Optimal pension funding for sophisticated managers

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Abstract

We consider the optimal management of a defined benefit stochastic pension fund where the participants have different rates of time preference. The fund manager invests in a portfolio with risky assets and one riskless asset. Its main objective is to select the amortization rate and the investment strategy minimizing both the contribution rate risk and the solvency risk. The problem is formulated as an optimal control problem with non-constant rate of discount and is solved analytically by means of the dynamic programming approach and the technical interest rate is selected in order to keep stable the fund evolution within prescribed targets.

Keywords: Optimization in Financial Mathematics; pension funding; dynamic programming; non–constant discount; risk management
1 Introduction

The world financial crisis and the increase in life expectancy have stressed the importance of alternative pension plans in order to complement public protection at retirement. As a consequence, the study of the management of pension plans is an important subject in the economic field, and also in the financial field because the fund managers use the financial markets to invest the fund assets of the pension plan.


We focus our attention in plans of defined benefit type where the benefits are fixed in advance by the sponsor and contributions are designed to amortize the fund according to a previously chosen actuarial scheme. The plan sponsor can built an investment portfolio of the fund. The main aim of the plan manager is the minimization of both the solvency risk and the contribution rate risk. This objective is generally accepted since the papers Haberman and Sung (1994) and Josa–Fombellida and Rincón–Zapatero (2001). These risk concepts are defined as quadratic deviations of fund wealth and amortization rates with respect to liabilities and normal cost, respectively. The solvency risk is related with the security of the pension fund in attaining the comprised liabilities and the contribution risk with its stability. UNDERFUNDED CASE

In this paper we study the optimal management of a defined benefit pension plan with stochastic benefits, where the aim of the sponsor is the minimization of the contribution and the solvency risks along an infinite horizon, as in Josa–Fombellida and Rincón–Zapatero (2004). We provide an extension of that paper, supposing that the discount function is not necessarily of exponential type with the aim of capturing the diversity in the temporal preferences of the...
participants in the plan. Collective temporal decisions for agents with different rates of time preferences lead to time inconsistent aggregate temporal preferences. As a consequence, standard optimization techniques (Pontryagin Maximum Principle (PMP) or Hamilton-Jacobi-Bellman (HJB)) fail in characterizing time consistent optimal policies. As preferences change with time, as long as the decision maker goes through the time horizon, they differ depending on the instant $t$ at which solutions are obtained, so a $t'$-agent, in general, will not find optimal the solutions computed by the $t$-agent, for any $t$ and $t'$ in the time horizon.

Karp (2007) introduced the analysis of dynamic optimization problems in continuous time setting with non-constant rate of discount, deriving in infinite time horizon a modified Hamilton-Jacobi-Belman (HJB) equation. Later Marín-Solano and Navas (2009) extended the approach to the finite horizon case and study the application to a non-renewable resource problem with non-constant discount. The methodology for stochastic control problems with non-constant discount in a finite time horizon is developed in Marín-Solano and Navas (2010). The classical optimal consumption and portfolio problem by Merton (1971) with non-constant discount is detailed studied for logarithmic, power and exponential utility functions in this last paper. It is important to comment that pension fund models with a non-constant rate of discount have not been considered yet in the specialized literature.

Our main finding is that the rate of discount function intervenes in the optimal strategies and in the optimal fund evolution and it is possible to select the technical rate of interest in order that the optimal contribution does not depend on the parameters of the benefit process and has the form of a spread method of funding, providing the stability of the plan.

The paper is organized as follows. Section 2 defines the elements of the pension scheme and describes the financial market where the fund operates. We consider the fund is invested in a portfolio with a risky asset and a riskless asset. The management of the defined benefit plan is formulated as a stochastic optimal control problem with non-constant discount where the objective is to minimize on a infinite horizon the contribution rate risk and the solvency risk. In Section 3 the optimal strategies of contribution and investment, and the optimal fund are obtained with dynamic programming techniques. Some properties of the optimal solutions
are found. A particular case where the technical rate of interest is selected to lead to a spread
method of funding is analyzed. Section 4 serves as a numerical illustration of previous results.
Finally, Section 5 establishes some conclusions. All proofs are developed in Appendix A.

