RACSAM manuscript No.
(will be inserted by the editor)

On a factorization of Riemann's ζ function with respect to a quadratic field and its computation

Xavier Ros-Oton

Received: date / Accepted: date

Abstract Let K be a quadratic field, and let ζ_K its Dedekind zeta function. In this paper we introduce a factorization of ζ_K into two functions, L_1 and L_2, defined as partial Euler products of ζ_K, which lead to a factorization of Riemann's ζ function into two functions, p_1 and p_2. We prove that these functions satisfy a functional equation which has a unique solution, and we give series of very fast convergence to them. Moreover, when $\Delta_K > 0$ the general term of these series at even positive integers is calculated explicitly in terms of generalized Bernoulli numbers.

Keywords Riemman's ζ function · factorization · functional equation · quadratic field

Mathematics Subject Classification (2000) 11M06 · 11M41

1 Introduction

Let K be a quadratic field and let χ be the Dirichlet character attached to $K/\mathbb{Q}$. Its Dedekind's zeta function can be written as

$$\zeta_K(s) = \zeta(s)L(s, \chi),$$

where ζ is Riemann's zeta function and L is the L-function associated with χ (see, for example, [1]). Hence, an alternative factorization, for $\mathfrak{Re}(s) > 1$, is the one given by the partial products

$$\zeta_K(s) = \prod_{p|d}(1 - p^{-s})^{-1} L_1(s)L_2(s),$$

Xavier Ros-Oton
Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
Tel.: +34 934015815
E-mail: xavier.ros.oton@upc.edu

where $d = |\Delta_K|$ is the absolute value of the discriminant of K, and

$$L_1(s) = \prod_{\chi(p)=1} (1-p^{-s})^{-2}, \qquad L_2(s) = \prod_{\chi(p)=-1} (1-p^{-2s})^{-1}.$$

Note that L_1 and L_2 are obtained as partial Euler products of $\zeta(s)^2$ and $\zeta(2s)$ respectively, so they converge and are non-zero for $\mathfrak{Re}(s) > 1$ and $\mathfrak{Re}(s) > 1/2$ respectively.

Define now

$$p_1(s) = \prod_{\chi(p)=1} (1-p^{-s})^{-1} \qquad \text{and} \qquad p_2(s) = \prod_{\chi(p)=-1} (1-p^{-s})^{-1}. \tag{1}$$

Then, we have that

$$L_1(s) = p_1(s)^2, \qquad L_2(s) = p_2(2s),$$

and thus it is equivalent to study L_1 and L_2 or p_1 and p_2. Note that

$$\zeta(s) = \prod_{p|d} (1-p^{-s})^{-1} p_1(s) p_2(s),$$

and hence, p_1 and p_2 give a factorization of Riemann's zeta function.

The plan of the paper is as follows. In section 2 we see that p_1 and p_2 satisfy a functional equation. More precisely, we prove

Theorem 1 *The functions p_1 and p_2 satisfy the functional equations*

$$\frac{p_i(2s)}{p_i(s)^2} = q_i(s), \qquad \lim_{\mathfrak{Re}(s)\to+\infty} p_i(s) = 1, \qquad \textit{for} \qquad i = 1, 2, \tag{2}$$

where

$$q_1(s) = \frac{\zeta(2s)}{\zeta(s)L(s,\chi)} \prod_{p|d} (1+p^{-s}), \qquad q_2(s) = \frac{L(s,\chi)}{\zeta(s)} \prod_{p|d} (1-p^{-s})^{-1}. \tag{3}$$

Furthermore, these functional equations have a unique solution, so they completely determine the functions p_1 and p_2.

Moreover, we shall see that the logarithm of the solution of this functional equation can be written as a series

$$\log p_i(s) = -\sum_{n=0}^{+\infty} \frac{\log q_i(2^n s)}{2^{n+1}}, \qquad i = 1, 2, \tag{4}$$

and hence, we will have an alternative expression of p_1 and p_2.

In section 3 we will see that the series given by (4) are of very fast convergence. We shall prove

Theorem 2 *Let s be complex number such that $\mathfrak{Re}(s) \geq 1$. Then,*

$$p_1(2s) = \exp\left\{ -\sum_{k=1}^{n} \frac{1}{2^k} \log q_1(2^k s) \right\} + o\left(2^{-2^n}\right),$$

and

$$p_2(2s) = \exp\left\{ -\sum_{k=1}^{n} \frac{1}{2^k} \log q_2(2^k s) \right\} + o\left(2^{-2^n}\right).$$

As a consequence, we will have a way to evaluate p_1 and p_2 at even positive integers when Δ_K is positive. This will be done by calculating explicitly the general term of the series in this case.

