

A semi-linear group which is not affine

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Group topology

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- ▶ The G -topology and the subspace topology coincide on a large subset V of G . ($\dim(G \setminus V) < \dim(G)$)

Affine embedding

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G is called *affine* if the G -topology and the subspace topology coincide on (the whole of) G . We say that G *admits an affine embedding* if there is a definable isomorphism of topological groups $\tau : G \rightarrow G' \subseteq M^r$ between G and an affine definable group G' .

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Equivalently:

- ▶ there a definable injective map $\tau : G \rightarrow M^r$, $r \in \mathbb{N}$, which is continuous with respect to the subspace topology in the range.

Known results

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3. (Edmundo, E.) If G is semi-linear and torsion-free.
(G is definably isomorphic to $\langle M^n, + \rangle$.)

Definition (Peterzil, Steinhorn, 1999)

G is definably compact if for every definable continuous $f : (a, b) \subseteq M \rightarrow G$, $-\infty \leq a, b \leq \infty$, the limit $\lim_{t \rightarrow b^-}^G f(t)$ exists.

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Fact

If $G \subseteq M^n$ is affine, then G is definably compact if and only if it is closed (in M^n) and bounded.

Semi-linear context

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Fact

Every definable function $f : A \subseteq M^n \rightarrow M^m$ is piecewise-linear (PL); that is, there is a partition of A into finitely many definable sets A_i , $i = 1, \dots, k$, such that for each of them:

- ▶ *there is an $n \times m$ matrix λ with entries from D , and an element $a \in M^m$, such that for every $x \in A_i$, $f(x) = \lambda x + a$.*

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$$G \cong_{\text{defly}} \langle S, +_L \rangle.$$

Moreover, there is a definable “parallelogram” $H \subseteq M^n$, such that

$$G \cong \langle H \rangle / L$$

Examples

Let $\mathcal{M} = \langle \mathbb{R}, <, +, 0 \rangle$.

- ▶ $G_1 = \langle [0, 1), +_L, 0 \rangle$, where $L = \mathbb{Z}$.
 $x +_L y = z \Leftrightarrow x + y - z \in \mathbb{Z} \Leftrightarrow x + y - z \in \{0, 1\}$.
- ▶ $G_2 = G_1 \times G_1 = \langle [0, 1) \times [0, 1), +_L, 0 \rangle$, where $L = \mathbb{Z}^2$.
- ▶ $G_3 = \langle [0, 1) \times [0, \pi/2), +_L, 0 \rangle$, where
 $L = \mathbb{Z}(0, 1) + \mathbb{Z}(1/2, \pi/2)$.

Let $\mathcal{M} = \langle M, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D . Define

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We let $x \prec_D y$ if $x \preceq_D y$ but not $y \preceq_D x$. We have:

$$x \prec_D y \Leftrightarrow \forall d \in D, d|x| < |y|.$$

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- ▶ Let $G = \langle [0, a) \times [0, b), +_L, 0 \rangle$, where $L = \mathbb{Z}(a, 0) + \mathbb{Z}(a - c, b) \leq M^2$.

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- ▶ Let $G = \langle [0, a) \times [0, b), +_L, 0 \rangle$, where $L = \mathbb{Z}(a, 0) + \mathbb{Z}(a - c, b) \leq M^2$.

Assume that $\tau : G \rightarrow M^r$ is an affine embedding.

For every $t \in [0, a - c)$, consider the one-to-one G -path

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$\tau(\phi_{nc}) : [0, b) \rightarrow M^r$ has endpoints $\tau(nc, 0)$ and $\tau((n + 1)c, 0)$,

▶ \exists definable onto

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▶ Hence, \exists definable onto $f : [0, b) \rightarrow [0, c)$,

a contradiction.

Theorem

Let $\mathcal{M} = \langle M, +, <, 0 \rangle$ be an ordered divisible abelian group.

Let $G = \langle [0, a) \times [0, b), +_L, 0 \rangle$, where

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Let $G = \langle [0, a) \times [0, b), +_L, 0 \rangle$, where

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Then G admits an affine embedding if and only if

- ▶ $\exists m, n \in \mathbb{Z}, m^2 + n^2 \neq 0, ma + nc \preceq_D b$.

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$$G \cong_{\text{defly}} \langle S, +_L \rangle \cong \langle H \rangle / L,$$

where

- ▶ $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \leq \langle M^n, + \rangle,$
- ▶ $H = \{ \lambda_1 t_1 + \cdots + \lambda_n t_n : -e_j < t_j < e_j \},$
with $e_j > 0$ in M , and $\lambda_j \in D^n$.

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with $e_j > 0$ in M , and $\lambda_j \in D^n$.

Up to a linear transformation, we may assume that H is a rectangle.

Theorem

Let $G = \langle H \rangle / L$ be a definably compact, definably connected semi-linear group of dimension n , where

- ▶ $L = \mathbb{Z}(a_1, a_2) + \mathbb{Z}(b_1, b_2) \leq \langle M^2, + \rangle$
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Then, G admits an affine embedding if and only if the following two conditions both hold:

1. $\exists m_1, n_1 \in \mathbb{Z}, m_1^2 + n_1^2 \neq 0, m_1 a_1 + n_1 b_1 \preceq_D |a_2| + |b_2|,$
2. $\exists m_2, n_2 \in \mathbb{Z}, m_2^2 + n_2^2 \neq 0, m_2 a_2 + n_2 b_2 \preceq_D |a_1| + |b_1|.$

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Corollary

If \mathcal{M} is Archimedean, then G admits an affine embedding.

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- ▶ (Burago, Zalgaller, 1995) Every orientable real PL-manifold of dimension 2 admits an isometric affine embedding into \mathbb{R}^3 . The generalization of this statement to manifolds of higher dimension is open.