

Semiabelian varieties over function fields, logarithmic derivatives, and exponentiation (Joint work with D. Bertrand)

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November 7, 2008

Aims of the talk

- ▶ I will talk about a functional/differential algebraic analogue of Lindemann's theorem, for semiabelian varieties G over function fields K , whose statement is still moving.
- ▶ Lindemann's theorem says that if x_1, \dots, x_n are \mathbb{Q} -linearly independent algebraic numbers, then e^{x_1}, \dots, e^{x_n} are algebraically independent. It is the “exponential side” of Schanuel's conjecture that $\text{tr.deg}(\mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})/\mathbb{Q}) \geq n$ for an arbitrary set $(x_i)_i$ of \mathbb{Q} -linearly independent complex numbers.
- ▶ The novelty, compared with say work of Ax on the function field case, is that we will allow “nonconstant” semiabelian varieties.
- ▶ I will always concentrate on the “exponential” side where the x_i 's are rational over the base field K , even though some methods give information on other cases such as the logarithmic side too.

The functional case for algebraic tori I

- ▶ Let K be an algebraically closed field of transcendence degree 1 over \mathbb{C} . We can equip K with a derivation ∂ with field of constants \mathbb{C} (e.g. ∂ extends d/dt .)
- ▶ If $x \in K$, $y = \exp(x)$ makes sense, as a point in a larger differential field F : $x \in K_0$ for some finitely generated differential subfield of K containing \mathbb{C} . So x can be viewed as a rational function on a complex curve S , so $\exp(x)$ lives in a differential field F_0 of meromorphic functions on some small disc in S , and can be jointly embedded with K over K_0 into suitable F .
- ▶ Moreover the differential relation $\partial y/y = \partial x$ is satisfied by any (y, x) for which $y = \exp(x)$.

The functional case for algebraic tori II

Theorem 1.1

(Exponential side of Ax) Suppose $x_1, \dots, x_n \in K$ are \mathbb{Q} -linearly independent modulo \mathbb{C} . Then

(i) if y_1, \dots, y_n are elements of a differential field $F > K$ such that $\partial y_i / y_i = \partial x_i$ for $i = 1, \dots, n$ then y_1, \dots, y_n are algebraically independent over K .

(ii) In particular if $y_i = \exp(x_i)$ for $i = 1, \dots, n$ then y_1, \dots, y_n are algebraically independent over K .

Note that in this functional setting, the “modulo \mathbb{C} ” part of the hypothesis is needed.

The functional case for algebraic tori III

Proof. (i)

- ▶ If not then we may choose such solutions y_1, \dots, y_n in K^{diff} with $tr.deg(K(y_1, \dots, y_n)/K) < n$.
- ▶ Let $a_i = \partial x_i \in K$. So (y_1, \dots, y_n) is a solution of the system $\partial y_i = a_i y_i$, $i = 1, \dots, n$ of linear differential equations.
- ▶ $L = K(y_1, \dots, y_n)$ is a Picard-Vessiot extension of K .
- ▶ In fact if $\sigma \in Aut(L/K)$ then $\sigma(y_i) = y_i \cdot b_i(\sigma)$ for some unique $b_i(\sigma) \in \mathbb{C}^*$, and the map which takes σ to $(b_1(\sigma), \dots, b_n(\sigma))$ is an isomorphism of $Aut(L/K)$ with a proper algebraic subgroup H of \mathbb{C}^{*n} .
- ▶ H is defined by equations $z_1^{k_1} \cdot \dots \cdot z_n^{k_n} = 1$ ($k_i \in \mathbb{Z}$, not all 0).
- ▶ Hence for some such k_1, \dots, k_n we have that $b_1(\sigma)^{k_1} \cdot \dots \cdot b_n(\sigma)^{k_n} = 1$ for all $\sigma \in Aut(L/K)$.

The functional case for algebraic tori IV

- ▶ Then check that $\sigma(y) = y$ for all $\sigma \in \text{Aut}(L/K)$, where $y = y_1^{k_1} \cdot \dots \cdot y_n^{k_n}$.
- ▶ But then $y \in K$.
- ▶ It is clear that $\partial y/y = \partial x$ where $x = k_1 x_1 + \dots + k_n x_n$, and $x \notin \mathbb{C}$ by hypothesis.
- ▶ So we have reduced the theorem to the case $n = 1$, which states essentially that a rational function $f(z)$ cannot be both a derivative and a logarithmic derivative, unless it is 0, And this is left to the reader.