2 The functional equation of p_1 and p_2

First we prove that the functional equation appearing in Theorem 1 has a unique solution and that this solution can be written as an infinite series. The statement of the result is the following.

Proposition 1 *Let $\Omega = \{s \in \mathbb{C} |\ \mathfrak{Re}(s) > 1\}$, and q an holomorphic function defined in Ω, with $q(s) \neq 0$ for all $s \in \Omega$ and $\lim_{\mathfrak{Re}(s)\to+\infty} q(s) = 1$. Then, the functional equation*

$$\frac{p(2s)}{p(s)^2} = q(s), \qquad \lim_{\mathfrak{Re}(s)\to+\infty} p(s) = 1$$

has a unique solution $p(s)$. In addition, the solution can be written as

$$p(s) = \exp\left\{ -\sum_{n\geq 0} \frac{\log q(2^n s)}{2^{n+1}} \right\},$$

and this series is absolutely convergent for all s in Ω.

Proof Suppose that $p(s)$ satisfies the functional equation. Then, $p(s) \neq 0$ for all $s \in \Omega$. This is because $p(s) = 0$ implies $p(2s) = 0$ and $p(2^k s) = 0$ for $k = 1, 2, ...,$ which contradicts the hypothesis $\lim_{\mathfrak{Re}(s)\to+\infty} p(s) = 1$. Thus, we can define

$$f(s) = \frac{\log p(s)}{s}, \qquad g(s) = \frac{\log q(s)}{2s},$$

where log is the principal branch of the complex logarithm. Taking logarithms to our functional equation and dividing by $2s$, we have that

$$f(2s) = f(s) + g(s), \qquad \lim_{\mathfrak{Re}(s)\to+\infty} f(s) = 0.$$

Writing this last equation for $s, 2s, 4s, 8s, ..., 2^N s$, and adding them, we obtain that

$$f(2^{N+1}s) = f(s) + \sum_{n=0}^{N} g(2^n s).$$

Since $\mathfrak{Re}(s) > 1$, then $\mathfrak{Re}(2^{N+1}s) \to +\infty$ when $N \to \infty$, so

$$f(s) + \sum_{n=0}^{\infty} g(2^n s) = \lim_{N\to\infty} f(2^{N+1}s) = 0,$$

and

$$\log p(s) = -\sum_{n\geq 0} \frac{\log q(2^n s)}{2^{n+1}}.$$

Since

$$\lim_{\mathfrak{Re}(s)\to+\infty} \log q(s) = 0,$$

the sequence $\{\log q(2^n s)\}_{n\in\mathbb{N}}$ converges (it tends to 0), and in particular it is bounded. Hence, there exists $M > 0$ such that $|\log q(2^n s)| < M$, and then

$$\sum_{n\geq 0} \left| \frac{\log q(2^n s)}{2^{n+1}} \right| \leq \sum_{n\geq 0} \frac{M}{2^{n+1}} = M,$$

so the series is absolutely convergent for all $s \in \Omega$.

Let us see that this function satisfies the functional equation. We have that

$$\begin{aligned} \log p(2s) - 2\log p(s) &= -\sum_{n\geq 0} \frac{\log q(2^{n+1}s)}{2^{n+1}} + 2\sum_{n\geq 0} \frac{\log q(2^n s)}{2^{n+1}} \\ &= -\sum_{n\geq 1} \frac{\log q(2^n s)}{2^n} + \sum_{n\geq 0} \frac{\log q(2^n s)}{2^n} \\ &= \log q(s), \end{aligned}$$

and then,

$$\frac{p(2s)}{p(s)^2} = q(s).$$

We now have to see that $\lim_{\mathfrak{Re}(s)\to+\infty} p(s) = 1$, or equivalently,

$$\lim_{\mathfrak{Re}(s)\to+\infty} \log p(s) = 0.$$

For it, fix $\varepsilon > 0$. Since $\lim_{\mathfrak{Re}(s)\to+\infty} q(s) = 1$, then $\lim_{\mathfrak{Re}(s)\to+\infty} \log q(s) = 0$, and exists $\sigma > 0$ such that

$$|\log q(s)| < \epsilon \quad \text{for all} \quad s \quad \text{with} \quad \mathfrak{Re}(s) \geq \sigma.$$

Hence, if $\mathfrak{Re}(s) \geq \sigma$, then

$$|\log p(s)| \leq \sum_{n\geq 0} \left| \frac{\log q(2^n s)}{2^{n+1}} \right| \leq \sum_{n\geq 0} \frac{\varepsilon}{2^{n+1}} = \varepsilon,$$

and $\lim_{\mathfrak{Re}(s)\to+\infty} \log p(s) = 0$, as claimed.

