REGULARITY OF MINIMIZERS UP TO DIMENSION 7 IN DOMAINS OF DOUBLE REVOLUTION

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Abstract. We consider the class of semi-stable positive solutions to semilinear equations $-\Delta u = f(u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ of double revolution, that is, a domain invariant under rotations of the first m variables and of the last $n-m$ variables. We assume $2 \le m \le n-2$. When the domain is convex, we establish a priori L^p and H^1_0 bounds for each dimension n, with $p = \infty$ when $n \le 7$. These estimates lead to the boundedness of the extremal solution of $-\Delta u = \lambda f(u)$ in every convex domain of double revolution when $n \le 7$. The boundedness of extremal solutions is known when $n \le 3$ for any domain Ω, in dimensions $n \le 4$ when the domain is convex, and in dimensions $n \le 9$ in the radial case. In dimensions $5 \le n \le 9$ it remains an open question.

1. Introduction and results

Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, and consider the problem

$$\left\{ \begin{array}{rcll} -\Delta u &=& \lambda f(u) & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where λ is a positive parameter and the nonlinearity $f : [0,\infty) \longrightarrow \mathbb{R}$ satisfies

$$f \text{ is } C^{1,\gamma},\ \gamma \in (0,1),\ f \text{ is nondecreasing},\ f(0) > 0, \text{ and } \lim_{\tau\to\infty} \frac{f(\tau)}{\tau} = \infty. \tag{1.2}$$

It is well known (see the excellent monograph [7] and references therein) that there exists an extremal parameter $\lambda^* \in (0,\infty)$ such that if $0 < \lambda < \lambda^*$ then the problem (1.1) admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no solution, even in the weak sense. Here, minimal means smallest. Moreover, the set $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is a weak solution of problem (1.1) with $\lambda = \lambda^*$. It is called the extremal solution of (1.1).

When $f(u) = e^u$, it is well known that $u \in L^\infty(\Omega)$ if $n \le 9$, while $u^*(x) = \log \frac{1}{|x|^2}$ if $n \ge 10$ and $\Omega = B_1$. An analogous result holds for $f(u) = (1+u)^p$, $p > 1$. In the nineties H. Brezis and J.L. Vázquez [1] raised the question of determining the regularity of u^*, depending on the dimension n, for general convex nonlinearities satisfying (1.2). The first general results were proved by G. Nedev [9, 10] – see [5] for the statement and proofs of the results of [10].

Theorem 1.1 ([9],[10]). *Let Ω be a smooth bounded domain, f be a convex function satisfying (1.2), and u^* be the extremal solution of (1.1).*

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i) *If* $n \leq 3$*, then* $u^* \in L^\infty(\Omega)$.
ii) *If* $n \geq 4$*, then* $u^* \in L^p(\Omega)$ *for every* $p < \frac{n}{n-4}$.
iii) *Assume either that* $n \leq 5$ *or that* Ω *is strictly convex. Then* $u^* \in H_0^1(\Omega)$.

In 2006, the first autor and A. Capella [3] studied the radial case. Their result establishes optimal regularity results for general f.

Theorem 1.2 ([3]). *Let* $\Omega = B_1$ *be the unit ball in* $\mathbb{R}^n$*,* f *be a function satisfying* (1.2)*, and* u^* *be the extremal solution of* (1.1).

i) *If* $n \leq 9$*, then* $u^* \in L^\infty(\Omega)$.
ii) *If* $n \geq 10$*, then* $u^* \in L^p(\Omega)$ *for every* $p < p_n$*, where*

$$p_n = 2 + \frac{4}{\frac{n}{2+\sqrt{n-1}} - 2}. \tag{1.3}$$

iii) *For every dimension* n*,* $u^* \in H^3(\Omega)$.

The best known result was established in 2010 by the first author [2] and establishes the boundedness of u^* in convex domains in dimension $n = 4$. Related ideas recently allowed the first author and M. Sanchón [5] to improve the L^p estimates of Theorem 1.1 for u^* when $n \geq 5$:

Theorem 1.3 ([2],[5]). *Let* $\Omega \subset \mathbb{R}^n$ *be a convex, smooth and bounded domain,* f *be a function satisfying* (1.2)*, and* u^* *be the extremal solution of* (1.1).

i) *If* $n \leq 4$*, then* $u^* \in L^\infty(\Omega)$.
ii) *If* $n \geq 5$*, then* $u^* \in L^p(\Omega)$ *for every* $p < \frac{2n}{n-4} = 2 + \frac{4}{\frac{n}{2}-2}$.

The boundedness of extremal solutions remains an open question in dimensions $5 \leq n \leq 9$, even in the case of convex domains.

The aim of this paper is to study the regularity of the extremal solution u^* of (1.1) in a class of domains that we call of double revolution. The class contains domains much more general than balls, but is much simpler than general convex domains.

Let $n \geq 4$ and

$$\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k \quad \text{with} \quad n = m + k, \ m \geq 2 \quad \text{and} \quad k \geq 2. \tag{1.4}$$

For each $x \in \mathbb{R}^n$ we define the variables

$$\begin{cases} s = \sqrt{x_1^2 + \cdots + x_m^2} \\ t = \sqrt{x_{m+1}^2 + \cdots + x_n^2}. \end{cases}$$

We say that a domain $\Omega \subset \mathbb{R}^n$ is a *domain of double revolution* if it is invariant under rotations of the first m variables and also under rotations of the last k variables. Equivalently, Ω is of the form $\Omega = \{x \in \mathbb{R}^n : (s,t) \in U\}$ where U is a domain in $\mathbb{R}^2$ symmetric with respect to the two coordinate axes. In fact, $U = \{(s,t) \in \mathbb{R}^2 : x = (x_1 = s, x_2 = 0, ..., x_m = 0, x_{m+1} = t, ..., x_n = 0) \in \Omega\}$ is the intersection of Ω with the (x_1, x_{m+1})-plane. Note that U is smooth if and only if Ω is smooth. Let us call $\widetilde{\Omega}$ the intersection of U with the positive quadrant of $\mathbb{R}^2$, i.e.,

$$\widetilde{\Omega} = \{(s,t) \in \mathbb{R}^2 : s > 0, t > 0, (x_1 = s, x_2 = 0, ..., x_m = 0, x_{m+1} = t, ..., x_n = 0) \in \Omega\}. \tag{1.5}$$