2 The pension model

Consider a pension plan of aggregated type where, at every instant of time, active participants
coexist with retired participants. The plan is of defined benefit type, that is to say, the benefits
paid to the participants at the age of retirement are fixed in advance by the manager. The
benefit is modeled by a stochastic process correlated with the financial market.

The main elements intervening in the pension plan are the following. We denote $F(t)$ the
value of the fund assets at time $t$ and $C(t)$ the contribution rate made by the sponsor at time $t$
to the funding process in order to accrue the benefit at the moment of retirement. The risk–free
market interest rate is the constant $r$. The technical rate of valuation $\delta$ is the constant used
for the valuation of the liabilities. This valuation is made using the distribution function $M$
on $[a,d]$, that is, $100M(x)\%$ is the percentage of the value of the future benefits accumulated until
age $x \in [a,d]$, where $a$ is the common age of entrance in the fund and $d$ is the common age of
retirement.

2.1 The actuarial functions

The main functions of the pension plan are described next. $P(t)$ denotes the benefits promised
to the participants at time $t$. They are related with the salary at the moment of retirement. $NC(t)$
is the normal cost at time $t$ for all participants. If the fund assets match the actuarial
liability, and if there are no uncertain elements in the plan, the normal cost is the value of
the contributions allowing equality between asset funds and liabilities. $AL(t)$ is the actuarial
liability at time $t$, that is, the total liabilities of the sponsor. The unfunded actuarial liability
at time $t$ (equal to $AL(t) - F(t)$) is denoted by $UAL(t)$, and the supplementary cost at time $t$
(the difference $C(t) - NC(t)$) by $SC(t)$. We suppose that these actuarial functions are stochastic.
processes.

We consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\mathbb{P}\) is a probability measure on \(\Omega\) and \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) is a complete and right continuous filtration generated by the \((n+1)\)-dimensional standard Brownian motion \((w_0, w_1, \ldots, w_n)\), that is to say, \(\mathcal{F}_t = \sigma\{w_0(s), w_1(s), \ldots, w_n(s); 0 \leq s \leq t\}\).

The stochastic actuarial liability and the stochastic normal cost are defined as in Josa–Fombellida and Rincón–Zapatero (2004):

\[
AL(t) = \int_a^d e^{-\delta(d-x)} M(x) \mathbb{E} \left( P(t + d - x) \mid \mathcal{F}_t \right) \, dx,
\]

\[
NC(t) = \int_a^d e^{-\delta(d-x)} M'(x) \mathbb{E} \left( P(t + d - x) \mid \mathcal{F}_t \right) \, dx.
\]

for every \(t \geq 0\), where \(\mathbb{E}(\cdot \mid \mathcal{F}_t)\) denotes conditional expectation with respect to the filtration \(\mathcal{F}_t\).

The actuarial liability \(AL(t)\) is the total expected value of the promised benefits accumulated according to the distribution function \(M\), and the normal cost \(NC(t)\) is the total expected value of the promised benefits accumulated according to the density function \(M'\), both discounted at the constant rate \(\delta\).

In order to get analytical tractability, we assume that the benefit \(P\) is given by a geometric Brownian motion, as in Josa–Fombellida and Rincón–Zapatero (2004). Thus the expected benefit grows exponentially which is coherent because the benefit depends on the salary and the population pension plan size.

**Assumption 1** The benefit \(P\) satisfies the stochastic differential equation (SDE therefore)

\[
dP(t) = \mu P(t) \, dt + \eta P(t) \, dB(t), \quad t \geq 0,
\]

where \(B\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\), and where \(\mu \in \mathbb{R}\) and \(\eta \in \mathbb{R}_+\). The initial condition \(P(0) = P_0\) is a random variable that represents the initial liabilities.

