STRATIFICATIONS FOR SEMI-LINEAR SETS
- PRELIMINARY VERSION -

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ABSTRACT. In this note we prove that every open semi-linear set is a finite union of open cells. The proof also provides a stratification theorem in this setting.

1. INTRODUCTION

In this section we provide the main definitions. Section 2 contains the main body of work, and Section 3 concludes with the results.

Let $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D}\rangle$ be an ordered vector space over an ordered division ring D. A *linear (affine) function* on $A \subseteq M^n$ is a function $f : A \to M$ of the form $f(x_1, \ldots, x_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n + a$, for some fixed $\lambda_i \in D$ and $a \in M$. In what follows, by 'definable' we mean 'definable in $\mathcal{M}$ possibly with parameters'. Traditionally, the definable sets in $\mathcal{M}$ are also called *semi-linear*.

Definition 1.1. For a definable set $X \subseteq M^n$, we set

$$L(X) = \{f : X \to M : f \text{ is linear}\},$$

$$L_\infty(X) = L(X) \cup \{\pm\infty\},$$

where we regard $-\infty$ and $+\infty$ as constant functions on X.

If $f \in L(X)$, we denote by $\Gamma(f)$ the graph of f. If $f, g \in L_\infty(X)$ with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and denote by $(f, g)_X$ the 'generalized cylinder' $(f, g)_X = \{(x, y) \in X \times M : f(x) < y < g(x)\}$ between f and g. Then,

- a *linear cell in M* is either a singleton subset of M, or an open interval with endpoints in $M \cup \{\pm\infty\}$,
- a *linear cell in M^{n+1}* is a set of the form $\Gamma(f)$, for some $f \in L(X)$, or $(f, g)_X$, for some $f, g \in L_\infty(X)$, $f < g$, where X is a linear cell in M^n.

In either case, X is called *the domain* of the defined cell.

We refer the reader to [vdD, Chapter 3, (2.10)] for the definition of a *decomposition* of M^n. A *linear decomposition of M^n* is then a decomposition $\mathcal{C}$ of M^n such that each $B \in \mathcal{C}$ is a linear cell. The following can be proved similarly to [vdD, Chapter 3, (2.11)].

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Linear Cell Decomposition Theorem (Linear CDT).

(1) *Given any definable sets* $A_1, \dots, A_k \subseteq M^n$*, there is a linear decomposition* $\mathcal{C}$ *of* M^n *that partitions each* A_i*.*

(2) *Given a definable function* $f : A \to M$*, there is a linear decomposition* $\mathcal{C}$ *of* M^n *that partitions* A *such that the restriction* $f_{\restriction B}$ *to each* $B \in \mathcal{C}$ *with* $B \subseteq A$ *is linear.*

The following definition is taken from [ElSt, Section 6], and it originates in [vdD, Chapter 8, (3.1)].

Definition 1.2. Let $f, g : M^m \supseteq X \to M^n$ be two definable continuous maps in M^n. A *homotopy between* f *and* g is a definable continuous map $F(t, s) : X \times [0, r] \to M^n$, for some $r > 0$ in M, such that $f = F_0$ and $g = F_r$, where $\forall s \in [0, r]$, $F_s := F(-, s)$.

Let $A \subseteq X \subseteq M^m$. We say that X *deformation retracts* to A if there is a homotopy $F(t, s) : X \times [0, r] \to X$ such that $F(X, 0) = A$, $F_1 = \mathbf{1}_X$, and $\forall s \in [0, r]$, $F_s \restriction_A = \mathbf{1}_A$.

We say that X is *definably contractible* if it deformation retracts to a point.

Definition 1.3. Let $\mathcal{C}$ be a linear decomposition of M^n and X a definable subset of M^n. Denote

$$Star_{\mathcal{C}}(X) = \{D \in \mathcal{C} : X \cap cl(D) \neq \emptyset\}.$$

The *star of* X *with respect to* $\mathcal{C}$, denoted by $st_{\mathcal{C}}(X)$, is then

$$st_{\mathcal{C}}(X) = \bigcup Star_{\mathcal{C}}(X).$$

We just write $Star(X)$ and $st(X)$ if $\mathcal{C}$ is fixed.

In what follows, if $k > 0$, then $\pi : M^{k+1} \to M^k$ denotes the usual projection map, and if $\mathcal{C}$ is a linear decomposition of M^{k+1}, then $\pi(\mathcal{C})$ denotes the linear decomposition $\{\pi(C) : C \in \mathcal{C}\}$ of M^k.

