Differential schemes for rings with a Hasse derivation

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Hasse derivations - definition

A Hasse derivation on a (commutative, unitary) ring A is a sequence $D = (D_i)_{i\in\mathbb{N}}$ of additive operators $D_i : A \longrightarrow A$ such that :

- $D_0 = id_A$
- $D_i(xy) = \sum_{m+n=i} D_m(x)D_n(y)$ (generalized Leibniz rule)
- $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ (iteration rule)

Remark : D_1 is a usual derivation.
If A is a $\mathbb{Q}$-algebra, a Hasse derivation is nothing else than D_1 since $D_i = \frac{1}{i!}D_1^i$.

A little bit of model-theory

There is a complete first order theory of existentially closed fields with a Hasse derivation, of fixed characteristic p (results by Robinson in char. zero, by Ziegler in positive char.).

These theories admit QE.
Consequence : type-definable sets are given, up to some boolean combinations, as zero sets of families of differential polynomials.

Analogy with affine varieties in algebraically closed fields.
Aim : describe the objects of differential algebraic geometry in the language of schemes.

Affine D-schemes

A a D-ring.
$V = Spec_D(A) :=$ set of prime D-ideals of A.

Topology on V : Closed sets are the $\mathcal{V}_D(B) := \{I \in Spec_D(A) \mid B \subseteq I\}$ for every $B \subseteq A$.

Sheaf of regular functions $\mathcal{O}_V^D$: for $U \subseteq V$ open, $\mathcal{O}_V^D(U)$ is the ring of functions f

$$I \in U \mapsto f(I) \in A_I$$

which can be written locally as $I \mapsto (a/b)_I$ for some $a, b \in A$.

Comparison with usual affine schemes

$Spec_D(A)$ has the same nice topological properties as the usual affine schemes (in particular, it is compact).
As in the usual case, the stalk of the sheaf in each I, $\mathcal{O}^D_{V,I}$, is isomorphic to the localized ring A_I. It gives a natural structure of sheaf of D-rings to $\mathcal{O}^D_V$.

We have a natural D-homomorphism

$$\begin{array}{rccc} \iota_A : & A & \longrightarrow & \widehat{A} := \mathcal{O}^D_V(V) \\ & a & \mapsto & (I \mapsto a_I) \end{array}$$

BUT, unlike the case of usual schemes, this is not an isomorphism in general.

An example (Kovacic)

Let $A := k[x, y, D_1(y), \ldots, D_i(y), \ldots]/(xy, \ldots, D_i(xy), \ldots)$ for k a D-field (of char. 0), with $D_1(x) = 1$.
y is not zero in A. But $\iota_A(y) = 0$: for each $I \in Spec_D(A)$, $x \notin I$ since $1 = D_1(x) \notin I$, and $xy = 0$, hence $y_I = 0 \in A_I$.

Note that $Ann(y)$, the annulator of y, is not a D-ideal, since $x \in Ann(y)$ but $1 = D_1(x) \notin Ann(y)$.

The “well-mixed” case

Assume that A is ”well-mixed”, i.e. the annulator $Ann(a)$ is a D-ideal for each $a \in A$ (this is the case for example if A is reduced, or if the Hasse derivation is trivial on A).
Then ι_A is injective.

Furthermore, ι_A may not be surjective, but the induced morphism

$$Spec_D(\iota_A) : Spec_D(\widehat{A}) \longrightarrow Spec_D(A)$$

is an isomorphism.

The general case

We wonder whether $Spec_D(\iota_A)$ is an isomorphism in general.

Theorem 1 *The map induced by ι_a, $Spec_D(\hat{A}) \longrightarrow Spec_D(A)$ as topological spaces, in an homeomorphism.*

A D-homomorphism $\phi : A \longrightarrow B$ is said to be "almost surjective" if for every $b \in B$, for every $I \in Spec_D(A)$, there are $a_1, a_2 \in A$, with $a_1 \notin I$ such that $\phi(a_1)b = \phi(a_2)$.

Theorem 2 *The following are equivalent :*

– $Spec_D(\iota_A)$ *is an isomorphism of affine D-schemes*

– ι_A *is almost surjective*

– $\iota_{\hat{A}}$ *is an isomorphism*

The category of entire affine D-schemes

An affine D-scheme V which satisfies these conditions is called "entire". It is an intrinsic property of V (does not depend of the choice of A such that $V = Spec_D(A)$).
We can exhibit affine D-schemes which are not entire.

A D-ring A such that ι_A is an isomorphism is called "entire".

Corollary 1 *The functors $Spec_D$ and "global sections" give an equivalence of categories between entire affine D-schemes and entire D-rings.*

Corollary 2 *The product of two entire affine D-schemes over an affine D-scheme exists.*

We don't know how to make such a basic construction in general !