Under Assumption 1 the actuarial functions satisfy \(AL(t) = \psi_{AL} P(t)\) and \(NC(t) = \psi_{NC} P(t)\), where \(\psi_{AL} = \int_a^d e^{(\mu - \delta)(d-x)} M(x) \, dx\) and \(\psi_{NC} = \int_a^d e^{(\mu - \delta)(d-x)} M'(x) \, dx\), and they are linked by the identity

\[
(\delta - \mu) AL(t) + NC(t) - P(t) = 0, \quad (1)
\]

for every \(t \geq 0\). See Proposition 2.1 in Josa–Fombellida and Rincón–Zapatero (2004).
2.2 The financial market and the fund wealth

In this section we describe the financial market where the fund is invested. The plan sponsor manages the fund by means of a portfolio formed by \( n \) risky assets \( S^1, \ldots, S^n \), which are geometric Brownian motions correlated with the benefit process, and a riskless asset \( S^0 \), as proposed Merton (1971), that is, whose evolutions are given by the equations:

\[
\begin{align*}
    dS^0(t) &= rS^0(t)dt, \quad S^0(0) = 1, \\
    dS^i(t) &= S^i(t)\left(b_i dt + \sum_{j=1}^n \sigma_{ij} dw_j(t)\right), \quad S^i(0) = s_i, \quad i = 1, \ldots, n.
\end{align*}
\]

We have denoted by \( r > 0 \) the short risk–free rate of interest, \( b_i > 0 \) the mean rate of return of the risky asset \( S^i \), and \( \sigma_{ij} > 0 \) the uncertainty parameters. We assume that \( b_i > r \), for each \( i = 1, \ldots, n \), so the sponsor has incentives to invest with risk. The matrix \( \sigma \) is denoted by \( \Sigma = \sigma \sigma^\top \) is positive definite, and that there exists correlation \( q_i \in [-1, 1] \) between \( B \) and \( w_i \), for \( i = 1, \ldots, n \). As a consequence, \( B \) is expressed in terms of \( (w_0, \ldots, w_n) \) as

\[
B(t) = \sqrt{1 - q^\top q} w_0(t) + q^\top w(t),
\]

where \( w = (w_1 \ldots, w_n)^\top \) and \( q = (q_1, \ldots, q_n)^\top \). In this way the influence of salary and inflation on the evolution of liabilities \( P \) is taken into account, as well as the effect of inflation on the prices of the assets.

By Assumption 1 and using \( AL = \psi_{AL} P \), the actuarial liability satisfies the SDE

\[
dAL(t) = \mu AL(t) dt + \eta AL(t) dB(t),
\]

with the initial condition \( AL(0) = AL_0 = \psi_{AL} P_0 \). And in terms of the Brownian motion generating \( \mathcal{F} \), \( AL \) is determined by

\[
dAL(t) = \mu AL(t) dt + \eta AL(t) \sqrt{1 - q^\top q} dw_0(t) + \eta AL(t) q^\top dw(t),
\]

with \( AL(0) = AL_0 \). Thus the benefit \( P \) and actuarial liability \( AL \) depend on the financial market.
The manager builds a portfolio based on the financial market and designs an amortization scheme varying with time. The amount of fund invested in time $t$ in the risky asset $S^i$ is denoted by $\pi_i(t)$, $i = 1, \ldots, n$. The remainder, $F(t) - \sum_{i=1}^{n} \pi_i(t)$, is invested in the bond. Borrowing and shortselling are allowed. A negative value of $\pi_i$ means that the sponsor sells a part of his risky asset $S^i$ short while, if $\pi_i$ is larger than $F$, he or she then gets into debt to purchase the corresponding stock, borrowing at the riskless interest rate $r$. $\pi$ denotes $(\pi_1, \ldots, \pi_n)^\top$. We suppose the investment strategy $\{\pi(t) : t \geq 0\}$ is a control process adapted to filtration $\{\mathcal{F}_t\}_{t\geq0}$, $\mathcal{F}_t$-measurable, markovian and stationary, satisfying

$$
\mathbb{E}_{F_0, AL_0} \int_0^T \pi^\top(t) \pi(t) dt < \infty, \quad \text{for every } T < \infty,
$$

and the contribution rate process $C(t)$ is also an adapted process with respect to $\{\mathcal{F}_t\}_{t\geq0}$ verifying

$$
\mathbb{E}_{F_0, AL_0} \int_0^T SC_2(t) dt < \infty, \quad \text{for every } T < \infty,
$$

and

$$
\mathbb{E}_{F_0, AL_0} \int_0^\infty e^{-\int_0^s \tilde{\rho}(v) dv} (\beta SC_2(s) + (1 - \beta)(AL(s) - F(s))^2) ds < \infty,
$$

In the above, $\mathbb{E}_{F_0, AL_0}$ denotes conditional expectation with respect to the initial conditions $(F_0, AL_0)$.