Since $\{s=0\}$ and $\{t=0\}$ have zero measure in $\mathbb{R}^2$, we have that

$$\int_\Omega v\; dx = c_{m,k}\int_{\widetilde{\Omega}} v(s,t)s^{m-1}t^{k-1}dsdt$$

for every $L^1(\Omega)$ function $v=v(x)$ which depends only on the radial variables s and t. Here, $c_{m,k}$ is a positive constant depending only on m and k.

We will see that any semi-stable classical solution u of (1.1), and more generally of (1.7) below, depends only on s and t, and hence we can identify it with a function $u=u(s,t)$ defined in $(0,\infty)^2$ which satisfies the equation

$$u_{ss}+u_{tt}+\frac{m-1}{s}u_s+\frac{k-1}{t}u_t+f(u)=0 \quad \text{for} \quad s,t\in\widetilde{\Omega}.$$

Moreover, in the case of convex domains we will also have $u_s\le 0$ and $u_t\le 0$ and hence, $u(0)=\|u\|_{L^\infty}$ (see Remark 2.2).

The following is our main result. We prove that, in convex domains of double revolution, the extremal solution u^* is bounded when $n\le 7$, and belongs to H_0^1 and certain L^p spaces when $n\ge 8$. We also prove that in dimension $n=4$ the convexity of the domain is not required for the boundedness of u^*.

Theorem 1.4. *Assume* (1.4)*. Let $\Omega\subset\mathbb{R}^n$ be a smooth and bounded domain of double revolution, f be a function satisfying* (1.2)*, and u^* be the extremal solution of* (1.1)*.*

a) *Assume either that $n=4$ or that $n\le 7$ and Ω is convex. Then, $u^*\in L^\infty(\Omega)$.*

b) *If $n\ge 8$ and Ω is convex, then $u^*\in L^p(\Omega)$ for all $p<p_{m,k}$, where*

$$p_{m,k}=2+\frac{4}{\frac{m}{2+\sqrt{m-1}}+\frac{k}{2+\sqrt{k-1}}-2}. \tag{1.6}$$

c) *Assume either that $n\le 6$ or that Ω is convex. Then, $u^*\in H_0^1(\Omega)$.*

Remark 1.5. Let $q_{m,k}=\frac{m}{2+\sqrt{m-1}}+\frac{k}{2+\sqrt{k-1}}$. Since $\frac{x}{2+\sqrt{x-1}}$ is concave in $(2,\infty)$, we have that $q_{n-2,2}\le q_{m,k}\le q_{\frac{n}{2},\frac{n}{2}}$, and therefore $p_{\frac{n}{2},\frac{n}{2}}\le p_{m,k}\le p_{n-2,2}$. Thus, asymptotically as $n\to\infty$,

$$2+\frac{2\sqrt{2}}{\sqrt{n}}\simeq p_{\frac{n}{2},\frac{n}{2}}\le p_{m,k}\le p_{n-2,2}\simeq 2+\frac{4}{\sqrt{n}}.$$

Instead, in a general convex domain, L^p estimates are only known for $p\simeq 2+\frac{8}{n}$ (see Theorem 1.3 ii) above), while in the radial case one has L^p estimates for $p\simeq 2+\frac{4}{\sqrt{n}}$ (see Theorem 1.2 ii). In fact, the optimal exponent (1.3) in the radial case can be obtained by setting $m=n$ and $k=0$ in (1.6), but recall that in this paper we always assume $m\ge 2$ and $k\ge 2$.

The proofs of the results in [9, 10, 3, 2, 5] use the semi-stability of the extremal solution u^*. In fact, one first proves estimates for any regular semi-stable solution u of

$$\left\{\begin{array}{rcll} -\Delta u &=& f(u) & \text{in } \Omega\\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.7}$$

then one applies these estimates to the minimal solutions u_λ, and finally by monotone convergence the estimates also hold for the extremal solution u^*.

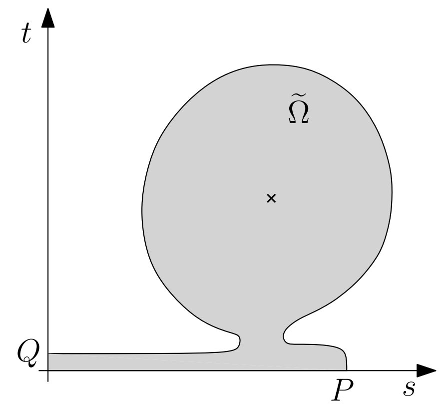

FIGURE 1. A non-convex domain for which the maximum of u^* will not be $u^*(0)$

Recall that a classical solution u of (1.7) is said to be *semi-stable* if the second variation of energy at u is nonnegative, i.e., if

$$Q_u(\xi) = \int_\Omega \left\{ |\nabla \xi|^2 - f'(u)\xi^2 \right\} dx \geq 0 \tag{1.8}$$

for all $\xi \in C^1_0(\overline{\Omega})$.

The proof of the estimates in [3, 2, 5] was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in $\mathbb{R}^n$ for $n \leq 7$. The key idea is to take $\xi = |\nabla u|\eta$ (or $\xi = u_r\eta$ in the radial case) and compute $Q_u(|\nabla u|\eta)$ in the semi-stability property satisfied by u. Then, the expression of Q_u in terms of η does not depend on f, and a clever choice of the test function η leads to L^p and L^∞ bounds depending on the dimension n.

