Small groups of odd type

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A small group of finite Morley rank

PSL$_2$
Groups and rank
PSL\(_2\)
Results

A closer view

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Small groups of odd type
1 Groups and rank
- Groups, rank, and algebraic groups
- Groups of low Morley rank
- Groups of finite MR and finite groups

2 $\text{PSL}_2$
- Early results
- Description
- Analysis

3 Results
- The notion of smallness and results
- Difficulties and solutions
- The main tool
Groups of finite Morley rank appeared as $\aleph_1$-categorical groups.

**Theorem (Baldwin, Zilber)**

A simple group has finite Morley rank iff it is $\aleph_1$-categorical.

In the 80’s, Borovik and Poizat suggested a more naive approach.

**Theorem (Poizat)**

A group has finite Morley rank iff there is a rank function $\text{rk}$ on the set of interpretable sets, which behaves like a dimension ought to.
Typical example of a group of finite Morley rank:

- an alg. group over an alg. closed field, equipped with the Zariski dimension.
- an infinite field of finite Morley rank is alg. closed (Macintyre)

slogan:

*groups of finite Morley rank generalize alg. groups ranked by the Zariski dimension*
Ranked groups and algebraic groups

- Analogies:
  - chain conditions
  - connected components for definable subgroups “$H^\circ$”
  - generation lemmas (in part., $G'$ is definable!)
  - presence of a field (sometimes)

Conjecture (Cherlin-Zilber)

A *simple* infinite group of finite Morley rank is (isomorphic to) an algebraic group over an algebraically closed field.
Let us attack the conjecture inductively.

Fact: There are no simple groups of Morley rank 1 or 2.

- Groups of Morley rank 1 are abelian (Reineke).
- Groups of Morley rank 2 are solvable (Cherlin).

Now what about groups of rank 3?
Some tapas:

- \( SL_2 = \{ M \in GL_2 : \det M = 1 \} \)
- \( Z(SL_2) = \{ \pm Id \} \)
- \( PSL_2 = SL_2/Z(SL_2) \)

\( PSL_2 \) is the smallest simple algebraic group:
Zariski dimension \( = 3 \), Lie rank \( = 1 \), Morley rank \( = 3 \text{rk} K \)

- \( PSL_2 \): only simple algebraic group of Zariski dimension 3
- \( PSL_2 \): only simple algebraic group of Lie rank 1
- \( PSL_2 \) is the basis of inductive arguments \( \rightarrow \) crucial piece

Main question of the talk:

Identify \( PSL_2 \) among small groups of finite Morley rank
Rank 3 and bad groups

Theorem (Cherlin)

A simple group of MR 3 is either $\text{PSL}_2(K)$ or a simple bad group.

- A **bad group** would be a weird non-algebraic configuration.
  - No fields involved.
  - Disjoint union of maximal subgroups.
  - No involutions.

- Open for 30 years!

- Moral:
  
  “low Morley rank” not a good notion of smallness
Conjecture (Cherlin-Zilber)

A simple infinite group of finite Morley rank is an algebraic group over an ACF.

Theorem (A logician’s CFSG)

A simple group of Morley rank 0 is

- the finite version of an algebraic group
- or something else.

Well... you know logicians.
A finite simple group is
- cyclic $\mathbb{Z}/p\mathbb{Z}$
- alternate $A_n$
- the finite version of an alg. group (Chevalley twists welcome)
- or one of 26 “sporadic” known exceptions.

- the only infinite cyclic group, $\mathbb{Z}$, is not $\omega$-stable
- the infinite version of $A_n$ is not stable (not $M_C$)
- fields of finite Morley rank do not allow Chevalley twists
- the sporadics may disappear when one goes to infinite objects
Borovik’s program

- The Cherlin-Zilber Conjecture looks like a simpler CFSG idea (Borovik): imitate CFSG
- (possible gain: a “generic”, simpler CFSG)
- Work with 2-elements, involutions, and their centralizers
- fortunately: good 2-Sylow theory
Four types

- Let $S$ be a Sylow 2-subgroup. Then $S^\circ = U \ast T$, with
  - $U$ of bounded exponent is 2-unipotent
    i.e. definable, connected, of exponent $2^k$
  - $T \cong \mathbb{Z}_{2^\infty}^d$ is a 2-torus of Prüfer rank $d$
    $\mathbb{Z}_{2^\infty}$ is the Prüfer 2-group $\{z \in \mathbb{C} : z^{2^k} = 1 \text{ for some } k \in \mathbb{N}\}$

- One thus defines 4 “types” depending on structure of $S^\circ$
  
<table>
<thead>
<tr>
<th>$U = 1$</th>
<th>$U \neq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 1$</td>
<td>$2^\perp$</td>
</tr>
<tr>
<td>even</td>
<td>mixt</td>
</tr>
</tbody>
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- correspond to the char. of the expected underlying field
State of the Case-Division

- Cases $U \neq 1$ have been solved (Altınel, Borovik, Cherlin).
- Cases $U = 1$ are open.
- The case $U = T = 1$ looks so hard the Conjecture might fail.
  - no Feit-Thompson Theorem
    - FT: finite simple groups have involutions... (would kill bad groups!)

