ON THE REGULARITY OF THE FREE BOUNDARY IN THE p-LAPLACIAN OBSTACLE PROBLEM

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Abstract. We study the regularity of the free boundary in the obstacle for the p-Laplacian, $\min\{-\Delta_p u,\, u-\varphi\} = 0$ in $\Omega \subset \mathbb{R}^n$. Here, $\Delta_p u = \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)$, and $p \in (1,2)\cup(2,\infty)$.

Near those free boundary points where $\nabla\varphi \neq 0$, the operator Δ_p is uniformly elliptic and smooth, and hence the free boundary is well understood. However, when $\nabla\varphi = 0$ then Δ_p is singular or degenerate, and nothing was known about the regularity of the free boundary at those points.

Here we study the regularity of the free boundary where $\nabla\varphi = 0$. On the one hand, for every $p \neq 2$ we construct explicit global 2-homogeneous solutions to the p-Laplacian obstacle problem whose free boundaries have a corner at the origin. In particular, we show that the free boundary is in general not C^1 at points where $\nabla\varphi = 0$. On the other hand, under the "concavity" assumption $|\nabla\varphi|^{2-p}\Delta_p\varphi < 0$, we show the free boundary is countably $(n-1)$-rectifiable and we prove a nondegeneracy property for u at all free boundary points.

1. Introduction

In this paper we study the obstacle problem

$$\min\big\{-\Delta_p u,\, u-\varphi\big\} = 0 \qquad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{1.1}$$

for the p-Laplacian operator

$$\Delta_p u = \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big), \qquad 1 < p < \infty.$$

The problem appears for example when considering minimizers of the constrained p-Dirichlet energy

$$\inf\left\{\int_\Omega |\nabla v|^p \,:\, v \in W^{1,p}(\Omega), \quad v \geq \varphi \text{ in } \Omega, \quad v = g \text{ on } \partial\Omega\right\},$$

where φ and g are given smooth functions and Ω is a bounded smooth domain.

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The regularity of solutions to (1.1) was recently studied by Andersson, Lindgren, and Shahgholian in [ALS15]. Their main result establishes that if $\varphi \in C^{1,1}$ then

$$\sup_{B_r(x_0)} (u - \varphi) \leq Cr^2 \qquad \text{for all} \quad r \in (0,1)$$

at any free boundary point $x_0 \in \partial\{u > \varphi\}$. Thus, solutions u leave the obstacle φ in a $C^{1,1}$ fashion at free boundary points x_0.

Notice that, near any free boundary point $x_0 \in \partial\{u > \varphi\}$ at which $\nabla\varphi(x_0) \neq 0$, the solution u will satisfy $\nabla u \neq 0$ as well and hence the operator $\Delta_p u$ is uniformly elliptic in a neighborhood of x_0. Therefore, by classical results [Caf77, Caf98, PSU12], the solution u is $C^{1,1}$ near x_0, and the structure and regularity of the free boundary is well understood.

Thus, the main challenge in problem (1.1) is to understand the regularity of solutions and free boundaries near those free boundary points $x_0 \in \partial\{u > \varphi\}$ at which $\nabla\varphi(x_0) = 0$. Our first main result is the following.

Theorem 1.1. *Let $p \in (1,2) \cup (2,\infty)$, and let $\varphi(x) = -|x|^2$ in $\mathbb{R}^2$. There exists a 2-homogeneous function $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying (1.1) in all of $\mathbb{R}^2$, and such that the set $\{u > \varphi\}$ is a cone with angle*

$$\theta_0 = 2\pi \left(1 - \sqrt{\frac{p-1}{2p}}\right) \neq \pi.$$

In particular, the free boundary has a corner at the origin.

Remark 1.2. Let u be a solution to (1.1) with $\varphi(x) = -|x|^2$ as in Theorem 1.1. For each $a \in \mathbb{R}$ and $b > 0$, $a - bu$ is a solution to (1.1) in $\mathbb{R}^2$ with $\varphi(x) = a - b|x|^2$ for which the contact set is a cone with angle $\theta_0 \neq \pi$.

Remark 1.3. Notice that for $p \in (2,\infty)$, $\theta_0 > \pi$ and thus u is not convex. This is in contrast with the classical result of Caffarelli on the classifications of global solutions to the obstable problem for the Laplacian [Caf98].

Remark 1.4. In the process of constructing the solutions u of Theorem 1.1, for $p = 9$ we will construct a global solution u to $\Delta_p u = 0$ in all of $\mathbb{R}^2$.

In view of the above result, no C^1 regularity can be expected for the free boundary at points at which $\nabla\varphi = 0$. Also, the lack of convexity of possible blow-up profiles seems to be a major obstacle of understanding the fine structure of the free boundary at these points.

Still, an interesting question is to decide whether the free boundary has finite $\mathcal{H}^{n-1}$ measure near points at which the gradient of the obstacle vanishes. A standard first step in this direction is to prove a nondegeneracy result stating that $u - \varphi$ cannot decay faster than quadratic at free boundary points. [ALS15] previously proved a similar nondegeneracy result at the free boundary points under the assumptions that $\varphi \in C^2$, $p > 2$, and $\Delta_p\varphi \leq -c_0 < 0$. However, if $\varphi \in C^2$ satisfies $\nabla\varphi(x_0) = 0$ then $\Delta_p\varphi(x_0) = 0$, and thus the result in [ALS15] can not be applied to free boundary points on $\{\nabla\varphi = 0\}$. We show the following.

Theorem 1.5. *Let $p \in (1,\infty)$, $\varphi \in C^2(B_1)$, and u be a solution of* (1.1) *in B_1. Assume that φ satisfies*

$$|\nabla\varphi|^{2-p}\mathrm{div}\big(|\nabla\varphi|^{p-2}\nabla\varphi\big) \le -c_0 < 0 \qquad in \quad \{\nabla\varphi \neq 0\}. \tag{1.2}$$

Then for any free boundary point $x_0 \in \partial\{u > \varphi\} \cap B_{1/2}$ there exists $c_1 > 0$ such that

$$\sup_{B_r(x_0)} (u-\varphi) \ge c_1 r^2 \qquad for \quad r \in (0, 1/2),$$

where the constant c_1 depends only on the modulus of continuity of $D^2\varphi$ and on the constant c_0 in (1.2).