In this paper we will proceed in a similar way, proving first results for general semi-stable solutions of (1.7) and then applying them to u_λ to deduce estimates for u^*. We will take $\xi = u_s\eta$ and $\xi = u_t\eta$ separately instead of $\xi = |\nabla u|\eta$, to obtain bounds for

$$\int_\Omega u_s^2 s^{-2\alpha-2} dx \qquad \text{and} \qquad \int_\Omega u_t^2 t^{-2\beta-2} dx, \tag{1.9}$$

where $\alpha < \sqrt{m-1}$ and $\beta < \sqrt{k-1}$. When the domain Ω is convex, we will have the additional information $\|u\|_{L^\infty} = u(0)$, $u_s \leq 0$, and $u_t \leq 0$, which combined with (1.9) will lead to L^∞ and L^p estimates for u^*.

Instead, when the domain Ω is not convex the maximum of u may not be achieved at the origin – see Figure 1 for an example in which $u(0)$ will be much smaller than $\|u\|_{L^\infty}$. Thus, in nonconvex domains we can not apply the same argument. However, if the maximum is away from $\{s = 0\}$ and $\{t = 0\}$ (as in Figure 1) then the problem is essentially two dimensional near the maximum, since $dx = c_{m,k}s^{m-1}t^{k-1}dsdt$ and both s and t will be positive and bounded below. We will still have to prove some boundary estimates, for instance estimates near the points P and Q in Figure 1. But, by the same reason as before, near P the coordinate s is positive and bonded below. Thus, the problem near P will be essentially $k+1$ dimensional, and $k = n - m \leq n - 2$. This will allow us, if $n - 1$ is small enough, to use Nedev's [9] $W^{2,p}$ estimates to obtain boundary estimates.

Our result for general semi-stable solutions of (1.7) reads as follows. It states global estimates controlled in terms of boundary estimates.

Proposition 1.6. *Assume* (1.4). *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain of double revolution, f be any $C^{1,\gamma}$ function, and u be a positive bounded semi-stable solution of* (1.7).

Let δ be a positive real number, and define

$$\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}.$$

Then, for some constant C depending only on Ω, δ, n, and also p in part b) below, one has:

a) *If $n \leq 7$ and Ω is convex, then $\|u\|_{L^\infty(\Omega)} \leq C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right)$.*

b) *If $n \geq 8$ and Ω is convex, then $\|u\|_{L^p(\Omega)} \leq C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right)$ for each $p < p_{m,k}$, where $p_{m,k}$ is given by* (1.6).

c) *For all $n \geq 4$, $\|u\|_{H_0^1(\Omega)} \leq C\|u\|_{H^1(\Omega_\delta)}$.*

To prove part b) of Proposition 1.6 we will need a new weighted Sobolev inequality in $(\mathbb{R}_+)^2 = \{(s,t) \in \mathbb{R}^2 : s > 0, t > 0\}$. It states the following.

Proposition 1.7. *Let $a > -1$ and $b > -1$ be real numbers, being positive at least one of them, and let*

$$D = 2 + a + b.$$

Let u be a nonnegative Lipschitz function with compact support in $\mathbb{R}^2$ such that

$$u_s \leq 0 \quad \text{and} \quad u_t \leq 0 \quad \text{in} \quad (\mathbb{R}_+)^2,$$

with strict inequality when $u > 0$. Then, for each $1 \leq q < D$ there exist a constant C, depending only on a, b, and q, such that

$$\left(\int_{(\mathbb{R}_+)^2} s^a t^b |u|^{q^*} dsdt\right)^{1/q^*} \leq C \left(\int_{(\mathbb{R}_+)^2} s^a t^b |\nabla u|^q dsdt\right)^{1/q}, \tag{1.10}$$

where $q^* = \frac{Dq}{D-q}$.

In section 4 we establish this weighted Sobolev inequality as a consequence of a new weighted isoperimetric inequality.

Remark 1.8. When a and b are nonnegative integers, inequality (1.10) is a direct consequence of the classical Sobolev inequality in $\mathbb{R}^D$. Namely, define in $\mathbb{R}^D = \mathbb{R}^{a+1} \times \mathbb{R}^{b+1}$ the variables s and t as before, with $m = a+1$ and $k = b+1$. Then, for functions u defined in $\mathbb{R}^D$ depending only on the variables s and t, write the integrals appearing in the classical Sobolev inequality in $\mathbb{R}^D$ in terms of s and t. Since $dx = c_{a,b} s^a t^b dsdt$, the obtained inequality is precisely the one given in Proposition 1.7.

Thus, the previous proposition extends the classical Sobolev inequality to the case of non-integer exponents a and b. In another article, [4], we prove inequality (1.10) with $(\mathbb{R}_+)^2$ replaced by $(\mathbb{R}_+)^d$ and with $s^a t^b$ replaced by the monomial weight $x^A = x_1^{A_1} \cdots x_d^{A_d}$. We also prove a related isoperimetric inequality with best constant, a weighted Morrey's inequality, and we determine extremal functions for some of these inequalities.

The paper is organized as follows. In section 2 we prove the estimates of Proposition 1.6. Section 3 deals with the regularity of the extremal solution of (1.1). Finally, in section 4 we prove the weighted Sobolev inequality of Proposition 1.7.

2. Proof of Proposition 1.6

In this section we prove the estimates of Proposition 1.6. For this, we will need two preliminary results.

Lemma 2.1 ([3, 2])**.** *Let u be a bounded semi-stable solution of* (1.7)*, and let c be a $C^2(\Omega)$ function. Then,*

$$\int_\Omega c\{\Delta c + f'(u)c\}\eta^2 dx \le \int_\Omega c^2|\nabla\eta|^2 dx$$

for all $\eta \in Lip(\Omega)$ with $\eta|_{\partial\Omega} = 0$.