Lemma 1.4. *Let* $\mathcal{C}$ *be a linear decomposition of* M^n *and* X *a definable subset of* M^n*. Then:*

(i) If $n > 1$*, then* $Star_{\pi(\mathcal{C})}(\pi(X)) = \pi(Star_{\mathcal{C}}(X))$*.*

(ii) If X *is an open union of cells in* $\mathcal{C}$*, and* $C \in \mathcal{C}$ *with* $C \subseteq X$*, then* $st(C) \subseteq X$*.*

Proof. (i) $\subseteq$. Let $D \in Star(\pi(X))$. Since π is open, for any open set U containing X, $\pi(U)$ is an open set containing $\pi(X)$. Thus $D \cap \pi(U) \neq \emptyset$, which implies $\pi^{-1}(D) \cap U \neq \emptyset$. Hence, by Linear CDT, there is some $D' \in Star(X)$ such that $\pi(D') = D$.

$\supseteq$. Let $D \in Star(X)$. For any open set U containing $\pi(X)$, $\pi^{-1}(U)$ is an open neighborhood of X. Therefore $\pi^{-1}(U) \cap D \neq \emptyset$, and $U \cap \pi(D) \neq \emptyset$. Hence $\pi(D)$ belongs to $Star(\pi(X))$.

(ii) Since X is open, for every $B \in Star(C)$, $B \cap X \neq \emptyset$, and hence $B \subseteq X$. □

One would expect that $st_{\mathcal{C}}(X)$ is an open set. However, the following example shows that this is not the case.

Example 1.5. Let $\mathcal{C}$ be a linear decomposition of M^2 that contains the following cells: $(-1, 0) \times (0, 2)$, $(0, 1) \times (0, 2)$, $\{0\} \times (0, 1)$, $\{0\} \times (1, 3)$ and the point $(0, 1)$. Then the star of the point $(0, 1)$ is the union of the above cells, which is not open.

In the next section we define a special kind of a linear decomposition $\mathcal{C}$ of M^n that remedies the above problem. In fact, such a $\mathcal{C}$ will give us that every $st_{\mathcal{C}}(X)$ is an open (usual) cell (see Proposition 2.7 below).

2. Special linear decompositions

The next definition is by induction on n. In what follows, if $h \in L(B)$, $B \subseteq M^n$, and $c \in cl(B)$, we denote $h(c) := \lim_{t\to c} h(t)$, which always exists.

Definition 2.1. A *special linear decomposition of* M is any linear decomposition of M. A *special linear decomposition of* M^{k+1}*,* $k > 0$*, is a linear decomposition* $\mathcal{C}$ *of* M^{k+1} *with the following two properties:*

- Let $C, C' \in \mathcal{C}$ be two linear cells of the form
$$C = (f, g)_B \text{ and } C' = (f', g')_{B'},$$
where $B, B' \subseteq M^k$ are disjoint, $f < g$ in $L_\infty(B)$ and $f' < g'$ in $L_\infty(B')$. Then, for every $c \in cl(B) \cap cl(B')$, if $\pi^{-1}(c) \cap cl(C) \cap cl(C')$ is infinite, then
$$\pi^{-1}(c) \cap cl(C) = \pi^{-1}(c) \cap cl(C').$$
Equivalently,
$$f(c) = f'(c) \text{ and } g(c) = g'(c).$$
- $\pi(\mathcal{C}) = \{\pi(D) : D \in \mathcal{C}\}$ is a special linear decomposition of M^k.

Before providing the nice consequences of special linear decompositions, we prove that they always exist.

Lemma 2.2. *For any linear decomposition* $\mathcal{D}$ *of* M^n*, there is a special linear decomposition* $\mathcal{C}$ *of* M^n *that refines* $\mathcal{D}$ *(that is, every linear cell in* $\mathcal{D}$ *is a union of linear cells in* $\mathcal{C}$*).*

