REGULARITY THEORY FOR GENERAL STABLE OPERATORS

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ABSTRACT. We establish sharp regularity estimates for solutions to $Lu = f$ in $\Omega \subset \mathbb{R}^n$, being L the generator of any stable and symmetric Lévy process. Such nonlocal operators L depend on a finite measure on S^{n-1}, called the spectral measure.

First, we study the interior regularity of solutions to $Lu = f$ in B_1. We prove that if f is C^α then u belong to $C^{\alpha+2s}$ whenever $\alpha + 2s$ is not an integer. In case $f \in L^\infty$, we show that the solution u is C^{2s} when $s \neq 1/2$, and $C^{2s-\epsilon}$ for all $\epsilon > 0$ when $s = 1/2$.

Then, we study the boundary regularity of solutions to $Lu = f$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, in $C^{1,1}$ domains Ω. We show that solutions u satisfy $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$ for all $\epsilon > 0$, where d is the distance to $\partial\Omega$.

Finally, we show that our results are sharp by constructing two counterexamples.

1. INTRODUCTION AND RESULTS

The regularity of solutions to integro-differential equations has attracted much interest in the last years, both in the Probability and in the PDE community. This type of equations arise naturally in the study of Lévy processes, which appear in many different situations, from Physics to Biology or Finance.

A very important class of Lévy processes are the α-stable processes, with $\alpha \in (0, 2)$; see [4] and [32]. These are processes satisfying self-similarity properties. More precisely, X_t is said to be α-stable if

$$X_1 \stackrel{d}{=} \frac{1}{t^{1/\alpha}} X_t \qquad \text{for all } t > 0.$$

These processes are the equivalent to Gaussian random processes when dealing with infinite variance random variables. Indeed, the Generalized Central Limit Theorem states that, under certain assumptions, the distribution of the sum of infinite variance random variables converges to a stable distribution (see for example [32] for a precise statement of this result).

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Stable processes can be used to model real-world phenomena [32, 20], and in particular they are commonly used in Mathematical Finance; see for example [26, 11, 27, 28, 29, 8] and references therein.

The infinitesimal generator of any symmetric stable Lévy process is of the form

$$Lu(x) = \int_{S^{n-1}} \int_{-\infty}^{+\infty} \left(u(x + \theta r) + u(x - \theta r) - 2u(x) \right) \frac{dr}{|r|^{1+2s}} \, d\mu(\theta), \qquad (1.1)$$

where μ is any nonnegative and finite measure on the unit sphere, called the *spectral measure*, and $s \in (0, 1)$.

The aim of this paper is to establish new and sharp interior and boundary regularity results for general symmetric stable operators (1.1).

Remarkably, the only ellipticity assumptions in all our results will be

$$0 < \lambda \leq \inf_{\nu \in S^{n-1}} \int_{S^{n-1}} |\nu \cdot \theta|^{2s} d\mu(\theta), \qquad \int_{S^{n-1}} d\mu \leq \Lambda < \infty. \qquad (1.2)$$

Notice that these hypotheses are satisfied for *any* symmetric stable operator whose spectral measure μ is n-dimensional, i.e., such that there is no hyperplane V of $\mathbb{R}^n$ such that μ is supported on V. Notice also that in case that the spectral measure μ is supported on an hyperplane V, then no regularity result holds.

When the spectral measure is absolutely continuous, $d\mu(\theta) = a(\theta)d\theta$, then these operators can be written as

$$Lu(x) = \int_{\mathbb{R}^n} \left(u(x + y) + u(x - y) - 2u(x) \right) \frac{a(y/|y|)}{|y|^{n+2s}} \, dy, \qquad (1.3)$$

where $a \in L^1(S^{n-1})$ is a nonnegative and even function.

