Lipschitz continuity properties

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Introduction

Definition

A function $f : X \to Y$ is called Lipschitz continuous with constant C if, for each $x_1, x_2 \in X$ one has

$$d(f(x_1), f(x_2)) \leq C \cdot d(x_1, x_2),$$

where d stands for the distance.

(Question)

When is a definable function piecewise C-Lipschitz for some $C > 0$?

Clearly

$$\mathbb{R}_{>0} \to \mathbb{R} : x \mapsto 1/x$$

is not Lipschitz continuous,
nor is

$$\mathbb{R}_{>0} \to \mathbb{R} : x \mapsto \sqrt{x},$$

because the derivatives are unbounded.

The real setting

Theorem (Kurdyka, subanalytic, semi-algebraic [1])

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a definable C^1-function such that

$$|\partial f/\partial x_i| < M$$

for some M and each i.
Then there exist a finite partition of X and $C > 0$ such that on each piece, the restriction of f to this piece is C-Lipschitz.
Moreover, this finite partition only depends on X and not on f.
(And C only depends on M and n.)

A whole framework is set up to obtain this (and more).

Krzysztof Kurdyka, *On a subanalytic stratification satisfying a Whitney property with exponent* 1, Real algebraic geometry (Rennes, 1991), Lecture Notes in Math., vol. 1524, Springer, Berlin, 1992, pp. 316–322.

For example, suppose that $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ is C^1 with $|f'(x)| < M$.
Then it suffices to partition X into a finite union of intervals and points.
Indeed, let $I \subset X$ be an interval and $x < y$ in I. Then

$$|f(x) - f(y)| = |\int_x^y f'(z)dz|$$

$$\leq \int_x^y |f'(z)|dz \leq M|y - x|.$$

(Hence one can take $C = M$.)

The real setting

A set $X \subset \mathbb{R}^n$ is called an *s-cell* if it is a cell for some affine coordinate system on $\mathbb{R}^n$.

An s-cell is called *L-regular* with constant M if all "boundary" functions that appear in its description as a cell (for some affine coordinate system) have partial derivatives bounded by M.

The real setting

Theorem (Kurdyka, subanalytic, semi-algebraic)

Let $A \subset \mathbb{R}^n$ be definable.
Then there exists a finite partition of A into L-regular s-cells with some constant M. (And M only depends on n.)

Lemma

Let $A \subset \mathbb{R}^n$ be an L-regular s-cell with some constant M.
Then there exists a constant N such that for any $x, y \in A$ there exists a path γ in A with endpoints x and y and with

$$length(\gamma) \leq N \cdot |x - y|$$

(And N only depends on n and M.)

Proof.

By induction on n. □

(Uses the chain rule for differentiation and the equivalence of the L_1 and the L_2 norm.)

Corollary (Kurdyka)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a definable function such that

$$|\partial f / \partial x_i| < M$$

for some M and each i.
Then f is piecewise C-Lipschitz for some C.

Proof.

One can integrate the (directional) derivative of f along the curve γ to obtain

$$f(x) - f(y)$$

as the value of this integral.
On the other hand, one can bound this integral by

$$c \cdot \text{length}(\gamma) \cdot M$$

for some c only depending on n, and one is done since

$$\text{length}(\gamma) \leq N \cdot |x - y|$$

□

Indeed, use

$$\int_0^1 \frac{d}{dt} f \circ \gamma(t) dt,$$

plus chain rule, and use that the Euclidean norm is equivalent with the L_1-norm.

Proof of existence of partition into L-regular cells.

By induction on n. If $\dim A < n$ then easy by induction. We only treat the case $n = 2$ here.

Suppose $n = \dim A = 2$. We can partition A into s-cells such that the boundaries are ε-flat (that is, the tangent lines at different points on the boundary move "ε-little"), by compactness of the Grassmannian. Now choose new affine coordinates intelligently. Finish by induction.

□

The p-adic setting

No notion of intervals, paths joining two points (let alone a path having endpoints), no relation between integral of derivative and distance.
Moreover, geometry of cells is more difficult to visualize and to describe than on reals.

A p-adic cell $X \subset \mathbb{Q}_p$ is a set of the form

$$\{x \in \mathbb{Q}_p \mid |a| < |x - c| < |b|, x - c \in \lambda P_n\},$$

where P_n is the set of nonzero n-th powers in $\mathbb{Q}_p$, $n \geq 2$.

c lies outside the cell but is called "the center" of the cell.

In general, for a family of definable subsets X_y of $\mathbb{Q}_p$,
a, b, c may depend on the parameters y and then the family X is still called a cell.

