Generic expansions of countable models

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joint work with Domenico Zambella

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Our aim: to compare two different notions of

generic models:

- genericity defined in terms of the topology on the space of expansions of a structure (à la Truss-Ivanov);
- genericity related to the existentially closed models of a theory (à la Lascar/Chatzidakis & Pillay).

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Plan:

- define two different notions of a generic **automorphism**, with an easy example;
- define two corresponding notions of generic **expansion**:
  - ‘generic’ (*rich*) expansions in the context of ‘inductive amalgamation classes’
  - Truss-Ivanov generic expansions
- explain the relationship between the two in an easy special case (i.e. the base structure is ω- categorical);
- sketch a generalization.

Truss-generic automorphisms

Let M be a countable structure.

$\mathrm{Aut}(M)$ is a Baire space (with the standard topology, generated by basic open sets of the form

$$\mathrm{Aut}(M)_{ab} \quad := \quad \{g \in \mathrm{Aut}(M) : a^g = b\},$$

where a, b are finite tuples from M)

Definition

$\alpha \in \mathrm{Aut}(M)$ is **Truss-generic** *if*

$$\alpha^{\mathrm{Aut}(M)} := \{\alpha^g : g \in \mathrm{Aut}(M)\}$$

is comeagre, i.e. it contains a countable intersection of dense open sets.

Intuition: α **exhibits any finite behaviour consistent in** $\mathrm{Aut}(M)$.

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Lascar-generic automorphisms

The setting:
T a complete theory with q.e. in a countable language L;
$L_0 = L \cup \{F\}$ an expansion of L by a unary function symbol;
$T_0 = T \cup \{$'F is an automorphism'$\}$.

Definition

Let $(M, \sigma) \models T_0$. Then σ is **Lascar-generic** *if for every partial isomorphism*

$$f : (N, \tau) \to (M, \sigma) \qquad \text{partial}$$

such that $(N, \tau) \models T_0$ is countable and $\mathrm{dom}(f) \subseteq N$ is algebraically closed, there is an embedding

$$\hat{f} : (N, \tau) \to (M, \sigma) \qquad \text{total}$$

extending f.

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Let $T_{rich} := \mathrm{Th}\{(M, \sigma) : M \models T,\ \sigma \text{ Lascar-generic}\}$.

If T is stable:

1. Lascar-generic automorphisms exist;
2. T_0 has a model companion $T_A \ \Rightarrow\ T_A = T_{rich}$;
3. T_{rich} is model-complete $\ \Rightarrow\ T_{rich}$ is the model companion of T_0.

$L = \emptyset$, Ω a countable set, $T = \mathrm{Th}(\Omega)$

Truss-generic automorphisms:

ω fixed points
ω cycles of length 2
ω cycles of length 3
$\vdots$
ω cycles of length n
$\vdots$

Lascar-generic automorphisms:

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Remark

The model companion T_{rich} of T_0 exists.
If $f \in \mathrm{Aut}(\Omega)$ is Truss-generic, $(\Omega, f) \models T_{\mathrm{rich}}$.

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Lascar genericity: richness

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T a complete L-theory
$L_0 = L \cup \{R\}$, where R is a finite tuple of function and relation symbols
T_0 an expansion of T to L_0.

Let κ be a class containing **models + morphisms**, where

- models: infinite models of T_0. Notation: (M, σ), where $M \models T$ and σ is an interpretation of R;
- morphisms are *partial* embeddings between models

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We say that κ is an **inductive amalgamation class** if:

- every morphism is a partial isomorphism;
- every partial elementary map is a morphism;
- (AP) every morphism
$$f : (M,\sigma) \to (N,\tau) \qquad \text{(partial)}$$
extends to a total morphism
$$\hat{f} : (M,\sigma) \to (N',\tau') \qquad \text{(total)}$$
- (JEP) for every $(M_1,\sigma_1),(M_2,\sigma_2) \in \kappa$ there are a model (N,τ) and total morphisms $f_i : (M_i,\sigma_i) \to (N,\tau)$;
- the class of morphisms is closed under inverse and composition;
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Definition

$(M, \sigma) \in \kappa$ is **rich**[a] *if every morphism*

$$f : (N, \tau) \longrightarrow (M, \sigma) \qquad \textbf{(partial)}$$

such that $|f| < |N| \leq |M|$ extends to a total morphism

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[a] Lascar's $\aleph_1$-generic if σ, τ are automorphisms and $|f| = \aleph_0$

Fact

All rich models have the same theory.

(JEP is essential in the proof!)

