

# On Normal Integer Parts of Real Closed Fields

Gurgen Asatryan  
Department of Mathematics,  
University of Mons-Hainaut,  
Place du Parc 20, 7000 Mons, Belgium

## Introduction

- SHEPHERDSON [6]: The models of **Open Induction (OI)** are the **Integer Parts (IP)** of real closed fields (RCF).
- WILKIE [7]: Each discretely ordered  $\mathbb{Z}$ -ring can be embedded in a model of OI.
- VAN DEN DRIES [3]: The previous result is extended for the normal (integrally closed) case.
- MACINTYRE & MARKER [4]: Some classical theorems on primes fail in some nonstandard (normal) models of OI.
- MOURGUES & RESSAYRE [5]: Each RCF has an IP (the IP they constructed is not normal for non-trivial cases).
- S. KUHLMANN & AL. [2]: *Does every RCF have a normal IP?*
- BERARDUCCI and OTERO [1] constructed a nonstandard normal model of OI with cofinal set of primes. This model was an IP of  $k(t)^{r.cl.}$  ( $t \ll 1$ ,  $k$  is recursive RCF,  $k \subseteq \mathbb{R}$ ,  $\text{trdeg}(k) = \aleph_0$ ).

## Objectives

- to give a recurrent construction which allows to generate new IP's based on the existed ones.
- to construct normal IP's for a class of RCF's (a partial answer to a question in [2]).
- to show that each field from that class possesses an IP which satisfies the same homogeneous existential formulae (in the language without  $<$ ) as a prescribed subfield  $k_0 \subseteq \mathbb{R}$  with  $\text{trdeg}(\mathbb{R}/k_0) = 2^{\aleph_0}$ .
- to present continuumly many elementary non-equivalent IP's for each field from the considered class.

# Main Results

**Proposition (Basic Construction).** Let  $K \subseteq F \subseteq_{\text{tr}} K((G))$ ,  $M$  is an IP and  $H$  is a subfield of  $K$  and  $M \subseteq H$ . Also, let  $\mu \stackrel{\text{def}}{=} \text{trdeg}(K/H) \geq |\text{Neg}(F)|$  and  $cf(\mu) > |\text{Supp}(u)|$  for all  $u \in \text{Neg}(F)$ . Then there exists  $T \subseteq F \cap K((G^{\leq}))$  such that the elements of  $T$  are algebraically independent over  $H$  and  $H[T]_0 \oplus M$  is an IP of  $F$ .

**Theorem 1.** Let  $G$  be a divisible ordered abelian (non-trivial) group with anti-well-ordered value set  $\alpha^*$ . Let

- $\mathbb{R}(G) \subseteq F \subseteq_{\text{tr}} \mathbb{R}((G))$  and
- $|F_\gamma| > |\gamma|$  ( $\gamma \leq \alpha$ ).

Then, assuming GCH, there exists an IP  $M$  of  $F$  such that  $\text{TH}_{\exists, h}(M) \equiv \text{TH}_{\exists, h}(k_0)$ .

**Theorem 2.** Under the same assumptions as in Theorem 1,

- The number of elementary non-equivalent IP's of  $F$  is continuum.
- $F$  has a normal IP.

**Theorem 3.** Let  $F$  be an RCF with the residue field  $\mathbb{R}$  and a value group  $G$ . Let  $G$  have an anti-well-ordered value set  $\alpha^*$  with  $\alpha \leq \omega_1$ . Then  $F$  has a normal IP, and the number of elementary non-equivalent IP's of  $F$  is continuum.

