

CHARACTERIZING NIP AND NTP_2 SEMANTICALLY

ARTEM CHERNIKOV

1. BASIC DEFINITIONS

Definition 1.1. 1) We say that $(\phi^i(x, y^i), \bar{a}^i : i < \kappa)$ with $|\bar{a}^i| = \omega$ is an independence pattern of depth κ if:

- for each $i < \kappa : \bigwedge_{j < \omega} \phi^i(x, a_j^i)$ is k^i -inconsistent for some $k^i < \omega$
- for each $f \in \omega^\kappa : \bigwedge_{i < \kappa} \phi^i(x, a_{f(i)}^i)$ is consistent.

2) For theory T and $n < \omega$ define $\kappa_{inp}^n(T)$ to be the maximal possible depth of an independence pattern with $|x| \leq n$ or ∞ if it does not exist. Note that $\forall n < m \ \kappa_{inp}^n(T) \leq \kappa_{inp}^m(T)$.

3) $\kappa_{inp}(T) := \sup_{n < \omega} \{\kappa_{inp}^n(T)\}$

4) T is *strong* ^{n} if there is no independence pattern of infinite depth with $|x| \leq n$ and *strong* if it is *strong* ^{n} for all n .

5) T has TP_2 (tree property of the second kind) if there is an independence pattern of infinite depth with some $\phi = \phi^i$ and $k = k^i$, and is NTP_2 if there is no such.

Note 1.2. 1) In the definition of independence pattern we can assume rows to be mutually indiscernible.

2) For any T : $\kappa_{inp}(T) \geq |T|^+ \iff \kappa_{inp}(T) = \infty \iff T$ has TP_2 .

Remark 1.3. Strong theories were defined by Hans Adler in [Adler], all the rest is due to Shelah of course.

2. BUNCH OF ACRONYMS

What is the place of all this in the complexity hierarchy of theories?

Fact 2.1. 1) *simple* $\implies NTP_2$

- 2) $NIP \implies NTP_2$
- 3) $strong \implies NTP_2$ (and is actually a “uniform NTP_2 ”, see further sections)
- 4) $NTP_2 + NSOP_2 = simple$
- 5) $strong + NIP = strongly\ dependent$
- 6) $strong + simple = every\ finitary\ type\ has\ finite\ weight.$

Proof. 1) and 2) is an easy exercise, 3) is a definition, 4) is in the book and 5), 6) are in [Adler].

□

3. EXAMPLES

Theories with NTP_2

- (1) T_1, T_2 are $NTP_2 \implies T_1 \times T_2$ is NTP_2 (e.g. product of simple and dependent groups)
- (2) Reduct of an NTP_2 theory is NTP_2
- (3) So RV valued fields are NTP_2 assuming residue field and the value group are. Ask [MartinHils] to find out the details.
- (4) Dividing-preserving expansion of NTP_2 theory is NTP_2 (we say that $T' \supseteq T$ is a dividing preserving expansion of T if $a \downarrow_c^d b$ in sense of $T \implies a \downarrow_c^d b$ in sense of T').
- (5) 1-minimal expansions preserve NTP_2 (T' is a 1-minimal expansion of T if every formula from $L(T')$ in one variable is equivalent to some $L(T)$ formula, possibly with parameters. E.g. T' is strongly minimal iff it is a 1-minimal expansion of the infinite set and \mathcal{o} -minimal if it is a 1-minimal expansion of DLO).

Reasonable examples with TP_2

- (1) Triangle-free random graph
Well-known to be non-simple, but actually is even worse.
consider sequences $\{a_i^j b_i^j\}_{i < \omega}$ for $j < \omega$ such that $\forall i_1 \neq i_2 a_{i_1}^j R b_{i_2}^j$

and these are the only edges around (easy to construct by genericity). Then it witnesses TP_2 for formula $xRy_1 \wedge xRy_2$ (where y_1 corresponds to a 's and y_2 to b 's).

(2) ω -free PAC fields

See [Chatzidakis] for the definitions and proof.

(3) Independent family of linear or circular orders

See [CheKap] for details.

(4) Any non-simple $NSOP_2$ theory.

4. LIFTING MUTUAL INDISCERNIBILITY

The following observation on extracting mutually-indiscernible sequences is due to Itai Ben Yaacov [BenYaacov].

Fact 4.1. *Let $C \subseteq \mathbb{M}$ and κ some cardinal. Then there is some λ such that for any $\bar{a}^{<n}$ with $|\bar{a}^i| \geq \lambda$ (and $|a_j^i| \leq \kappa$) we can find some mutually-indiscernible over C sequences $\bar{b}^{<n}$, $|b^i| = \omega$ such that $b_{<m}^0, \dots, b_{<m}^n \equiv_C a_{\bar{j}_0}^0, \dots, a_{\bar{j}_n}^n$ for some $\bar{j}_0 \in |\bar{a}_0|^m, \dots, \bar{j}_n \in |\bar{a}_n|^m$.*

Note that for $n = 1$ this is normal ‘‘Erdős-Rado’’.

This and some more effort allows to show that

Theorem 4.2. *TFAE:*

1) $\kappa_{inp}^n(T) \leq \kappa$

2) $(*)_n^\kappa$: for any indiscernible array $\bar{a}^{<\kappa^+}$ and $c \in \mathbb{M}$, $|c| \leq n$ there is $h \in \kappa^+$ and an $a_0^{[h, \kappa^+]}$ -automorphic image of $\bar{a}^{[h, \kappa^+]}$ indiscernible over c .

