Our aim is to compare two different notions of

generic models:

- genericity defined in terms of the topology on the space of expansions of a structure (à la Truss-Ivanov);
- genericity related to the existentially closed models of a theory (à la Lascar/Chatzidakis & Pillay).

The motivation comes from the similarity of two notions of **generic automorphisms**.

1 Generic automorphisms

1.1 Truss-generic automorphisms

Let M be a countable structure.

$\mathrm{Aut}(M)$ is a Baire space (with the standard topology, generated by basic open sets of the form

$$\mathrm{Aut}(M)_{ab} \quad := \quad \{g \in \mathrm{Aut}(M) : a^g = b\},$$

where a, b are finite tuples from M)

Definition. *$\alpha \in \mathrm{Aut}(M)$ is **Truss-generic** if*

$$\alpha^{\mathrm{Aut}(M)} := \{\alpha^g : g \in \mathrm{Aut}(M)\}$$

is comeagre, i.e. it contains a countable intersection of dense open sets.

The intuition is that α exhibits any finite behaviour consistent in $\mathrm{Aut}(M)$.

The existence of Truss generic automorphisms in $\mathrm{Aut}(M)$ has several applications, among which:

- definability of subgroups;
- the existence of generic *tuples* yields the small index property for the random graph and for ω-categorical ω stable structures [Hodkinson, Hodges, Lascar & Shelah];
- group theoretic properties of $\mathrm{Aut}(M)$ [Macpherson & Thomas];
- topological properties of Polish groups [Kechris & Rosendal.

1.2 Lascar-generic automorphisms

The setting: T a complete theory with q.e. in a countable language L;
$L_0 = L \cup \{F\}$ an expansion of L by a unary function symbol;
$T_0 = T \cup \{$'F is an automorphism'$\}$.

Definition. *Let* $(M, \sigma) \models T_0$. *Then* σ *is* ***Lascar-generic*** *if for every partial isomorphism*

$$f : (N, \tau) \to (M, \sigma) \qquad \textbf{\textit{partial}}$$

such that $(N, \tau) \models T_0$ *is countable and* $\mathrm{dom}(f) \subseteq N$ *is algebraically closed, there is an embedding*

$$\hat{f} : (N, \tau) \to (M, \sigma) \qquad \textbf{\textit{total}}$$

extending f.

Let $T_{rich} := \mathrm{Th}\{(M, \sigma) : M \models T,\ \sigma \text{ Lascar-generic}\}$.

Fact (Lascar; Chatzidakis & Pillay). *If* T *is stable:*

- *Lascar-generic automorphisms exist;*
- T_0 *has a model companion* $T_A \ \Rightarrow\ T_A = T_{rich}$*;*
- T_{rich} *is model-complete* $\ \Rightarrow\ T_{rich}$ *is the model companion of* T_0.

1.3 An example

Generic automorphisms on the two definitions are similar in a pure countable set: let $L = \emptyset$, Ω a countable set, $T = \mathrm{Th}(\Omega)$. Then:

Truss-generic automorphisms	Lascar-generic automorphisms
ω fixed points	ω fixed points
ω cycles of length 2	ω cycles of length 2
ω cycles of length 3	ω cycles of length 3
$\vdots$	$\vdots$
ω cycles of length n	ω cycles of length n
$\vdots$	$\vdots$
(no infinite cycles)	ω cycles of length ω

Remark. *The model companion* T_{rich} *of* T_0 *exists.*

If $f \in \mathrm{Aut}(\Omega)$ *is Truss-generic,* $(\Omega, f) \models T_{\mathrm{rich}}$.

This example is a very special case: Truss-generic automorphisms exists, and T_{rich} has a model companion. In general, one of these two conditions fail:

2 Genericity generalised

2.1 Lascar genericity: richness

We seek a common framework to compare the two notions of generic automorphisms, and we shall look more generally at expansions of a given model via a tuple of relations and functions, rather than a single automorphism. This is especially relevant to the question of whether *generic tuples* of automorphisms exist, and thus to the reconstruction of ω-categorical structures from their automorphism group.

Throughout this section we shall work in the following setting:

T is a complete L-theory
$L_0 = L \cup \{R\}$, where R is a finite tuple of function and relation symbols
T_0 is an expansion of T to L_0.