End of proof.

The functional case for arbitrary semiabelian varieties over \mathbb{C}

- ▶ For G a commutative connected n -dimensional algebraic group over \mathbb{C} and $LG = \mathbb{G}_a^n$ its Lie algebra, we have $\exp_G : LG(\mathbb{C}) = \mathbb{C}^n \rightarrow G(\mathbb{C})$, an analytic surjective homomorphism between the two complex Lie groups, characterized by its differential at 0 being the identity.
- ▶ We have Kolchin's logarithmic derivative $\partial \ell n_G : G \rightarrow LG$. This is a first order differential rational homomorphism, surjective when considering points in a differentially closed field, and with kernel the constants in whichever differential field the map is being evaluated.
- ▶ For example if G is an elliptic curve over \mathbb{C} in standard form $\partial \ell n_G$ is $\partial x/y$.
- ▶ We just write $\partial : \mathbb{G}_a^n \rightarrow \mathbb{G}_a^n$ for the map taking (x_1, \dots, x_n) to $(\partial(x_1), \dots, \partial(x_n))$.

The functional case for arbitrary semiabelian varieties over \mathbb{C} II

- ▶ If K is as before (tr.deg 1 algebraically closed extension of \mathbb{C} with derivation ∂), and $x \in LG(K) = K^n$, then $y = \exp_G(x) \in G(F)$ for suitable $F > K$ makes sense, and we have:
- ▶ $\partial \ell n_G(y) = \partial(x)$
- ▶ We consider a semiabelian variety G defined over \mathbb{C} , namely we have an exact sequence $T \rightarrow G \rightarrow A$ of commutative algebraic groups over \mathbb{C} with T an algebraic torus and A an abelian variety.
- ▶ Let \tilde{G} be the “universal vectorial extension” of G . Namely \tilde{G} is an extension of G by some vector group $W = \mathbb{G}_a^m$ and for any other such extension H of G there unique $\tilde{G} \rightarrow H$ with everything commuting.

The functional case for arbitrary semiabelian varieties over \mathbb{C} III

Theorem 1.2

(Exponential side of Ax-Kirby-Bertrand) Let G be a semiabelian variety over \mathbb{C} , and let $x \in LG(K)$ be such that

$x \notin LH(K) + LG(\mathbb{C})$ for any proper algebraic subgroup H of G .

(i) Let y be any solution of $\partial \ell n(y) = \partial(x)$ in a differential field F extending K . Then $\text{tr.deg}(K(y)/K) = \dim(G)$. In particular

$\text{tr.deg}(K(\exp_G(x))/K) = \dim(G)$.

(ii) Let $\tilde{x} \in L\tilde{G}(K)$ be any lift of x . Then again for any solution \tilde{y} of $\partial \ell n(-) = \partial(\tilde{x})$ we have that $\text{tr.deg}(K(\tilde{y})/K) = \dim(\tilde{G})$. In particular $\text{tr.deg}(K(\exp_{\tilde{G}}(\tilde{x}))/K) = \dim(\tilde{G})$.

Again this result reduces, via differential Galois theory, to showing that $y \notin G(K)$ in some “irreducible” contexts.

Nonconstant case - background I

- ▶ Let K be as before and we will consider commutative connected algebraic groups G defined over K .
- ▶ We call G constant if G is isomorphic as an algebraic group to one defined over \mathbb{C} .
- ▶ G always has a maximal constant algebraic subgroup, denoted by $G_{(0)}$.
- ▶ There are at least two sources of nonconstant G ; first nonconstant abelian varieties, such as the elliptic curve $y^2 = x(x-1)(x-t)$ where $t \in K \setminus \mathbb{C}$.
- ▶ Secondly nonconstant extensions of a constant abelian varieties A by an algebraic torus: the extensions of A by \mathbb{G}_m have a moduli space (which is the dual abelian variety \hat{A}).

Nonconstant case - background II

- ▶ If A is an abelian variety over K then up to isogeny $A = A_0 \times A_1$ where A_0 is constant, and A_1 of \mathbb{C} -trace 0 (totally nonconstant).
- ▶ If $T \rightarrow G \rightarrow A$ is a semiabelian variety, let G_0 denote the preimage in G of A_0 and call it the *semiconstant* part of G . So $G_{(0)} \subseteq G_0$.

