Lipschitz continuity properties

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Introduction

Definition

A function \( f : X \rightarrow Y \) is called Lipschitz continuous with constant \( C \) if, for each \( x_1, x_2 \in X \) one has

\[
d(f(x_1), f(x_2)) \leq C \cdot d(x_1, x_2),
\]

where \( d \) stands for the distance.

(Question)

When is a definable function piecewise \( C \)-Lipschitz for some \( C > 0 \)?
Clearly

\[ \mathbb{R}_0^+ \rightarrow \mathbb{R} : x \mapsto \frac{1}{x} \]

is not Lipschitz continuous,

nor is

\[ \mathbb{R}_0^+ \rightarrow \mathbb{R} : x \mapsto \sqrt{x}, \]

because the derivatives are unbounded.
The real setting

Theorem (Kurdyka, subanalytic, semi-algebraic [1])

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a definable $C^1$-function such that

$$\left| \frac{\partial f}{\partial x_i} \right| < M$$

for some $M$ and each $i$.

Then there exist a finite partition of $X$ and $C > 0$ such that on each piece, the restriction of $f$ to this piece is $C$-Lipschitz. Moreover, this finite partition only depends on $X$ and not on $f$. (And $C$ only depends on $M$ and $n$.)

A whole framework is set up to obtain this (and more).
For example, suppose that $X \subset \mathbb{R}$ and $f : X \rightarrow \mathbb{R}$ is $C^1$ with $|f'(x)| < M$.

Then it suffices to partition $X$ into a finite union of intervals and points.

Indeed, let $I \subset X$ be an interval and $x < y$ in $I$. Then

\[
|f(x) - f(y)| = |\int_x^y f'(z)dz| \\
\leq \int_x^y |f'(z)|dz \leq M|y - x|.
\]

(Hence one can take $C = M$.)
A set $X \subset \mathbb{R}^n$ is called an \textit{s-cell} if it is a cell for some affine coordinate system on $\mathbb{R}^n$.

An \textit{s-cell} is called \textit{L-regular} with constant $M$ if all “boundary” functions that appear in its description as a cell (for some affine coordinate system) have partial derivatives bounded by $M$. 
The real setting

Theorem (Kurdyka, subanalytic, semi-algebraic)

Let $A \subset \mathbb{R}^n$ be definable.

Then there exists a finite partition of $A$ into $L$-regular $s$-cells with some constant $M$. (And $M$ only depends on $n$.)
Lemma

Let $A \subset \mathbb{R}^n$ be an $L$-regular $s$-cell with some constant $M$. Then there exists a constant $N$ such that for any $x, y \in A$ there exists a path $\gamma$ in $A$ with endpoints $x$ and $y$ and with

$$\text{length}(\gamma) \leq N \cdot |x - y|$$

(And $N$ only depends on $n$ and $M$.)

Proof.

By induction on $n$.

(Uses the chain rule for differentiation and the equivalence of the $L_1$ and the $L_2$ norm.)
Corollary (Kurdyka)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a definable function such that

$$|\frac{\partial f}{\partial x_i}| < M$$

for some $M$ and each $i$. Then $f$ is piecewise $C$-Lipschitz for some $C$. 
Proof.

One can integrate the (directional) derivative of $f$ along the curve $\gamma$ to obtain

$$f(x) - f(y)$$

as the value of this integral.

On the other hand, one can bound this integral by

$$c \cdot \text{length}(\gamma) \cdot M$$

for some $c$ only depending on $n$, and one is done since

$$\text{length}(\gamma) \leq N \cdot |x - y|$$
Indeed, use

\[ \int_0^1 \frac{d}{dt} f \circ \gamma(t) dt, \]

plus chain rule, and use that the Euclidean norm is equivalent with the \( L_1 \)-norm.
Proof of existence of partition into $L$-regular cells.

By induction on $n$. If $\dim A < n$ then easy by induction. We only treat the case $n = 2$ here. Suppose $n = \dim A = 2$. We can partition $A$ into $s$-cells such that the boundaries are $\varepsilon$-flat (that is, the tangent lines at different points on the boundary move “$\varepsilon$-little”), by compactness of the Grassmannian. Now choose new affine coordinates intelligently. Finish by induction.
The \( p \)-adic setting

No notion of intervals, paths joining two points (let alone a path having endpoints), no relation between integral of derivative and distance. Moreover, geometry of cells is more difficult to visualize and to describe than on reals.
A $p$-adic cell $X \subset \mathbb{Q}_p$ is a set of the form

$$\{ x \in \mathbb{Q}_p \mid |a| < |x - c| < |b|, x - c \in \lambda P_n \},$$

where $P_n$ is the set of nonzero $n$-th powers in $\mathbb{Q}_p$, $n \geq 2$. $c$ lies outside the cell but is called “the center” of the cell.

In general, for a family of definable subsets $X_y$ of $\mathbb{Q}_p$, $a, b, c$ may depend on the parameters $y$ and then the family $X$ is still called a cell.
A cell $X \subset \mathbb{Q}_p$ is naturally a union of balls. Namely, (when $n \geq 2$) around each $x \in X$ there is a unique biggest ball $B$ with $B \subset X$.