Definition (Inductive amalgamation class)**.** *Let κ be a class containing **models** and **morphisms**, where*

- *models: infinite models of T_0. They shall be denoted in the form (M, σ), where $M \models T$ and σ is an interpretation of R;*
- *morphisms are* partial *embeddings between models*

*We say that κ is an **inductive amalgamation class** if:*

1. *every morphism is a partial isomorphism;*
2. *every partial elementary map is a morphism;*
3. *(AP) every morphism*
   $f : (M, \sigma) \to (N, \tau)$ ***(partial)***
   extends to a total morphism
   $\hat{f} : (M, \sigma) \to (N', \tau')$ ***(total)***
4. *(JEP) for every $(M_1, \sigma_1), (M_2, \sigma_2) \in \kappa$ there are a model (N, τ) and total morphisms $f_i : (M_i, \sigma_i) \to (N, \tau)$;*
5. *the class of morphisms is closed under inverse and composition;*
6. *the class of morphisms is closed under restrictions;*
7. *κ is closed under unions of chains.*

A *rich* model is a more general version of an expansion of a structure via a Lascar-generic automorphism:

Definition. *$(M,\sigma) \in \kappa$ is* ***rich***[1] *if every morphism*

$$f : (N,\tau) \to (M,\sigma) \qquad \textit{(partial)}$$

such that $|f| < |N| \leq |M|$ extends to a total morphism

$$\hat{f} : (N,\tau) \to (M,\sigma) \qquad \textit{(total).}$$

Fact. *All rich models have the same theory.*

Proof. JEP is essential! □

Definition. *Let κ be an inductive amalgamation class. Then*

$$T_{\text{rich}} := \text{Th}(\{U \in \kappa : U \textit{ is rich}\}).$$

The following result is due to Chatzidakis & Pillay in the case where σ is a single automorphism (see the next section). It holds in the context of inductive amalgamation classes.

Notation: $M \leq N$ if $\text{id}_M : M \to N$ is a morphism.

Theorem. *Let κ be an inductive amalgamation class, and suppose further that:*

$$\textit{if } M, N \models T_{\text{rich}}, \textit{ then } M \subseteq N \iff M \leq N.$$

Tfae:

- *T_{rich} is model complete;*
- *all rich models are saturated;*
- *T_{rich} is the model companion of T_0.*

Viceversa, if T_0 has a model companion, then T_{rich} is this model companion.

These results hold even when:

- the models in κ are not necessarily the models of a theory (although we need models to be structures in a given language and κ be closed under elementary equivalence);
- JEP does not hold (then κ can be partitioned into 'connected components', within each of which JEP holds).

[1] if σ, τ are automorphisms and $|f| = \aleph_0$, σ is Lascar-generic in our previous definition

2.2 Examples

1. Let:

- T be a complete L-theory with q.e. and with the PAPA (cf. Lascar; e.g. T stable);
- $L_0 = L \cup \{f\}$, with f a unary function symbol;
- $T_0 := T \cup \{\text{`}\sigma \text{ is an automorphism'}\}$;
- $(M, \sigma) \models T_0$;
- models in κ:
  $\{(N, \tau) : N \models T,\ \tau \in \mathrm{Aut}(N),\ (N, \tau) \equiv_{\mathrm{acl}(\emptyset)} (M, \sigma)\}$;
- morphisms in κ: partial isomorphisms between models s.t. their domain is a.c.

Then (M, σ) is rich iff σ is Lascar-generic.

JEP does not hold if we take $\kappa := \{(N, \tau) : (N, \tau) \models T_0)\}$.

2. Let:

- T be a complete L-theory with q.e.;
- $L_0 = L \cup \{R\}$, with R a unary predicate;
- $T_0 = T$;
- (M, R) a model of T_0;
- models in κ:
  $\{(N, Q) : N \models T,\ (\mathrm{acl}_T(\emptyset), Q \cap \mathrm{acl}_T(\emptyset)) \simeq (\mathrm{acl}_T(\emptyset), R \cap \mathrm{acl}_T(\emptyset))\}$;
- morphisms in κ: partial isomorphisms between models s.t. their domain is a.c.

If T eliminates the quantifier $\exists^\infty$, T_{rich} is the model companion of T_0.

2.3 Truss/Ivanov genericity

The setting:
T a complete L-theory
N a countable model of T
$L_0 = L \cup \{R\}$, where R is a finite tuple of function and relation symbols
T_0 an $\forall$–axiomatizable expansion of T to L_0.

Definition. *The space of expansions of N is*

$$\mathrm{Exp}(N, T_0) := \{\sigma : (N, \sigma) \models T_0\}.$$

We can topologise $\mathrm{Exp}(N, T_0)$ by taking as basic open sets those of the form

$$[\phi]_N := \{\sigma : (N, \sigma) \models T_0 \cup \{\phi\}\},$$

where ϕ is a quantifier–free N–sentence.

