

# Abelian categories and imaginaries

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October 20, 2008

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More generally, make the same definitions but now with  $R$  replaced by any skeletally small preadditive category  $\mathcal{R}$ .

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and if  $\mathcal{D}$  is a definable subcategory of  $\text{Mod-}\mathcal{R}$  then  $\mathbb{L}(\mathcal{D})^{\text{eq}+}$  is the quotient category/localisation of  $(\text{mod-}\mathcal{R}, \mathbf{Ab})^{\text{fp}}$  by the Serre subcategory of those functors which are 0 on  $\mathcal{D}$ .

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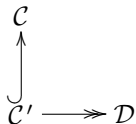
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Let **ABEX** denote the 2-category whose objects are the skeletally small abelian categories, whose (1-)arrows are the exact functors and whose 2-arrows are the natural transformations; let **DEF** denote the 2-category whose objects are the definable additive categories, whose (1-)arrows are the functors which commute with direct products and direct limits (i.e. the interpretation functors) and whose 2-arrows are the natural transformations.

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*The above gives an equivalence of 2-categories.*