The ball around $x$ depends only on $\text{ord}(x - c)$ and the $m$ first $p$-adic digits of $x - c$.

Hence, these balls have a nice description using the center of the cell.

Let’s call these balls “the balls of the cell”.
Let $f : X \to \mathbb{Q}_p$ be definable with $X \subset \mathbb{Q}_p$.

From the study in the context of $b$-minimality we know that we can find a finite partition of $X$ into cells such that $f$ is $C^1$ on each cell, and either injective or constant on each cell.

Moreover, $|f'|$ is constant on each ball of any such cell.

Moreover, if $f$ is injective on a cell $A$, then $f$ sends any ball of $A$ bijectively to a ball in $\mathbb{Q}_p$, with distances exactly controlled by $|f'|$ on that ball.
(Question)

Can we take the cells $A$ such that each $f(A)$ is a cell?
Main point: is there a center for $f(A)$?

Answer (new): Yes. (not too hard.)
Corollary

Let \( f : X \subset \mathbb{Q}_p \rightarrow \mathbb{Q}_p \) be such that \( |f'| \leq M \) for some \( M > 0 \). Then \( f \) is piecewise \( C \)-Lipschitz continuous for some \( C \).

Proof.

On each ball of a cell, we are ok since \( |f'| \) exactly controls distances. A cell \( A \) has of course only one center \( c \), and the image \( f(A) \) too, say \( d \). Only the first \( m \) \( p \)-adic digits of \( x - c \) and \( \text{ord}(x - c) \) are fixed on a ball, and similarly in the “image ball” in \( f(A) \). Hence, two different balls of \( A \) are send to balls of \( f(A) \) with the right size, the right description (centered around the same \( d \)). Hence done.

(easiest to see if only one \( p \)-adic digit is fixed.)
The same proof yields:
Let $f_y : X_y \subset \mathbb{Q}_p \to \mathbb{Q}_p$ be a (definable) family of definable functions in one variable with bounded derivative. Then there exist $C$ and a finite partition of $X$ (yielding definable partitions of $X_y$) such that for each $y$ and each part in $X_y$, $f_y$ is $C$-Lipschitz continuous thereon.
Theorem

Let $Y$ and $X \subset \mathbb{Q}_p^m \times Y$ and $f : X \rightarrow \mathbb{Q}_p$ be definable. Suppose that the function $f_y : X_y \rightarrow \mathbb{Q}_p$ has bounded partial derivatives, uniformly in $y$.

Then there exists a finite partition of $X$ making the restrictions of the $f_y$ C-Lipschitz continuous for some $C > 0$.

(This theorem lacked to complete another project by Loeser, Comte, C. on $p$-adic local densities.)
We will focus on \( m = 2 \). The general induction is similar. Use coordinates \((x_1, x_2, y)\) on \( X \subset \mathbb{Q}_p^2 \times Y \). By induction and the case \( m = 1 \), we may suppose that \( f_{x_1,y} \) and \( f_{x_2,y} \) are Lipschitz continuous.

We can’t make a path inside a cell, but we can “jump around” with finitely many jumps and control the distances under \( f \) of the jumps.

So, recapitulating, if we fix \((x_1, y)\), we can move \( x_2 \) freely and control the distances under \( f \), and likewise for fixing \((x_2, y)\).
But, a cell in two variables is not a product of two sets in one variable!

Idea: simplify the shape of the cell.

We may suppose that $X$ is a cell with center $c$.

Either the derivative of $c$ w.r.t. $x_1$ is bounded, and then we may suppose that it is Lipschitz by the case $m = 1$ (induction).

Problem: what if the derivative is not bounded?

(Surprising) answer (new): switch the order of $x_1$ and $x_2$ and use $c^{-1}$, the compositional inverse. This yields a cell!

By the chain rule, the new center has bounded derivative.
Hence, we may suppose that the center is identically zero, after the bi-Lipschitz transformation

$$(x_1, x_2, y) \mapsto (x_1, x_2 - c(x_1, y), y).$$

Do inductively the same in the $x_1$-variable (easier since it only depends on $y$).

The cell $X_y$ has the form

$$\{x_1, x_2 \in \mathbb{Q}_p^2 \mid |a(x_1, y)| < |x_2| < |b(x_1, y)|, x_2 \in \lambda P_n, (x_1, y) \in A'\},$$

Now jump from the begin point $(x_1, x_2)$ to $(x_1, a(x_1))$.

jump to $(x'_1, a(x'_1))$

jump to $(x'_1, x'_2)$.

We have connected $(x_1, x_2)$ with $(x'_1, x'_2)$.

Problem: Does $a(x_1)$ have bounded derivative? (recall Kurdyka $L$-regular).

Solution: if not, then just “switch” “certain aspects” of role of $x_1$ and $x_2$. Done.
Open questions:

1) Can one do it based just on the compactness of the Grassmannian?

2) Uniformity in $p$?