

Abelian categories and imaginaries

Mike Prest
Department of Mathematics
Alan Turing Building
University of Manchester
Manchester M13 9PL
UK
mprest@manchester.ac.uk

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More generally, make the same definitions but now with R replaced by any skeletally small preadditive category \mathcal{R} .

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$\mathcal{D} \simeq \text{Ex}(\mathbb{L}(\mathcal{D})^{\text{eq}+}, \mathbf{Ab})$, the category of exact functors from $\mathbb{L}(\mathcal{D})^{\text{eq}+}$ to \mathbf{Ab} .

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and if \mathcal{D} is a definable subcategory of $\text{Mod-}\mathcal{R}$ then $\mathbb{L}(\mathcal{D})^{\text{eq}+}$ is the quotient category/localisation of $(\text{mod-}\mathcal{R}, \mathbf{Ab})^{\text{fp}}$ by the Serre subcategory of those functors which are 0 on \mathcal{D} .

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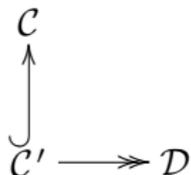
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Let **ABEX** denote the 2-category whose objects are the skeletally small abelian categories, whose (1-)arrows are the exact functors and whose 2-arrows are the natural transformations; let **DEF** denote the 2-category whose objects are the definable additive categories, whose (1-)arrows are the functors which commute with direct products and direct limits (i.e. the interpretation functors) and whose 2-arrows are the natural transformations.

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The above gives an equivalence of 2-categories.