Now it is easy to see that $(*)_1^\kappa \implies (*)_2^\kappa \implies \dots \implies (*)_\kappa^\kappa$, and so we can answer a question of Shelah from [ShC]:

Corollary 4.3. $\kappa_{inp}(T) = \kappa_{inp}^1(T)$ (so in particular if a theory has TP_2 then there is some formula with a single variable witnessing it).

Analogously we get

Corollary 4.4. 1) T is strong \iff for any indiscernible array $\bar{a}^{<\omega}$ and finite c there is $h < \omega$ and an $a_0^{[h,\omega]}$ -automorphic image of $\bar{a}^{[h,\omega]}$ indiscernible over c .
 2) Thus, strong = strong¹.

Problem 4.5. What strengthenings of this are true in simple or *NIP* theories?

5. PSEUDO-LOCAL CHARACTER

There are several equivalent pseudo-local characters characterizing *NTP₂*. It is still not clear which is the most useful.

Definition 5.1. We say that dividing in T has pseudo-local character with respect to a pre-independence relation \perp when:
 Let $p \in S(A)$, $B \subseteq A$. Then there is some $C \subseteq A$, $|C| \leq |T|$ such that for each $D \subseteq A$: $D \perp_B BC \implies p|_{BD}$ does not divide over BC .

Note 5.2. Of course local character of dividing (so simplicity of a theory) implies pseudo-local character with respect to any pre-independence relation.

Example 5.3. If $M \models DLO$ and $p \in S(M)$, then p actually corresponds to a Dedekind cut of M . So though in general p is not definable, $p|_{\{c \in M : c \notin (a,b)\}}$ is definable whenever (a,b) is an interval of M which includes the cut. So p is definable in large pieces.

The following claim stated in a somewhat wrong way appeared first in [Sh783] and was proved there assuming *NIP*. Reading the proof shows that it actually follows from *NTP₂*, and is moreover equivalent to it.

Theorem 5.4. *TFAE:* 1) T is *NTP₂*

2) Dividing has pseudo-local character with respect to \downarrow^{ist} (invariant+non-coforking, for more on this see [CheKap])

3) Dividing has pseudo-local character with respect to \downarrow^u for types over saturated enough models (where enough means $p \in S(M)$ and M is $|B|^+$ -saturated)

4) Dividing has pseudo-local character with respect to \downarrow^i for types over saturated enough models

5) If $(a_i : i < |T|^+)$ is a two ways- \downarrow^u -free sequence over A (or M) and b some tuple then there is some $h < |T|^+$ such that $b \downarrow_A^d a_h$.

Problem 5.5. Does *NTP₂* imply that dividing has pseudo-local character over arbitrary sets w.r.t. \downarrow^u ?

Remark 5.6. 1) Recall that super-simple means that every type does not fork over some finite subset of its domain. Analogously strongness is characterized by being able to find not just small but actually a finite subset C as above.

2) In *NIP* we have in addition that dividing has pseudo-local character with respect to \downarrow^d over saturated enough models.

It also allows to characterize *NTP₂* by counting small partial contradictory types with some extra technical conditions (which I hope to get rid of soon).

6. ABSTRACT PRE-INDEPENDENCE RELATIONS

There is a well-known and useful characterization of simple theories due to Kim and Pillay as exactly those which admit an independence relation satisfying amalgamation of types. More precisely:

Theorem 6.1. *T is simple \iff there is an invariant relation \downarrow on the monster satisfying*

- right extension
- local character
- amalgamation over model (actually what one needs from it is that any pre-independence relation satisfying it implies non-dividing)

and \perp is exactly non-dividing/non-forking.

There are several variants to give analogous characterizations for NTP_2 and NIP . Definitely none of them is final, but this is what I know is true at the moment

Theorem 6.2. 1) T is NIP \iff there are two invariant relations \perp and \perp' satisfying following

- \perp' exists over models and has extension on the right
- If $(a_i : i < |T|^+)$ is an \perp' -free sequence over A and b arbitrary tuple, then for some $h < |T|^+$ $b \perp_A a_h$ (or any pseudo-local characters)
- \perp is bounded by the exponent, that is each type over model M has at most 2^M \perp -free extensions

2) If in addition \perp it satisfies

- generic amalgamation of types: assume $p_i(x, a_i)$ is \perp -free over A for all $i < (2^{|T|+|A|})^+$. Then for some i, j we have that $p_i(x, a_i) \cup p_j(x, a_j)$ is \perp -free over A .

then it is exactly non-dividing.

3) If we put ω instead of $|T|^+$ then we get strongly dependent.

Of course it can be stated in terms of just \perp by letting \perp' be a shortcut for strict \perp -freeness.

Remark 6.3. What I call generic amalgamation of types is actually what is known as a chain condition. It is true both in simple theories and in NIP . Of course it should be true in NTP_2 as well, but no proof is known so far.

7. REFERENCES

- [Adler] Hans Adler, “Strong theories, burden, and weight”.
- [BenYaacov] Itai Ben Yaacov, “Simplicity in compact abstract theories”

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- [MartinHils] Martin Hils, Université Paris 7
- [ShC] Saharon Shelah, “Classification theory”
- [Sh783] Saharon Shelah, “Dependent first order theories, continued”