Second cohomology groups and finite covers

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Abstract

In this paper we show several applications of group cohomology to finite covers. In the first part we give a criterion for determining if a given profinite G-module is the kernel for a finite cover of some first order structure with automorphism group G. In the second part we provide several computations of certain cohomology groups and then, as an example of the effectivity of the criterion, we describe precisely all the covering expansions of certain free finite covers. In the first part we give a criterion for determining if a given profinite group is a finite cover with binding groups. In the second part we give the topology whose open sets are arbitrary unions of cosets of pointwise stabilizers of finite subsets of C. In the case when η is a principal cover with abelian kernel K0 was studied (in this case, Aut(C)) splits over K0 which makes the problem less hard.

Preliminaries

If C is any set, then the full symmetric group Sym(C) on C can be considered as a topological group by giving it the topology whose open sets are arbitrary unions of cosets of pointwise stabilizers of finite subsets of C. Closed subgroups in this topology are precisely automorphism groups of relational structures on C.

Let C and W be two first-order structures. A finite-to-one surjection π : C → W is a finite cover if its fibers form an Aut(C)-invariant partition of C and the induced map μ : Aut(C) → Sym(W) has image Aut(W). The kernel of μ, K, is keeps, which turns out to be a profinite group. The map μ is continuous and open and, thus,

1 → K → Aut(C) → Aut(W) → 1

is an exact sequence of topological groups.

Continuous cohomology groups

The appropriate cohomological machinery needed in order to work with finite covers is the continuous one (see [5]).

Let G be a permutation group considered with the canonical topology and K a continuous G-module. We denote by C2(G, K) the additive group of continuous functions ϕ : G → K. The usual cohomology operator H2 sends C2(G, K) to C2+1(G, K), so that (C2(G, K), C2+1(G, K)) is a cochain complex. The cohomology of this complex, H2(G, K), is the continuous cohomology of G with coefficients in K.

The closed G-invariant subspaces of F2+1 are precisely the annihilators X0 in the Pontrjagin duality of G-invariant subspaces of F2 (see [4] and [6]).

The lattices of the closed submodules of F2+1 for k = 2, 3 are the following.

Cohomological computations

We make use of the continuous versions of some results in the theory of cohomology of discrete groups, as the Shapiro’s Lemma and long exact sequences (using the dimension shifting trick).

Lemma 4 Let G be Sym(Ω) and F be a finite field. Then:

1. H2(F, G; F) = 0 for every n ∈ N (see [9]).
2. H2(F, G; ker(λ2)) = H2(F, G; ker(λ3)) = H1(F, G; F) = 0 for every n > 0 ∈ N.
3. H2(F, G; ker(λ2)) = H2(F, G; F) = F2 for every n > 0 ∈ N.

Lemma 5 Suppose G is a closed permutation group on Ω with a smooth strong type p. Suppose Q is a finite group and F a finite abelian group (equalised as a trivial G-module and Q-module). Then, for a ≤ b, F2(F × Q, F) = a(F, Q, F).

Theorem 2 Let W be a first-order structure with automorphism group G. Suppose η : C0 → W is a finite cover with abelian kernel K0 and K a closed G-submodule of K0. Let

0 → K → K0 → K → 0

be the natural short exact sequence where ι is the inclusion map. Consider

... H3(G, K) → H3(G, K) → H2(G, K)

part of the long exact sequence, where i* is the induced map in cohomology. Then there exists a covering expansion of η with kernel K if and only if there exists an element e ∈ H2(G, K) such that i*(e) = e, where e0 is the element in H2(G, K) which gives rise, up to isomorphism, to Aut(C) (an extension of K0 by K).

Corollary 1 Let G = Sym(Ω). Then, for every k ∈ N, H2(F, G; F2+k) = H2(F, Symk(F)). In particular, H2(F, G; F2+1) = H2(F, Sym2(F)).

The main result

Lemma 1 If K is a profinite continuous G-module, cohomology of low degree, retains its familiar applications: H2(G, K) classifies closed covers in the split extension and H2(G; K) classifies all permutation group extensions of K by G.

First continuous cohomology groups have been previously used in the context of model theory. Indeed, several results in [1] and [5] show how H2(G, K) makes us to parametrize finite covers with kernel K of a fixed structure W with automorphism group G.

Theorem 6 The free finite cover η2 : C2 → [Ω]2 with binding groups Z2 and fibre groups Z2 is minimal.

Theorem 7 The free finite cover η3 : C3 → [Ω]3 with binding groups Z2 and fibre groups Z3 has only two distinct proper covering expansions (up to isomorphism). The kernels of the expansions are (ker(λ1) × Ω) and (S3 × Ω). Moreover the expansion with kernel (ker(λ1) × Ω) is the only minimal one.