Remark 1.6. The hypothesis (1.2) is nontrivial in the sense that (1.2) implies that either φ is identically constant on B_1 or $\{\nabla\varphi \neq 0\}$ is an open dense subset of B_1, see Lemma 3.1 below. In the case that φ is identically constant on B_1, by the Hopf boundary point lemma [Váz84, Theorem 5] either $u \equiv \varphi$ in B_1 or $\Delta_p u = 0$ and $u > \varphi$ in B_1 and in particular the free boundary is an empty set.

As a consequence of Theorem 1.5 we can deduce that, under the hypotheses of Theorem 1.5, the free boundary is porous: i.e., there exists a $\delta > 0$ such that for every $B_r(x_0) \subseteq B_1$, there exists $B_{\delta r}(x) \subset B_r(x_0) \setminus \partial\{u > \varphi\}$. The proof is standard and follows from combining the optimal regularity of solutions in [ALS15] with Theorem 1.5 above. Porosity of the free boundary implies that the free boundary has zero Lebesgue measure. We in fact prove the stronger result that, under the hypotheses of Theorem 1.5, the free boundary $\partial\{u > \varphi\}$ is an $(n-1)$-dimensional rectifiable set.

Definition 1.7. Let $0 \le k \le n$ be an integer. We say a set $S \subseteq \mathbb{R}^n$ is countably k-rectifiable if there exists a set $E_0 \subset \mathbb{R}^n$ with $\mathcal{H}^k(E_0) = 0$ and a countable collection of Lipschitz maps $f_j : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$S \subseteq E_0 \cup \bigcup_{j=1}^{\infty} f_j(\mathbb{R}^k).$$

Theorem 1.8. *Let $p \in (1,\infty)$, $\varphi \in C^2(B_1)$, and u be a solution of* (1.1) *in B_1. Assume that φ satisfies* (1.2). *Then, the free boundary $\partial\{u > \varphi\}$ is countably $(n-1)$-rectifiable.*

Related obstacle-type problems for the p-Laplacian have been studied in [KKPS00, LS03, CLRT14, CLR12]. In those works, however, they studied the different problem

$$\Delta_p u = f(x)\chi_{\{u>0\}} \qquad \text{in} \quad \Omega \subset \mathbb{R}^n. \tag{1.3}$$

It is important to notice that, when $p \neq 2$, the obstacle problems (1.1) and (1.3) are of quite different nature. For example, when $f \equiv 1$ solutions to (1.3) are *not* $C^{1,1}$ but $C^{1,\frac{1}{p-1}}$ near all free boundary points.

The paper is organized as follows. In Section 2 we prove Theorem 1.1. Then, in Section 3 we prove Theorems 1.5 and 1.8.

2. Homogeneous degree-two solutions

We construct here the homogeneous solutions of Theorem 1.1.

Proof of Theorem 1.1. Let $1 < p < \infty$ and $p \neq 2$, and let $\varphi(x) = -|x|^2$ in $\mathbb{R}^2$. We will show that there exists a global solution $u(x_1, x_2)$ to (1.1) which is homogeneous of degree 2 and such that the free boundary consists of two rays meeting at an angle $\theta_0 \neq \pi$.

We use polar coordinates $re^{i\theta}$ on $\mathbb{R}^2$ where $r > 0$ and $\theta \in [0, 2\pi]$. We want to construct $u \in C^1(\mathbb{R}^2)$ such that

$$\begin{gathered} \Delta_p u(re^{i\theta}) = 0, \quad u(re^{i\theta}) \geq -r^2 \quad \text{for } \theta \in (0, \theta_0), \\ u(re^{i\theta}) = -r^2 \quad \text{for } \theta \in [\theta_0, 2\pi], \\ \frac{\partial u}{\partial \theta}(r) = \frac{\partial u}{\partial \theta}(re^{i\theta_0}) = 0. \end{gathered} \tag{2.1}$$

Assume that $u(re^{i\theta}) = r^2 v(\theta)$ for some 2π-periodic function $v \in C^1(\mathbb{R}) \cap C^\infty([0, \theta_0]) \cap C^\infty([\theta_0, 2\pi])$. We want to express $\Delta_p u(re^{i\theta}) = 0$ for $\theta \in (0, \theta_0)$ as a ordinary differential equation of v. We compute

$$\nabla u = 2\, r\, v(\theta)\, \frac{\partial}{\partial r} + v'(\theta)\, \frac{\partial}{\partial \theta}$$

and thus

$$\begin{aligned} \Delta_p u &= \frac{1}{r} \frac{\partial}{\partial r} \left(r\, |\nabla u|^{p-2} \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial \theta} \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r^p \,(4\, v^2 + (v')^2)^{(p-2)/2} \cdot 2v \right) + r^{p-2} \frac{\partial}{\partial \theta} \left((4\, v^2 + (v')^2)^{(p-2)/2}\, v' \right) \\ &= r^{p-2} \,(4\, v^2 + (v')^2)^{(p-2)/2} \left(2p\, v + v'' + (p-2)\, \frac{4\, v\, (v')^2 + (v')^2\, v''}{4\, v^2 + (v')^2} \right). \end{aligned}$$

Thus we can rewrite $\Delta_p u = 0$ as

$$2p\, v + v'' + (p-2)\, \frac{4\, v\, (v')^2 + (v')^2\, v''}{4\, v^2 + (v')^2} = 0\,.$$

Solving for v'',

$$v'' = -v \frac{8p\, v^2 + (6p-8)\, (v')^2}{4\, v^2 + (p-1)\, (v')^2}\,. \tag{2.2}$$