Proof. It suffices to set $\xi = c\eta$ in the semi-stability condition (1.8) and then integrate by parts. The fact that we can take $\eta \in \mathrm{Lip}(\Omega)$ can be deduced by density arguments. □

Remark 2.2. Note that when the domain is of double revolution, any bounded semi-stable solution u of (1.7) will depend only on the variables s and t. To prove this, define $v = x_i u_{x_j} - x_j u_{x_i}$, with $i \neq j$. Note that u will will depend only on s and t if and only if $v \equiv 0$ for each $i, j \in \{1, ..., m\}$ and for each $i, j \in \{m+1, ..., n\}$.

We first see that, for such indexes i, j, v is a solution of the linearized equation of (1.7):

$$\begin{aligned} \Delta v &= \Delta(x_i u_{x_j} - x_j u_{x_i}) \\ &= x_i \Delta u_{x_j} + 2\nabla x_i \cdot \nabla u_{x_j} - x_j \Delta u_{x_i} - 2\nabla x_j \cdot \nabla u_{x_i} \\ &= x_i(\Delta u)_{x_j} - x_j(\Delta u)_{x_i} \\ &= -f'(u)\{x_i u_{x_j} - x_j u_{x_i}\} \\ &= -f'(u)v. \end{aligned}$$

Moreover, since $u = 0$ on $\partial\Omega$ then $v = 0$ on $\partial\Omega$. Thus, multiplying the equation by v and integrating by parts, we obtain

$$\int_\Omega \{|\nabla v|^2 - f'(u)v^2\} dx = 0.$$

But since u is semi-stable, $\lambda_1(\Delta + f'(u); \Omega) \ge 0$.

If $\lambda_1(\Delta + f'(u); \Omega) > 0$, the preious inequality leads to $v \equiv 0$.

If $\lambda_1(\Delta + f'(u); \Omega) = 0$, then we must have $v = K\phi_1$, where K is a constant and ϕ_1 is the first eigenfunction, which we may take to be positive in Ω. But since v is the derivative of u along the vector field $\partial_t = x_i\partial_{x_j} - x_j\partial_{x_i}$, and its integral curves are closed, v can not have constant sign. Thus, $K = 0$, that is, $v \equiv 0$.

Hence, we have seen that any classical semi-stable solution u of (1.7) depend only on the variables s and t. Moreover, by the classical result of Gidas-Ni-Nirenberg [8], when Ω is even and convex with respect each coordinate, we have $u_{x_i} > 0$ when $x_i > 0$, for $i = 1, ..., n$. In particular, when Ω is a convex domain of double revolution, we have that $u_s < 0$ and $u_t < 0$ for $s > 0$, $t > 0$. In particular,

$$\|u\|_{L^\infty(\Omega)} = u(0).$$

On the other hand, since $f \in C^{1,\gamma}$ then a bounded solution u of (1.7) satisfies $u \in C^{3,\gamma}(\Omega)$. In particular, $u \in C^3(\Omega)$, and $u_s, u_t \in C^2(\Omega)$. Finally, note that since u is an even function of s and t, then

$$u_s = 0 \text{ when } s = 0 \text{ and } u_t = 0 \text{ when } t = 0.$$

We now apply Lemma 2.1 separately with $c = u_s$ and with $c = u_t$, and then we choose appropriately the test function η to get the following result. This estimate is the key ingredient in the proof of Proposition 1.6.

Lemma 2.3. *Assume* (1.4). *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain of double revolution, f be any $C^{1,\gamma}$ function, and u be a positive bounded semi-stable solution of* (1.7). *Let α and β be such that*

$$\alpha^2 < m-1 \quad \textit{and} \quad \beta^2 < k-1.$$

Then, for each $\delta > 0$ there exists a constant C, which depends only on Ω, δ, n, α, and β, such that

$$\left(\int_\Omega \left\{ u_s^2 s^{-2\alpha-2} + u_t^2 t^{-2\beta-2} \right\} dx \right)^{1/2} \leq C \left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)} \right), \tag{2.1}$$

where

$$\Omega_\delta = \{x \in \Omega : \mathit{dist}(x, \partial\Omega) < \delta\}.$$

Proof. We will prove only the estimate for $u_s^2 s^{-2\alpha-2}$; the other term can be estimated similarly.

Differentiating the equation $-\Delta u = f(u)$ with respect to s, we obtain

$$-\Delta u_s + (m-1)\frac{u_s}{s^2} = f'(u) u_s,$$

and hence, setting $c = u_s \in C^2(\Omega)$ in Lemma 2.1, we have that

$$(m-1)\int_\Omega u_s^2 \frac{\eta^2}{s^2} dx \leq \int_\Omega u_s^2 |\nabla \eta|^2 dx$$

for all $\eta \in \mathrm{Lip}(\Omega)$, $\eta|_{\partial\Omega} = 0$. Let us set in the last inequality $\eta = \eta_\epsilon$, where

$$\eta_\epsilon = \begin{cases} s^{-\alpha}\rho & \text{if } \ s > \epsilon \\ \epsilon^{-\alpha}\rho & \text{if } \ s \leq \epsilon \end{cases} \qquad \text{and} \qquad \rho = \begin{cases} 0 & \text{in } \Omega_{\delta/3} \\ 1 & \text{in } \Omega \backslash \Omega_{\delta/2}, \end{cases}$$

and ρ is a smooth function. Then, since $\alpha^2 < \frac{1}{2}(\alpha^2 + m - 1) < m - 1$,

$$|\nabla \eta_\epsilon|^2 \leq \begin{cases} \frac{1}{2}(\alpha^2 + m - 1) s^{-2\alpha-2}\rho^2 & \text{in } (\Omega\backslash\Omega_{\delta/2}) \cap \{s > \epsilon\} \\ \frac{1}{2}(\alpha^2 + m - 1) s^{-2\alpha-2}\rho^2 + C s^{-2\alpha} & \text{in } \Omega_{\delta/2} \cap \{s > \epsilon\} \\ C\epsilon^{-2\alpha} & \text{in } \Omega \cap \{s \leq \epsilon\}, \end{cases}$$

