $D(\pi \cup \{\varphi(x,a)\},\varphi) \geq \alpha$*
3. *$D(\pi,\varphi) \geq \alpha$ iff $D(\pi,\varphi) \geq \beta$ for all $\beta < \alpha$ if α is a limit ordinal number.*

Remark 18.17 *1. $D(\pi,\varphi,k) \leq D(\pi,\varphi) \leq D(\pi)$*

2. *If $\pi(x)$ is a partial type over A, then $D(\pi, \varphi) \geq \alpha + 1$ iff for some a, $\varphi(x, a)$ divides over A and $D(\pi \cup \{\varphi(x, a)\}, \varphi) \geq \alpha$*

Proposition 18.18 *Let T be simple and let $\varphi(x, y) \in L$. Those following are equivalent:*

1. *$\varphi(x, y)$ is low.*
2. *There is some $k < \omega$ such that for all π, $D(\pi, \varphi) = D(\pi, \varphi, k)$.*
3. *$D(x = x, \varphi) < \omega$.*
4. *There is some $n < \omega$ such that for all $k < \omega$, $D(x = x, \varphi, k) < n$.*
5. *There is some $n < \omega$ such that φ divides at most n times.*
6. *$\{(a, b) \in \mathfrak{C} : \varphi(x, a)$ divides over $b\}$ is type-definable over $\emptyset$.*

Proof: *1 $\Rightarrow$ 2.* Fix $n < \omega$ as in the definition of low. If $\varphi(x, a)$ divides over A, it divides over A with respect to n. Hence $D(\pi, \varphi) = D(\pi, \varphi, n)$.

2 $\Rightarrow$ 3. By simplicity, $D(x = x, \varphi, k) < \omega$.

3 $\Rightarrow$ 4 is clear since $D(x = x, \varphi, k) \leq D(x = x, \varphi)$.

4 $\Rightarrow$ 5. Fix n as in *4*. If φ divide m times, there are sequences $(a_i : i < m)$ and $(k_i : i < m)$ such that $\{\varphi(x, a_i) : i < m\}$ is consistent and for each $i < m$, $\varphi(x, a_i)$ divides over $(a_j : j < i)$ with respect to k_i. If $k = \max_{i<m} k_i$ then $\varphi(x, a_i)$ divides over $(a_j : j < i)$ with respect to k and hence $m \leq n$.

5 $\Rightarrow$ 3. By Proposition 3.7.

5 $\Rightarrow$ 1. If $(a_i : i < \omega)$ is indiscernible, and $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent but not $k+1$-inconsistent, then $(a_i : i < k)$ witnesses that $\varphi(x, y)$ divides k times.

1 $\Rightarrow$ 6. It follows from *1* that there is some $k < \omega$ such that for all sequences a, b if $\varphi(x, a)$ divides over b, then $\varphi(x, a)$ divides over b with respect to k. But $\{(a, b) \in \mathfrak{C} : \varphi(x, a)$ divides over b with respect to $k\}$ is type-definable over $\emptyset$.

6 $\Rightarrow$ 1. Assume $\varphi(x, y)$ is not low. For each $k < \omega$ let $(a_i^k : i < \omega+\omega)$ be an indiscernible sequence such that $\{\varphi(x, a_i^k) : i < \omega + \omega\}$ is inconsistent but not k-inconsistent. Choose a nonprincipal ultrafilter D over ω and let $(c_i : i < \omega+\omega)$ be a realization of the ultraproduct of types $\prod_D(p_k : k < \omega)$ where $p_k = \mathrm{tp}(a_i^k : i < \omega+\omega)$. Then $(c_i : i < \omega+\omega)$ is indiscernible and $\{\varphi(x, c_i) : \omega \leq i < \omega + \omega\}$ is k-consistent for every $k < \omega$ and hence it is consistent. Let $c = (c_i : i < \omega)$. By Lemma 10.1, $(c_i : \omega \leq i < \omega + \omega)$ is a Morley sequence over c. By Proposition 5.13 $\varphi(x, c_\omega)$ does not divide over c. Assume $\pi(x, y)$ is a partial type over $\emptyset$ defining dividing as in *6*. Since, for each $k < \omega$, $\varphi(x, a_\omega^k)$ divides over $b_k = (a_i^k : i < \omega)$, we have $\models \pi(a_\omega^k, b_k)$ and therefore $\models \pi(c_\omega, c)$. But then $\varphi(x, c_\omega)$ divides over c. $\square$

