GROUPS DEFINABLE IN LINEAR O-MINIMAL STRUCTURES: 
THE NON-COMPACT CASE

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Abstract. Let \( M = \langle M, +, <, 0, S \rangle \) be a linear o-minimal expansion of an ordered group, and \( G = \langle G, \oplus, e_G \rangle \) an \( n \)-dimensional group definable in \( M \). We show that if \( G \) is definably connected with respect to the \( t \)-topology, then it is definably isomorphic to a definable quotient group \( U/L \), for some convex \( \bigvee \)-definable subgroup \( U \) of \( \langle M^n, + \rangle \) and a lattice \( L \) of rank equal to the dimension of the ‘compact part’ of \( G \).

1. Introduction

This paper is a natural continuation of [ElSt]. Let \( M = \langle M, +, <, 0, \{ \lambda \}_{\lambda \in D} \rangle \) be an ordered vector space over an ordered division ring \( D \). It was shown in [ElSt, Theorem 1.4] that a definably compact group \( G \) definable in \( M \), satisfying the assumptions of the above abstract, is definably isomorphic to a definable quotient group \( U/L \), for some convex \( \bigvee \)-definable subgroup \( U \subseteq \langle M^n, + \rangle \) and a lattice \( L \) of rank \( n \). Here we consider the case where \( G \) is not necessarily definably compact, and generalize [ElSt, Theorem 1.4] towards a structure theorem analogous to the following classical theorem (see, for example, [Bour]).

Fact 1.1. Every connected abelian real Lie group is isomorphic to a direct sum of copies of the additive group \( \langle \mathbb{R}, + \rangle \) of the reals and the torus \( S^1 \).

Moreover, we prove our theorem in the more general setting where \( M \) is any linear o-minimal expansion of an ordered group.

Definition 1.2 ([LP]). An o-minimal expansion \( M = \langle M, +, <, 0, \ldots \rangle \) of an ordered group is called linear if for every \( M \)-definable function \( f : A \subseteq M^n \to M \), there is a partition of \( A \) into finitely many \( A_i \), such that for each \( i \), if \( x, y, x+t, y+t \in A_i \), then

\[
f(x + t) - f(x) = f(y + t) - f(y).
\]

For the rest of this introduction, let \( M \) be a linear o-minimal expansion of an ordered group. By ‘definable’ we mean ‘definable in \( M \) with parameters’.

By [Pil], we know that every definable group \( G \) can be equipped with a unique definable manifold topology that makes it into a topological group, called \( t \)-topology. In the rest of this introduction, all topological notions about such a \( G \) are referring to this \( t \)-topology.

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It is known that a definably connected definable group $G$ is abelian. (See, for example, [PeSt, Corollary 5.1].)

**Fact 1.3 ([EdEl]).** Every definable group $G$ is a definable extension of a definably compact definable group $B$ by $M^r = \langle M^r, +, 0 \rangle$, for some $r \in \mathbb{N}$. That is, there is a short exact sequence

$$0 \to M^r \to G \to B \to 0,$$

where all maps involved are definable homomorphisms.

We let $\sigma : B \to G$ be a definable global section; that is, a definable map such that $q \circ \sigma = \text{id}_B$. Let $K = \sigma(B)$ be the topological group with the structure induced by $\sigma$. We call $K$ the compact part of $G$. Clearly, $G$ is in definable bijection with $M^r \times K$ as abstract sets. As we know by examples in [PeS] and [Str], however, we cannot always expect $G$ to be definably isomorphic to the direct sum of $M^r$ and $K$. We show:

**Theorem 1.4.** Let $G$ be a definably connected definable group of dimension $n$. Assume that the compact part of $G$ has dimension $s$. Then $G$ is definably isomorphic to a definable quotient group $U/L$, for some convex $\forall$-definable subgroup $U \leq \langle M^n, +, 0 \rangle$, and a lattice $L$ of rank $s$.

The terminology in Theorem 1.4 was introduced in [ElSt], and we briefly recall it in Section 2 below. We obtain two corollaries. The first one is a generalization of Pillay’s Conjecture in the present context.

**Proposition 4.1.** Assume that $M$ is sufficiently saturated, and let $G$ be as in Theorem 1.4. Then there is a smallest type-definable subgroup $G^{00}$ of $G$ of bounded index, and $G/G^{00}$ equipped with the logic topology is a compact Lie group of dimension $s$.

**Proposition 5.8.** Let $G$ be as in Theorem 1.4. Then the o-minimal fundamental group of $G$ is isomorphic to $L$.

**Structure of the paper.**

In Sections 2, 3 and 4, we handle the case where $M$ is an ordered vector space over an ordered division ring.

Section 2 contains some definitions and basic results that were proved in [ElSt] without (using) the assumption that $G$ is definably compact.

Section 3 contains the proof of Theorem 1.4, which we outline here. In analogy with [ElSt, Theorem 1.4], the proof consists of three steps. In Step I, we compare the two group operations $\oplus$ and $\cdot$. In Step II, we find a suitable generic open $s$-parallelogram $H$ in $K$. We then let $U$ be the subgroup of $\langle M^n, +, 0 \rangle$ generated by the set $H^2 = M^r \times H$, and define a surjective group homomorphism $\phi : U \to G$. In Step III, we prove that the kernel $L = \ker(\phi)$ is a lattice of rank $s$.

In Section 4, we prove Proposition 4.1.

In Section 5, we extend our results to the case where $M$ is any linear o-minimal expansion of an ordered group, and prove Proposition 5.8.

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2. Preliminaries

Until Section 5, we fix a big sufficiently saturated ordered vector space $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D} \rangle$ over an ordered division ring $D = \langle D, +, \cdot, <, 0, 1 \rangle$.