Notice that (2.1) is equivalent to v satisfying (2.2) for $\theta \in (0, \theta_0)$ and

$$\begin{gathered} v(\theta) \geq -1 \quad \text{for } \theta \in (0, \theta_0), \\ v(\theta) = -1 \quad \text{for } \theta \in [\theta_0, 2\pi], \\ v'(0) = v'(\theta_0) = 0. \end{gathered} \tag{2.3}$$

Moreover, by integration by parts, (2.2), and the homogeneity of u, for all $\zeta \in C^1_c(\mathbb{R}^2 \setminus \{0\})$ it holds

$$
\begin{aligned}
-\int_{B_1} &|\nabla u|^{p-2}\, \nabla u \cdot \nabla \zeta \\
&= -\int_0^{\theta_0}\int_0^{\infty} |\nabla u|^{p-2}\, \nabla u \cdot \nabla\zeta\, r\, dr\, d\theta - \int_{\theta_0}^{2\pi}\int_0^{\infty} |\nabla u|^{p-2}\, \nabla u \cdot \nabla \zeta\, r\, dr\, d\theta \\
&= \int_0^{\theta_0}\int_0^{\infty} \Delta_p u\, \zeta\, r\, dr\, d\theta + \int_{\theta_0}^{2\pi}\int_0^{\infty} \Delta_p u\, \zeta\, r\, dr\, d\theta + \lim_{r\downarrow 0}\int_0^{2\pi} |\nabla u|^{p-2}\, D_r u\, \zeta\, r\, d\theta \\
&= \int_{\theta_0}^{2\pi}\int_0^{\infty} \Delta_p(-r^2)\, \zeta\, r\, dr\, d\theta.
\end{aligned}
$$

Hence, $\Delta_p u = \Delta_p(-r^2)\,\chi_{\{\theta_0<\theta<2\pi\}} \le 0$ weakly in $\mathbb{R}^2 \setminus \{0\}$. This together with (2.1) implies that u is a solution to (1.1).

Now let us solve (2.2). Set

$$
X = v(\theta) \qquad \text{and} \qquad Y = v'(\theta)
$$

so that we transform (2.2) into the first order system

$$
\begin{cases} X' = Y, \\ Y' = -X\, \dfrac{8p\, X^2 + (6p-8)\, Y^2}{4\, X^2 + (p-1)\, Y^2} \qquad \text{on } (0,\theta_0). \end{cases} \tag{2.4}
$$

Now, (2.3) implies that

$$
\begin{aligned}
X(\theta) \ge -1 \quad &\text{for } \theta \in (0,\theta_0), \\
(X(0), Y(0)) = (X(\theta_0)&, Y(\theta_0)) = (-1, 0)\,.
\end{aligned} \tag{2.5}
$$

Notice that (2.4) states that $X' = Y$ and Y' equals a homogeneous degree one function of (X, Y). Thus it is convenient to set

$$
X = \rho(\theta)\, \cos(\psi(\theta)) \qquad \text{and} \qquad Y = \rho(\theta)\, \sin(\psi(\theta))
$$

for some functions ρ and ψ, so that (2.4) is equivalent to

$$
\begin{cases} \dfrac{\rho'}{\rho}\, \cos(\psi) - \psi'\, \sin(\psi) = \sin(\psi), \\ \dfrac{\rho'}{\rho}\, \sin(\psi) + \psi'\, \cos(\psi) = -\cos(\psi)\, \dfrac{8p\, \cos^2(\psi) + (6p-8)\, \sin^2(\psi)}{4\, \cos^2(\psi) + (p-1)\, \sin^2(\psi)} \end{cases}
$$

for all $\theta \in (0,\theta_0)$. Let

$$
F_p(\psi) := \frac{8p\, \cos^2(\psi) + (6p-8)\, \sin^2(\psi)}{4\, \cos^2(\psi) + (p-1)\, \sin^2(\psi)} - 1 = \frac{(8p-4)\, \cos^2(\psi) + (5p-7)\, \sin^2(\psi)}{4\, \cos^2(\psi) + (p-1)\, \sin^2(\psi)}
$$

so that

$$\begin{cases} \dfrac{\rho'}{\rho} \cos(\psi) - \psi' \sin(\psi) = \sin(\psi), \\ \dfrac{\rho'}{\rho} \sin(\psi) + \psi' \cos(\psi) = -\cos(\psi) - \cos(\psi)\, F(\psi) \end{cases} \tag{2.6}$$

for all $\theta \in (0, \theta_0)$. Note that (2.6) can be rewritten as

$$\frac{\rho'}{\rho} = -\cos(\psi) \sin(\psi)\, F_p(\psi), \tag{2.7}$$

$$\psi' = -1 - \cos^2(\psi)\, F_p(\psi), \tag{2.8}$$

and this system can be solved by first solving (2.8) to find ψ, and then integrating (2.7) to find ρ. We compute that

$$\begin{aligned} \frac{\partial}{\partial p} F_p(\psi) &= \frac{\partial}{\partial p} \left(\frac{(8p-4) \cos^2(\psi) + (5p-7) \sin^2(\psi)}{4 \cos^2(\psi) + (p-1) \sin^2(\psi)} \right) \\ &= \frac{\partial}{\partial p} \left(\frac{(3p+3) \cos^2(\psi) + 5p - 7}{(-p+5) \cos^2(\psi) + p - 1} \right) \\ &= \frac{(3 \cos^2(\psi) + 5)\,(5 \cos^2(\psi) - 1) - (3 \cos^2(\psi) - 7)\,(-\cos^2(\psi) + 1)}{((-p+5) \cos^2(\psi) + p - 1)^2} \\ &= \frac{18 \cos^4(\psi) + 12 \cos^2(\psi) + 2}{((-p+5) \cos^2(\psi) + p - 1)^2} > 0 \end{aligned}$$