and therefore

$$\frac{m-1-\alpha^2}{2} \int_{\Omega\cap\{s>\epsilon\}} u_s^2 s^{-2\alpha-2}\rho^2 dx \leq C \int_{\Omega_{\delta/2}\cap\{s>\epsilon\}} u_s^2 s^{-2\alpha} dx + C\epsilon^{-2\alpha} \int_{\Omega\cap\{s\leq\epsilon\}} u_s^2 dx,$$

where C denote different constants depending only on the quantities appearing in the statement of the lemma. Now, since $u_s \in L^\infty(\Omega)$ the last term is bounded by $C\|u_s\|_{L^infty}\epsilon^{m-2\alpha}$. Making $\epsilon \to 0$ and using that $2\alpha < 2\sqrt{m-1} \leq m$, we deduce

$$\int_\Omega u_s^2 s^{-2\alpha-2}\rho^2 dx \leq C \int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha} dx.$$

Hence, since $\rho \equiv 1$ in $\Omega\backslash\Omega_{\delta/2}$,

$$\int_{\Omega\backslash\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2} dx \leq C \int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha} dx \leq C \int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2} dx. \tag{2.2}$$

From this we deduce that, for another constant C,

$$\int_\Omega u_s^2 s^{-2\alpha-2}dx \le C\int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2}dx. \tag{2.3}$$

Let $\nu<1$ to be chosen later. On the one hand using that $u_s(0,t)=0$ we obtain that, if δ is small enough, $|u_s(s,t)|\le Cs^\nu\|u_s\|_{C^{0,\nu}}$ in $\Omega_{\delta/2}\cap\{s<\delta\}$. Moreover, since $-\Delta u=f(u)$ in Ω_δ then $\|u\|_{C^{1,\nu}(\Omega_{\delta/2})}\le C\left(\|u\|_{L^\infty(\Omega_\delta)}+\|f(u)\|_{L^\infty(\Omega_\delta)}\right)$, and therefore,

$$\|s^{-\nu}u_s\|_{L^\infty(\Omega_{\delta/2}\cap\{s<\delta\})}\le C\left(\|u\|_{L^\infty(\Omega_\delta)}+\|f(u)\|_{L^\infty(\Omega_\delta)}\right).$$

Thus, also in all $\Omega_{\delta/2}$ we have

$$\|s^{-\nu}u_s\|_{L^\infty(\Omega_{\delta/2})}\le C\left(\|u\|_{L^\infty(\Omega_\delta)}+\|f(u)\|_{L^\infty(\Omega_\delta)}\right). \tag{2.4}$$

On the other hand, taking ν sufficiently close to 1 such that $m-2\alpha-2+2\nu>0$, we will have

$$\int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2}dx \le \|s^{-\nu}u_s\|^2_{L^\infty(\Omega_{\delta/2})}\int_{\Omega_{\delta/2}} s^{-2\alpha-2+2\nu}dx\le C\|s^{-\nu}u_s\|^2_{L^\infty(\Omega_{\delta/2})},$$

and hence by (2.3) and (2.4),

$$\int_\Omega u_s^2 s^{-2\alpha-2}dx\le C\left(\|u\|_{L^\infty(\Omega_\delta)}+\|f(u)\|_{L^\infty(\Omega_\delta)}\right)^2.$$

□

Using Lemma 2.3 we can now establish Proposition 1.6.

Proof of Proposition 1.6. a) We assume Ω to be convex. Recall that in this case $\|u\|_{L^\infty}=u(0)$; see Remark 2.2.

On the one hand, making the change of variables $\sigma=s^{2+\alpha}$, $\tau=t^{2+\beta}$ in the integral in (2.1), one has

$$\left\{\begin{array}{rcl} s^{m-1}ds &=& c_\alpha\sigma^{\frac{m}{2+\alpha}-1}d\sigma\\ t^{k-1}dt &=& c_\beta\tau^{\frac{k}{2+\beta}-1}d\tau,\end{array}\right.$$

and thus,

$$\int_\Omega \sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}(u_\sigma^2+u_\tau^2)d\sigma d\tau\le C\left(\|u\|^2_{L^\infty(\Omega_\delta)}+\|f(u)\|^2_{L^\infty(\Omega_\delta)}\right). \tag{2.5}$$

Here, to simplify, we are abusing notation and calling again Ω the image of the two dimensional domain $\widetilde{\Omega}$ in (1.5) after the transformation $(s,t)\mapsto(\sigma,\tau)$. Therefore, setting $\rho=\sqrt{\sigma^2+\tau^2}$ and taking into account that in $\{\tau<\sigma<2\tau\}$ we have $\sigma>\frac{\rho}{2}$ and $\tau\ge\frac{\rho}{3}$, we obtain

$$\int_{\Omega\cap\{\tau<\sigma<2\tau\}}\rho^{\frac{m}{2+\alpha}+\frac{k}{2+\beta}-2}(u_\sigma^2+u_\tau^2)d\sigma d\tau\le C\left(\|u\|^2_{L^\infty(\Omega_\delta)}+\|f(u)\|^2_{L^\infty(\Omega_\delta)}\right). \tag{2.6}$$

On the other hand, for each angle θ we have

$$u(0)\le\int_{l_\theta}|\nabla u|d\rho,$$

where l_θ is the segment of angle θ in the (σ,τ)-plane from the origin to $\partial\Omega$. Integrating in $\arctan\frac{1}{2}<\theta<\arctan\frac{1}{2}=\frac{\pi}{4}$,

$$u(0)\le C\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}}\int_{l_\theta}|\nabla u|d\rho d\theta=C\int_{\Omega\cap\{\tau<\sigma<2\tau\}}\frac{|\nabla u|}{\rho}d\sigma d\tau. \tag{2.7}$$