Note that, in fact, the branch of the logarithm is irrelevant, since when we take exponentials, we will have

$$p(s) = \exp\left\{ -\sum_{n\geq 0} \frac{\log q(2^n s)}{2^{n+1}} \right\},$$

independently of the chosen branch. □

We can now give the:

Proof of Theorem 1. On the one hand, it is clear that $\lim_{\mathfrak{Re}(s)\to+\infty} p_i(s) = 1$, $i = 1, 2$.

On the other hand, we have that

$$\begin{aligned} p_1(s)\frac{p_2(2s)}{p_2(s)} &= \prod_{\chi(p)=1}(1-p^{-s})^{-1}\frac{\prod_{\chi(p)=-1}(1-p^{-2s})^{-1}}{\prod_{\chi(p)=-1}(1-p^{-s})^{-1}} \\ &= \prod_{\chi(p)=1}(1-p^{-s})^{-1}\prod_{\chi(p)=-1}\left(\frac{1-p^{-2s}}{1-p^{-s}}\right)^{-1} \\ &= \prod_{\chi(p)=1}(1-p^{-s})^{-1}\prod_{\chi(p)=-1}(1+p^{-s})^{-1} \\ &= L(s,\chi), \end{aligned}$$

and since

$$p_1(s) = \frac{1}{p_2(s)}\zeta(s)\prod_{p|d}(1-p^{-s}),$$

then

$$\frac{p_2(2s)}{p_2(s)^2} = \frac{L(s,\chi)}{\zeta(s)}\prod_{p|d}(1-p^{-s})^{-1}.$$

Using now that

$$p_2(s) = \frac{1}{p_1(s)}\zeta(s)\prod_{p|d}(1-p^{-s}),$$

we obtain

$$\frac{p_1(2s)}{p_1(s)^2} = \frac{p_2(s)^2}{p_2(2s)}\cdot\frac{\zeta(2s)\prod_{p|d}(1-p^{-2s})}{\zeta(s)^2\prod_{p|d}(1-p^{-s})^2} = \frac{\zeta(2s)}{\zeta(s)L(s,\chi)}\prod_{p|d}(1+p^{-s}).$$

The fact that these functional equations have an unique solution follows from Proposition 1. □

As a consequence of Proposition 1 and Theorem 1, we obtain the following expression for $p_1(s)$ and $p_2(s)$.

Corollary 1 *Let p_1 and p_2 be given by* (1). *Then,*

$$p_i(s) = \exp\left\{-\frac{1}{2}\sum_{n\geq 0}\frac{\log q_i(2^n s)}{2^n}\right\} \qquad \text{for} \qquad i = 1, 2,$$

where

$$q_1(s) = \frac{\zeta(2s)}{\zeta(s)L(s,\chi)}\prod_{p|d}(1+p^{-s}), \quad \text{and} \quad q_2(s) = \frac{L(s,\chi)}{\zeta(s)}\prod_{p|d}(1-p^{-s})^{-1}.$$

These expressions will be used in the next section.

3 Evaluating p_1 and p_2

In this section we will calculate the order of convergence of the series given by Corollary 1. We will see that this convergence is of order 2^{-2^n}, i.e.,

$$p_i(2s) = \exp\left\{-\sum_{k=1}^{n} \frac{1}{2^k} \log q_i(2^k s)\right\} + o\left(2^{-2^n}\right),$$

and therefore this will be a better way to evaluate the functions p_1 and p_2 than the one given by the infinite products

$$p_1(s) = \prod_{\chi(p)=1} (1-p^{-s})^{-1} \qquad \text{and} \qquad p_2(s) = \prod_{\chi(p)=-1} (1-p^{-s})^{-1}.$$

Moreover, we will provide the general term of these series at even positive integers in the case $\Delta_K > 0$. For it, we will use generalized Bernoulli numbers.

Remark 1 Recall that

$$f(n) = o\left(g(n)\right) \text{ means that } \lim_{n\to+\infty} \frac{f(n)}{g(n)} = 0,$$

and

$$a(n) = b(n) + o\left(g(n)\right) \text{ means that } a(n) - b(n) = o\left(g(n)\right).$$

In order to prove Theorem 2, we will need two lemmata.