The dynamic fund evolution under the investment policy $\pi$ is:

$$
dF(t) = \sum_{i=1}^{n} \pi_i(t) \frac{dS^i(t)}{S^i(t)} + \left( F(t) - \sum_{i=1}^{n} \pi_i(t) \right) \frac{dS^0(t)}{S^0(t)} + (C(t) - P(t)) dt.
$$

By substituting (2) and (3) in (8), we obtain the SDE that determines the fund evolution,

$$
dF(t) = \left( rF(t) + \pi^\top(t)(b - rT) + C(t) - P(t) \right) dt + \pi^\top(t)\sigma dw(t),
$$

with initial condition $F(0) = F_0 > 0$.

### 2.3 The optimization problem

The manager wishes to minimize a convex combination of the contribution rate risk and the solvency risk in a infinite horizon. Thus, the objective functional to be minimized over the class
of admissible controls $\mathcal{A}_{F_0,AL_0}$, is given by

$$J((F_0, AL_0); (SC, \pi)) = \mathbb{E}_{F_0,AL_0} \int_0^\infty e^{-\int_0^s \bar{\rho}(v)dv} \left( \beta SC^2(s) + (1 - \beta)(AL(s) - F(s))^2 \right) ds. \quad (10)$$

Note that we choose $SC = C - NC$ as the control variable instead of $C$, leading to an equivalent control problem. Here, $\mathcal{A}_{F_0,AL_0}$ is the set of Markovian processes $(SC, \pi)$, adapted to the filter $\{\mathcal{F}_t\}_{t \geq 0}$ where $C$ satisfies (6), $\pi$ satisfies (5), and where $F$ and $AL$ satisfy (9) and (4), respectively.

The parameter $\beta$, $0 < \beta \leq 1$, is a weighting factor reflecting the relative importance for the manager of the two different types of risks. Note that the specification (10) implies that the fund manager assigns the same importance to over and under deviations of the fund’s assets and contributions from their respective targets.

The dynamic programming approach will be used to solve the problem. The value function is defined as

$$\hat{V}(F, AL) = \min_{(SC, \pi) \in \mathcal{A}_{F,AL}} \left\{ J((F, AL); (SC, \pi)) : \text{s.t. } (4), (9) \right\}. \quad (11)$$

Since the problem is autonomous and the horizon unbounded, we may suppose that $\hat{V}$ is time independent. It is clear that the value function so defined is non-negative and strictly convex.

The connection between value functions and optimal feedback controls in stochastic control theory with non-constant discount is accomplished by a modified HJB. Marín–Solano and Navas (2010) analyse the finite horizon case, that is easily translated to the unbounded case. Tomamos solución sofisticada! The HJB equation is

$$-\rho V - K + \min_{SC, \pi} \left\{ \beta SC^2 + (1 - \beta)(F - AL)^2 + (rF + \pi^T(b - r\mathbf{1}) + SC + NC - P)V_F + \mu AL V_{AL} + \frac{1}{2} \pi^T \Sigma^{-1} \pi V_{FF} + \frac{1}{2} \eta^2 AL^2 V_{AL, AL} + \eta \pi^T \sigma q AL V_{F, AL} \right\} = 0, \quad (11)$$

where

$$K(F, AL) = \int_0^\infty e^{-\int_0^s \bar{\rho}(v)dv} (\bar{\rho}(s) - \rho)\mathbb{E}_{F,AL} \left\{ \beta SC^2(s) + (1 - \beta)(AL(s) - F(s))^2 \right\} ds, \quad (12)$$

and where $\mathbb{E}_{F,AL}$ denotes the conditional expectation to $F(0) = F$ and $AL(0) = AL$. 