Yet one can work in odd type $S^\circ \simeq \mathbb{Z}_{2^\infty}^d$ ($U = 1$ but $T \neq 1$).

Problem: Identify $\text{PSL}_2$ among small groups of odd type.
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The Hrushovski analysis

**Theorem (Hrushovski)**

Let a non-solvable group of finite MR $G$ act definably and faithfully on a strongly minimal set. Then $G \simeq \text{PSL}_2$ and $\text{rk } G = 3$.

In practice, actions arise from coset spaces.

**Corollary (Cherlin)**

Let $G$ be a non-solvable group of finite Morley rank with a definable subgroup of corank 1. Then $G \simeq \text{PSL}_2$ (and $\text{rk } G = 3$).

Moral: try to understand the action on coset spaces
Caution: this slide contains technical material.

Another identification result using actions.

**Theorem (Delahan-Nesin)**

Let $G$ be a group of finite Morley rank. Assume that $G$ is an infinite split Zassenhaus group. Assume further that the stabilizer of two points contains an involution. Then $G \cong \PSL_2$.

A Zassenhaus group is a 2-transitive group $(G, X)$ s.t. $G_{x,y,z} = 1$. It is split if there is $N \triangleleft G_x$ s.t. $G_x = N \rtimes G_{x,y}$. 
The setting

- Moral of last slide: useful abstract identification results exist
- From now on it will suffice to
  - fix an involution \( i \in G \)
  - fix a Borel \( B \supseteq C^\circ(i) \)
    - Recall that a Borel is a maximal definable, connected, solvable subgroup
  - split \( B \cong \mathbb{K}_+ \rtimes \mathbb{K}^\times \)
  - understand \( G/B \)
- Nesin’s machinery can then recognize \( \text{PSL}_2 \)
  - Question: find natural properties of \( \text{PSL}_2 \) characterizing it
- Latin letters for the abstract group; Greek for the true \( \text{PSL}_2 \).
Study of $\text{PSL}_2$

Let $\mathbb{K} \models \text{ACF} \neq 2$. Let’s have a look at $\text{PSL}_2(\mathbb{K})$.

- $\iota = \begin{pmatrix} i & \  \\ \ -i & \end{pmatrix}$
- $\beta = \left\{ \begin{pmatrix} t & a \\ t^{-1} & \end{pmatrix}, a \in \mathbb{K}, t \in \mathbb{K}^\times \right\} > C^\circ(\iota)$ is a Borel
- $\beta' = F^\circ(\beta) = \left\{ \begin{pmatrix} 1 & a \\ 1 & \end{pmatrix}, a \in \mathbb{K} \right\} \cong \mathbb{K}_+$
- $\Theta = \left\{ \begin{pmatrix} t \\ \ t^{-1} \end{pmatrix}, t \in \mathbb{K}^\times \right\} \cong \mathbb{K}^\times$
- Then $\beta = F^\circ(\beta) \times \Theta \cong \mathbb{K}_+ \times \mathbb{K}^\times$
Modelling the torus

- **Observations** in $\text{PSL}_2$:

  Let $\iota = \begin{pmatrix} i & \ 0 \\ 0 & -i \end{pmatrix} \in \Sigma^\circ$. Note that $\iota$ inverts $F^\circ(\beta)$.

  One has $\Theta = \left\{ \begin{pmatrix} t \\ t^{-1} \end{pmatrix}, t \in K^\times \right\} = C^\circ(\iota)$.

  Let $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Sigma \setminus \Sigma^\circ$. Note that $\omega$ inverts $\Theta$.

- **Modelisation** in $G$: for an involution $w \notin B$, let

  $$T[w] := \{ b \in B, b^w = b^{-1} \}$$

  $T[w]$ will be our model of the torus.

- **Target**: $B = (F^\circ(B))^{-i} \rtimes T[w]$. 
Using $T[w]$

$i \in G$, $B \supseteq C^o(i)$ a Borel.

For an involution $w \notin B$, $T[w] = \{b \in B, b^w = b^{-1}\}$

For generic $w$, $\text{rg } T[w] \geq \text{rg } (F^o(B))^{-i}$.