Proof. By induction on n. For $n = 1$, take $\mathcal{C} = \mathcal{D}$. Now assume that $n = k + 1$ and the lemma holds for $k > 0$. Let $\mathcal{D}$ be a linear decomposition of M^{k+1}. Let $\mathcal{F}$ be the collection of linear maps that appear in the definitions of the linear cells that are contained in $\mathcal{D}$. For every $f \in \mathcal{F}$ with domain $X \subseteq M^k$ and form $f(x_1, \ldots, x_k) = \lambda_1 x_1 + \cdots + \lambda_k x_k + c$, we define *the extension of* f *to* M^k to be the linear map $g : M^k \to M$ with $g(x_1, \ldots, x_k) = \lambda_1 x_1 + \cdots + \lambda_k x_k + c$. We say that g *extends* f. Now let
$$\bar{\mathcal{F}} = \{g : M^k \to M \ : \ g \text{ extends some } f \in \mathcal{F}\}.$$
Let
$$\mathcal{G} = \{\Gamma(f) \cap \Gamma(g) : f, g \in \bar{\mathcal{F}}\} \text{ and } \mathcal{G}' = \{\pi(A) : A \in \mathcal{G}\}.$$
Clearly, $\mathcal{G}'$ is a finite collection of definable subsets of M^k. By (Linear CDT and) the inductive hypothesis, there is a special linear decomposition $\mathcal{C}'$ of M^k that partitions each $B \in \mathcal{G}'$.

Claim. *For every two distinct* $f, g \in \bar{\mathcal{F}}$*, and* $X \in \mathcal{C}'$*,*
$$f_{\restriction X} < g_{\restriction X} \text{ or } f_{\restriction X} = g_{\restriction X} \text{ or } f_{\restriction X} > g_{\restriction X}.$$

Proof of Claim. Indeed, let $A = \Gamma(f) \cap \Gamma(g) \neq \emptyset$. Since $\pi(A) \in \mathcal{G}'$ and $\mathcal{C}'$ partitions $\pi(A)$, we have either $X \subseteq \pi(A)$ or $X \subseteq M^k \setminus \pi(A)$. The former implies $f_{\restriction X} = g_{\restriction X}$, whereas the latter implies one of the two other cases. □

We can thus write $\mathcal{C}' = \{X_1, \dots, X_m\}$, such that for each $i \in \{1, \dots, m\}$, $f_{i1} < \cdots < f_{in(i)}$ are the distinct functions in $L(X_i)$, each being a restriction of some $f \in \overline{F}$, and exhausting all possible such. Then

$$\mathcal{C}_i = \{(-\infty, f_{1\restriction X_i}), (f_{1\restriction X_i}, f_{2\restriction X_i}), \dots, (f_{l\restriction X_i}, \infty), \Gamma(f_{1\restriction X_i}), \dots, \Gamma(f_{l\restriction X_i})\}$$

is a partition of $X_i \times M$, and $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$ is a linear decomposition of M^{k+1} which refines $\mathcal{D}$. We show that $\mathcal{C}$ is special. Let $C = (f, g)_B$ and $C' = (f', g')_{B'}$ be as in Definition 2.1. We need to check that

$$f(c) = f'(c) \text{ and } g(c) = g'(c).$$

If not, then since $\pi^{-1}(c) \cap cl(C) \cap cl(C')$ is infinite, we have either

$$f(c) < f'(c) < g(c) \text{ or } f'(c) < g(c) < g'(c).$$

In the first case, the extension h of f' restricted to B satisfies:

$$f < h_{\restriction B} < g.$$

This contradicts the definition of $\mathcal{C}_i = B$.

Finally, notice that $\pi(\mathcal{C}) = \mathcal{C}'$ is a special linear decomposition of M^k. $\square$

We now aim towards Proposition 2.7 below. Observe that for any linear cell $C \subseteq M^n$, $\pi(cl(C)) = cl(\pi(C))$. This is proved for the case C is any bounded cell in [vdD, Chapter 6, (1.7)]. It is remarked there that the boundedness assumption is necessary. However, in the linear case, it is not. For example, following the proof of [vdD, Chapter 6, (1.7)], and if $C = (f, g)_B$, one would define

$$\lambda(t) = \begin{cases} \big(f(\gamma(t)) + g(\gamma(t))\big)/2 & \text{if } f, g \in L(B), \\ f(\gamma(t)) + 1 & \text{if } f \in L(B), g = +\infty, \\ g(\gamma(t)) - 1 & \text{if } f = -\infty, g \in L(B), \\ 0 & \text{if } f = -\infty, g = +\infty, \end{cases}$$

We leave the complete proof to the reader.