for all $\psi \in [0, 2\pi]$, so

$$1 + \cos^2(\psi)\, F_p(\psi) \geq 1 + \cos^2(\psi)\, F_1(\psi) = 1 + \cos^2(\psi) - \frac{1}{2} \sin^2(\psi) = \frac{1}{2} + \frac{3}{2} \cos^2(\psi) > 0$$

for all $\psi \in [0, 2\pi] \setminus \{\pi/2, 3\pi/2\}$. Note that, when $\psi = \pi/2, 3\pi/2$, $F_p(\psi)$ degenerates as $p \downarrow 1$, but $1 + \cos^2(\psi)\, F_p(\psi) = 1$ for all $p > 1$. Thus, solving (2.8), we find that $\psi(\theta) = \Theta^{-1}(\theta)$ for all $\theta \in [0, \theta_0]$ where $\Theta : \mathbb{R} \to \mathbb{R}$ is the strictly decreasing function defined by

$$\Theta(\psi) := -\int_{\psi(0)}^{\psi} \frac{d\sigma}{1 + \cos^2(\sigma)\, F_p(\sigma)} \tag{2.9}$$

for $\psi(0)$ to be determined. Integrating (2.7) over $[0, \theta]$, we obtain

$$\rho(\theta) = \rho(0) \exp\left(-\int_0^{\theta} \cos(\psi(\tau)) \sin(\psi(\tau))\, F_p(\psi(\tau))\, d\tau \right) \tag{2.10}$$

for all $\theta \in [0, \theta_0]$ and for $\rho(0)$ to be determined. Notice that $\psi, \rho \in C^\infty([0, \theta_0])$ and thus $v = \rho \cos(\psi) \in C^\infty([0, \theta_0])$.

It remains to determine θ_0, $\psi(0)$, and $\rho(0)$, and to verify that (2.5) holds true.

To this aim, we observe that (2.5) is equivalent to

$$\rho(0) = \rho(\theta_0) = 1, \quad \psi(0) = \pi, \quad \psi(\theta_0) = -(2k-1)\pi \tag{2.11}$$

for some integer $k \geq 1$. Then, in view of the fact that $X' = Y$ (and so $X(\theta)$ attains its minimum value when $\psi(\theta) = -(2j-1)\pi$ for some integer j), we see that

$$\rho(\theta) \leq 1 \text{ whenever } \psi(\theta) = -(2j-1)\pi, \text{ for } j = 1, 2, \ldots, k-1. \tag{2.12}$$

Hence, by (2.11), we should choose $\rho(0) = 1$ and $\psi(0) = \pi$. To choose θ_0 observe that, by (2.9), $\psi(\theta_0) = -(2k-1)\,\pi$ if and only if

$$\theta_0 = \int_{-(2k-1)\,\pi}^{\pi} \frac{d\sigma}{1 + \cos^2(\sigma)\, F_p(\sigma)} = 4k \int_0^{\pi/2} \frac{d\sigma}{1 + \cos^2(\sigma)\, F_p(\sigma)},$$

where the last step follows by symmetry. We compute that

$$\begin{aligned}
&\int_0^{\pi/2} \frac{d\sigma}{1 + \cos^2(\sigma)\, F_p(\sigma)} \\
&\quad = \int_0^{\pi/2} \frac{4\,\cos^2(\sigma) + (p-1)\,\sin^2(\sigma)}{4\,\cos^2(\sigma) + (p-1)\,\sin^2(\sigma) + (8p-4)\,\cos^4(\sigma) + (5p-7)\,\cos^2(\sigma)\,\sin^2(\sigma)}\, d\sigma \\
&\quad = \int_0^{\pi/2} \frac{(-p+5)\,\cos^2(\sigma) + p - 1}{(3p+3)\,\cos^4(\sigma) + (4p-2)\,\cos^2(\sigma) + p - 1}\, d\sigma \\
&\quad = \int_0^{\pi/2} \frac{(-p+5)\,\cos^2(\sigma) + p - 1}{(3\,\cos^2(\sigma) + 1)\,((p+1)\,\cos^2(\sigma) + p - 1)}\, d\sigma \\
&\quad = \int_0^{\pi/2} \left(\frac{2}{3\,\cos^2(\sigma) + 1} - \frac{p-1}{(p+1)\,\cos^2(\sigma) + p - 1} \right) d\sigma \\
&\quad = \int_0^{\pi/2} \left(\frac{2}{\tan^2(\sigma) + 4} - \frac{p-1}{(p-1)\,\tan^2(\sigma) + 2p} \right) \sec^2(\sigma)\, d\sigma \\
&\quad = \int_0^{\infty} \left(\frac{2}{t^2 + 4} - \frac{p-1}{(p-1)\,t^2 + 2p} \right) dt \\
&\quad = \left[\arctan\left(\frac{t}{2}\right) - \sqrt{\frac{p-1}{2p}}\, \arctan\left(\sqrt{\frac{p-1}{2p}}\, t \right) \right]_0^{\infty} \\
&\quad = \frac{\pi}{2} \left(1 - \sqrt{\frac{p-1}{2p}} \right),
\end{aligned}$$

where we let $t = \tan(\sigma)$. Thus, we need to choose

$$\theta_0 = 2k\pi \left(1 - \sqrt{\frac{p-1}{2p}} \right) \tag{2.13}$$

for some integer $k \geq 1$. Since

$$\frac{k\pi}{2} < 2k\pi \left(1 - \sqrt{\frac{p-1}{2p}} \right) = \theta_0 < 2\pi,$$

we deduce that $k \leq 3$.