Definition

Let κ be an inductive amalgamation class. Then

$$T_{\mathrm{rich}} := \mathrm{Th}(\{U \in \kappa : U \text{ is rich}\}).$$

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A result (Chatzidakis & Pillay, adapted to our context):

Notation: $M \leq N$ if $\mathrm{id}_M : M \to N$ is a morphism.

Theorem

Let κ be an inductive amalgamation class, and suppose further that:

$$\text{if } M, N \models T_{\mathrm{rich}}, \text{ then } M \subseteq N \iff M \leq N.$$

Tfae:

- T_{rich} *is model complete;*
- *all rich models are saturated;*
- T_{rich} *is the model companion of* T_0.

Viceversa, if T_0 has a model companion, then T_{rich} is this model companion.

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These results hold even when:

- the models in κ are not necessarily the models of a theory (although we need models to be structures in a given language and κ be closed under elementary equivalence);
- JEP does not hold (then κ can be partitioned into 'connected components', within each of which JEP holds).

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- T be a complete L-theory with q.e. and with the PAPA (cf. Lascar; e.g. T stable);
- $L_0 = L \cup \{f\}$, with f a unary function symbol;
- $T_0 := T \cup \{\text{'}\sigma \text{ is an automorphism'}\}$;
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Truss/Ivanov genericity

The setting:
T a complete L-theory
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The space of expansions of N is

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where ϕ is a quantifier–free N–sentence.

Fact

With this topology $\mathrm{Exp}(N, T_0)$ is a Baire space.

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An expansion $\sigma \in \mathrm{Exp}(N, T_0)$ is **Truss-generic** *if*

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Further assumptions:

- T is ω-categorical;
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Let N be the countable model of T. Then:

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An expansion $\sigma \in \mathrm{Exp}(N, T_0)$ is **atomic** *if it is an atomic model of T_{rich}.*

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Fact

1. *The set of countable models of* T_{rich} *is comeagre in* $\mathrm{Exp}(N, T_0)$*;*
2. *if* (M, σ) *is atomic, then* $(M, \sigma) \models T_{\mathrm{rich}}$*;*
3. *if an atomic expansion exists, then the set of atomic expansions is comeagre in* $\mathrm{Exp}(N, T_0)$*.*

Fact

Let $\alpha \in \mathrm{Exp}(N, T_0)$, i.e. $(N, \alpha) \in \kappa$. Tfae:

- *α is atomic;*
- *α is Truss-generic.*

Proof.

($\Rightarrow$): let α be an atomic expansion. Then the set of atomic expansions is of the form $Y := \{\alpha^g \ : \ g \in \mathrm{Aut}(N)\}$. By the previous fact, Y is comeagre. But two comeagre sets of this form coincide. Hence Y is exactly the set of Truss-generic expansions.

($\Leftarrow$): longer! □

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How to generalize the comparison

Idea: get rid of the assumptions

- T ω-categorical;
- T_{rich} model complete.

Idea:

1. If $N \models T$ is countable and saturated, T_{rich} model-complete, then

Truss-generic expansions of N = 'smooth' prime models of T_{rich}.

2. If $N \models T$ is countable and saturated, then

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Definition

- *A partial T_0–type is* **quasifinite** *if it contains only finitely many formulae not in L.*
- (M, σ) *is a* **smooth** *model (or σ is a smooth expansion) if it realizes every quantifier free quasifinite type which:*
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Let X be the set of existentially closed smooth expansions of N.

Fact

X is a comeagre subset of $\mathrm{Exp}(N, T_0)$.

Definition

Let $p(x) \in S(\emptyset)$ be realized in some $(N, \sigma) \in X$, and let $p_{\restriction_1}(x)$ be the set of universal and existential formulae in p. Then p is **e-isolated** *if there is a quasifinite type $\pi(x)$ such that the set*

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is the singleton $\{p_{\restriction_1}(x)\}$.
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An expansion $\alpha \in X$ is **e-atomic** *if every finite tuple in N is e-isolated.*

Remark

If T is ω-categorical, any expansion is smooth.
If T_{rich} is model-complete, every model of T_{rich} is e.c. and any formula is equivalent to an existential one.
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Theorem

Let T be small, $N \models T$ the countable saturated model, $\alpha \in \mathrm{Exp}(N, T_0)$.
Tfae:

1. α *is e-atomic;*
2. α *is Truss-generic.*

Theorem

Let S_x be the set of types of the form $p_{\restriction 1}(x)$, where $p(x)$ is some complete parameter free type realized in some e.c. smooth expansion of N. Then S_x can be equipped with a topology so that the following are equivalent:

- *Truss-generic expansions exist;*
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