8
3 The optimal strategies

In this section we show how the sponsor may select the rate contribution and the proportion of fund assets put into the risky assets. We analyze some properties of these optimal strategies and study the optimal fund evolution. We have the following result.

**Theorem 3.1** Suppose that Assumption 1 holds. If the inequalities

\[ 2 \mu + \eta^2 < \rho, \]  
\[ 2r - 2 \frac{\alpha_{FF}}{\beta} \theta^\top \theta < \rho, \]  

are satisfied, then the optimal rate of contribution and the optimal investment in the risky assets are given by

\[ C^* = NC - \frac{\alpha_{FF}}{\beta} F - \frac{\alpha_{F,AL}}{2\beta} AL, \]  
\[ \pi^* = -\Sigma^{-1}(b - r I)F - \frac{\alpha_{F,AL}}{2\alpha_{FF}} (\Sigma^{-1}(b - r I) + \eta \sigma^{-\top} q) AL, \]

respectively, where \( \alpha_{FF} \) is the unique positive solution to the equation

\[ -\frac{\alpha_{FF}^2}{\beta} + \left( -\rho + 2r - \theta^\top \theta \right) \alpha_{FF} + (1 - \beta) - \kappa_{FF} = 0, \]

where

\[ \kappa_{FF} = \left( \frac{\alpha_{FF}^2}{\beta} + 1 - \beta \right) \int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv} (\tilde{\rho}(s) - \rho) e^{\left(2r - 2 \frac{\alpha_{FF}}{\beta} \theta^\top \theta\right)s} ds, \]

and \( \alpha_{F,AL} \) is the unique solution to the equation

\[ -\frac{\alpha_{FF}}{\beta} \alpha_{F,AL} + \left( -\rho + r - \theta^\top \theta - \eta q^\top \theta + \mu \right) \alpha_{F,AL} + 2(\mu - \delta) \alpha_{FF} - 2(1 - \beta) - \kappa_{F,AL} = 0, \]

where

\[ \kappa_{F,AL} = \left( \frac{\alpha_{FF}^2}{\beta} + 1 - \beta \right) \left( \frac{\alpha_{F,AL}}{\beta} + 2(\delta - \mu) \right) \int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv} (\tilde{\rho}(s) - \rho) e^{\left(2r - 2 \frac{\alpha_{FF}}{\beta} \theta^\top \theta\right)s} ds \]

\[ + \left( \frac{\alpha_{FF}}{\beta} \alpha_{F,AL} - 2(1 - \beta) - \frac{\alpha_{FF}^2}{\beta} + 1 - \beta \right) \left( \frac{\alpha_{F,AL}}{\beta} + 2(\delta - \mu) \right) \int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv} (\tilde{\rho}(s) - \rho) e^{(r - \theta^\top \theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta \theta^\top \theta)s} ds. \]
The optimal fund is the solution of the system (4),

\[
dF(t) = \left( \left( r - \theta^\top \theta - \frac{\alpha_{FF}}{\beta} \right) F(t) - \left( \frac{\alpha_{F,AL}}{2\alpha_{FF}} (\theta^\top \theta + \eta q^\top \theta + \frac{\alpha_{FF}}{\beta}) + \delta - \mu \right) AL(t) \right) dt \]

\[-\left( \theta^\top F(t) + \frac{\alpha_{F,AL}}{2\alpha_{FF}} (\theta^\top + \eta q^\top) AL(t) \right) dw(t),
\]

with \( F(0) = F_0, AL(0) = AL_0 \).

The optimal strategies \( C^\ast, \pi^\ast \) are linear functions of the fund assets \( F \) and the actuarial liability \( AL \), and depend on the parameters of the financial market and the benefit process, and also, through \( \alpha_{F,AL} \), depend on the rate of discount \( \tilde{\rho} \), the technical rate of interest \( \delta \) and the benefit drift parameter \( \mu \).