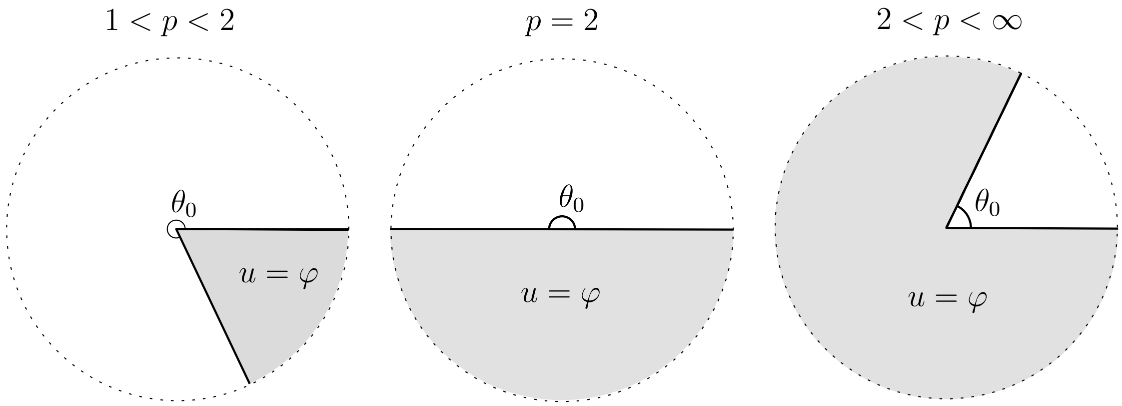

FIGURE 1. The angle θ_0 and the contact set $\{u = \varphi\}$ of the homogeneous solution for $1 < p < 2$, $p = 2$, and $2 < p < \infty$, respectively.

Notice that, for each $k = 1, 2, 3$, θ_0 given by (2.13) is decreasing as a function of p. In particular, when $k = 1$, $\theta_0 = 2\pi$ for $p = 1$, $\pi < \theta_0 < 2\pi$ for $1 < p < 2$, $\theta_0 = \pi$ for $p = 2$, and $0 < \theta_0 < \pi$ for $2 < p < \infty$, see Figure 1.

Hence if $p \in (1, 2) \cup (2, \infty]$, by setting $k = 1$ we can construct u for which

$$\theta_0 = 2\pi \left(1 - \sqrt{\frac{p-1}{2p}}\right) \in (0, \pi) \cup (\pi, 2\pi).$$

When $k = 2$, for all $1 < p < 2$ we have $\theta_0 > 2\pi$ and consequently we do not obtain a solution u, for $p = 2$ we have $\theta_0 = 2\pi$, and for all $2 < p < \infty$ we obtain a solution u with $\pi < \theta_0 < 2\pi$. Similarly, when $k = 3$, we do not obtain a solution u for $1 < p < 9$, $\theta_0 = 2\pi$ for $p = 9$, and we obtain a solution u with $\pi < \theta_0 < 2\pi$ for $9 < p < \infty$.

To conclude, we need to verify $\rho(\theta_0) = 1$. For this, suppose that θ is such that $\psi(\theta) = -(2j - 1)\pi$ for an integer $j \geq 1$. Observe that by (2.9) and symmetry,

$$\int_0^\theta \cos(\psi(\tau)) \sin(\psi(\tau)) F_p(\psi(\tau)) \, d\tau = \int_{-(2j-1)\pi}^{\pi} \frac{\cos(\sigma) \sin(\sigma) F_p(\sigma)}{1 + \cos^2(\sigma) F_p(\sigma)} \, d\sigma = 0 \,,$$

where $\sigma = \psi(\tau)$. Therefore by (2.10) $\rho(\theta) = \rho(0) = 1$. In particular, when $j = k$, we get $\rho(\theta_0) = \rho(0) = 1$.

Notice that for $k = 1$ the contact set $\{u > \varphi\}$ is precisely $\{\theta_0 \leq \theta \leq 2\pi\}$, whereas for $k = 2, 3$ the contact set $\{u > \varphi\}$ is the union of $\{\theta_0 \leq \theta \leq 2\pi\}$ and the rays $\psi(\theta) = -(2j - 1)\pi/2$, i.e. $\theta = 2\pi j \left(1 - \sqrt{\frac{p-1}{2p}}\right)$, for $j = 1, \ldots, k - 1$. □

Remark 2.1. Observe that when $k = 1$ and $p = 2$, the above argument produces a solution to u to (2.1) with $\theta_0 = \pi$. In other words, the contact set $\{u = \varphi\}$ is a half-space. On the other hand, when $k = p = 2$, or when $k = 3$ and $p = 9$, the above argument produces solutions ρ and ψ to (2.6) with $\theta_0 = 2\pi$ so that

$$\rho(0) = \rho(2\pi), \quad \psi(2\pi) - \psi(0) = -2k\pi,$$

and we thereby obtain $u \in C^1(\mathbb{R}^2)$ such that $\Delta_p u = 0$ in all of $\mathbb{R}^2$. Note that $\rho(0) > 0$ and $\psi(0)$ are arbitrary and this corresponds to the invariance of $\Delta_p u = 0$ in $\mathbb{R}^2$ under scaling and rotations.

While this solution for $p = 9$ is new (at least to our knowledge), these solutions for $p = 2$ are well-known. Indeed, when $p = 2$, (2.2) reduces to

$$v'' = -4v,$$

which obviously has the solution

$$v(\theta) = A\cos(2\theta) + B\sin(2\theta) \tag{2.14}$$

for constants $A, B \in \mathbb{R}$. Assuming that v is given by (2.14) for all $\theta \in [0, \theta_0]$ and v satisfies the boundary conditions $v(0) = v(\theta_0) = -1$ and $v'(0) = v'(\theta_0) = 0$, we obtain $\theta_0 = \pi$, $A = -1$, and $B = 0$ so that

$$u(x) = r^2 v(re^{i\theta}) = -|x|^2 + 2(x_2)_+^2$$

so that $w = u - \varphi$ is the well-known global solution $w = 2(x_2)_+^2$ to the obstacle problem $\min\{\Delta w, w\} = 0$ in $\mathbb{R}^2$. If instead we assume that v is given by (2.14) for all $\theta \in [0, 2\pi]$, then

$$u(x) = r^2 v(re^{i\theta}) = A(x_1^2 - x_2^2) + 2Bx_1x_2,$$

giving us the usual homogeneous degree two harmonic polynomials.