The terminology and notation of this paper was introduced in [ElSt]. We recall the definitions that play an important role here, and refer the reader to [ElSt] for the rest.

**Definition 2.1** ([ElSt], Section 1). Let $L \subseteq U \subseteq \langle M^n, + \rangle$. The group $U/L$ is called a definable quotient group if there is a definable complete set $S \subseteq U$ of representatives for $U/L$, such that the induced group structure $\langle S, +_S \rangle$ is definable. In this case, we identify $U/L$ with $\langle S, +_S \rangle$.

In [ElSt, Section 2], the $t$-topology of a definable group $G$ and several notions relevant to it were fixed. In what follows, an index ‘$t$’ will indicate that the corresponding notion is taken with respect to the $t$-topology of $G$, if ambiguity would otherwise arise. In [ElSt, Section 3], basic facts about the definable structure of $\mathcal{M}$ were established. In particular, the Linear Cell Decomposition Theorem (Linear CDT) was stated therein. Another important notion was that of an ‘open parallelogram’, which we now recall. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in D^n$ and $x \in M$, let us denote $\lambda x := (\lambda_1 x, \ldots, \lambda_n x)$.

**Definition 2.2** ([ElSt], Definition 3.5). Let $0 < m \leq n$ and $c \in M^n$. An open $m$-parallelogram $H$ with center $c$ is a definable subset $H \subseteq M^n$ of the form

$$H = c + \{ \lambda_1 t_1 + \cdots + \lambda_m t_m : -e_1 < t_i < e_1 \},$$

for some fixed $e_i > 0$ in $M$ and $\lambda_i \in D^n$, $1 \leq i \leq m$.

For the rest of this section, let $G = \langle G, \odot, e_G \rangle$ be a definable group, with $G \subseteq M^n$. We recall some basic results which were proved in [ElSt] without (using) the assumption that $G$ is definably compact or $t$-connected.

**Definition 2.3** ([ElSt], Definition 2.4). Let $W^G$ be a fixed definable large $t$-open subset of $G$ on which the $\mathcal{M}$- and $t$-topologies coincide. Let

$$V^G = \{ a \in G : \text{there is a } t\text{-open neighborhood } V_a \text{ of } a \text{ in } G,$$

such that $\forall x, y \in V_a, x \odot a \odot y = x - a + y \cap W^G$.}

By [ElSt, Lemma 2.5], $V^G$ is a definable, large, open and $t$-open subset of $G$. By cell decomposition, $V^G$ is the disjoint union of finitely many open ($t$-)connected components.

**Fact 2.4** ([ElSt], Lemma 4.7). For all $u, v$ in the same definably connected component of $V^G$, there is $r > 0$ in $M$, such that for all $\varepsilon \in (-r, r)^n$, we have $u + \varepsilon, v + \varepsilon \in G$, and

$$(u + \varepsilon) \odot u = (v + \varepsilon) \odot v.$$

It will often be convenient to assume that $e_G = 0 \in V^G$. In [ElSt, Lemma 4.9] it was shown that we may do so. Namely, it was shown there that $(G, \odot, e_G)$ is definably isomorphic to a topological group $(G', +_G, 0)$ with $0 \in V^{G'}$. The critical lemma is the following fact, which we will also use here later.
Fact 2.5. Let $b \in V^G$ and $f : G \to M^n$ with
$$f(x) = (x \oplus b) - b.$$ If $G' = \langle f(G), +_1, 0 \rangle$ is the induced group, then $V^{G'} = V^G - b.$

Fact 2.5 was proved in [ElSt, Lemma 4.9]. In that proof, a generic element $b$ was chosen, but only the property that $b \in V^G$ was used.

Finally, we state the main theorem from [ElSt], in a form that will be useful in the present paper (and whose proof can easily be extracted from [ElSt]).

Fact 2.6. Let $K = \langle K, \oplus_K, 0 \rangle \subseteq M^s$ be a definably compact, $t$-connected, definable group of dimension $s$. Assume that $H$ is an open $s$-parallelogram, generic in $K$, such that
- $H$ has center 0,
- $H$ is contained in $V^K$.

Let $U_H$ be the subgroup of $\langle M^s, + \rangle$ generated by $H$:
$$U_H = \langle H \rangle = \bigcup_{k<\omega} H + \cdots + H \leq M^s.$$ Then:
- the following map $\phi^K: U_H \to K$ is a well-defined, continuous, surjective group homomorphism: for all $x_1, \ldots, x_k \in H$, if $x = x_1 + \cdots + x_k$, then $\phi^K(x) = x_1 \oplus_K \cdots \oplus_K x_k$.
- $L^K = \ker(\phi^K)$ is a lattice of rank $s$,
- $U_H/L^K = \langle S, +_S \rangle$ is a definable quotient group and $\phi_{H|S}: \langle S, +_S \rangle \to K$ is a definable isomorphism.

That is, $K \cong_{df} U_H/L^K$.

The following fact can also be extracted from [ElSt, Step II], but we include its proof for completeness.

Fact 2.7. Let $K \subseteq M^s$ be a definably compact group of dimension $s$. Let $W$ be a large definable subset of $K$. Then there is an open $s$-parallelogram $H \subseteq W$ which is generic in $K$.

Proof. Since $W$ is large in $K$, it is also generic. By Linear CDT ([ElSt]), $W$ is a finite union of linear cells, and by [PePi, Corollary 3.9], one of them, call it $Y$, must be generic. By [ElSt, Lemma 3.10], $Y$ has dimension $s$. By [ElSt, Lemma 3.6], $\overline{Y}$ is a finite union of closed $s$-parallelograms, say $W_1, \ldots, W_l$. For $i \in \{1, \ldots, l\}$, let $Y_i = \overline{Y} \cap W_i$. Then $Y = Y_1 \cup \cdots \cup Y_l$. By [ElSt, Lemma 3.10] again, one of the $Y_i$’s must be generic, say $Y_1$. Let $H = \text{Int}(Y_1)$. By [ElSt, Lemma 3.10], $H$ is generic.