Lemma 1 *Let σ be a real number, $\sigma > 1$. Then,*

$$\frac{2^\sigma - 1}{2^\sigma - 2} < \zeta(\sigma) < \frac{2^\sigma}{2^\sigma - 2}.$$

Proof We make a partition of $\mathbb{N}$ in the sets $A_k = \{n \in \mathbb{N} : 2^k \le n < 2^{k+1}\}$, $k \ge 1$. It is clear that $|A_k| = 2^k$, and that if $n \in A_k$, then $n^{-\sigma} \le 2^{-k\sigma}$. Hence,

$$\begin{aligned}\zeta(\sigma) &= \sum_{n\in\mathbb{N}} n^{-\sigma} = \sum_{k\ge 0}\sum_{n\in A_k} n^{-\sigma} < \sum_{k\ge 0}\sum_{n\in A_k} 2^{-k\sigma} \\ &= \sum_{k\ge 0} |A_k| \cdot 2^{-k\sigma} = \sum_{k\ge 0} 2^k \cdot 2^{-k\sigma} = \sum_{k\ge 0} (2^{1-\sigma})^k \\ &= \frac{1}{1-2^{1-\sigma}} = \frac{2^\sigma}{2^\sigma - 2}.\end{aligned}$$

Using that if $n \in A_k$ then $n^{-\sigma} \le 2^{-(k+1)\sigma}$, we obtain the other side of the inequality. □

Lemma 2 *Let $s = \sigma + it$, with $\sigma \ge 2$, and let q_1 and q_2 be given by* (3). *Then,*

$$|\log q_i(s)| \le \frac{16}{2^\sigma - 2} \qquad \textit{for} \qquad i = 1, 2,$$

where log *denotes the principal branch of the complex logarithm.*

Proof First we claim that

$$|\log(1+z)| \le -\log(1-|z|), \tag{5}$$

for each $|z|<1$. To see it, it suffices to compare its power series:

$$|\log(1+z)| = \left| z - \frac{z^2}{2} + \cdots \right| \le |z| + \frac{|z|^2}{2} + \cdots = -\log(1-|z|).$$

Now, using (5) and that

$$\left| \frac{1-p^{-s}}{1+p^{-s}} - 1 \right| = \frac{2p^{-\sigma}}{1-p^{-\sigma}},$$

we get

$$\begin{aligned} |\log q_i(s)| &= \left| \log \prod_{\chi(p)=\pm 1} \left(\frac{1-p^{-s}}{1+p^{-s}} \right) \right| \\ &\le \sum_{\chi(p)=\pm 1} \left| \log \left(\frac{1-p^{-s}}{1+p^{-s}} \right) \right| \\ &\le \sum_{\chi(p)=\pm 1} -\log\left(1 - \frac{2p^{-\sigma}}{1-p^{-\sigma}} \right) \\ &= \sum_{\chi(p)=\pm 1} \log \left(\frac{1-p^{-\sigma}}{1-3p^{-\sigma}} \right). \end{aligned}$$

Moreover, since $\log(1+x) \le x$ for each $x>0$, then

$$|\log q_i(s)| \le \sum_{\chi(p)=\pm 1} \left(\frac{1-p^{-\sigma}}{1-3p^{-\sigma}} - 1 \right) = \sum_{\chi(p)=\pm 1} \frac{2}{p^\sigma - 3}.$$

But since $\sigma \ge 2$ then

$$p^\sigma - 3 \ge \frac{1}{4} p^\sigma$$

for each $p \ge 2$, and therefore

$$|\log q_i(s)| \le 8 \sum_{\chi(p)=\pm 1} p^{-\sigma}, \qquad i=1,2.$$

Finally, by Lemma 1 we have that

$$|\log q_i(s)| \le 8 \sum_{n \ge 2} n^{-\sigma} \le \frac{16}{2^\sigma - 2}, \qquad i = 1,2,$$

and we are done. □

By using the last Lemma, we will be able to bound the general term of the series which give p_1 and p_2, and from this, we will deduce Theorem 2.