Remark 18.19 *Conditions* 3, 4, 5 *of Proposition 18.18 are equivalent in any theory T. Moreover, if they hold for any φ, the theory T is simple.*

Remark 18.20 *If T is low, then $T(A)$ is also low for any set A.*

Proof: This is clear, for instance, from point *6* of Proposition 18.18 since $\varphi(x, a, b, c)$ divides over bc in T if and only if $\varphi(x, a, b, c)$ divides over b in $T(c)$. $\square$

Proposition 18.21 *1. Any stable theory is low.*

2. Any supersimple theory of finite D rank is low.

Proof: By Proposition 18.18 *2* is clear, since $D(x = x, \varphi) \leq D(x = x)$. For *1*, assume $\{\varphi(x, a_i) : i < n\}$ is consistent and $\varphi(x, a_i)$ divides over $(a_j : j < i)$ for each $i < n$. Let $b \models \bigwedge_{i<n} \varphi(x, a_i)$ and let $p_i(x) = \mathrm{tp}(b/\{a_j : j < i\})$ for $i = 1, \dots, n$. By Corollary 8.7, $CB_\varphi(p_i) > CB_\varphi(p_{i+1})$ for all $i < n$ and therefore $\omega > CB_\varphi(x = x) \geq D(x = x, \varphi)$. $\square$

Example 18.22 *(Casanovas-Kim) There are supersimple nonlow theories.*

Definition 18.23 *The* Zariski topology *in $\mathfrak{C}^I$ is the topology whose closed sets are the type-definable (over subsets of $\mathfrak{C}$) subsets of $\mathfrak{C}^I$. If E is a 0-type-definable equivalence relation in a type-definable over $\emptyset$ subclass of $\mathfrak{C}^I$, the* logic topology *or the* Kim-Pillay topology *is the quotient topology of the Zariski topology. If $\pi(x)$ is the type defining the domain of E and $X = \pi(\mathfrak{C})/E$ is the quotient, then $A \subseteq X$ is closed iff $\{a \models \pi : a_E \in A\}$ is type-definable. In this context we will always identify E with the type defining it and we will assume that the type $E(x, y)$ is closed under finite conjunctions and that $E(x, y) \vdash \pi(x) \cup \pi(y)$.*

Proposition 18.24 *Let $\pi(x)$ be a type over $\emptyset$, let E be a 0-type-definable equivalence relation on $\pi(\mathfrak{C})$, and consider the Kim-Pillay space $X = \pi(\mathfrak{C})/E$.*

1. *$Y \subseteq X$ is closed iff for some type-definable class A, $Y = \{a_E : a \in A\}$.*
2. *X is Hausdorff.*
3. *A basis of open sets is given by the collection of all*

$$U_{a\varphi} = \{b_E : \models \varphi(a', b') \text{ for all } a', b' \text{ such that } E(a, a'), E(b, b')\}$$

   where $a \models \pi$ and $\varphi = \varphi(x, y) \in E$.

4. *X is compact iff E is bounded.*

Proof: *1.* If $A = \Phi(\mathfrak{C})$, then $\{a : a_E \in Y\}$ is defined by the type

$$\Psi(x) = \exists y(E(x, y) \land \Phi(y))$$

2 is clear. We check *3*. Note that $\{b \models \pi : b_E \notin U_{a\varphi}\}$ is type-definable and hence $U_{a\varphi}$ is open. Let U be open and $a_E \in U$. We will show that $a_E \in U_{a\varphi} \subseteq U$ for some $\varphi(x, y) \in E(x, y)$. We may assume that $\{b \models \pi : b_E \notin U\} = \{b \models \pi : \models \psi(b)\}$ for some formula $\psi(x) \in L(\mathfrak{C})$. Then $E(x, y) \land \psi(x) \vdash \psi(y)$. By compactness $\varphi(x, y) \land \psi(x) \vdash \psi(y)$ for some $\varphi(x, y) \in E(x, y)$. It is easily seen that φ works.