3. The proof of Theorem 1.4

As described in the introduction, the proof runs in three steps. Let $G = \langle G, \oplus, e_G \rangle$ be a definable group, with $G \subseteq M^n$.

Step I. Comparing $\oplus$ with $+$.

Lemma 3.1. Assume that $e_G = 0$. Assume that $H$ is an open $m$-parallelogram, $m \leq n$, contained in $G$ such that:
This is [ElSt, Lemma 4.27], with identical proof. The assumption that

\[ H \]

is contained in \( V^G \).

Then, for every \( x, y \in H \) such that \( x + y \in H \), we have:

\[ x \oplus y = x + y. \]

**Proof.** We first notice that:

**Claim 1.** For all \( u, v \in H \), such that \( u \oplus v \in H \), there is \( r > 0 \) in \( M \), such that for all \( \varepsilon \in (-r, r)^n \), \( v + \varepsilon \in H \) and

\[ u \oplus (v + \varepsilon) = (u \oplus v) + \varepsilon. \]

Indeed, by Fact 2.4, there is \( r > 0 \) in \( M \), such that \( \forall \varepsilon \in (-r, r)^n \),

\[ (v + \varepsilon) \oplus v = [(u \oplus v) + \varepsilon] \oplus (u \oplus v). \]

**Claim 2.** Let \( \varepsilon(t) : [0, p] \to H, \varepsilon(0) = 0 \), be a path, such that \( \forall t \in [0, p], u + \varepsilon(t) \in H \). Then:

\[ u \oplus \varepsilon(p) = u + \varepsilon(p). \]

Indeed, consider the function

\[ f : G \to M^n \text{ with } x \mapsto (u \oplus x) - x. \]

We show that \( f \) is locally constant on \( \text{Im}(\varepsilon) \). Indeed, by Claim 1, \( \forall s \in [0, p], \exists z > 0 \), such that \( \forall t \in [s - z, s + z] \cap [0, p], \)

\[ u \oplus \varepsilon(t) = u \oplus (\varepsilon(s) + \varepsilon(t) - \varepsilon(s)) = (u \oplus \varepsilon(s)) + \varepsilon(t) - \varepsilon(s). \]

Thus \( \forall t \in [s - z, s + z], f(\varepsilon(t)) = (u \oplus \varepsilon(t)) - \varepsilon(t) = (u \oplus \varepsilon(s)) - \varepsilon(s) = f(u + \varepsilon(s)) \).

It follows that \( f \) is constant on \( \text{Im}(\varepsilon) \) and equal to \( (u \oplus 0) - 0 = u \). Thus, \( \forall t \in [0, p], (u \oplus \varepsilon(t)) - \varepsilon(t) = u \), that is, \( u \oplus \varepsilon(t) = u + \varepsilon(t) \). This proves Claim 2.

Now, let \( x, y \in H \), such that \( x + y \in H \). By [ElSt, Lemma 4.25], there is a path \( \varepsilon(t) \) in \( H \) from 0 to \( y \), such that the path \( x + \varepsilon(t) \) lies entirely in \( H \), as well. By Claim 2 for \( u = x \), we have: \( x \oplus y = x + y \).

**Corollary 3.2.** Let \( H \) be as in Lemma 3.1. Let \( x_1, \ldots, x_l \in H \) be such that for any subset \( \sigma \) of \( \{1, \ldots, l\} \), \( \sum_{j \in \sigma} x_j \in H \). Then, \( x_1 + \cdots + x_l = x_1 \oplus \cdots \oplus x_l \).

**Proof.** By induction on \( l \).

1 = 2. By Lemma 3.1.

1 > 2. \( x_1 + \cdots + x_l = x_1 + (x_2 + \cdots + x_l) = x_1 \oplus (x_2 \oplus \cdots \oplus x_l) = x_1 \oplus \cdots \oplus x_l \).

**Corollary 3.3.** Let \( H \) be as in Lemma 3.1. For every \( x_1, \ldots, x_l, y_1, \ldots, y_m \in H \), if \( x_1 + \cdots + x_l = y_1 + \cdots + y_m \), then \( x_1 \oplus \cdots \oplus x_l = y_1 \oplus \cdots \oplus y_m \).

**Proof.** This is [ElSt, Lemma 4.27], with identical proof. The assumption that \( G \) is definably compact in that proof is not used.

**Step II.** A generic open \( s \)-parallelogram of \( K \). As we saw in the introduction, \( G \) is a definable extension of a definably compact definable group \( B \) by

\[ M' = \langle M^r, +, 0 \rangle, \]

for some \( r \in \mathbb{N} \):

\[ 0 \to M^r \xrightarrow{\sigma} G \xrightarrow{\phi} B \to 0. \]

Let \( \sigma : B \to G \) be a definable global section and \( K = \sigma(B) \). Then \( K \subseteq G \) is a definable complete set of representatives for \( G/M' \). We may choose \( \sigma \) so that
$e_G \in K$. We let $K = (K, \oplus_K, e_G)$ be the topological group with the structure induced by $\sigma : B \to K$. We have:

$$G = \{i(a) \oplus u : a \in M^r, u \in K\}.$$  

Clearly, $i(\langle M \rangle) \cap K = \{e_G\}$. Thus there is a definable bijection $G \to \langle M \rangle \times K$, with $i(a) \oplus u \mapsto (a, u)$. Moreover, since $K$ is definably compact of dimension $n - r$, using [ElSt, Lemma 3.7] we can find a definable injective map $f : K \to M^n - r$ with $f_2(e_G) = 0$. By taking the composition $i(a) \oplus u \mapsto (a, f(u))$, we may assume that:

1. $G = \langle M \rangle \times K \subseteq M^n$
2. $M^r \subseteq G$
3. $G = (G, \oplus, 0)$
4. $K = (K, \oplus_K, 0)$ is a definable complete set of representatives for $G/M^r$ equipped with the induced group structure
5. for every $a \in M^r$ and $u \in K$, $a \oplus u = (a, u)$.