**Theorem (Zilber)**

Let $A \rtimes T$ be a group of finite Morley rank with $A, T$ two abelian definable infinite subgroups s.t. $T$ is faithful and $A$ is $T$-minimal.

Then there is a definable field $K$ s.t. $A \simeq K_+$ and $T \hookrightarrow K^\times$.

- If $A \subseteq F^o(B)^{-i}$, ranks would force $T[w] \simeq K^\times$...
- ... but $T[w]$ has no reason to be a group!
- As $T[w] \subseteq B \cap B^w$, it would be good to control intersections of Borel subgroups
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Locally solvable groups

- Recall MR is no suitable notion of smallness (as we are unable to solve MR = 3)
- Observation in (P)SL$_2$:
  - if $A < G$ is infinite and abelian, $N^G_\circ(A)$ is solvable.
  - Fails for finite $A$ (e.g. $A = Z(SL_2)$)
  - characterizes (P)SL$_2$ among non-solvable alg. groups

Definition

A group $G$ is locally solvable if: whenever $A < G$ is infinite and abelian, $N^G_\circ(A)$ is solvable.

- Nothing to do with f.g. subgroups; follows another tradition...
- ...from finite group theory and Thompson’s papers.
Theorem

Let $G$ be a locally solvable non-solvable connected group of finite MR. Assume:

- $S^\circ \simeq \mathbb{Z}_{2^\infty}^d$ with $d \geq 1$
- and for any involution $i$ $C_G^\circ(i)$ solvable.
- $G \nsubseteq PSL_2(\mathbb{K})$ for $\mathbb{K} \models ACF \neq 2$.

Then $C_G^\circ(i)$ is always a Borel and either:

1. $S \simeq \mathbb{Z}_{2^\infty}$
2. $S \simeq \mathbb{Z}_{2^\infty} \rtimes \langle i^g \rangle$ and $C^\circ(i)$ is abelian
3. $S \simeq \mathbb{Z}_{2^\infty}^2$ and the three involutions are conjugate
Complications

- Since the first counting arguments involving $T[w]$, the proofs have continuously grown more complex.
- Works by Nesin, J., Cherlin and J., D.
- Main issue: control intersections of Borel subgroups
Keywords

Here are some ingredients of a proof:

- strongly real elements and $T[w]$ sets
- $(0, d)$-Sylow subgroups
- Rigidity Lemmas
- The Bender method, Burdges’ style, revisited
- concentration of semi-simple elements and contradiction!
A key observation

- Fact:
  \[
  \text{In } (\text{P})\text{SL}_2, \text{ Borel subgroups meet on tori}
  \]
  (whatever that means)

- Question: can one mimic this fact in locally solvable groups?

- More precisely: can one prove that distinct Borel subgroups don’t share unipotent elements?

- Subtlety: “unipotent elements” is non-sense to us. Work with unipotent subgroups. Define them first!
Torsion unipotence

Observation:
If $\mathbb{K} \models ACF_p$, then $F^\circ(\beta) = \begin{pmatrix} 1 & * \\ 1 & 1 \end{pmatrix} = \{ g \in \beta : g^p = 1 \}$.

Definition
$U \leq G$ is $p$-unipotent if it is definable, connected, nilpotent, of exponent $p^k$.

Fact (Intersection control)
If $G$ is locally $^\circ$ solvable$^\circ$ and $U \leq G$ is $p$-unipotent, then $U$ lies in a unique Borel, and actually in its Fitting subgroup.

(In $\text{PSL}_2$, $\beta \cap \beta^\omega$ is a torus indeed, thus so is $T[\omega]$)
Burdges’ unipotence

**Fact (Burdges)**

*For each integer* $d \geq 1$, *there is a notion of* $(0, d)$-unipotence *(gradual unipotence)* *and a* $d$-*unipotence radical*

- $d$ is a unipotence degree (more or less heavy)
- problems
  - the $d$-unipotence radical is not always in the Fitting!
  - the heaviest radical (last non-trivial) is in it.
- Caution! two Borels can share $d$-unipotence.
- two Borels of degree $d$ can even share $d$-unipotence!
Rigidity Lemma

Fact (intersection control)

If $G$ is locally solvable and $U \leq G$ is $p$-unipotent, then $U$ is in a unique Borel, and actually in its Fitting subgroup.

Lemma

Let $G$ be locally solvable and $B$ a Borel with unipotence degree $d$. Let $U \triangleleft B$ be a $(0, d)$-unipotent subgroup. Then $B$ is the only Borel of degree $d$ that contains $U$.

- controlling the intersection $B \cap B^w$ is possible...
- ... which will enable us to split $B$. We’re done!
- Moral: Burdges’ 0-unipotence allows intersection control
Acknowledgments

Thank you!