Lemma 2.3. *Let $\mathcal{C}$ be a special linear decomposition of M^n, $n > 1$, $D, E \in \mathcal{C}$ two linear cells of the form*

$$D = \Gamma(f) \textit{ and } E = \Gamma(g),$$

where $f \in L(B)$, $g \in L(B')$, and $A = cl(B) \cap cl(B') \neq \emptyset$. Then:

$$f_{\restriction A} < g_{\restriction A} \textit{ or } f_{\restriction A} = g_{\restriction A} \textit{ or } f_{\restriction A} > g_{\restriction A}.$$

Proof. Assume not. Then there are points $c, d \in A$, such that $f(c) = g(c)$ and $f(d) \neq g(d)$. Say, $f(d) < g(d)$. Let $F, G \in \mathcal{C}$ be linear cells of the form $F = (h, k)_B$, $G = (l, m)_{B'}$ such that

$$f_{\restriction A} = h_{\restriction A} < k_{\restriction A} \text{ and } g_{\restriction A} = l_{\restriction A} < m_{\restriction A}.$$

We next claim that there is a point $e \in A$, such that $f(e) < g(e) < k(e)$. If $g(d) < k(d)$, then let $e = d$. So assume $k(d) \leq g(d)$. We will choose e to be 'between' c and d. We first see that there is $q_0 \in (0, 1] \cap \mathbb{Q}$, such that

$$q_0 g(d) + (1 - q_0) g(c) < q_0 k(d) + (1 - q_0) k(c)$$

Indeed, if not, then $k(c) \leq g(c)$. But $g(c) = f(c) < k(c)$, a contradiction. On the other hand, since $f(d) < g(d)$ and $f(c) = g(c)$, we have that for every $q \in (0, 1] \cap \mathbb{Q}$,

$$q f(d) + (1 - q) f(c) < q g(d) + (1 - q) g(c).$$

Hence, if we let $e = q_0 d + (1 - q_0)c$, we have $f(e) < g(e) < k(e)$, proving our claim.

Now, since $f(e) = h(e)$ and $g(e) = l(e)$, we have $h(e) < l(e) < k(e)$. This implies that $\pi^{-1}(e) \cap cl(F) \cap cl(G)$ is infinite, but $\pi^{-1}(e) \cap cl(F) \neq \pi^{-1}(e) \cap cl(G)$, contradicting the fact that $\mathcal{C}$ is special. $\square$

Lemma 2.4. *Let $\mathcal{C}$ be a special linear decomposition of M^n, $n > 1$, and $D, E \in \mathcal{C}$ such that $D \cap cl(E) \neq \emptyset$. Then:*

$$\pi(D) \subseteq cl(\pi(E)) \Rightarrow D \subseteq cl(E).$$

Proof. The statement trivially holds if $D = E$, hence assume $D \neq E$. Let $E = (f, g)_B$ or $E = \Gamma(f)$, for some $f, g \in L_\infty(B)$. If D has domain B, then $E = (f, g)_B$, and $D = \Gamma(f)$ or $D = \Gamma(g)$. Hence, $D \subseteq cl(E)$. So now assume that D has domain $B' = \pi(D)$, disjoint from $B = \pi(E)$, and, for a contradiction, that $B' \subseteq cl(B)$ but $D \nsubseteq cl(E)$. Let $A = cl(B) \cap cl(B') \neq \emptyset$.

Case A: $D = \Gamma(g')$, for some $g' \in L(B')$. Then one of the pairs f, g' or g, g' must contradict Lemma 2.3.

Case B: $D = (f', g')_{B'}$, for some $f', g' \in L_\infty(B)$. Then, again by Lemma 2.3, applied to each of the four pairs $\{f, f'\}$, $\{f, g'\}$, $\{g, f'\}$, $\{g, g'\}$ that are involved, the only remaining possibilities are the following:

$$f'_{\restriction A} < g_{\restriction A} < g'_{\restriction A} \text{ or } f'_{\restriction A} < f_{\restriction A} < g'_{\restriction A}.$$

In the first case, let $F \in \mathcal{C}$ be a linear cell of the form $F = (h, k)_B$, such that

$$g_{\restriction A} = h_{\restriction A} < k_{\restriction A}.$$

Then for any $c \in A$, $f'(c) < h(c) < g'(c)$. This implies that $\pi^{-1}(c) \cap cl(D) \cap cl(F)$ is infinite, but $\pi^{-1}(c) \cap cl(D) \neq \pi^{-1}(c) \cap cl(F)$, contradicting the fact that $\mathcal{C}$ is special. Similarly for the second case. $\square$