Now, applying Schwarz's inequality and taking into account (2.6) and (2.7),

$$u(0)\le C\left(\|u\|^2_{L^\infty(\Omega_\delta)}+\|f(u)\|^2_{L^\infty(\Omega_\delta)}\right)\left(\int_{\Omega\cap\{\tau<\sigma<2\tau\}}\rho^{-\left(\frac{m}{2+\alpha}+\frac{k}{2+\beta}\right)}d\sigma d\tau\right)^{1/2}.$$

This integral is finite when

$$\frac{m}{2+\alpha}+\frac{k}{2+\beta}<2.$$

Therefore, if

$$\frac{m}{2+\sqrt{m-1}}+\frac{k}{2+\sqrt{k-1}}<2 \tag{2.8}$$

then we can choose $\alpha<\sqrt{m-1}$ and $\beta<\sqrt{k-1}$ such that the integral is finite. Hence, since $\|u\|_{L^\infty(\Omega)}=u(0)$, if condition (2.8) is satisfied then

$$\|u\|_{L^\infty(\Omega)}\le C\left(\|u\|^2_{L^\infty(\Omega_\delta)}+\|f(u)\|^2_{L^\infty(\Omega_\delta)}\right).$$

Let

$$q_{m,k}=\frac{m}{2+\sqrt{m-1}}+\frac{k}{2+\sqrt{k-1}}.$$

If $n\le 7$ then by Remark 1.5 we have that $q_{m,k}\le q_{\frac{n}{2},\frac{n}{2}}\le q_{\frac{7}{2},\frac{7}{2}}<2$, while if $n\ge 8$ then $q_{m,k}\ge q_{n-2,2}\ge q_{6,2}>2$. Hence, (2.8) is satisfied if and only if $n\le 7$.

b) We assume that Ω is convex and that $n\ge 8$. Thus, $q_{m,k}\ge q_{n-2,2}\ge q_{6,2}>2$. Moreover, without loss of generality we assume that $p\ge\frac{2n}{n-4}$. Then,

$$\frac{2n}{n-4}\le p<p_{m,k}$$

and we can choose nonnegative numbers α and β such that $\alpha^2<m-1$, $\beta^2<k-1$,

$$p=2+\frac{4}{\frac{m}{2+\alpha}+\frac{k}{2+\beta}-2}, \tag{2.9}$$

and such that one of the numbers $\frac{m}{2+\alpha}-1$ or $\frac{k}{2+\beta}-1$ is positive. This is because the expression (2.9) is increasing in α and β, and its value for $\alpha=\beta=0$ is $\frac{2n}{n-4}$.

Making the change of variables $\sigma=s^{2+\alpha}$, $\tau=t^{2+\beta}$ we obtain inequality (2.5), and hence, using Proposition 1.7 with $a=\frac{m}{2+\alpha}-1$, $b=\frac{k}{2+\beta}-1$ and $q=2$, we deduce that

$$\left(\int_\Omega\sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}|u|^p d\sigma d\tau\right)^{1/p}\le C\left(\|u\|^2_{L^\infty(\Omega_\delta)}+\|f(u)\|^2_{L^\infty(\Omega_\delta)}\right).$$

Here we have extended u by zero outside Ω, obtaining a nonnegative Lipschitz function. By Remark 2.2 it satisfies $u_s<0$ and $u_t<0$ for $s>0$, $t>0$, since Ω is convex. Note also that $q^*=2^*=\frac{2D}{D-2}=2+\frac{4}{D-2}=p$, since $D=\frac{m}{2+\alpha}+\frac{k}{2+\beta}$.

Finally, since

$$\int_{\Omega} \sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}|u|^p d\sigma d\tau = c_{\alpha,\beta}\int_{\Omega} s^{m-1}t^{k-1}|u|^p ds dt = c_{\alpha,\beta,m,k}\|u\|^p_{L^p(\Omega)},$$

we conclude

$$\|u\|_{L^p(\Omega)} \le C\left(\|u\|^2_{L^\infty(\Omega_\delta)} + \|f(u)\|^2_{L^\infty(\Omega_\delta)}\right).$$

c) Setting $\alpha = 0$ in (2.2), we obtain

$$\int_{\Omega\setminus\Omega_{\delta/2}} u_s^2 s^{-2} dx \le C\int_{\Omega_{\delta/2}} u_s^2 dx,$$

and therefore, for a different constant C,

$$\int_{\Omega} u_s^2 dx \le C\int_{\Omega_{\delta/2}} u_s^2 dx.$$

□

3. Regularity of the extremal solution

This section is devoted to give the proof of Theorem 1.4. The estimates for convex domains will follow easily from Proposition 1.6 and the boundary estimates in convex domains of DeFigueireda, Lions, and Nussbaum [6]; see also [2]. These boundary estimates follow easily from the moving planes method [8].

Theorem 3.1 ([8],[6]). *Let Ω be a smooth, bounded, and convex domain, f be any Lipschitz function, and u be a bounded positive solution of* (1.7). *Then, there exist constants $\delta > 0$ and C, both depending only on Ω, such that*

$$\|u\|_{L^\infty(\Omega_\delta)} \le C\|u\|_{L^1(\Omega)},$$

where $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$.

We can now give the proof of Theorem 1.4. The main part of the proof are the estimates for non-convex domains. They will be proved by interpolating the $W^{1,p}$ and $W^{2,p}$ estimates of Nedev [9] and our estimate of Lemma 2.3, and by applying the classical Sobolev inequality as explained in Remark 1.8.

Proof of Theorem 1.4. As we have pointed out, the estimates for convex domains are a consequence of Proposition 1.6 and Theorem 3.1. Namely, we can apply estimates of Proposition 1.6 to the bounded solutions u_λ of (1.1) for $\lambda < \lambda^*$, and then by monotone convergence the estimates holds for the extremal solution u^*. Note that $\|u_\lambda\|_{L^1(\Omega)} \le \|u^*\|_{L^1(\Omega)} < \infty$ for all $\lambda < \lambda^*$.