The optimal investment decisions, (16), are composed by two terms. The first is proportional to \( F \), with coefficient proportional to \( \theta \), but the second is proportional to \( AL \) and depends on the rate of discount, and the parameters containing the correlation between benefit and risky assets. The constant of proportionality in the first term, \( \Sigma^{-1}(b - r\bar{1}) \), is the so called optimal–growth portfolio strategy, that appears in the Merton model where a CRRA utility of consumption is maximized.

An interesting consequence is that there exists a linear relationship between the optimal supplementary cost and the optimal investment strategy,

\[
\pi^\ast = \Sigma^{-1}(b - r\bar{1}) \frac{\beta}{\alpha_{FF}} SC^\ast - \eta \sigma^{-\top} q \frac{\alpha_{F,AL}}{2\alpha_{FF}} AL,
\]

thus for each unit of additional amortization with respect to the normal cost the manager must invest \( \Sigma^{-1}(b - r\bar{1}) \frac{\beta}{\sigma_{FF}} \) units in the risky assets, plus an additional quantity of \( \eta \sigma^{-\top} q \frac{\alpha_{F,AL}}{2\alpha_{FF}} AL \) units.

**Remark 3.1** The manager must borrow money at rate \( r \) to invest in the risky asset \( S^i \), that is to say \( \pi^i_1 > F^i \), when the level of the fund is below \( \lambda_i AL \), where the constant \( \lambda_i \) is defined as

\[
\lambda_i = \frac{\sigma_i \Sigma^{-1}(b - r\bar{1}) + \eta \sigma_i \sigma^{-\top} q \alpha_{F,AL}}{1 + \sigma_i \Sigma^{-1}(b - r\bar{1})} \frac{\alpha_{F,AL}}{2\alpha_{FF}}, \quad \sigma_i = (0, \ldots, 1, 0, \ldots, 0),
\]
for all \( i = 1, 2, \ldots, n \), and he/she need to short sell asset, that is to say \( \pi^*_i < 0 \), when the fund is above the value \( \lambda'_i AL \), where

\[
\lambda'_i = \frac{\bar{\varepsilon}_i \Sigma^{-1} (b - r \bar{I}) + \eta \bar{e} \sigma^{-\top} \bar{q}}{\bar{\pi}_i \Sigma^{-1} (b - r \bar{I})} \frac{\alpha_{F,AL}}{2 \alpha_{FF}}.
\]

Thus the manager does not need short–selling neither borrowing, \( 0 \leq \pi^*_i \leq F^* \), when the fund \( F^* \) is between \( \lambda_i AL \) and \( \lambda'_i AL \).

In order to maintain in the long term the fund assets near the actuarial liability and the rate of contribution near the normal cost, we give a valuation of the technical rate of actualization \( \delta \), consisting in a spread method of fund amortization, as in Josa–Fombellida and Rincón–Zapatero (2004). These spread methods, widely used in pension funding (see Owadally and Haberman (1999)), assume that the supplementary cost \( SC \) is proportional to the unfunded actuarial liability \( UAL \).

From (15), in order to achieve a spread method the identity \( \alpha_{F,AL} = -2 \alpha_{FF} \) must be satisfied. Substituting in (19) we obtain

\[
-\frac{\alpha_{FF}^2}{\beta} + \left( -\rho + r - \theta^\top \theta - \eta q^\top \theta + \delta \right) \alpha_{FF} + 1 - \beta - \frac{\alpha_{FF}}{\beta} \frac{\mu - \delta}{\eta q^\top \theta} e_{FF} \\
+ \frac{\alpha_{FF}^2}{\beta} + \mu - r - \eta q^\top \theta \int_0^\infty e^{-\int_{\rho(s)}^{\infty} \tilde{\rho}(v) dv \tilde{\rho}(s) - \rho} e^{(r - \frac{\alpha_{FF}}{\beta} \theta - \mu - \eta q^\top \theta)^s} ds = 0, \quad (22)
\]

and comparing (22) with (17), we obtain that the technical interest rate must coincide with the rate of return of the bond modified to get rid of the sources of uncertainty. Specifically we assume that the valuation of liabilities \( \delta \) is \( r \) plus the product of the Sharpe ratio of the portfolio and a term depending on the parameters containing the correlations and the diffusion parameter of the benefit process, as in Josa–Fombellida and Rincón–Zapatero (2004).