4. Assume first E is bounded and let $(F_i : i \in I)$ be a family of closed sets with the finite intersection property. For each $i \in I$ choose a type $\Phi_i(x)$ such that $F_i = \{a_E : a \models \Phi_i\}$. If the number of E-classes is bounded by κ, the number of closed sets in X is bounded by 2^κ and hence $|I| \leq 2^\kappa$. Therefore $\bigcup_{i \in I} \Phi_i$ is a partial type over a subset of $\mathfrak{C}$ and we can realize it by some $a \in \mathfrak{C}$. Clearly, $a_E \in F_i$ for all $i \in I$. For the other direction, assume now X is compact. Fix $\varphi(x, y) \in E(x, y)$. We will show that φ is finite on π, that is, there is no infinite sequence $(a_i : i \in \omega)$ of realizations a_i of π such that $\models \neg\varphi(a_i, a_j)$ for all $i < j < \omega$. From this it follows that E is bounded. Assume there is such a sequence $(a_i : i \in \omega)$. We can extend it to a maximal one $(a_i : i \in I)$. Then for any $a \models \pi$ there is some $i \in I$ such that $\models \varphi(a, a_i)$, that is $X \subseteq \bigcup_{i \in I} U_{a_i\varphi}$. By compactness of X, for some finite $I_0 \subseteq I$, $X \subseteq \bigcup_{i \in I_0} U_{a_i\varphi}$. This contradicts the choice of the sequence. $\square$

Proposition 18.25 *Let $\pi(x)$ be a type over $\emptyset$, let E be a 0-type-definable equivalence relation on $\pi(\mathfrak{C})$, and consider the Kim-Pillay space $X = \pi(\mathfrak{C})/E$. The following conditions are equivalent.*

1. *X is 0-dimensional.*
2. *E is an intersection of 0-definable equivalence relations.*
3. *For each $\varphi(x,y) \in E$ there is some $\varphi'(x,y) \in E$ such that*
   (a) *$\pi(x) \cup \pi(y) \vdash \varphi'(x,y) \to \varphi(x,y)$.*
   (b) *$E(x,x') \cup E(y,y') \vdash \varphi'(x,y) \to \varphi'(x',y')$.*

Proof: *1 ⇒ 2.* Let $(O_i : i \in I)$ be a basis of clopen sets. For each $i \in I$ there is some formula $\varphi_i(x) \in L(\mathfrak{C})$ such that $\{a \models \pi : \models \varphi_i(a)\} = \{a \models \pi : a_E \in O_i\}$. Let $a \models \pi$. Since $\{a_E\}$ is closed, there is a subset $I_a \subseteq I$ such that $\{a_E\} = \bigcap_{i \in I_a} O_i$. For each $i \in I$, $(\varphi_i(x) \leftrightarrow \varphi_i(y))$ defines an equivalence relation. It is easy to check that E can be defined by

$$\bigwedge_{a \models \pi} \bigwedge_{i \in I_a} (\varphi_i(x) \leftrightarrow \varphi_i(y))$$

2 ⇒ 3. Let $E = \bigcap_{i \in I} E_i$ where each E_i is a 0-definable equivalence relation. If $\varphi(x,y) \in E(x,y)$, then for some $i \in I$, $E_i(x,y) \vdash \varphi(x,y)$ and clearly $\varphi'(x,y) = E_i(x,y)$ satisfies all the requirements.

3 ⇒ 1. Let $(U_{a\varphi} : a \models \pi,\ \varphi \in E)$ be the basis of open sets described in Proposition 18.24. For each $\varphi \in E$ choose φ' as in *3*. Then $(U_{a\varphi'} : a \models \pi,\ \varphi \in E)$ is again a basis of open sets. It is easy to check that in fact each $U_{a\varphi'}$ is clopen. □

Proposition 18.26 *If T is simple, then T eliminates all bounded hyperimaginaries if and only if* Lstp = stp*, that is, if and only if for all sequences a, b: $a \overset{\text{Ls}}{\equiv} b$ iff $a \overset{\text{s}}{\equiv} b$.*

Proof: It is clear that if T eliminates all bounded hyperimaginaries, then $\mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(\emptyset)) = \mathrm{Aut}(\mathfrak{C}/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ and therefore Lstp = stp. For the other direction, let $e = a_E$ be a bounded hyperimaginary. By Proposition 15.21 we can assume E is a bounded equivalence relation. By Corollary 10.14, $\overset{\text{Ls}}{\equiv}$ is the least bounded 0-type-definable equivalence relation and therefore a_E splits into a bounded number of Lascar strong types. By assumption and by Proposition 18.2 for each bEa there is a sequence of imaginaries b' such that $b_{\overset{\text{Ls}}{\equiv}} \sim b'$. Let $(b_i : i \in I)$ be a sequence of representatives of Lascar strong types of elements in a_E. Then $e \in \mathrm{dcl}^{\mathrm{heq}}(b'_i : i \in I)$ and $(b'_i : i \in I) \in \mathrm{bdd}(e)$. By Lemma 18.6 e is equivalent to a sequence of imaginaries. □

Lemma 18.27 *Let T be simple, $p(y) \in S(\emptyset)$ and let $\psi_1(x,y), \ldots, \psi_n(x,y) \in L$. Then*

$$\{(a_1, \ldots, a_n) : a_1, \ldots, a_n \textit{ are independent realizations of } p \textit{ and the formula}$$

$$\psi_1(x,a_1) \wedge \ldots \wedge \psi_n(x,a_n) \textit{ does not fork over } \emptyset\}$$

is type-definable over $\emptyset$.