Furthermore, since $V^G$ is a large definable subset of $G$, we may assume that

6. $V^G \cap K$ is large in $K$.

Indeed, if needed, we may take another section $\tau : K \to G$, $\tau(0) = 0$, such that a large subset of its image $K'$ is contained in $V^G$. Then, we may project $K'$ onto $M^n$. If $\pi$ denotes this projection (which is a bijection), it remains to check that $V^G \cap \pi(K')$ is large in $\pi(K')$. Since $\pi$ is piecewise linear, it suffices to show that for every $A \subseteq K'$ on which $\pi$ is linear, $V^G \cap \pi(A)$ is large in $\pi(A)$. We show that $\pi(V^G \cap \pi(A)) \subseteq V^G \cap \pi(A)$, which is clearly enough. Assume $\pi(G) = (\pi(G), \ast, 0)$ is the induced group via $\pi$, and let $c \in \pi(V^G) \cap \pi(A)$. Since $\pi^{-1}(c) \in V^G$, there is a neighborhood $V_c$ of $c$ in $\pi(A)$ such that for all $x, y \in V_c$,

$$x - a \ast y = \pi(\pi^{-1}(x) \oplus_K \pi^{-1}(a) \oplus_K \pi^{-1}(y)) = \pi(\pi^{-1}(x) - \pi^{-1}(a) + \pi^{-1}(y)).$$

But since $\pi$ is linear on $A$, the latter is equal to $\pi(\pi^{-1}(x)) = \pi(\pi^{-1}(a)) + \pi^{-1}(y) = x - a + y$.

Until Section 5, we fix an $n$-dimensional, $t$-connected definable group $G$, and its compact part $K$, such that conditions (1)-(6) above hold. Let $s = \dim(K) = n - r$.

As mentioned in the Introduction, $G$ is abelian.

Observe that the $t$-topology of $K$ is the same as the quotient topology induced by the canonical surjection $q : G \to G/M^r$, by [ElSt, Fact 2.1].

Lemma 3.4. For all $x_1, x_2 \in K$, there is $a \in M^r$, such that

$$x_1 \oplus_K x_2 = x_1 \oplus x_2 \oplus a.$$

Proof. Since $q(x_1 \oplus_K x_2) = q(\sigma(x_1 \oplus x_2)) = q(\sigma(x_1 \oplus x_2)) = q(x_1 \oplus x_2)$, we have $(x_1 \oplus_K x_2) \oplus (x_1 \oplus x_2) \in \ker(q) = M^r$. \(\square\)

We now proceed to define a suitable generic open $s$-parallelogram $H$ in $K$. By Condition (6) above, $V^K \cap V^G \cap K$ is large in $K$. By Fact 2.7, there is an open $s$-parallelogram $H \subseteq K$, generic in $K$, contained in $V^G$ and in $V^K$. We are going to show that $H$ may be assumed to have center 0.

Lemma 3.5. The group $G$ is definably isomorphic to a group $G' = (G', +_{G'}, 0)$, and there is a definably compact group $K' = (K', +_{K'}, 0)$, such that:
(1) \( G' = M^r \times K' \subseteq M^n \)

(2) \( M^r \subseteq G' \)

(3) (a) \( K' \subseteq G' \) is a definable complete set of representatives for \( G'/M^r \), and
(b) \( +_{K'} \) coincides with the group operation induced by the canonical surjection \( q : G' \twoheadrightarrow G'/M^r \).

(4) for every \( a \in M^r \) and \( u \in K' \), \( a +_{G'} u = (a, u) \)

(5) there is an open s-parallelagram \( H' \subseteq K' \), generic in \( K' \), such that:
(a) \( H' \) has center 0,
(b) \( H' \) is contained in \( V^{G'} \) and in \( V^K \).

Proof. Let \( c \) be the center of \( H \). Consider the following two definable bijections:

\[ f_G : G \ni x \mapsto (x + c) - c \in f_G(G) \subseteq M^n, \]

\[ f_K : K \ni x \mapsto (x +_K c) - c \in f_K(K) \subseteq M^n. \]

Now, let \( G' = f_G(G), K' = f_K(K), H' = f_K(H \cap_K c) = H - c. \) Let \( G' = (G', +_{G'}, 0) \) and \( K' = (K', +_{K'}, 0) \) be the induced topological group structures induced by \( f_G \) and \( f_K \), respectively. By [ElSt, Remark 2.2], \( f_G \) and \( f_K \) are definable isomorphisms, for all \( x, y \in G' \),

\[ x +_{G'} y = [(x + c) \oplus c \oplus (y + c)] - c, \]

and for all \( x, y \in K' \),

\[ x +_{K'} y = [(x + c) \oplus_0 c \oplus_K (y + c)] - c. \]

(1) For every \( x = (a, u) = a \oplus u \in G = M^r \times K \), we have:

\[ f_G(x) = [(a \oplus u \oplus c) - c = [a \oplus u \oplus c \oplus (u \oplus_K c)] - c = (a \oplus u \oplus c \oplus (u \oplus_K c), u \oplus_K c) - (0, c) = (a \oplus u \oplus (u \oplus_K c), (u \oplus_K c) - c) \in M^r \times K', \]

by Lemma 3.1. Hence \( G' \subseteq M^r \times K' \). On the other hand, if \( y = (b, v) \in M^r \times K' \), let \( x = (a, u) \in K \) and \( a = b \oplus (u \oplus c) \oplus (u \oplus_K c) \in M^r \). It can be checked that \( f_G(x) = y \). Hence \( G' \supseteq M^r \times K' \).