3. Structure of the free boundary

In this section we prove Theorem 1.5 and Theorem 1.8. First we will use the implicit function theorem to show that (1.2) implies that either φ is a constant function or $\{\nabla\varphi = 0\}$ is countably $(n-1)$-rectifiable. One immediate consequence is that $\{\nabla\varphi \neq 0\}$ is either empty or an open dense subset, which we use to prove Theorem 1.5. Another immediate consequence is Theorem 1.8.

Lemma 3.1. *Let $p \in (1, \infty)$ and $\varphi \in C^2(B_1)$ such that (1.2) holds true. Then either φ is identically constant on B_1 or $\{\nabla\varphi = 0\}$ is countably $(n-1)$-rectifiable.*

Proof. First we will show that (1.2) implies that either φ is identically constant on B_1 or

$$|D^2\varphi| \geq \frac{c_0}{n+p-2} \quad \text{in} \quad B_1, \tag{3.1}$$

where $|D^2\varphi(x)|$ denotes the operator norm of the matrix $D^2\varphi(x)$.

By (1.2),

$$c_0 \leq \left|\Delta\varphi + (p-2)\frac{\langle\nabla\varphi, D^2\varphi\nabla\varphi\rangle}{|\nabla\varphi|^2}\right| \leq (n+p-2)\,|D^2\varphi| \quad \text{in} \quad \{\nabla\varphi \neq 0\}.$$

Hence, noting that $\varphi \in C^2(B_1)$, we can express B_1 as the union of the disjoint sets

$$\left\{|D^2\varphi| \geq \frac{c_0}{n+p-2}\right\} \quad \text{and} \quad \text{int}\{\nabla\varphi = 0\},$$

which are both relatively open and closed in B_1, and use the connectedness of B_1 to reach our desired conclusion.

Now, suppose (3.1) holds true. Let $x_0 \in B_1 \cap \{\nabla\varphi = 0\}$. By (3.1), $D^2\varphi(x_0)$ has rank $k \geq 1$. Hence after an orthogonal change of variables, we may assume that

$$D^2\varphi(x_0) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

for some diagonal $k \times k$ matrix A with full rank. By the implicit function theorem, there is an open neighborhood of x_0 in which $M = \{D_i\varphi = 0 \text{ for } i = 1, 2, \ldots, k\}$ is a C^1 $(n-k)$-dimensional submanifold and $\{\nabla\varphi = 0\} \subseteq M$. Therefore $\{\nabla\varphi = 0\}$ is countably $(n-1)$-rectifiable. □

Next we will prove Theorem 1.5. For this, we will need the following Lemma.

Lemma 3.2. *Let $\varphi \in C^2(B_1)$ be a function satisfying* (1.2)*. Let $x_0 \in B_{1/2}$ be such that $\nabla\varphi(x_0) = 0$. Then, there exists $\varepsilon > 0$ and $\delta > 0$ such that*

$$\Delta_p\left(\varphi(x) + \varepsilon\frac{|x|^2}{2}\right) \leq 0 \qquad \text{in} \quad B_\delta(x_0).$$

The constants ε and δ depend only on the modulus of continuity of $D^2\varphi$ and on the constant c_0 in (1.2)*.*

Proof. We may assume $x_0 = 0$. Let us denote

$$\begin{aligned}\widetilde{\Delta}_p w := |\nabla w|^{2-p}\Delta_p w &= |\nabla w|^{2-p}\text{div}\big(|\nabla w|^{p-2}\nabla w\big) \\ &= \Delta w + (p-2)\frac{\langle\nabla w, D^2 w\nabla w\rangle}{|\nabla w|^2} \\ &= \Delta w + (p-2)\Delta_\infty w\end{aligned}$$

wherever $\nabla w \neq 0$. We know that by (1.2)

$$\widetilde{\Delta}_p\varphi \leq -c_0 < 0 \qquad \text{in} \quad B_1 \cap \{\nabla\varphi \neq 0\}, \tag{3.2}$$

and we want to show that $\widetilde{\Delta}_p(\varphi + \frac{1}{2}\varepsilon|x|^2) \leq 0$ in $B_\delta \cap \{\nabla\varphi + \varepsilon x \neq 0\}$.

Let

$$\lambda_{\min}(x) = \lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x) = \lambda_{\max}(x)$$

denote the eigenvalues of $D^2\varphi(x)$ and $\lambda_i = \lambda_i(0)$, $\lambda_{\min} = \lambda_{\min}(0)$, and $\lambda_{\max} = \lambda_{\max}(0)$. By continuity of $D^2\varphi$, we have that $\lambda_i(x)$ are continuous in x.

Case 1. Assume first that $\lambda_{\max} \leq 0$, i.e., $\lambda_i \leq 0$ for all $i = 1, ..., n$.

Noting that

$$\lambda_{\min}(x) \leq \Delta_\infty\varphi(x) \leq \lambda_{\max}(x)$$

and using (3.2), we obtain for every $x \in B_\delta \cap \{\nabla\varphi \neq 0\}$ that

$$(n+p-2)\lambda_{\min}(x) \leq \big(\lambda_1(x)+...+\lambda_n(x)\big)+(p-2)\lambda_{\min}(x) \leq -\frac{1}{2}c_0 \qquad \text{if} \quad p>2$$

and

$$(n+p-2)\lambda_{\min}(x) \leq \big(\lambda_1(x)+...+\lambda_n(x)\big)+(p-2)\lambda_{\max}(x) \leq -\frac{1}{2}c_0 \qquad \text{if} \quad p<2,$$

provided that $\delta > 0$ is small enough. In any case, we find $\lambda_{\min}(x) \leq \frac{-1}{2(n+p-2)}c_0$ in B_δ. Moreover, if δ is small, then $\lambda_{\max}(x) \leq \varepsilon$ in B_δ. Hence, for all $x \in B_\delta$ such that $\nabla\varphi(x) \neq 0$ and $\nabla\varphi(x)+\epsilon x \neq 0$ we have