Corollary 2.5. *Let $\mathcal{C}$ be a special linear decomposition of M^n, $n > 0$, and $D, E \in \mathcal{C}$ such that $D \cap cl(E) \neq \emptyset$. Then $D \subseteq cl(E)$.*

Proof. The statement trivially holds if $D = E$, hence assume $D \neq E$. We work by induction on n. For $n = 1$, the assumption $D \cap cl(E) \neq \emptyset$ implies that E is an open interval and D is one of its endpoints. So now assume $n > 1$. Clearly, $\pi(D) \cap cl(\pi(E)) \neq \emptyset$, and hence by inductive hypothesis, $\pi(D) \subseteq cl(\pi(E))$. By Lemma 2.4, $D \subseteq cl(E)$. $\square$

Lemma 2.6. *Let $\mathcal{C}$ be a special linear decomposition of M^n, $n > 0$. Then, for any definable $X \subseteq M^n$, $st(X)$ is open.*

Proof. Let $x \in st(X)$, and assume that x is not in the interior of $st(X)$. Then there is a linear cell $E \in \mathcal{C}$, $E \subseteq M^n \setminus st(X)$, such that $x \in cl(E)$. We split cases:

Case A: $x \in X$. Then $x \in X \cap cl(E)$, contradicting the fact that $E \subseteq M^n \setminus st(X)$.

Case B: $x \in D$, for some linear cell $D \in \mathcal{C}$ such that $X \cap cl(D) \neq \emptyset$. Hence $x \in D \cap cl(E) \neq \emptyset$, and thus by Corollary 2.5, $D \subseteq cl(E)$. Therefore $cl(D) \subseteq cl(E)$, and hence $X \cap cl(E) \neq \emptyset$, contradicting the fact that $E \subseteq M^n \setminus st(X)$. $\square$

Proposition 2.7. *Let $\mathcal{C}$ be a special linear decomposition of M^n, $n > 0$, and $C \in \mathcal{C}$. Then $U = st(C)$ is an open (usual) cell.*

Proof. By Lemma 2.6, U is open and thus has dimension n.

Let $\dim(C) = k \leq n$. If $k = n$, then $U = C$ and the statement holds trivially. We may thus assume that $k < n$. We work by induction on n. If $n = 1$, then C is a point and U is an open interval. Now assume that $n = k + 1$ and the Claim holds for $k > 0$.

Assume first that C is a linear cell in $\mathcal{C}$ of dimension k which is the graph of a linear function $h : D \to M$. Then clearly

$$st(C) = (f, g)_D,$$

for some $f, g \in L_\infty(D)$ with $f < h < g$.

In all other cases, $\dim(\pi(C)) < \dim(\pi(U))$. Since $\mathcal{C}$ is a linear decomposition, for every $B \in Star(\pi(C))$, $\pi^{-1}(B) \cap U$ is a union of linear cells in $\mathcal{C}$ which are either graphs of linear maps, or cylinders between linear maps, with domain B. By Lemma 1.4(i), $U \subseteq \bigcup\{\pi^{-1}(B) : B \in Star(\pi(C))\}$, and hence

$$U = \bigcup\{\pi^{-1}(B) \cap U : B \in Star(\pi(C))\}.$$

Since U is open, we easily obtain that for every $B \in Star(\pi(C))$,

$$\pi^{-1}(B) \cap U = (f_B, g_B)_B,$$

for some $f_B, g_B \in L_\infty(B)$ with $f_B < g_B$. Let $D = st(\pi(C))$, $f = \bigcup_{B \in Star(\pi(C))} f_B$ and $g = \bigcup_{B \in Star(\pi(C))} g_B$. Then

$$U = (f, g)_D.$$

By inductive hypothesis, D is a usual cell. To show that f, g are continuous, we need to show that for every $B, B' \in Star(\pi(C))$, and $c \in cl(B) \cap cl(B')$,