Nonconstant case - exp

- ▶ For G a commutative connected algebraic group over K and $LG = \mathbb{G}_a^n$ its Lie algebra, and for $x \in LG(K)$ we can speak of $exp_G(x)$, as a point in a larger differential field:
- ▶ Again $x \in LG(K_0) = \mathbb{C}(S)$ for some complex curve S with all data defined over K_0 .
- ▶ G is the “generic fibre” of a fibration $\mathbf{G} \rightarrow S$ of complex varieties, where the fibres \mathbf{G}_s are complex algebraic groups.
- ▶ Likewise there is a corresponding complex vector bundle $\mathbf{LG} \rightarrow S$ whose generic fibre is LG .
- ▶ $x \in LG(K_0)$ is then a rational *section* of $\mathbf{LG} \rightarrow S$, holomorphic on some small S_0 .
- ▶ Applying appropriate exp 's in the fibres, gives us a holomorphic section $exp_{\mathbf{G}}(x)$ of $\mathbf{G} \rightarrow S$ above S_0 , which we call $exp_G(x)$, and lives in the differential field of meromorphic functions on S_0 , which extends K_0 .

Nonconstant case - logarithmic derivatives I

- ▶ Let now G be a possibly nonconstant semiabelian variety over K
- ▶ To obtain an appropriate analogue of the differential relation $\partial \ln(y) = \partial(x)$ which was satisfied by the graph of exponentiation in the constant case, we are in general *forced* to pass to the universal vectorial extension \tilde{G} of G .
- ▶ The point is that \tilde{G} has a (unique) so-called D -group structure, namely an extension ∂' of ∂ on K to a derivation of the “coordinate ring” of \tilde{G} which respects co-multiplication.
- ▶ Equivalently, a K -rational homomorphic section $s : G \rightarrow T_{\partial}(\tilde{G})$, where the latter is the “first prolongation” or “shifted tangent bundle” of \tilde{G} .
- ▶ This yields our logarithmic derivative $\partial \ln_{\tilde{G}} : \tilde{G} \rightarrow L\tilde{G}$:

Nonconstant case - logarithmic derivatives II

- ▶ For F a differential field extending K and $g \in \tilde{G}(K)$, $\partial \ell n_{\tilde{G}}(g) = \partial(g) - s(g)$ where $-$ is in the sense of the canonical group structure on $T_{\partial} \tilde{G}$. (Same definition works to give Kolchin's log.derivative in the constant case, taking $s = 0$.)
- ▶ The D -structure on \tilde{G} gives rise to the “connection” $\partial_{L\tilde{G}}$ on $L\tilde{G}$:
- ▶ Either by differentiating (in the sense of Kolchin) $\partial \ell n_{\tilde{G}}$ at the identity, or by considering the map from the cotangent space of \tilde{G} at the identity to itself, induced by the derivation ∂' (as in the work with Ziegler).
- ▶ In any case $\partial_{L\tilde{G}} : L\tilde{G} \rightarrow L\tilde{G}$ is additive and satisfies the Leibniz rule with respect to scalar multiplication, namely equips the vector space $L\tilde{G}$ with a ∂ -module structure, but now possibly nontrivial.

Nonconstant case - logarithmic derivatives III

- ▶ When A is an abelian variety over K , then $L\tilde{A}$ identifies with the dual of the de Rham cohomology group $H_{dR}^1(A)$, and $\partial_{L\tilde{G}}$ coincides with the dual of the standard Gauss-Manin connection on $H_{dR}^1(A)$.
- ▶ In any case for $\tilde{x} \in L\tilde{G}(K)$, and $\tilde{y} = \exp_{\tilde{G}}(\tilde{x})$ it is again the case that $\partial \ell n_{\tilde{G}}(\tilde{y}) = \partial_{L\tilde{G}}(\tilde{x})$, although with our differential algebraic definitions above, this requires some work to verify.
- ▶ We are now in a position to state the main theorem, of which Theorem 1.2 above is a special case.

Theorem 2.1

Let G be a semiabelian variety over K . Let $x \in LG(K)$. Assume that

Hyp_x: $x \notin LH(K) + LG_{(0)}(\mathbb{C})$ for any proper algebraic subgroup H of G , and moreover for any algebraic subgroup G_1 of G , the same holds for the image of x in $L(G/G_1)$.

