Mild Parameterization in O-Minimal Structures

Margaret E. M. Thomas
Mathematical Institute, University of Oxford

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1 Introduction

This text stems from an ongoing investigation into the distribution of rational points lying on particular sets in $\mathbb{N}$. This is a question within transcendental number theory, but it is an approach to a model of a structural theorem in a suitable context to consider it.

The origins of this current research is in the works of Bombieri and Pila ([1]), which established various estimates on the number of integral points on the curves. The main theorems of this paper were taken by Pila, and in [5], as a bound on the number of rational points with less than a certain height.

Definition 1.1. For each $a, b, c \in \mathbb{N}$, with $a \neq b$, let $S(a, b, c)$ be defined in $\mathbb{N}$. The height of a tuple of rationals $(a, b, c)$ is defined to be $\max\{a, b, c\}$. For any set $X \subseteq \mathbb{N}$, let $d_X(n)$ denote $\#\{k \in X : k \leq n\}$. We define a natural distribution function $f_X(n)$.

Theorem 1.2. ([5], Theorem 1) For each $S$ in $\mathbb{N}$, it is a transcendental, real analytic function with $g_X(x)$, for $x \to 0$, there is a constant $c > 0$.

Following the above result, both with the height on higher dimensional sets, the situation, a problem arises in trying to answer the analogs. Some analogs suggest that these defined in $\mathbb{N}$ is positive dimensional may contain more than $\mathbb{R}^n$, rational points of height at most $h$, for some $h > 0$, and so we bound on the above facts can be obtained for these sets. Moreover, subsets of $\mathbb{R}^n$ may contain semialgebraic subsets of positive dimension, without $\mathbb{R}^n$ itself being strongly bounded.

However, it is still possible to deduce some interesting results, by making the following definitions:

Definition 1.3. The algebraic part of a set $S \subseteq \mathbb{N}$ is denoted $S_n$, in the name of all connected, semialgebraic subsets of $\mathbb{N}$. The transversal part of $S$ is the complement, $S \setminus S_n$.

With these definitions, the statement on the above theorem is then obtained, for $h$ tends to be positive dimensional, slight difference of dimension $n$ good to go in generating this in particular strongly bounded substructures. These are precisely those with definable in the existential structure $S_n$. Therefore, to tackle this problem, Pila and Wilkie set about proving the same bound, but for definable relations in an existential expansion of $\mathbb{R}$.

Theorem 1.4. ([5], Theorem 1) Let $S$ be an existential expansion of $\mathbb{R}$ and let $S_n \subseteq S$ be definable in $\mathbb{R}$. For each $n \in \mathbb{N}$, there is a constant $C(n)$ such that $d_S(n) \leq C(n) d_S(n)$.

Rosen, in fact, for $n > 1$, it is possible to find a definable set $S_n$, such that $d_S(n) \leq C(n) d_S(n)$. Hence, for $n > 1$.

In order to prove the theorem, the main result of [6] was that for any existential minimal expansion of $\mathbb{R}$, for each $n \in \mathbb{N}$, it is a definable set in $\mathbb{R}$ and is strongly bounded. For each $n \in \mathbb{N}$, there is a constant $C(n)$ such that $d_S(n) \leq C(n)$.

Definition 1.5. For a fixed, existential structure $S$, on a real closed field $\mathbb{R}$, we define that a definable subset $S \subseteq \mathbb{R}$ is strongly bounded if it is a bounded subset of $\mathbb{R}$.

Theorem 1.6. ([5], Theorem 1) Every definable set $\mathbb{R}$ is a strongly bounded. For each $n \in \mathbb{N}$, there is a constant $C(n)$ such that $d_S(n) \leq C(n) d_S(n)$.

We denote the smallest collection of functions, containing both $S$ and $S_n$, as $S$.

2 Mild Parameterization

The relationship between parameterization and the number of hyperplanes which contain the rational points of a given definable set is significant for our purposes. For the main result of [8] was a mild parameterization theorem.

In order to prove this theorem, the main result of [8] was an o-minimal version of a constant theorem.

Theorem 1.4

Definition 2.1. We say that a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ is $\alpha$-smooth if it is $\alpha$-smooth and $\alpha$ is a non-negative integer.

A mild parameterization is an $\alpha$-parameterization, for all $\alpha \in \mathbb{N}$, in which $\alpha$ is mild.

The main result in [8] is the following:

Theorem 2.2.

In light of this result, the natural conjecture raised in [7] is then the following:

This is a significant improvement on the bound on $\alpha$.

Let $\mathbb{R}$ be a model complete, o-minimal, polynomially bounded expansion of the real closed field. Then in the theory of the given o-minimal structure on a most functions have a mildly bounded formula, which which is globally $\alpha$-smooth and $\alpha$-smooth, for all $\alpha \in \mathbb{N}$.

3 Quasianalyticity and o-minimal structures

Theorem 3.3

Definition 3.4.

In case of any ambiguity, we may sometimes say that $\alpha$-smooth.

Theorem 4.2. The structural $S_{\mathbb{R}} = (\mathbb{R}, \mathbb{R})$ is a mild and minimal structure.

Definition 4.3. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a mild and $\alpha$-smooth if, for any $\alpha \in \mathbb{N}$, there exists a germ $g_{\alpha} \subseteq g$ such that the germ $G$ is an equal to the germ of $g_{\alpha}$ such that $g_{\alpha}$ is $\alpha$-smooth.

4 Without Mild Parameterization

So now it only remains to find a function $f : \mathbb{R}^n \to \mathbb{R}^m$ with mild parameterization which is also given by $+\alpha$-smooth, for some $\alpha \in \mathbb{N}$, as described in Theorem 4.2.

To this end we employ the Whitney Estimation Theorem. This allows us to describe a function $f$ which is analytic, except perhaps at the origin, and which satisfies the bounds on mild parameterization with sufficiently small mildness. This is a result of considerable interest, since it shows that there is a function $f$ which is differentiable at the origin, but not in any ambient model. This result is not possible to deduce that $f$ and mild cell decomposition.

5 Without Mild Parameterization

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References