Proof of Theorem 2. Let x_n and y_n be the general term of the series which give $\log p_1(2s)$ and $\log p_2(2s)$, i.e.

$$x_n = \frac{1}{2^{n+1}} \log q_1(2^n s), \qquad y_n = \frac{1}{2^{n+1}} \log q_2(2^n s).$$

By Lemma 2, we have that

$$|x_n| = \frac{1}{2^{n+1}} |\log q_1(2^n s)| \le \frac{1}{2^{n+1}} \frac{16}{2^{2^n \sigma} - 2} = o\left(2^{-2^n}\right).$$

Analogously,

$$y_n = o\left(2^{-2^n}\right).$$

Thus,

$$\begin{aligned}
p_i(2s) &= \exp\left\{-\sum_{k=1}^{n} x_k - \sum_{k=n+1}^{\infty} o\left(2^{-2^k}\right)\right\} \\
&= \exp\left\{-\sum_{k=1}^{n} x_k - o\left(\sum_{k=n+1}^{\infty} 2^{-2^k}\right)\right\} \\
&= \exp\left\{-\sum_{k=1}^{n} x_k + o\left(2^{-2^n}\right)\right\} \\
&= \exp\left\{-\sum_{k=1}^{n} x_k\right\} \exp\left\{o\left(2^{-2^n}\right)\right\} \\
&= \exp\left\{-\sum_{k=1}^{n} x_k\right\} \left(1 + o\left(2^{-2^n}\right)\right) \\
&= \exp\left\{-\sum_{k=1}^{n} x_k\right\} + o\left(2^{-2^n}\right),
\end{aligned}$$

and we are done. □

Let us see now how can we evaluate the general term $2^{-n-1} \log q_i(2^n s)$ of the series at even positive integers when $\Delta_K > 0$.

Recall that given a Dirichlet character χ mod d, the generalized Bernoulli numbers [2] are given by

$$\sum_{a=1}^{d} \chi(a) \frac{te^{at}}{e^{dt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

Moreover,

$$L(1-n, \chi) = -\frac{B_{n,\chi}}{n},$$

and using the functional equation of the L-function one can evaluate L at some positive integers, as given in the following Theorem.

Theorem 3 **([2])** *Let χ be a nontrivial primitive character modulo d, and let a be 0 if χ is even and 1 if χ is odd. Then, if $n \equiv a \pmod 2$,*

$$L(n,\chi) = (-1)^{1+\frac{n-a}{2}} \frac{g(\chi)}{2i^a} \left(\frac{2\pi}{m}\right)^n \frac{B_{n,\overline{\chi}}}{n!},$$

where $g(\chi)$ is the Gauss sum of the character.

Let now be $d = \Delta_K > 0$. Then, χ is an even quadratic character *mod d*. Therefore, for each $n \in \mathbb{N}$ even, one has

$$L(n,\chi) = (-1)^{1+\frac{n}{2}} \frac{\sqrt{d}}{2} \left(\frac{2\pi}{d}\right)^n \frac{B_{n,\chi}}{n!}, \tag{6}$$

and

$$\zeta(n) = (-1)^{1+\frac{n}{2}} \frac{(2\pi)^n}{2} \frac{B_n}{n!}. \tag{7}$$

From these equalities, we deduce the following.

Proposition 2 *Assume that $d = \Delta_K > 0$. Then, for each even natural number $n \geq 2$, we have*

$$q_1(n) = \frac{2d^n}{\binom{2n}{n}\sqrt{d}} \frac{B_{2n}}{B_{n,\chi} B_n} \prod_{p|d} (1 + p^{-n}), \tag{8}$$

$$q_2(n) = \frac{\sqrt{d}}{d^n} \frac{B_{n,\chi}}{B_n} \prod_{p|d} (1 - p^{-n})^{-1}. \tag{9}$$

Proof It follows immediately from (6), (7), and the definition of q_1 and q_2 (3). □

Hence, by using Proposition 2 and Theorem 2 we obtain series of very fast convergence to evaluate p_1 and p_2 at even positive integers.

To see an example, let χ be the primitive character modulo 5, and let us evaluate $p_1(2)$. One the one hand, Taking the first 10 terms of the infinite product one obtains 2 correct digits. On the other hand, taking also the first 10 terms in our series one obtains 619 correct digits. The following table shows the aproximate error when taking n terms of our series.

N	$p_1(2) - \exp\left\{-\sum_{k=1}^{N} \frac{1}{2^k} \log q_1(2^k)\right\}$
1	10^{-2}
2	10^{-3}
3	10^{-6}
4	10^{-11}
5	10^{-21}
6	10^{-41}
7	10^{-79}
8	10^{-157}
9	10^{-311}
10	10^{-620}
11	10^{-1237}
12	10^{-2470}

Acknowledgements The author thanks Joan-C. Lario for his comments and suggestions.

References

1. Davenport H.: Multiplicative number theory. Springer-Verlag, 1980.
2. Zagier D. B.: Zetafunktionen und quadratische Körper. Springer-Verlag, 1981.