A cell $X \subset \mathbb{Q}_p$ is naturally a union of balls. Namely, (when $n \geq 2$) around each $x \in X$ there is a unique biggest ball B with $B \subset X$.

The ball around x depends only on $\operatorname{ord}(x - c)$ and the m first p-adic digits of $x - c$.

Hence, these balls have a nice description using the center of the cell.

Let's call these balls "the balls of the cell".

Let $f : X \to \mathbb{Q}_p$ be definable with $X \subset \mathbb{Q}_p$.

$>$From the study in the context of b-minimality we know that we can find a finite partition of X into cells such that f is C^1 on each cell, and either injective or constant on each cell.

Moreover, $|f'|$ is constant on each ball of any such cell.

Moreover, if f is injective on a cell A, then f sends any ball of A bijectively to a ball in $\mathbb{Q}_p$, with distances exactly controlled by $|f'|$ on that ball.

(Question)

Can we take the cells A such that each $f(A)$ is a cell?
Main point: is there a center for $f(A)$?

Answer (new): Yes. (not too hard.)

Corollary

Let $f : X \subset \mathbb{Q}_p \to \mathbb{Q}_p$ be such that $|f'| \leq M$ for some $M > 0$. Then f is piecewise C-Lipschitz continuous for some C.

Proof.

On each ball of a cell, we are ok since $|f'|$ exactly controls distances. A cell A has of course only one center c, and the image $f(A)$ too, say d. Only the first m p-adic digits of $x - c$ and $\mathrm{ord}(x - c)$ are fixed on a ball, and similarly in the "image ball" in $f(A)$. Hence, two different balls of A are send to balls of $f(A)$ with
the right size,
the right description (centered around the same d).
Hence done.
(easiest to see if only one p-adic digit is fixed.)

□

The same proof yields:
Let $f_y : X_y \subset \mathbb{Q}_p \to \mathbb{Q}_p$ be a (definable) family of definable functions in one variable with bounded derivative.
Then there exist C and a finite partition of X (yielding definable partitions of X_y) such that for each y and each part in X_y, f_y is C-Lipschitz continuous thereon.

Theorem

Let Y and $X \subset \mathbb{Q}_p^m \times Y$ and $f : X \to \mathbb{Q}_p$ be definable. Suppose that the function $f_y : X_y \to \mathbb{Q}_p$ has bounded partial derivatives, uniformly in y.
Then there exists a finite partition of X making the restrictions of the f_y C-Lipschitz continuous for some $C > 0$.

(This theorem lacked to complete another project by Loeser, Comte, C. on p-adic local densities.)

We will focus on $m = 2$. The general induction is similar.
Use coordinates (x_1, x_2, y) on $X \subset \mathbb{Q}_p^2 \times Y$.
By induction and the case $m = 1$, we may suppose that $f_{x_1,y}$ and $f_{x_2,y}$ are Lipschitz continuous.

We can't make a path inside a cell, but we can "jump around" with finitely many jumps and control the distances under f of the jumps.

So, recapitulating, if we fix (x_1, y), we can move x_2 freely and control the distances under f, and likewise for fixing (x_2, y).

But, a cell in two variables is not a product of two sets in one variable!
Idea: simplify the shape of the cell.
We may suppose that X is a cell with center c.
Either the derivative of c w.r.t. x_1 is bounded, and then we may suppose that it is Lipschitz by the case $m = 1$ (induction).

Problem: what if the derivative is not bounded?

(Surprizing) answer (new): switch the order of x_1 and x_2 and use c^{-1}, the compositional inverse. This yields a cell!
By the chain rule, the new center has bounded derivative.

Hence, we may suppose that the center is identically zero, after the bi-Lipschitz transformation

$$(x_1, x_2, y) \mapsto (x_1, x_2 - c(x_1, y), y).$$

Do inductively the same in the x_1-variable (easier since it only depends on y).
The cell X_y has the form

$$\{x_1, x_2 \in \mathbb{Q}_p^2 \mid |a(x_1, y)| < |x_2| < |b(x_1, y)|, x_2 \in \lambda P_n,\ (x_1, y) \in A'\},$$

Now jump from the begin point (x_1, x_2) to $(x_1, a(x_1))$.
jump to $(x_1', a(x_1'))$
jump to (x_1', x_2').
We have connected (x_1, x_2) with (x_1', x_2').
Problem: Does $a(x_1)$ have bounded derivative? (recall Kurdyka L-regular).
Solution: if not, then just "switch" "certain aspects" of role of x_1 and x_2. Done.

Open questions:

1) Can one do it based just on the compactness of the Grassmannian?

2) Uniformity in p?

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