(2) Observe that \( f_G(M^r) = M^r \). Indeed, for every \( x \in M^r \), \( (x + c) - c = (x, c) - (0, c) = (x, 0) = x \).

(3) (a) We first show that \( K' \subseteq G' \). Let \( (g \oplus_K c) - c \in K' \), for some \( g \in K \). Then \( g_1 = (g \oplus_K c) \oplus c \in G \) and \( (g \oplus_K c) - c = (g_1 \oplus c) - c \in G' \).

We next show that \( K' \) is a definable set of representatives for \( G'/M^r \). Let \( g' = f_G(g) = (g \oplus c) - c \in G' \), for some \( g \in G \). Since \( G = \{ a \oplus k : a \in M^r, k \in K \} \), there are \( a \in M^r \) and \( k \in K \) such that \( g \oplus c = a \oplus k \). Then \( f_G(a) +_{G'} f_K(k \oplus_K c) = [(a \oplus c) - c] +_{G'} (k - c) = (a \oplus c \oplus c \oplus k) - c = (a \oplus k) - c = (g \oplus c) - c = g' \).

Finally, \( K' \) is complete: assume \( f_K(k_1) = f_G(a) +_{G'} f_K(k_2) \), for some \( k_1, k_2 \in K \) and \( a \in M^r \). We show \( k_1 = k_2 \) and, thus, \( f_K(k_1) = f_K(k_2) \). We have \( f_K(k_1) = (k_1 \oplus_K c) - c \) and \( f_G(a) +_{G'} f_K(k_2) = [(a \oplus c) - c] +_{G'} [(k_2 \oplus_K c) - c] = [a \oplus (k_2 \oplus_K c)] - c \). Thus, \( k_1 \oplus_K c = a \oplus (k_2 \oplus_K c) \). Since \( K \) is a complete set of representatives for \( G/M^r \), \( k_1 \oplus_K c = k_2 \oplus_K c \) and, thus, \( k_1 = k_2 \).

(3) (b) We show that for every \( x, y \in K' \), there is \( a \in M^r \) such that \( x +_{K'} y = x +_{G'} y +_{G'} a \). We have \( x +_{K'} y = [(x + c) \oplus_K c \oplus_K (y + c)] - c = [(x + c) \oplus c \oplus_E (y + c) \oplus b] - c \), for some \( b \in M^r \). Let \( a = f_G(b) = (b \oplus c) - c \in M^r \). Then we have \( x +_{G'} y +_{G'} a = [(x + c) \oplus c \oplus (y + c)] - c +_{G'} [(b \oplus c) - c] = [(x + c) \oplus c \oplus (y + c) \oplus c \oplus b \oplus c] - c = x +_{K'} y. \)
(4) We have
\[ a + G, u = [(a + c) \ominus c \oplus (u + c)] - c = [(a, c) \ominus c \oplus (u, c)] - c = [(a, u) - (0, c)] = (a, u). \]

(5)(a) It is clear that \( H' = H - c \) is an open \( s \)-parallelogram with center 0. Since \( H \) is generic in \( K \), \( H \ominus_K c \) is generic in \( K \), and, thus, \( H' = f_K(H \ominus_K c) \) is generic in \( K' \). (b) By Fact 2.5 applied to \( f_G \) and \( f_K \) separately.

We may thus assume that: \( H \subseteq K \) is an open \( s \)-parallelogram, generic in \( K \), such that:

- \( H \) has center 0,
- \( H \) is contained in \( V^G \) and in \( V^K \).

We let \( H^G = \{ a \oplus u : a \in M', u \in H \} = M' \times H \).

Since \( H \) is generic in \( K \), it is easy to see that \( H^G \) is generic in \( G \).

**Lemma 3.6.** There is \( \Xi \in \mathbb{N} \), such that \( G = \underbrace{H^G \oplus \cdots \oplus H^G}_{\Xi \text{-times}} \).

**Proof.** By [ElSt, Lemma 4.29], there is \( \Xi \in \mathbb{N} \), such that \( K = \underbrace{H \oplus \cdots \oplus H}_\Xi \). Since \( G = \{ a \oplus u : a \in M', u \in K \} \), Lemma 3.4 gives \( G = \underbrace{H^G \oplus \cdots \oplus H^G}_{\Xi \text{-times}} \).

**Definition 3.7.** Let \( U_H \) be the subgroup of \( \langle M^s, +, 0 \rangle \) generated by \( H \); that is, \( U_H = \langle H \rangle = \bigcup_{k<\omega} H^k \), where \( H^k = \underbrace{H + \cdots + H}_k \). Let \( U \) be the subgroup of \( M^n = \langle M^n, +, 0 \rangle \) generated by \( H^G \); that is,

\[ U = \langle H^G \rangle = \bigcup_{k<\omega} (H^G)^k. \]

Equivalently, \( U = M^r \times U_H \). By Corollary 3.3, the following function \( \phi : U \to G \) is well-defined. For all \( x_1 = (a_1, u_1), \ldots, x_k = (a_k, u_k) \in H^G = M^r \times H \), if \( x = x_1 + \cdots + x_n \), then
\[ \phi(x) = x_1 \oplus \cdots \oplus x_k = (a_1 + \cdots + a_k) \oplus u_1 \oplus \cdots \oplus u_k. \]

Since \( M^r \) and \( U_H = (U_H, + | U_H, 0) \) are subgroups of \( M^n \), so is their direct product \( U = M^r \times U_H \). Easily, \( U \) is a \( \bigvee \)-definable group, and convexity of \( H \) implies convexity of \( U \).