$$f_B(c) = f_{B'}(c) \text{ and } g_B(c) = g_{B'}(c).$$

Let $H = (h, g_B)_B$ be the upper-most linear cell in $\mathcal{C}$ contained in $(f_B, g_B)_B$ and $H' = (h', g_{B'})_{B'}$ the upper-most linear cell in $\mathcal{C}$ contained in $(f_{B'}, g_{B'})_{B'}$. By Corollary 2.5, $C \subseteq cl(H) \cap cl(H')$. Hence, if $C = (l, m)_A$, for some $l, m \in L(A)$, then $\pi^{-1}(c) \cap cl(H) \cap cl(H')$ is infinite. On the other hand, if $C = \Gamma(l)$ for some $l \in L(A)$, then by Lemma 2.3,

$$h_{\restriction A} \leq l \text{ and } h'_{\restriction A} \leq l,$$

and hence $\pi^{-1}(c) \cap cl(H) \cap cl(H')$ is again infinite. Since $\mathcal{C}$ is special,

$$h(c) = h'(c) \text{ and } g_B(c) = g_{B'}(c).$$

Similarly, we can show that $f_B(c) = f_{B'}(c)$.

It follows that $U = (f, g)_D$ is a cell. □

3. Applications

3.1. **Nice coverings.** Contractibility of bounded cells already appears in [BF, Lemmas 5.1-5.2] and [ElSt, Lemma 6.6].

Fact 3.1. *Every bounded (usual) cell is definably contractible.*

Corollary 3.2. *Every open definable set $X \subseteq M^n$ is a finite union of open cells.*[1] *Hence, if X is moreover bounded, then X is a finite union of contractible definable sets.*

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[1] An independent proof of this statement has been claimed by Simon Andrews.

Proof. Let $\mathcal{C}$ be special linear decomposition of M^n that partitions X. By Lemma 1.4(ii),

$$X = \bigcup_{C \in \mathcal{C},\, C \subseteq X} st(C).$$

Then apply Proposition 2.7 and Fact 3.1. $\square$

3.2. **Stratifications.** Among others, Lemma 2.1 and Corollary 2.5 yield a stratification theorem, namely [vdD, Chapter 4, Lemma (1.13)], in the this setting.

3.3. **Homology.** The following lemma will be used in developing homological aspects in the linear setting in the future.

Lemma 3.3. *Let $\mathcal{C}$ be a special linear cell decomposition of M^n, and $C, D \in \mathcal{C}$. Then, either*

$$cl(C) \cap cl(D) = \emptyset$$
$$\text{or } cl(C) \cap cl(D) = cl(E),$$

for some $E \in \mathcal{C}$.

Proof. Let $C, D \in \mathcal{C}$ such that $cl(C) \cap cl(D) \neq \emptyset$. If $C = D$, then just let $E = C$. So assume $C \neq D$. We work by induction on n. For $n = 1$, it is obvious. Assume now that $n = k + 1$ and that the statement holds for $n = k$.

Pick any $x \in cl(C) \cap cl(D)$, and let $E \in \mathcal{C}$ be such that $x \in E$. Then $E \cap cl(C) \cap cl(D) \neq \emptyset$, and hence by Corollary 2.5, $E \subseteq cl(C) \cap cl(D)$. It follows that $cl(E) \subseteq cl(C) \cap cl(D)$, and, after projecting, $cl(\pi(E)) \subseteq cl(\pi(C)) \cap cl(\pi(D))$. By Inductive Hypothesis,

$$cl(\pi(E)) = cl(\pi(C)) \cap cl(\pi(D)). \tag{1}$$

We now split two cases:

Case A: $E = (f, g)_B$, for some $f, g \in L_\infty(B)$. Then, for every $c \in cl(B)$, $\pi^{-1}(c) \cap cl(E) \cap cl(C)$ is infinite, and hence $\pi^{-1}(c) \cap cl(E) = \pi^{-1}(c) \cap cl(C)$. Similarly, $\pi^{-1}(c) \cap cl(E) = \pi^{-1}(c) \cap cl(D)$. Now let $y \in cl(C) \cap cl(D)$. By (1), $\pi(y) \in cl(B)$, and hence $y \in cl(E)$. This shows that $cl(C) \cap cl(D) = cl(E)$.

Case B: $E = \Gamma(f)$, for some $f \in L(B)$. If $cl(E)$ is properly contained in $cl(C) \cap cl(D)$, then (1) implies that there is a cell $E' = (h, k)_B$ with $h = f$ or $k = f$, such that $E' \cap cl(C) \cap cl(D) \neq \emptyset$. The above argument then shows that $cl(E') = cl(C) \cap cl(D)$. $\square$

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