## Definitions

- $\text{Neg}(F) \stackrel{\text{def}}{=} F \cap K((G^<)), G^< \stackrel{\text{def}}{=} \{g \in G \mid g < 0\}$ .
- $H[X]_0$  is the set of those polynomials of  $H[X]$  whose constant term is 0, and  $H[T]_0 = \{p(a_1, \dots, a_n) \mid a_i \in T, p \in H[X]_0\}$ .
- $\alpha^*$  is an ordinal with its order reversed. We consider the groups  $G$  with value set  $\alpha^*$ . We identify the natural valuation on  $G$  by the corresponding surjective map  $v: G \rightarrow \alpha^*$ .
- $k_0 \subseteq \mathbb{R}, \text{trdeg}(\mathbb{R}/k_0) = 2^{\aleph_0}$ .
- Given  $\mathbb{R}(G) \subseteq F \subseteq_{\text{tr}} \mathbb{R}((G))$ , we define  $C_\gamma = \{g \in G \mid v(g) \leq \gamma\}$  ( $\gamma \leq \alpha$ ) and  $F_\gamma = F \cap \mathbb{R}((C_\gamma))$ .
- $\text{TH}_{\exists, h}(M)$  denotes the part of existential theory of  $M$  (in the language without  $<$ ) consisting of homogeneous formulae.

## Remarks

- $\mathbb{R}((G))$  (with  $G$  as above) satisfies the hypotheses of Theorem 1.
- The field  $k(t)^{r.cl.}$  has infinitely many elementary non-equivalent IP's.

# Outline of the Proofs

**1) Basic Construction.**  $E$  is a transcendence base of  $K/H$ .

- choose a suitable well-order  $\prec$  on  $E$ ,
- induce functions  $\|\cdot\| : K \rightarrow E$ ,  $\|\cdot\| : \text{Neg}(F) \rightarrow E$ ,
- construct a suitable well-order on  $\text{Neg}(F)$ ,
- define  $\lambda : \text{Neg}(F) \rightarrow E$  so that  $\|u\| \prec \lambda(u)$  ( $u \in \text{Neg}(F)$ ).
- define  $T_1 \subseteq \text{Neg}(F)$  by transfinite induction.
- prove that  $T \stackrel{\text{def}}{=} \{u + \lambda(u) \mid u \in T_1\}$  is the desired set.

**2)** The proof of Theorem 1 is by transfinite induction and is mainly based on Proposition. Set  $k_0 = \mathbb{Q}(\{p^{1/p} \mid p \in A\})$  ( $A$  is a subset of primes) to prove Theorem 2 (for normality, set  $A = \emptyset$ ). Theorem 3 is a direct consequence of Proposition.

## References

- [1] Berarducci A., Otero M., A recursive nonstandard model of normal open induction, J. of Symbolic Logic, 61 (1996), no. 4, 1228–1241.
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## Definitions

- $K$  is an ordered field,  $G$  is an ordered abelian group (orders are total).
- A discretely ordered ring (**DOR**) is an ordered ring having 1 as its least positive element.
- A discretely ordered subring  $M \subseteq K$  is called an **Integer Part (IP)** of  $K$  if for each  $x \in K$  there exists  $z \in M$  such that  $z \leq x < z + 1$ .
- $M$  is called normal if  $(x, y, c_1, \dots, c_n \in M)$ :

$$x^n + c_1 x^{n-1} y + \dots + c_n y^n = 0 \\ \Rightarrow \exists z \in M (x = yz).$$

- A subfield  $F \subseteq K((G))$  is called truncation closed if:  $\sum a_g t^g \in F$ ,  $g_0 \in G \Rightarrow \sum_{g < g_0} a_g t^g \in F$ . In symbols  $F \subseteq_{\text{tr}} K((G))$ .
- **OI** is a first order theory in the language  $\{0, 1, +, -, \cdot, <\}$  which has the axioms of **DOR** and the following scheme of axioms ( $\psi$  is quantifier-free):

$$\left( \psi(\vec{x}, 0) \wedge \forall y \geq 0 [\psi(\vec{x}, y) \rightarrow \psi(\vec{x}, y + 1)] \right) \\ \rightarrow \forall y \geq 0 \psi(\vec{x}, y)$$