**Proposition 3.8.** \( \phi \) is a \( t \)-continuous group homomorphism from \( U \) onto \( G \).

**Proof.** \( \phi \) is a group homomorphism, because if \( x = x_1 + \cdots + x_l \) and \( y = y_1 + \cdots + y_m \), with \( x_i, y_i \in K \), then \( \phi(x + y) = \phi(x_1 + \cdots + x_l + y_1 + \cdots + y_m) = x_1 + \cdots + x_l + y_1 + \cdots + y_m = \phi(x) + \phi(y) \). It is onto, by Lemma 3.6. Since \( \oplus \) is \( t \)-continuous, so is \( \phi \).

Thus, if we let \( L = \ker(\phi) \), we know that \( U/L \cong G \) as abstract groups.

**Step III.** \( L \) is a lattice of rank \( s \).

We begin with an easy lemma.
For every $x \in H_G$, $\phi(x) = x$.
(ii) For $x \in U$, since $\phi(x) \in G$, there are $x_1, \ldots, x_\Xi \in H_G$ such that $\phi(x) = x_1 + \cdots + x_\Xi$. Clearly, if $y = x_1 + \cdots + x_\Xi \in (H_G)^\Xi$, then $\phi(y) = (y)$.

**Proof.** The proof is quite standard, but we include it for completeness. Assume that $(a_1, u_1) \oplus \cdots \oplus (a_k, u_k) = (a, u)$. By taking $\sigma \circ q$ on both sides, we have $u_1 \oplus K \cdots K u_k = u$. On the other hand,$$egin{align*}
a_1 \oplus u_1 + \cdots + a_k \oplus u_k &= (a_1, u_1) + \cdots + (a_k, u_k) = (a, u) = a + u,
\end{align*}$$
and hence $a = a_1 + \cdots + a_k + h(u_1, \ldots, u_k)$. Finally, by Lemma 3.4, $h(u_1, \ldots, u_k) \in M^r$, allowing us to replace $\oplus$ by $+$ in the last equation.

By Fact 2.6, $\phi^K : U_H \to K$ is well-defined and $L^K = \ker(\phi)$ has rank $s$. Let $\{w_1, \ldots, w_s\}$ be a fixed set of generators for $L^K$. For every $i \in \{1, \ldots, s\}$, define
$$v_i = -h(w_i^1, \ldots, w_i^k), w_i \in M^r \times U_H = U, \text{ where } w_i = w_i^1 + \cdots + w_i^k, w_i^j \in H.$$

**Claim 3.11.** \{v_1, \ldots, v_s\} is a Z-independent set of generators for $L$.

**Proof.** We first show that each $v_i$ belongs to $L = \ker(\phi)$. By Lemma 3.10, if $x = (a, u) \in M^r \times U_H$, where $u = u_1 + \cdots + u_k$, $u \in H$, we have

$$(1) \quad \phi(x) = (a + h(u_1, \ldots, u_k), \phi^K(u)).$$

It follows that $\phi(v_i) = 0$.

Next we show that for every $x \in L$, there are $l_1, \ldots, l_s \in Z$ such that $x = l_1 v_1 + \cdots + l_s v_s$. Denote by $\tau : U \to U_H$ the group homomorphism $(a, u) \mapsto u$. Observe then, by (1), that $\phi(x) = 0$ implies $\phi^K(\tau(x)) = 0$. Hence, there are $l_1, \ldots, l_s \in Z$ such that
$$\tau(x) = l_1 w_1 + \cdots + l_s w_s = l_1 \tau(v_1) + \cdots + l_s \tau(v_s) = \tau(l_1 v_1 + \cdots + l_s v_s).$$

That is, $\tau(x) = (l_1 v_1 + \cdots + l_s v_s) = 0 \in H$. Hence $x = (l_1 v_1 + \cdots + l_s v_s) \in H^G = M^r \times H$. Since $L \cap H^G = \{0\}$, we have $x - (l_1 v_1 + \cdots + l_s v_s) = 0$.

Finally, if $v_1, \ldots, v_s$ were not Z-independent, then $l_1 v_1 + \cdots + l_s v_s = 0$, for some $l_i \in Z$. Hence $l_1 w_1 + \cdots + l_s w_s = \tau(l_1 v_1 + \cdots + l_s v_s) = 0$, a contradiction.

**Proof of Theorem 1.4.** In Definition 3.7, we defined a convex $\bigvee$-definable subgroup $U \leq M^n$, and an onto group homomorphism $\phi : U \to G$ (Proposition 3.8). In Claim 3.11 we showed that $L = \ker(\phi) \leq U$ is a lattice of rank $s$.

Let $\Sigma = (H_G)^\Xi$, where $\Xi$ is as in Lemma 3.6. Then $\Sigma$ and $\phi|_\Sigma$ are definable. Moreover, the coset equivalence relation induced by $U/L$ on $\Sigma$ is definable, since, for all $x, y \in \Sigma$, we have $x - y \in L \Leftrightarrow \phi(x) = \phi(y)$. By Lemma 3.9(ii), $\Sigma$ contains a complete set $S$ of representatives for $U/L$, and thus, by definable choice, there is a definable such set $S$. By [ElSt, Claim 2.7], $U/L = \langle S, +_S \rangle$ is a definable quotient group. The restriction of $\phi$ on $S$ is a definable group isomorphism between $\langle S, +_S \rangle$ and $G$. By [ElSt, Remark 2.2(ii)], we are done.
Corollary 3.12. For every \( k \in \mathbb{N} \), the \( k \)-torsion subgroup \( G[k] \) of \( G \) is isomorphic to \( (\mathbb{Z}/k\mathbb{Z})^s \).