The most simple example of stable Lévy process X_t in $\mathbb{R}^n$ is the one corresponding to $d\mu(\theta) = c \, d\theta$, with $c > 0$. In this case, the operator L is a multiple of the fractional Laplacian $-(-\Delta)^s$. Another simple example is given by $X_t = (X_t^1, ..., X_t^n)$, being X_t^i independent symmetric stable processes in dimension 1. In this case, the infinitesimal generator of X_t is

$$- Lu = (-\partial_{x_1 x_1})^s u + \cdots + (-\partial_{x_n x_n})^s u, \qquad (1.4)$$

and its spectral measure consist on $2n$ delta functions. For example, when $n = 2$ we have $\mu = \delta_{(1,0)} + \delta_{(0,1)} + \delta_{(-1,0)} + \delta_{(0,-1)}$ (up to a multiplicative constant).

The regularity of solutions to $Lu = f$ (or $Lu = 0$) for operators L like (1.3), (1.1), or related ones, has been widely investigated; see the works by Bass, Kassmann, Schwab, Silvestre, Sztonyk, and Bogdan, among others [1, 23, 22, 3, 39, 21, 2, 37, 5, 6, 7, 9, 24]. A typical assumption in some of these results is that

$$0 < c \leq a(\theta) \leq C \qquad \text{in } S^{n-1}. \qquad (1.5)$$

Still, the results in [22], [5], [23], and [21] do not require this assumption, and they apply to all operators of the form (1.3) satisfying

$$a(\theta) \geq c > 0 \quad \text{in a subset } \Sigma \subset S^{n-1} \text{ of positive measure}; \qquad (1.6)$$

see also [33]. Furthermore, the results of [23] and [1] do not assume the spectral measure to be absolutely continuous, and apply also to the operator (1.4) (and also to x-dependent operators of the type (1.4)).

An important difficulty when studying the regularity for operators (1.1) is that no Harnack inequality holds in general; see Bogdan-Sztonyk [6]. Also, the Fourier symbols of these operators are in general only Hölder continuous, so that the usual Fourier multiplier theorems [38], [17, page 168], or [25] can not be used to show our results.

Here we establish sharp regularity results in Hölder spaces for *all* stable operators (1.1)-(1.2).

Our first result reads as follows.

Theorem 1.1. *Let $s \in (0,1)$, and let L be any operator of the form* (1.1)-(1.2). *Let u be any bounded weak solution to*

$$Lu = f \quad in \ B_1. \tag{1.7}$$

Then,

(a) *If $f \in L^\infty(B_1)$ and $u \in L^\infty(\mathbb{R}^n)$,*

$$\|u\|_{C^{2s}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)}\right) \quad if \ s \neq \frac{1}{2},$$

and

$$\|u\|_{C^{2s-\epsilon}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)}\right) \quad if \ s = \frac{1}{2},$$

for all $\epsilon > 0$.

(b) *If $f \in C^\alpha(B_1)$ and $u \in C^\alpha(\mathbb{R}^n)$ for some $\alpha > 0$, then*

$$\|u\|_{C^{\alpha+2s}(B_{1/2})} \leq C\left(\|u\|_{C^\alpha(\mathbb{R}^n)} + \|f\|_{C^\alpha(B_1)}\right) \tag{1.8}$$

whenever $\alpha + 2s$ is not an integer.

The constant C depends only on n, s, α, ϵ, and the ellipticity constants (1.2).

Notice that when $s \neq \frac{1}{2}$ we obtain a C^{2s} estimate in part (a), and not only a $C^{2s-\epsilon}$ one.

Note also that in part (b) it is required that $u \in C^\alpha(\mathbb{R}^n)$ in order to have a $C^{\alpha+2s}$ estimate for u in $B_{1/2}$. When the spectral measure μ is not regular, the estimate is not true anymore if u is not C^α in all of $\mathbb{R}^n$: we can construct a solution to $Lu = 0$ in B_1, which satisfies $u \in C^{\alpha-\epsilon}(\mathbb{R}^n)$ but $u \notin C^{\alpha+2s}(B_{1/2})$; see Proposition 6.1.