Next we prove the estimates in parts a) and c) for non-convex domains.

We start by proving part a) when Ω is not convex. We have that $n = 4$, i.e. $m = k = 2$.

In [9] (see its Remark 1) it is proved that the extremal solution satisfies $u^* \in W^{1,p}(\Omega)$ for all $p < \frac{n}{n-3}$. Thus, in our case for each $p < 4$ we have

$$\int_{\Omega} |u_s^*|^p dx \le C \qquad \text{and} \qquad \int_{\Omega} |u_t^*|^p dx \le C.$$

Assume $\|u^*\|_{L^\infty(\Omega_\delta)} \le C$, which we left to be proven later. Then, by Lemma 2.3, for all $\gamma < 4$ we have

$$\int_\Omega s^{-\gamma}|u_s^*|^2 dx \le C \qquad \text{and} \qquad \int_\Omega t^{-\gamma}|u_t^*|^2 dx \le C.$$

Hence, for each $\lambda \in [0,1]$,

$$\int_\Omega (s^{-\lambda\gamma}|u_s^*|^{p-\lambda(p-2)} + t^{-\lambda\gamma}|u_t^*|^{p-\lambda(p-2)})dx \le C.$$

Setting now $\sigma = s^\alpha$, $\tau = t^\alpha$ and

$$\alpha = 1 + \frac{\lambda\gamma}{p-\lambda(p-2)},$$

we obtain

$$\int_\Omega \sigma^{\frac{2}{\alpha}-1}\tau^{\frac{2}{\alpha}-1}|\nabla_{(\sigma,\tau)}u^*|^{p-\lambda(p-2)}d\sigma d\tau \le C,$$

and taking $p = 3$, $\gamma = 3$ and $\lambda = 3/4$, we obtain

$$\int_\Omega |\nabla_{(\sigma,\tau)}u^*|^{9/4}d\sigma d\tau \le C.$$

Finally, applying Sobolev's inequality in dimension 2, $u^* \in L^\infty(\Omega)$.

It remains to prove that $\|u^*\|_{L^\infty(\Omega_\delta)} \le C$. Since $u^* \in W^{1,p}$ for $p < 4$, we have

$$\int_{\Omega_\delta} st|\nabla u^*|^p dsdt \le C.$$

Since the domain is smooth, then it has to be $0 \notin \partial\Omega$ (otherwise the boundary will have an isolated point) and hence, there exist $r_0 > 0$ and $\delta > 0$ such that $\Omega_\delta \cap B_{r_0}(0) = \emptyset$. Thus, $s \ge c$ in $\Omega_\delta \cap \{s > t\}$ and $t \ge c$ in $\Omega_\delta \cap \{s < t\}$, and

$$\int_{\Omega_\delta\cap\{s>t\}} t|\nabla u^*|^p dsdt \le C, \qquad \int_{\Omega_\delta\cap\{s<t\}} s|\nabla u^*|^p dsdt \le C.$$

If $p > 3$ then we can apply Sobolev's inequality in dimension 3 (as explained in Remark 1.8, to obtain $u^* \in L^\infty(\Omega_\delta\cap\{s > t\})$ and $u^* \in L^\infty(\Omega_\delta\cap\{s < t\})$. Therefore $u^* \in L^\infty(\Omega_\delta)$, as claimed.

To prove part c) in the non-convex case, let $n \le 6$. By Proposition 1.6, it suffices to prove that $u^* \in H^1(\Omega_\delta)$ for some $\delta > 0$. Take r_0 and δ such that $\Omega_\delta \cap B_{r_0}(0) = \emptyset$, as in part a).

In [9] is proved that $u^* \in W^{2,p}(\Omega)$ for $p < \frac{n}{n-2}$. Thus,

$$\int_{\Omega_\delta\cap\{s>t\}} t^{k-1}|D^2u^*|^p dsdt \le C, \qquad \int_{\Omega_\delta\cap\{s<t\}} s^{m-1}|D^2u^*|^p dsdt \le C.$$

Taking $p = \frac{2k+2}{k+3}$ and $p = \frac{2m+2}{m+3}$ respectively, and applying Sobolev's inequality in dimension $k+1$ and $m+1$ respectively, we obtain $u^* \in H^1(\Omega_\delta \cap \{s > t\})$ and $u^* \in H^1(\Omega_\delta \cap \{s < t\})$, and therefore, $u^* \in H^1(\Omega_\delta)$. $\square$

4. Weighted Sobolev inequality

It is well known that the classical Sobolev inequality can be deduced from an isoperimetric inequality. This is done by applying first the isoperimetric inequality to the level sets of the function and then using the coarea formula. In this way one deduces the Sobolev inequality with exponent 1 on the gradient. Then, by applying Hölder's inequality one deduces the general Sobolev inequality.

Since in our case we have $u_s \le 0$ and $u_t \le 0$, with strict inequality when $u > 0$, it suffices to prove a weighted isoperimetric inequality for domains $\Omega \subset (\mathbb{R}_+)^2$ satisfying the following properties:

P1) If $(s,t) \in \Omega$, then $(s',t') \in \Omega$ for all s' and t' with $0 < s' < s$ and $0 < t' < t$.
P2) $\Omega_t = \{s > 0 : \ (s,t) \in \Omega\}$ and $\Omega_s = \{t > 0 : \ (s,t) \in \Omega\}$ are strictly decreasing in t and s, respectively.

We will denote

$$m(\Omega) = \int_\Omega s^a t^b ds dt \qquad \text{and} \qquad m(\partial\Omega) = \int_{\partial\Omega \cap (\mathbb{R}_+)^2} s^a t^b ds dt.$$

Proposition 4.1. *Let $\Omega \subset (\mathbb{R}_+)^2$ be a bounded Lipschitz domain satisfying P1 and P2, and let $a > -1$ and $b > -1$ be real numbers, being positive at least one of them. Then, there exists a constant C, depending only on a and b such that*

$$m(\Omega)^{\frac{D-1}{D}} \le C m(\partial\Omega),$$

where $D = a + b + 2$.