$$\begin{aligned}\widetilde{\Delta}_p\left(\varphi(x)+\frac{1}{2}\varepsilon|x|^2\right) &\leq \lambda_{\min}(x)+(n-1)\lambda_{\max}(x)+(p-2)\lambda_{\max}(x)+(n+p-2)\varepsilon\\ &\leq -\frac{1}{2(n+p-2)}c_0+(2n+2p-5)\varepsilon \leq 0 \qquad \text{if} \quad p>2, \end{aligned} \tag{3.3}$$

provided ε is sufficiently small. Since $B_\delta \cap \{\nabla\varphi \neq 0\}$ is an open dense subset of B_δ (thanks to Lemma 3.1), we have (3.3) for all $x \in B_\delta$ such that $\nabla\varphi(x)+\epsilon x \neq 0$. Similarly, for all $x \in B_\delta$ such that $\nabla\varphi(x)+\epsilon x \neq 0$ we obtain

$$\begin{aligned}\widetilde{\Delta}_p\left(\varphi(x)+\frac{1}{2}\varepsilon|x|^2\right) &\leq \lambda_{\min}(x)+(n-1)\lambda_{\max}(x)+(p-2)\lambda_{\min}(x)+(n+p-2)\varepsilon\\ &\leq -\frac{p-1}{2(n+p-2)}c_0+(2n+p-3)\varepsilon \leq 0 \qquad \text{if} \quad p<2\end{aligned}$$

provided ε is sufficiently small, as desired.

Case 2. Let us assume now that $\lambda_{\max} > 0$.

Since $\varphi \in C^2(B_1)$, there is a modulus of continuity ω such that

$$\begin{aligned}\left|D^2\varphi(x)-D^2\varphi(0)\right| &\leq \omega(|x|),\\ \left|\nabla\varphi(x)-D^2\varphi(0)x\right| &\leq |x|\,\omega(|x|).\end{aligned} \tag{3.4}$$

After an affine change of variables, we may assume that $D^2\varphi(0)$ is a diagonal matrix,

$$D^2\varphi(0) = \begin{pmatrix} \lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n \end{pmatrix}. \tag{3.5}$$

Notice that by (3.4) and (3.5) we have

$$\left|\nabla\varphi(x)-(\lambda_1x_1,...,\lambda_nx_n)\right| \leq |x|\,\omega(|x|). \tag{3.6}$$

By (3.4), (3.5), and (3.6), for any $x \in B_1$ such that $\nabla\varphi(x) \neq 0$ we have

$$\widetilde{\Delta}_p\varphi(x) = (\lambda_1+...+\lambda_n)+o(1)+(p-2)\frac{\sum_i \lambda_i^3x_i^2+o(|x|^2)}{\sum_i \lambda_i^2x_i^2+o(|x|^2)}.$$

In particular, since $\{\nabla\varphi \neq 0\}$ is dense in B_1 (as a consequence of Lemma 3.1), for each $\lambda_i \neq 0$ there is a sequence of points $x^{(k)} \to 0$ such that

$$\widetilde{\Delta}_p \varphi(x^{(k)}) \to (\lambda_1 + ... + \lambda_n) + (p-2)\lambda_i.$$

This together with (3.2) means that

$$\begin{aligned} (\lambda_1 + \cdots + \lambda_n) + (p-2)\lambda_{\max} &\le -c_0 \qquad \text{if} \quad p > 2, \\ (\lambda_1 + \cdots + \lambda_n) + (p-2)\lambda_{\min} &\le -c_0 \qquad \text{if} \quad p < 2. \end{aligned} \tag{3.7}$$

Let $\varepsilon > 0$ to be chosen later, and let $\delta > 0$ small so that $\omega(|x|) < \varepsilon^5$ for $|x| < \delta$. Then for all $x \in B_\delta$ such that $\nabla\varphi(x) + \varepsilon x \neq 0$ we have

$$\begin{aligned} &\widetilde{\Delta}_p \left(\varphi(x) + \frac{1}{2}\varepsilon|x|^2 \right) \\ &\qquad = \Delta\varphi(x) + n\varepsilon + (p-2)\frac{\langle \nabla\varphi(x) + \varepsilon x, (D^2\varphi(x) + \varepsilon \mathrm{Id})(\nabla\varphi(x) + \varepsilon x)\rangle}{|\nabla\varphi + \varepsilon x|^2}. \end{aligned} \tag{3.8}$$

We must now be careful and choose $\varepsilon > 0$ such that the denominator is not zero for any $|x| < \delta$. For this, let $\theta > 0$ to be chosen later, and $k \in \{1, ..., n+1\}$ be such that no λ_i satisfies $-(k+1)\theta < \lambda_i < -k\theta$. Take $\varepsilon = (k + \frac{1}{2})\theta$, and notice that $\frac{1}{2}\theta \le \varepsilon \le (n+2)\theta$ and

$$|\lambda_i + \varepsilon| \ge \frac{\varepsilon}{2(n+2)} \qquad \text{for all} \quad i = 1, ..., n. \tag{3.9}$$

Suppose $p > 2$. By (3.4), (3.5), and (3.6), we find that for $|x| < \delta$

$$\begin{aligned} \frac{\langle \nabla\varphi(x) + \varepsilon x, (D^2\varphi(x) + \varepsilon \mathrm{Id})(\nabla\varphi(x) + \varepsilon x)\rangle}{|\nabla\varphi + \varepsilon x|^2} &\le \frac{\sum_i (\lambda_i + \varepsilon)^3 x_i^2 + \varepsilon^4 |x|^2}{\sum_i (\lambda_i + \varepsilon)^2 x_i^2 - \varepsilon^4 |x|^2} \\ &= \frac{\sum_i \left[(\lambda_i + \varepsilon)^3 - \varepsilon^4\right] x_i^2}{\sum_i \left[(\lambda_i + \varepsilon)^2 - \varepsilon^4\right] x_i^2} \\ &\le \max_i \frac{(\lambda_i + \varepsilon)^3 + \varepsilon^4}{(\lambda_i + \varepsilon)^2 - \varepsilon^4} \\ &= \max_i \left(\lambda_i + \varepsilon + \frac{(\lambda_i + \varepsilon)\varepsilon^4 + \varepsilon^4}{(\lambda_i + \varepsilon)^2 - \varepsilon^4} \right) \\ &\le \lambda_{\max} + \varepsilon + C\varepsilon^2 \end{aligned} \tag{3.10}$$

