On $\omega$-categorical simple theories

Daniel Palacín

Universitat de Barcelona

Logic Colloquium Paris
July 2010
Basic definitions

Definition
Let $\alpha$ be an ordinal. A formula $\varphi(x, y) \in L$ divides $\alpha$ times if there is a sequence $(a_i : i < \alpha)$ of parameters (in the monster model) such that

1. the set $\{\varphi(x, a_i) : i < \alpha\}$ is consistent, and
2. for every $i < \alpha$ $\varphi(x, a_i)$ divides over $a_{<i} = \{a_j : j < i\}$ (with respect to some $k_i < \omega$).

Fact
A theory is simple iff no formula divides $\omega_1$ times iff no formula divides $\omega$ times with respect to some fixed $k < \omega$.

Definition
Let $\varphi(x, y)$ be a formula. It is short if it does not divide $\omega$ times; it is low if it does not divide $n$ times for some $n < \omega$. A theory is low (short) if all formulas are low (short).
All ‘natural’ examples are low

Remark

1. Stable theories are low.
2. Supersimple theories are short. Moreover, supersimple theories of finite degree are low.

Question

Is every $\omega$-categorical simple theory low?
What do we know?

Fact (Casanovas-Wagner, 2002)
An ω-categorical short theory is low.

Lemma (Casanovas-Wagner, 2002)
Let $T$ be an ω-categorical simple theory and assume that $\varphi(x, y)$ is nonlow. Then $\varphi(x, y)$ is nonshort and there is some indiscernible sequence $(a_i : i < \omega)$ witnessing it.

Indeed, with the same proof we can prove:

First Lemma
Let $T$ be an ω-categorical simple theory and assume that $\varphi(x, y)$ is nonlow. Then there are a finite tuple $c$ and some $c$-indiscernible sequence $(a_i : i < \omega)$ such that $c$ realizes $\{\varphi(x, a_i) : i < \omega\}$ and for every $i < \omega$ $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$. 
We need another lemma

Second Lemma
Let $T$ be an $\omega$-categorical theory. Let $a$ be a finite tuple and let $A$ be an arbitrary set. Then there is a single imaginary element $e \in \mathcal{C}^{eq}$ such that

$$\text{bdd}(e) = \text{bdd}(a) \cap \text{bdd}(A).$$

Proof.
Let $a'$ be such that $a' \equiv_{\text{bdd}(a) \cap \text{bdd}(A)} a$ and

$$\text{bdd}(a) \cap \text{bdd}(a') = \text{bdd}(a) \cap \text{bdd}(A).$$

Define

$$R(xy; uv) \iff \text{bdd}(x) \cap \text{bdd}(y) = \text{bdd}(u) \cap \text{bdd}(v).$$

It is an invariant equivalence relation and by $\omega$-categoricity it is definable. Then $e = (aa')_R$ works.
Main result

Theorem

An \( \omega \)-categorical simple CM-trivial theory is low.

Definition

A simple theory is CM-trivial if for every tuple \( a \) and for every sets \( A \subseteq B \) with \( \mathsf{bdd}(aA) \cap \mathsf{bdd}(B) = \mathsf{bdd}(A) \) we have \( \mathsf{Cb}(a/A) \subseteq \mathsf{bdd}(\mathsf{Cb}(a/B)) \).

Equivalently, if for every tuple \( a \) and for every set \( A, B \) with \( a \downarrow_A B \) we have that \( a \downarrow_{\mathsf{bdd}(aB) \cap \mathsf{bdd}(A)} B \).
Proof

We will prove it by contradiction. Assume there is a nonlow formula, say $\varphi(x; y) \in L$.

- **First step.** We choose a ‘convenient’ dividing chain. By First Lemma there is a $c$-indiscernible sequence $(a_i : i < \omega)$ such that $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$ and $c \models \bigwedge_{i<\omega} \varphi(x, a_i)$. We extend the sequence to $(a_i : i \leq \omega)$. Then $a_\omega \downarrow_{(a_i : i < \omega)} c$.

- **Second step.** Applying CM-triviality and Second Lemma we get an imaginary element $e \in \mathcal{C}^{eq}$ such that $a_\omega \downarrow_{e} c$ and $e \in acl(a_i : i < n)$ for some $n < \omega$.

- **Third (and final) step.** We get a contradiction. By $c$-indiscernibility, $c \models \varphi(x, a_\omega)$ and also $\varphi(x, a_\omega)$ divides over $\{a_i : i < n\}$; so, it does over $acl(a_i : i < n)$. Thus, $\varphi(x, a_\omega)$ divides over $e$. Hence, $c \not\models_{e} a_\omega$. 
Some comments

Question
The same proof works if for all finite tuples $a, b$ and for any set $A$ such that $a \downarrow_A b$ there is some finite subset $A_0 \subseteq \text{acl}(A)$ with $a \downarrow_{A_0} b$. Is this true in general?

Remark
Recently, on joint work with Frank O. Wagner, we have proved that a simple $CM$-trivial theory eliminates all hyperimaginaries whenever it eliminates finitary ones. In particular, an $\omega$-categorical simple $CM$-trivial theory eliminates all hyperimaginaries. Hence, in our context we may work in $C^{eq}$. 