Let $\tilde{x} \in L\tilde{G}(K)$ be any lift of x . Then

(i) If \tilde{y} is any solution of $\partial \ln_{\tilde{G}}(-) = \partial_{L\tilde{G}}(\tilde{x})$ in a differential field $(F, \partial) \supseteq (K, \partial)$ then $\text{tr.deg}(K(\tilde{y})/K) = \dim(\tilde{G})$.

(ii) In particular $\text{tr.deg}(K(\exp_{\tilde{G}}(\tilde{x}))/K) = \dim(\tilde{G})$, and so $\text{tr.deg}(K(\exp_G(x))/K) = \dim(G)$.

Main theorem and remarks II

- ▶ The hypothesis Hyp_x is easily seen to be necessary. But when the semiconstant part G_0 of G coincides with the constant part $G_{(0)}$, then the moreover clause in Hyp_x follows from the first clause, so can be dispensed with.
- ▶ But in the simplest case where the semiconstant part of G is not constant, namely when G is a nonconstant extension of a constant elliptic curve E by \mathbb{G}_m , the moreover clause canNOT be dropped. Even to see this counterexample requires results around variation of mixed Hodge structure.
- ▶ Note that when $G = A$ is an abelian variety with \mathbb{C} -trace 0 then Hyp_x says simply that $x \notin LB(K)$ for any proper abelian subvariety of A , and is a *direct* translation of the hypothesis on x_1, \dots, x_n in the number theoretic situation (Theorems 1.1, 1.2).

Main theorem and remarks III

- ▶ Applying Theorem 2.1 to the case where G is a power of a nonconstant elliptic curve, one obtains:
- ▶ If \wp is an elliptic function with nonconstant invariant $j \in \mathbb{C}(z)$ and zeta function ζ , and if $x_1(z), \dots, x_n(z)$ are \mathbb{Z} -linearly independent algebraic functions, then the $2n$ analytic functions defined on some open domain in \mathbb{C} by $\wp(x_1(z)), \dots, \wp(x_n(z)), \zeta(x_1(z)), \dots, \zeta(x_n(z))$ are algebraically independent over $\mathbb{C}(z)$.

Comments on the proof I

- ▶ The proof of Theorem 2.1 is inductive in nature and takes us into the category of “almost semiabelian D -groups”.
- ▶ Deligne’s theorem of the fixed part (that the set of K -rational solutions of the linear DE $\partial_{L\tilde{A}}(-) = 0$ is trivial when A is abelian and traceless) plays a role.
- ▶ There are essentially two base cases of the inductive proof. The first can be taken to be the case when G is constant (so Theorem 1.2).
- ▶ The second is a kind of $n = 1$ case of the other extreme: and says that when $G = A$ is simple and of \mathbb{C} -trace 0, $x \in LA(K)$ is nonzero, and $\tilde{x} \in L\tilde{A}(K)$ is an arbitrary lift of x , then there is NO $\tilde{y} \in \tilde{A}(K)$ satisfying $\partial \ell n_{\tilde{A}}(\tilde{y}) = \partial_{L\tilde{A}}(\tilde{x})$.
- ▶ The latter is precisely Manin’s “theorem of the kernel” in the form discussed by Coleman and proved by Chai.

Comments on the proof II

- ▶ We call \tilde{G} K -large, if working in the differential closure K^{diff} of K , the kernel of $\partial \ln_{\tilde{G}}$ is contained in $\tilde{G}(K)$.
- ▶ If \tilde{G} is K -large, then the reduction to the two special cases above can be effected via (generalized) differential Galois theory, as in our proof of Theorem 1.1 above.
- ▶ However K -largeness of \tilde{G} is a rather restrictive condition. But it holds for example if G is a product of a torus, a constant A_0 and a “general” traceless A_1 .
- ▶ To effect the inductive proof in general we need the “socle theorem” (from the jet space paper): If G is a connected finite-dimensional differential algebraic group and X is an irreducible differential algebraic subvariety of G with trivial stabilizer, then X is contained in a coset of the maximal “split” or “algebraic” connected differential algebraic subgroup of G .

Comments on the proof III

- ▶ Even in this exponential side of nonconstant Ax , our statement is not optimal. One would like for example, for arbitrary $x \in LG(K)$ a geometric object attached to x which governs the relevant transcendence degrees (as in the usual statements of Ax).
- ▶ One would again look for such statements in the logarithmic and mixed cases, although some work on the logarithmic case already appears in Bertand's paper in the Newton volume.