Proof. First, by symmetry we can suppose $a > 0$.

Properties $P1$ and $P2$ ensure the existence of a non-increasing function $\psi \in W^{1,1}(\mathbb{R})$ with compact support such that $\Omega = \{(s,t) \in (\mathbb{R}_+)^2 : t < \psi(s)\}$. Then,

$$m(\Omega) = \int_0^{+\infty} s^a \psi^{b+1} ds, \qquad m(\partial\Omega) = \int_0^{+\infty} s^a \psi^b \sqrt{1 + \dot{\psi}^2} ds,$$

where $\dot{\psi}$ is the derivative of ψ.

Let $\lambda > 0$ be such that $m(\Omega) = \frac{\lambda^D}{a+1}$. Then, we claim that $\psi(s) < \lambda$ for $s > \lambda$. Assume that it is false. Then, we would have $v(s') \ge \lambda$ for some $s' > \lambda$, and hence

$$m(\Omega) \ge \int_0^{s'} s^a \psi^{b+1} ds > \int_0^\lambda s^a \lambda^{b+1} = \frac{\lambda^D}{a+1},$$

a contradiction. On the other hand, since $a > 0$, $b + 1 > 0$, and $\dot{\psi} \le 0$,

$$\begin{aligned} m(\partial\Omega) &= \int_0^{+\infty} s^a \psi^b \sqrt{1 + \dot{\psi}^2} ds \\ &\ge c \int_0^{+\infty} s^a \psi^b \left[1 - \frac{b+1}{a} \dot{\psi} \right] ds \\ &= c \int_0^{+\infty} s^a \left[\psi^b - \frac{d}{ds} \left(\frac{\psi^{b+1}}{a} \right) \right] ds \\ &= c \int_0^{+\infty} s^a \psi^{b+1} \left(\frac{1}{\psi} + \frac{1}{s} \right) ds. \end{aligned}$$

Finally, taking into accout that $\psi(s) < \lambda$ for $s > \lambda$, we obtain that $\frac{1}{\psi} + \frac{1}{s} \geq \lambda^{-1}$ for each $s > 0$, and

$$m(\partial\Omega) \geq c \int_0^{+\infty} s^a \psi^{b+1} \left(\frac{1}{\psi} + \frac{1}{s} \right) ds \geq c\lambda^{-1} m(\Omega) = cm(\Omega)^{\frac{D-1}{D}},$$

as claimed. □

Now we are able to prove our Sobolev inequality.

Proof of Proposition 1.7. We will prove first the case $q = 1$.

Letting χ_A the characteristic function of the set A, we have

$$u(x) = \int_0^{+\infty} \chi_{[u(x)>\tau]} d\tau,$$

where $x = (s, t)$.

So, by Minkowski's integral inequality

$$\begin{aligned} \left(\int_{(\mathbb{R}_+)^2} s^a t^b |u|^{\frac{D}{D-1}} dsdt \right)^{\frac{D-1}{D}} &\leq \int_0^{+\infty} \left(\int_{(\mathbb{R}_+)^2} s^a t^b \chi_{[u(x)>\tau]} dsdt \right)^{\frac{D-1}{D}} d\tau \\ &= \int_0^{+\infty} m(\{u(x) > \tau\})^{\frac{D-1}{D}} d\tau. \end{aligned}$$

Since $u_s \leq 0$ and $u_t \leq 0$, with strict inequality when $u > 0$, then the level sets $\{u(x) > \tau\}$ satisfy P1 and P2, so Proposition 4.1 implies that

$$m(\{u(x) > \tau\})^{\frac{D-1}{D}} \leq Cm(\{u(x) = \tau\}),$$

whence

$$\left(\int_{(\mathbb{R}_+)^2} s^a t^b |u|^{\frac{D}{D-1}} dsdt \right)^{\frac{D-1}{D}} \leq C \int_0^{+\infty} m(\{u(x) = \tau\}) d\tau = C \int_{(\mathbb{R}_+)^2} s^a t^b |\nabla u| dsdt,$$

where we have used the coarea formula.

It remains to prove the case $1 < q < D$. Take u satisfying the hypothesis of Proposition 1.7, and define $v = u^\gamma$, where $\gamma = \frac{q^*}{1^*}$. In particular, $\gamma > 1$, so that $v \in C^1$, and we can apply the weighted Sobolev inequality with $q = 1$ to get

$$\left(\int_{(\mathbb{R}_+)^2} s^a t^b |u|^{q^*} dsdt \right)^{1/1^*} = \left(\int_{(\mathbb{R}_+)^2} s^a t^b |v|^{\frac{D}{D-1}} dsdt \right)^{\frac{D-1}{D}} \leq C \int_{(\mathbb{R}_+)^2} s^a t^b |\nabla v| dsdt.$$

Now, $|\nabla v| = \gamma u^{\gamma-1} |\nabla u|$, and by Hölder's inequality it follows that

$$\int_{(\mathbb{R}_+)^2} s^a t^b |\nabla v| dsdt \leq C \left(\int_{(\mathbb{R}_+)^2} s^a t^b |\nabla u|^q dsdt \right)^{1/q} \left(\int_{(\mathbb{R}_+)^2} s^a t^b |u|^{(\gamma-1)q'} dsdt \right)^{1/q'}.$$

But from the definition of γ and q^* it follows that

$$\frac{1}{1^*} - \frac{1}{q^*} = \frac{1}{q'}, \qquad (\gamma - 1)q' = q^*,$$

and hence

$$\left(\int_{(\mathbb{R}_+)^2} s^a t^b |u|^{q^*} dsdt \right)^{1/q^*} \leq C \left(\int_{(\mathbb{R}_+)^2} s^a t^b |\nabla u|^q dsdt \right)^{1/q},$$

as desired. □

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