**Assumption 2** The technical rate of actualization is \( \delta = r + \eta q^\top \theta \).

Note that \( \delta \) does not depend on parameter \( \mu \) associated to the benefit \( P \). If there is no correlation between the benefit and the financial market, then \( \delta \) is the risk–free rate of interest.
Note that the existence of a non–constant discount rate does not influence in the selection of the technical rate of interest.

Besides this valuation it provides, this selection of $\delta$ will allow us to simplify the explicit solution of the problem in the following result.

**Corollary 3.1** Suppose that Assumptions 1 and 2 hold, and the inequalities (13), (14), are satisfied. The optimal rate of contribution and the optimal investment in the risky assets are given by

$$C^* = NC + \frac{\alpha_{FF}}{\beta} UAL,$$

$$\pi^* = \Sigma^{-1}(b - \bar{r})UAL + \eta\sigma^{-1}q.AL,$$  \hspace{1cm} (23)

respectively, where $\alpha_{FF}$ is the unique positive solution to the equation (17). The optimal fund is the solution of the system (4),

$$dF^*(t) = \left(\left(r - \theta^{\top}\theta - \frac{\alpha_{FF}}{\beta}\right) F^*(t) - \left(r - \theta^{\top}\theta - \frac{\alpha_{FF}}{\beta} - \mu\right) AL(t)\right) dt$$

$$+ \left( - \theta^{\top} F^*(t) + (\theta^{\top} + \eta q^{\top}) AL(t)\right) dw(t),$$ \hspace{1cm} (24)

with $F(0) = F_0$, $AL(0) = AL_0$.

The supplementary cost $SC^*$ is proportional to the unfunded actuarial liability $UAL$, with constant of proportionality depending on the rate of discount. The optimal investment decisions $\pi^*$, (23), are the same than in Josa–Fombellida and Rincón–Zapatero (2004), thus it does not depend on the rate of discount. They are composed by two terms. The first is again proportional to $UAL$, but the second is a correction term, depending on the risk parameters of the model and $AL$. This second term is zero when there is no uncertainty in the benefits, as in Josa–Fombellida and Rincón–Zapatero (2001), and when there is no correlation between benefit and risky asset.

We obtain that $C^*$ and $\pi^*$ do not depend on $\mu$. Also we observe that the manager takes a greater risk when the wealth of the fund is far below the actuarial liability than when it is closer. $C^*$ and fund evolution $F^*$ depend on $\bar{\rho}$ through $\alpha_{FF}$. However, from (21), all parameters of the benefit process influence linearly in the optimal fund evolution.
This selection of \( \delta \) allows the fulfillment in the long term of one objective of the pension plan manager, that is the maintenance of the fund \( F^* \) and the contribution \( C^* \) close to their ideal values \( AL \) and \( NC \), respectively.

**Proposition 3.1** Suppose that Assumptions 1, 2 and the inequalities (13), (14) and

\[
\alpha_{FF} > \beta (r - \theta^\top \theta),
\]

are satisfied. Then the expected unfunded actuarial liability and the expected supplementary cost converge in the long term to zero, that is to say,

\[
\lim_{t \to \infty} \mathbb{E}_{F_0, AL_0} UAL^*(t) = \lim_{t \to \infty} \mathbb{E}_{F_0, AL_0} SC^*(t) = 0.
\]

And the total expected contribution is

\[
SC = \int_0^\infty \mathbb{E}_{F_0, AL_0} SC(t) dt = \frac{\alpha_{FF}/\beta}{\alpha_{FF}/\beta + \theta^\top \theta - r} UAL_0.
\]

Notice that inequality (25) is automatically fulfilled if \( r \geq \rho + \theta^\top \theta \). However, when \( r < \rho + \theta^\top \theta \), it reduces to

\[
\beta < \frac{1}{1 + r(\rho + \theta^\top \theta - r)}.
\]

In the underfunded case \( SC > 0 \).