When L is of the form (1.3) and $a \in C^\alpha(S^{n-1})$, then it is easy to see that one can replace the $C^\alpha(\mathbb{R}^n)$ norm of u in (1.8) by the $L^\infty(\mathbb{R}^n)$ norm; see Corollary 3.5. Also, when the equation is posed in the whole $\mathbb{R}^n$ then there is no such problem, and one has the estimate $\|u\|_{C^{\alpha+2s}(\mathbb{R}^n)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{C^\alpha(\mathbb{R}^n)})$ —which follows easily from (1.8).

Concerning the boundary regularity of solutions, our main result reads as follows.

Theorem 1.2. *Let $s \in (0,1)$, L be any operator of the form (1.1)-(1.2), and Ω be any bounded $C^{1,1}$ domain. Let $f \in L^\infty(\Omega)$, and u be a weak solution of*

$$\begin{cases} Lu &= f & in \ \Omega \\ u &= 0 & in \ \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.9}$$

Let d be the distance to $\partial\Omega$. Then, $u \in C^s(\mathbb{R}^n)$, and

$$\|u/d^s\|_{C^{s-\epsilon}(\overline{\Omega})} \le C\|f\|_{L^\infty(\Omega)}$$

for all $\epsilon > 0$. The constant C depends only on n, s, Ω, ϵ, and the ellipticity constants (1.2).

For general stable operators (1.1), we expect this result to be optimal. Indeed, we can construct a C^∞ domain Ω for which $L(d^s)$ does not belong to $L^\infty(\Omega)$; see Proposition 6.2. Thus, even in C^∞ domains and with $f \in C^\infty$, we do not expect solutions u to satisfy $u/d^s \in C^s(\overline{\Omega})$.

The estimate of Theorem 1.2 was only known in case that the spectral measure μ is absolutely continuous and satisfies quite strong regularity assumptions. Indeed, when (1.5) holds, $a \in C^{1,\alpha}(S^{n-1})$, and Ω is $C^{2,\alpha}$, then the result is a particular case from our estimates in [31] for fully nonlinear equations. Also, when Ω is C^∞ and $a \in C^\infty(S^{n-1})$ then Theorem 1.2 follows from the results of Grubb [18, 19] for pseudodifferential operators satisfying the μ-transmission property.

Even for the fractional Laplacian, the proof we present here is new and completely independent with respect to the ones in [30] and [18, 19]. Let us explain briefly the main ideas in the proofs of our results.

To prove Theorems 1.1 and 1.2 we use some ideas introduced in [34, 31, 35]. Namely, all the proofs of the present paper have a similar structure in which we first establish a Liouville-type theorem in $\mathbb{R}^n$ (or $\mathbb{R}^n_+$ in case of boundary regularity), and then we deduce by a blow up and compactness argument an estimate for solutions to $Lu = f$ in, say, B_1. An important difference with respect to the proofs [34, 31, 35] is that here we do not have any C^γ estimate that we can iterate in order to prove a Liouville theorem, and hence the proofs of the present Liouville theorems must be completely different from the ones in [34, 31, 35].

For example, in case of Theorem 1.1, to prove the Liouville-type Theorem 2.1 we need to establish first a C^γ estimate in $\mathbb{R}^n$ via the heat kernel of the operator, to then iterate it and deduce the Liouville theorem. Recall that even this first C^γ estimate is new for general operators (1.1). In case of Theorem 1.2, we also prove the Liouville-type Theorem 4.1 in a different way with respect to [31]. Indeed, in [31] we first established a C^γ estimate for u/d^s by using a method of Caffarelli, which relies mainly on the Harnack inequality, and then we deduced from this a Liouville theorem in $\mathbb{R}^n_+$. However, in the present context we do not have any Harnack inequality, and we have to establish Theorem 4.1 using only the interior estimates for u previously proven in Theorem 1.1.