Proof. By Theorem 1.4, we may assume that \( G \) is a definable set of representatives for \( U/L \). For every \( x \in G \) then, we have
\[
x \oplus \cdots \oplus x = 0 \iff \phi(kx) = 0 \iff kx \in L \cong \mathbb{Z}^s.
\]
Hence, \( x \in G[k] \) if and only if there are unique \( l_1, \ldots, l_n \in \mathbb{Z} \) such that \( x = l_1 \frac{x_1}{k} + \cdots + l_n \frac{x_n}{k} \). Equivalently, since \( x \in G \),
\[
x = \phi(x) = \phi \left( \underbrace{\frac{x_1}{k}}_{l_1 \text{-times}} \oplus \cdots \oplus \underbrace{\frac{x_n}{k}}_{l_n \text{-times}} \right).
\]
Clearly then, the map \( f : G[k] \to (\mathbb{Z}/k\mathbb{Z})^s \), defined by
\[
f(x) = (l_1 \mod k, \ldots, l_s \mod k),
\]
is a well-defined, surjective group homomorphism. To see that it is injective, check that \( f(x) = f(y) \) implies \( x - y \in L \) and, hence, \( x = \phi(x) = \phi(y) = y \). \( \square \)

4. On Pillay’s Conjecture

In [BOPP], the existence of \( G^{00} \) was established for a group \( G \) definable in any \( \omega \)-minimal structure. Here, we compute the dimension of the compact Lie group \( G/G^{00} \), for our fixed \( G \) and \( \mathcal{M} \). The special case where \( G \) is definably compact constitutes Pillay’s Conjecture for \( \mathcal{M} \), proved separately in [ElSt, Proposition 5.1] and [Ons]. The reader is referred to [Pi2] for any terminology.

Proposition 4.1. There is a smallest type-definable subgroup \( G^{00} \) of \( G \) of bounded index, and \( G/G^{00} \) equipped with the logic topology is a compact Lie group of dimension \( s \). Namely, assuming Conditions (1)-(6) from page 6, \( G^{00} = M^r \times K^{00} \).

Proof. For \( i < \omega \), we define \( H_i \) inductively as follows: \( H_0 = H \), and \( H_{i+1} = \frac{1}{2} H_i \). Let also for every \( i < \omega \), \( (H^G)_i = M^r \times H_i \). Denote
\[
B = \bigcap_{i < \omega} (H^G)_i = \bigcap_{i < \omega} (M^r \times H_i) = M^r \times \left( \bigcap_{i < \omega} H_i \right).
\]
Note that, by [ElSt, Proof of Proposition 5.1], \( \bigcup_{i < \omega} H_i = K^{00} \). Now, by Lemma 3.2, it is easy to see that \( B \) is a subgroup of \( G \). By induction and [ElSt, Lemma 4.28], each \( H_i \) is generic in \( K \). It follows that each \( (H^G)_i \) is generic in \( G \), and, thus, \( B \) has bounded index in \( G \). Moreover, it is not hard to see that \( B \) is torsion-free, and, thus, by [BOPP], it must be the smallest type-definable subgroup \( G^{00} \) of \( G \) of bounded index, and \( G/G^{00} \) with the logic topology is a connected compact abelian Lie group. Hence \( G^{00} = B \) is torsion-free. By [BOPP], \( G^{00} \) is also divisible. It follows that for all \( k \), the \( k \)-torsion subgroup of \( G/G^{00} \) is isomorphic to the \( k \)-torsion subgroup of \( G \), which is isomorphic to \( (\mathbb{Z}/k\mathbb{Z})^s \), by Corollary 3.12. Thus, \( G/G^{00} \) is isomorphic to the real \( s \)-torus and has dimension \( s \). \( \square \)
5. Linear o-minimal expansions of ordered groups

Here we show that Theorem 1.4 and Proposition 4.1 hold for a group $G$ definable in a sufficiently saturated linear o-minimal expansion of an ordered group (see Propositions 5.7 and 5.5, respectively). The relation with the context of the previous sections is the following.

Fact 5.1 ([LP]). Let $\mathcal{M} = \langle M, +, <, 0, S \rangle$ be a linear o-minimal expansion of an ordered group. Then $\mathcal{M}$ can be elementarily embedded into a reduct of an ordered vector space $\mathcal{N} = \langle N, +, <, 0, \{ \lambda \}_{\lambda \in D} \rangle$ over an ordered division ring $D$.

Let $\mathcal{M}$ and $\mathcal{N}$ be as above, sufficiently saturated, and $G$ a $t$-connected, $\mathcal{M}$-definable group of dimension $n$. We may assume that $\mathcal{M}$ is a reduct of $\mathcal{N}$, and, thus, $G$ is also $\mathcal{N}$-definable. Then Theorem 1.4 and Proposition 4.1 are true but with all definability stated with respect to $\mathcal{N}$. Namely, since $H$ is $\mathcal{N}$-definable, $U = \langle M^r \times H \rangle$ is $\mathcal{N}$-definable, and $G^{00} = \bigcap_{\lambda < \omega} (M^r \times H_\lambda)$ is type-definable in $\mathcal{N}$. We show however in Proposition 5.5 below that $G^{00}$ is 'absolute'.

For a group $G$ definable in a sufficiently saturated o-minimal structure $\mathcal{M}$, we denote by $G_{00}^{\mathcal{M}}$ the smallest type-definable in $\mathcal{M}$ subgroup of $G$ of bounded index (which exists by [BOPP, Theorem 1.1]). The following fact was pointed out by Pillay. (See [HPP] for any terminology.)