Fact. *With this topology* $\mathrm{Exp}(N, T_0)$ *is a Baire space.*

Definition. *An expansion* $\sigma \in \mathrm{Exp}(N, T_0)$ *is* ***Truss-generic*** *if*

$$\{\tau^g : g \in \mathrm{Aut}(N)\}$$

is comeagre in $\mathrm{Exp}(N, T_0)$.

3 Comparing generic expansions

The setting:

T a complete L-theory with q.e.
T is small, $N \models T$ is (the) countable saturated model
$L_0 = L \cup \{R\}$, where R is a finite tuple of function and relation symbols
T_0 an $\forall$–axiomatizable (modulo T) expansion of T to L_0
κ is an inductive amalgamation class whose models are the models of T_0

Is there a relationship between models of T_{rich} and Truss–generic expansions of N?

3.1 A special case

The relationship between models of T_{rich} and Truss-generic expansions is clear in the special case where:

- T is ω-categorical;
- T_{rich} is model complete.

Let N be the countable model of T. Then:

Definition. *An expansion* $\sigma \in \mathrm{Exp}(N, T_0)$ *is* ***atomic*** *if it is an atomic model of* T_{rich}.

Fact. *1. The set of countable models of* T_{rich} *is comeagre in* $\mathrm{Exp}(N, T_0)$*;*

2. if (M, σ) *is atomic, then* $(M, \sigma) \models T_{\text{rich}}$*;*

3. if an atomic expansion exists, then the set of atomic expansions is comeagre in $\mathrm{Exp}(N, T_0)$.

Fact. *Let* $\alpha \in \mathrm{Exp}(N, T_0)$*, i.e.* $(N, \alpha) \in \kappa$*. Tfae:*

- α *is atomic;*
- α *is Truss-generic.*

Proof. ($\Rightarrow$): let α be an atomic expansion. Then the set of atomic expansions is of the form $Y := \{\alpha^g \ : \ g \in \mathrm{Aut}(N)\}$. By the previous fact, Y is comeagre. But two comeagre sets of this form coincide. Hence Y is exactly the set of Truss-generic expansions.

($\Leftarrow$): longer! □

3.2 How to generalize the comparison

Idea: get rid of the assumptions

- T ω-categorical;
- T_{rich} model complete.

We shall get that:
1. If $N \models T$ is countable and saturated, T_{rich} model-complete, then

$$\text{Truss-generic expansions of } N = \text{`smooth' prime models of } T_{\text{rich}}.$$

2. If $N \models T$ is countable and saturated, then

$$\text{Truss-generic expansions of } N = \text{`smooth', `e-atomic' models of } T_{\text{rich}}.$$

Definition.
- *A partial* T_0*–type is* ***quasifinite*** *if it contains only finitely many formulae not in* L*.*
- (M, σ) *is a* ***smooth*** *model (or* σ *is a smooth expansion) if it realizes every quantifier free quasifinite type which:*
  1. *has finitely many parameters;*
  2. *is finitely consistent in* (M, σ)*.*

Let X be the set of existentially closed smooth expansions of N.

Fact. X *is a comeagre subset of* $\mathrm{Exp}(N, T_0)$.

Definition. *Let* $p(x) \in S(\emptyset)$ *be realized in some* $(N, \sigma) \in X$*, and let* $p_{\restriction_1}(x)$ *be the set of universal and existential formulae in* p*. Then* p *is* ***e-isolated*** *if there is a quasifinite type* $\pi(x)$ *such that the set*

$$\{q_{\restriction_1}(x) : q \text{ is realized in some } (N, \sigma) \in X \text{ and } \pi(x) \subseteq q(x)\}$$

is the singleton $\{p_{\restriction_1}(x)\}$.

A tuple is e-isolated if its type is.

Definition. *An expansion* $\alpha \in X$ *is* ***e-atomic*** *if every finite tuple in* N *is e-isolated.*

Remark. *If* T *is* ω*-categorical, any expansion is smooth.*

If T_{rich} *is model-complete, every model of* T_{rich} *is e.c. and any formula is equivalent to an existential one.*

Hence, when both hypotheses hold a model is e-atomic if and only if it is atomic.

Theorem. *Let* T *be small,* $N \models T$ *the countable saturated model,* $\alpha \in \text{Exp}(N, T_0)$.

Tfae:

1. α *is e-atomic;*
2. α *is Truss-generic.*

Theorem. *Let* S_x *be the set of types of the form* $p_{\restriction_1}(x)$*, where* $p(x)$ *is some complete parameter free type realized in some e.c. smooth expansion of* N*. Then* S_x *can be equipped with a topology so that the following are equivalent:*

- *Truss-generic expansions exist;*
- *for every finite* x*, the isolated points are dense in* S_x.