provided that $\varepsilon > 0$ is small enough, where in the last inequality we used (3.9). Now using (3.4), (3.5), (3.10), and (3.7), it follows by(3.8) that

$$\widetilde{\Delta}_p \left(\varphi(x) + \frac{1}{2}\varepsilon|x|^2 \right) \le (\lambda_1 + ... + \lambda_n) + (p-2)\lambda_{\max} + n\varepsilon + C\varepsilon^2 \le 0$$

on $B_\delta \setminus \{0\}$, provided that $\varepsilon > 0$ is small enough. (Recall from the argument above that $\nabla\varphi(x) + \varepsilon x = 0$ if and only if $x = 0$.)

On the other hand, if $p < 2$, then the same argument yields

$$\widetilde{\Delta}_p \left(\varphi(x) + \frac{1}{2}\varepsilon|x|^2 \right) \leq (\lambda_1 + ... + \lambda_n) + (p-2)\lambda_{\min} + n\varepsilon + C\varepsilon^2 \leq 0$$

on $B_\delta \setminus \{0\}$, and thus we are done. □

Using the previous Lemma, we can now establish the following nondegeneracy property.

Proof of Theorem 1.5. We claim that for every free boundary point $y_0 \in B_{1/2} \cap \partial\{u > \varphi\}$ there exists $\delta = \delta(y_0) > 0$ and $c = c(y_0) > 0$ such that

$$\sup_{B_r(y)} (u - \varphi) \geq cr^2 \qquad \text{for} \quad r \in (0, \delta),\, y \in B_\delta(y_0) \cap \overline{\{u > \varphi\}}. \tag{3.11}$$

The conclusion of Theorem 1.5 then follows from a standard covering argument.

In the case where $\nabla\varphi(y_0) \neq 0$, we may choose δ so that $\nabla\varphi \neq 0$ in $B_{4\delta}(y_0)$. In this way $\Delta_p u$ is uniformly elliptic in $B_{4\delta}(y_0)$ and (3.11) follows by the classical theory (see for instance [Caf98, Lemma 5]).

Suppose $\nabla\varphi(y_0) = 0$. By Lemma 3.2, there are $\varepsilon > 0$ and $\delta > 0$ such that

$$v(x) = \varphi(x) + \varepsilon\frac{|x-y|^2}{2}$$

satisfies $\Delta_p v \leq 0$ in $B_{2\delta}(y_0)$. By continuity, we may assume $y \in B_\delta(y_0) \cap \{u > \varphi\}$. Then, for any $r < \delta$, we have $\Delta_p u \geq \Delta_p v$ in $\{u > \varphi\} \cap B_r(y)$. Moreover, $u(y) \geq \varphi(y) = v(y)$. It follows from the comparison principle that there is $z_y \in \partial(\{u > \varphi\} \cap B_r(y))$ such that $u(z_y) \geq v(z_y)$. Since $u < v$ on $\{u = \varphi\}$ it follows that $z_y \in \{u > \varphi\} \cap \partial B_r(x_0)$, and so

$$u(z_y) - \varphi(z_y) = u(z_y) - v(z_y) + \frac{\varepsilon r^2}{2} \geq \frac{\varepsilon r^2}{2}.$$

□

As a direct consequence of Lemma 3.1 and the classical theory of the obstacle problem for uniformly elliptic operators, we obtain Theorem 1.8.

Proof of Theorem 1.8. Let us express the free boundary $\Gamma = \partial\{u > \varphi\}$ as

$$\Gamma = \Gamma_1 \cup \Gamma_2 \quad \text{where} \quad \Gamma_1 = \Gamma \cap \{\nabla\varphi \neq 0\} \quad \text{and} \quad \Gamma_2 = \Gamma \cap \{\nabla\varphi = 0\}. \tag{3.12}$$

In order to show that the free boundary Γ is countably $(n-1)$-rectifiable, it suffices to show that each of the sets Γ_1 and Γ_2 are countably $(n-1)$-rectifiable. For every $x_0 \in \Gamma_1$ there exists a $\delta > 0$ such that $\nabla\varphi \neq 0$ in $B_\delta(x_0)$ and thus $\Delta_p u$ is uniformly elliptic in $B_\delta(x_0)$. Hence $\Gamma_1 \cap B_{\delta/2}(x_0) = \Gamma \cap B_{\delta/2}(x_0)$ is a countably $(n-1)$-rectifiable set with finite $(n-1)$-dimensional measure (see for instance [Caf98, Corollary 4]). It follows from a covering argument that Γ_1 is countably $(n-1)$-rectifiable. By Lemma 3.1, $\{\nabla\varphi = 0\}$ is countably $(n-1)$-rectifiable and thus Γ_2 is countably $(n-1)$-rectifiable. □

Remark 3.3. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be as in (3.12). Our argument show that, for each $i = 1, 2$ and $x_0 \in \Gamma_i$, there exists a $\delta > 0$ such that $\Gamma_i \cap B_\delta(x_0)$ is a relatively closed, countably $(n-1)$-rectifiable subset of $B_\delta(x_0)$, with $\mathcal{H}^{n-1}(\Gamma_i \cap B_\delta(x_0)) < \infty$. However, since Γ_1 might be badly behaved near free boundary points x_0 at which $\nabla\varphi(x_0) = 0$, we cannot conclude that $\mathcal{H}^{n-1}(\Gamma \cap K) < \infty$ for all compact subsets $K \subset B_1$.

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