Fact 5.2 ([HPP]). Let $T$ be an o-minimal theory, $\mathcal{M}$ a sufficiently saturated model of $T$, and $G$ a group definable in $\mathcal{M}$. Assume:

1. For all definable $X \subseteq G$, either $X$ or $G \setminus X$ is generic.
2. There is a left-invariant Keisler measure on $G$.

Then ($G^{00}$ exists and) $G^{00}$ is torsion-free.

Fact 5.3 ([BOPP], Corollary 1.2). Let $G$ be a group definable in some sufficiently saturated o-minimal structure $\mathcal{M}$. Assume that $X$ is a torsion-free, type-definable in $\mathcal{M}$, subgroup of $G$ of bounded index. Then $X = G_{00}^{\mathcal{M}}$.

Corollary 5.4. Let $K$ be an abelian, definably compact group, definable in a sufficiently saturated o-minimal expansion $\mathcal{M}$ of an ordered group. Let $\mathcal{N}$ be a sufficiently saturated o-minimal expansion of $\mathcal{M}$. Then $K_{00}^{\mathcal{M}}$ is torsion-free and $K_{00}^{\mathcal{M}} = K_{00}^{\mathcal{N}}$.

Proof. We first verify that the assumptions of Fact 5.2 hold for $K$: (1) holds by [ElSt, Lemma 3.10], and (2) holds because $K$ is abelian. It follows that $K_{00}^{\mathcal{M}}$ is torsion-free. By Fact 5.3, $K_{00}^{\mathcal{M}} = K_{00}^{\mathcal{N}}$. $\square$

In what follows, let $\mathcal{M} = \langle M, +, <, 0, S \rangle$ be a sufficiently saturated linear o-minimal expansion of an ordered group, $G$ a $t$-connected, $\mathcal{M}$-definable group of dimension $n$, and $\mathcal{N}$ a sufficiently saturated ordered vector space over an ordered division ring expanding $\mathcal{M}$ as in Fact 5.1.

We may assume that there is a $\mathcal{M}$-definable group $K$ of dimension $s$ such that Conditions (1)-(6) from page 6 hold. Indeed, those conditions were established directly using the general Fact 1.3 (and not the assumption that $\mathcal{M}$ were a vector space.)

Proposition 5.5. $G_{00}^{\mathcal{M}} = G_{00}^{\mathcal{N}}$. Therefore, $G/G_{00}^{\mathcal{M}}$ equipped with the logic topology is a compact Lie group of dimension $s$. 

Proof. Since $G$ and $K$ are also $\mathcal{N}$-definable, we can find $H \subseteq K$ as in Step II of Section 3, which is $\mathcal{N}$-definable. By Proposition 4.1, $G_{\mathcal{N}}^{00} = M' \times K_{\mathcal{N}}^{00}$. Since $K$ is abelian, $K_{\mathcal{M}}^{00}$ is torsion-free, by Corollary 5.4. Therefore, $M' \times K_{\mathcal{M}}^{00}$ is torsion-free. Since $K_{\mathcal{M}}^{00}$ has bounded index in $K$, easily $M' \times K_{\mathcal{M}}^{00}$ has bounded index in $G$. By Fact 5.3, $G_{\mathcal{M}}^{00} = M' \times K_{\mathcal{M}}^{00}$. But, by Corollary 5.4, $K_{\mathcal{M}}^{00} = K_{\mathcal{M}}^{00}$. It follows that $G_{\mathcal{M}}^{00} = M' \times K_{\mathcal{M}} = M' \times K_{\mathcal{M}}^{00} = G_{\mathcal{M}}^{00}$.

The rest follows from Proposition 4.1. □

In case $G$ is definably compact, we obtain Pillay’s Conjecture in the linear setting.

Corollary 5.6 (Pillay’s Conjecture). Assume $G$ is a $t$-connected, definably compact, $\mathcal{M}$-definable group of dimension $s$. Then there is a smallest type-definable in $\mathcal{M}$ subgroup $G^{00}$ of $G$ of bounded index, and $G/G^{00}$ equipped with the logic topology is a compact Lie group of dimension $s$.

Proposition 5.7. $U = < M' \times H >$ is $\sqrt{s}$-definable in $\mathcal{M}$. Therefore, $G$ is definably isomorphic to a definable quotient group $U/L$, where $U$ is a $\sqrt{s}$-definable in $\mathcal{M}$ subgroup of $M^n$ and $L$ is a lattice of rank $s$.

Proof. Since $K_{00}$ is type-definable in $\mathcal{M}$ and it is contained in the $\mathcal{N}$-definable $H$, by compactness, there exists some $\mathcal{M}$-definable subset $S$ of $H$ that contains $K_{00}$. On the other hand, since $K_{00} = \bigcap_{k \subseteq H_k}$ contained in $X$, by compactness again, there exists some $H_k$ contained in $X$. We have $H_k \subseteq X \subseteq H$, and therefore $U H = < X >$ is a $\sqrt{s}$-definable in $\mathcal{M}$ subgroup of $M^s$. We have that $U = < M' \times X >$ is a $\sqrt{s}$-definable in $\mathcal{M}$ subgroup of $M^n$.

The rest follows from Theorem 1.4. □

5.1. O-minimal fundamental group. The o-minimal fundamental group $\pi_1(G)$ of $G$ can be defined as in the classical case except that all paths and homotopies are taken to be definable. We refer the reader to [ElSt, Section 6] for precise definitions. An adapted argument from that reference would show that $\pi_1(G) \cong L$, but the result in fact follows directly from [EdEl].

Proposition 5.8. $\pi_1(G) \cong L$.

Proof. By [ElSt, Corollary 1.5], there is $l \in \mathbb{N}$ such that $\pi_1(G) \cong \mathbb{Z}^l$ and $G[k] \cong (\mathbb{Z}/k\mathbb{Z})^l$. By Corollary 3.12, $l = s$. □

References


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