4 A numerical illustration

Next, we illustrate the dynamic behaviour of the optimal fund and the optimal policies (contribution rate and investment policies) by conducting some simulations for a specific example. As the benchmark case we consider that benefits have mean return \( \mu = 0.03 \) and standard deviation \( \eta = 0.1 \). Without loss of generality, we consider that there is one risky asset with mean \( \mu = 0.09 \) and standard deviation \( \sigma = 0.2 \) (this implies a Sharpe ratio of \( \theta = 0.3 \)), which has a correlation coefficient with benefits of \( q = 0.5 \), while the risk free rate of interest is equal to \( r = 0.03 \). Initial values for the actuarial liability and the fund are, respectively, \( AL_0 = 100 \) and \( F_0 = 80 \), so we consider that at \( t = 0 \) the actuarial liability is not totally covered by the fund. For the
objective function we take as weight parameter $\beta = 0.5$, i.e., the manager gives equal value to the contribution rate risk and the solvency risk to be minimized. Finally, we select the technical rate of interest leading a spread method of funding ($\delta = 0.045$), and as discount function for the manager in the objective function a linear combination between two exponential functions, i.e.,

$$\theta(s - t_0) = 0.5e^{-0.1(s-t_0)} + 0.5e^{-0.3(s-t_0)}$$

which will allow us to obtain parameters of the value function as described in Remark 2.1.

These parameters verify the transversality condition (13) and the convergence condition (25), assuring the stability of the pension plan.

![Figure 1: Expected actuarial liability and fund dynamics](image)

Figure 1: Expected actuarial liability and expected fund assets dynamics for different values of (a) $\lambda$ and (b) for different initial $F_0$ values

Figure 1 shows the expected fund actuarial liability ($AL(t)$) and the expected fund assets for $\lambda = 0.5$ (baseline case: $F(t)$), $\lambda = 0.9$ ($F_1(t)$) and $\lambda = 0.1$ ($F_2(t)$) for a total of 1000 realizations over a time horizon of 20 years (with a step of 1/12). In the baseline case we consider two kinds of participants (patient participants with $\rho_1 = 0.08$ and impatient participants with $\rho_2 = 0.3$) with equal weight (i.e., $\lambda = 0.5$), while the case of $\lambda = 0.9$ corresponds to a situations where patient participants are majority or $\lambda = 0.1$ where impatient agents are majority. Despite the patient or impatient majority in the whole group, we can observe that the fund dynamics are very similar. As a result, we can conclude that the heterogeneity in the temporal preferences inside the group has little effect in the optimal investment policy, leading to similar results. This points to a nice result as long as the manager, without taking into account this heterogeneity,
will act in a satisfactory way for the entire group independently of the relative weight between patient and impatient participants. Moreover, if we consider the extreme cases where $\lambda = 1$, which corresponds to a standard discount problem with $\rho = 0.08$ or $\lambda = 0$, which corresponds to the case of $\rho = 0.3$, differences are also small.

Next, in Fig. 2 we compare the differences between the expected fund dynamics between the baseline case and cases, i.e., $F(t) - F_1(t)$ and $F(t) - F_2(t)$. We can observe that $F_1 > F > F_2$, i.e., for a more patient collective the unfunded actuarial liability is slightly lower at initial periods but differences converge to zero after initial periods of the plan.

![Figure 2: Expected Fund Differences](image)

5 Conclusions

We have analyzed by means of dynamic programming techniques the management of an aggregated defined benefit pension plan where the rate of discount is non–constant. The objective is to determine the contribution rate and the investment strategy minimizing both the contribution and the solvency risk in an infinite time horizon. We have found that there is a linear relationship between the optimal supplementary cost and the optimal investment strategy, and between this strategy and the optimal fund, with correction terms due to the random behavior of benefits.

The rate of discount function intervenes in the optimal strategies and in the optimal fund evolution. It is possible to select the technical rate of interest such that the optimal investment
only depends on the parameters of the benefit due to the correlations, and the optimal contribution does not depend on the parameters of the benefit process, getting a spread amortization and the plan stability in the long term.

A numerical illustration shows how, in the long term, the expected values of the optimal fund and the optimal contribution are closed, respectively, to the expected values of the actuarial liability and to the normal cost.

References