Proof: The case $n = 1$ is clear by Proposition 5.13 and Corollary 5.20. For the general case, notice that it is enough to deal with Morley sequences in p since we only are interested in independent $a_1, \ldots, a_n$. □

Theorem 18.28 *Let T be a low theory.*

1. *T eliminates all bounded hyperimaginaries.*
2. *For any set A, for any sequences a, b: $a \stackrel{\mathrm{s}}{\equiv}_A b$ iff $a \stackrel{\mathrm{Ls}}{\equiv}_A b$.*

Proof: *1* follows from *2* and Proposition 18.26.

2. By Remark 18.20, we can assume $A = \emptyset$. Let $E = \stackrel{\mathrm{Ls}}{\equiv}$ and consider $e = a_E$, a bounded hyperimaginary. We will show that we can eliminate e using Proposition 18.2. Let $p(x) = \mathrm{tp}(a)$. We use point *3* of Proposition 18.25 to show that on $E \upharpoonright p$ is an intersection of 0-definable equivalence relations. Let $\varphi(x,y) \in E \upharpoonright p$. We need to find $\varphi'(x,y) \in E \upharpoonright p$ such that $\varphi'(x,y) \vdash \varphi(x,y)$ and $E(x,x') \wedge E(y,y') \wedge \varphi'(x,y) \vdash \varphi'(x',y')$. Choose $\bar{\varphi} \in E(x,y) \upharpoonright p$ such that

$$\bar{\varphi}(x,y) \wedge \bar{\varphi}(y,z) \wedge \bar{\varphi}(z,u) \wedge \bar{\varphi}(u,v) \vdash \varphi(x,v)$$

Consider the following binary relation $R(b,c)$ on realizations b, c of p :

$\bar{\varphi}(x,b') \wedge \bar{\varphi}(x,c')$ does not fork over $\emptyset$ for some $b', c' \models p$ such that $E(b,b'), E(c,c'), b' \mathop{\smile\!\!\!\!\!\!\;|}\, c'$

We will check that R is definable by some formula φ' as above. Since $e \in \mathrm{bdd}(\emptyset)$, for any $b \models p$ the type $E(x,b)$ does not fork over $\emptyset$. This implies that we can find an independent sequence b_1, b_2, b_3 in the E-class b_E, which shows that $\bar{\varphi}(x,b_1) \wedge \bar{\varphi}(x,b_2)$ does not fork over $\emptyset$. It follows that whenever $b, c \models p$ and $E(b,c)$ then $R(b,c)$. By choice of $\bar{\varphi}$, whenever $R(b,c)$ then $\models \varphi(b,c)$. Finally, it is obvious that if $E(b,b')$, $E(c,c')$, and $R(b,c)$, then $R(b',c')$.

To check the definability of R we show that R and its complement are type-definable. Type-definability of R follows from Lemma 18.27 and Corollary 5.20. For the complement $\bar{R}$ of R we need to use lowness of T. First note that, since $E = \stackrel{\mathrm{Ls}}{\equiv}$, by Corollary 10.5, for all b, c, b', c' realizing p, if $b \mathop{\smile\!\!\!\!\!\!\;|}\, c$, $b' \mathop{\smile\!\!\!\!\!\!\;|}\, c'$, $E(b,b')$, and $E(c,c')$, if $\bar{\varphi}(x,b) \wedge \bar{\varphi}(x,c)$ does not fork over $\emptyset$, then also $\bar{\varphi}(x,b') \wedge \bar{\varphi}(x,c')$ does not fork over $\emptyset$. Hence for $b, c \models p$, $\bar{R}(b,c)$ if and only if there are $b', c' \models p$ such that $E(b,b')$, $E(c,c')$, $b' \mathop{\smile\!\!\!\!\!\!\;|}\, c'$ and $\bar{\varphi}(x,b') \wedge \bar{\varphi}(x,c')$ forks over $\emptyset$. By Proposition 18.18 and Corollary 5.20 it is easily seen that this relation is type-definable over $\emptyset$. $\square$

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