All the regularity estimates of this paper are for translation invariant equations. Still, the methods presented here can be used to establish similar regularity results

for non translation invariant equations (with continuous dependence on x), and also for parabolic equations $\partial_t u + Lu = f$ in $\Omega \times (0, T)$. We plan to do this in a future work.

The paper is organized as follows. In Section 2 we establish a Liouville-type theorem in the entire space, Theorem 2.1. In Section 3 we prove Theorem 1.1. Then, in Section 4 we establish a Liouville-type theorem in the half-space, Theorem 4.1, and in Section 5 we prove Theorem 1.2. Finally, in Section 6 we prove Proposition 6.2.

2. A Liouville theorem in the entire space

The aim of this section is to prove the following.

Theorem 2.1. *Let $s \in (0, 1)$, and let L be any operator of the form* (1.1)-(1.2). *Let u be any weak solution of*
$$Lu = 0 \quad in \ \mathbb{R}^n$$
satisfying the growth condition
$$\|u\|_{L^\infty(B_R)} \leq CR^\beta \qquad for \ all \ R \geq 1,$$
for some $\beta < 2s$.

Then, u is a polynomial of degree at most $\lfloor \beta \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x.

This Liouville theorem will be used in the proof of Theorem 1.1. For related Liouville theorems, see [14, 13, 10].

Definition 2.2. Given $f \in L^\infty(\Omega)$, we say that u is a weak solution of $Lu = f$ in $\Omega \subset \mathbb{R}^n$ if: $|u(x)| \leq C(1 + |x|^{2s-\delta})$ in $\mathbb{R}^n$ for some $\delta > 0$, and
$$\int_{\mathbb{R}^n} u\, Lv\, dx = \int_{\Omega} fv\, dx$$
for all $v \in C_c^\infty(\Omega)$.

Remark 2.3. Thanks to the translation invariance of the operator, we have the following useful fact. If u satisfies $Lu = f$ in Ω in the weak sense, and $\eta_\epsilon \in C_c^\infty(B_\epsilon)$ is a mollifier, then $u_\epsilon := u * \eta_\epsilon$ satisfies $Lu_\epsilon = f * \eta_\epsilon$ in Ω_ϵ in the weak sense, where $\Omega_\epsilon = \Omega \cap \{\text{dist}(x, \partial\Omega) > \epsilon\}$.

2.1. **Heat kernel: regularity and decay in average.** The heat kernel of L is defined via Fourier transform as
$$p(t, \cdot) = \mathcal{F}^{-1}\big(\exp(-A(\xi)t)\big), \tag{2.1}$$
where $A(\xi)$ is the Fourier symbol of L.

The symbol $A(\xi)$ of L can be explicitly written in terms of s and the spectral measure μ. Indeed, it is given by
$$A(\xi) = c \int_{S^{n-1}} |\xi \cdot \theta|^{2s} d\mu(\theta); \tag{2.2}$$

see for example [32]. Notice that $A(\xi)$ is homogeneous of order $2s$.

In order to prove Theorem 2.1, we will need to show some kind of decay for the heat kernel of L.

The decay of the heat kernel has been studied in [12] and [16] in case that $d\mu(\theta) = a(\theta)d\theta$ (see also [7, 40]). It turns out that, when $a \in L^\infty(S^{n-1})$, the heat kernel $p(t,x)$ associated to the operator (1.3) satisfies

$$p(1,x) \leq \frac{C}{1 + |x|^{n+2s}}. \tag{2.3}$$

However, for general operators (1.1), the heat kernel does not satisfy in general (2.3). For example, when $X_t = (X_t^1, ..., X_t^n)$, being X^i independent symmetric stable processes in dimension 1, p satisfies

$$p(t,x) = p_1(t,x_1) \cdots p_1(t,x_n),$$

and thus it does not satisfy (2.3).

We prove here that for general operators (1.1), even if there is no decay of the form (2.3), the heat kernel $p(1,x)$ decays "in average" faster than $|x|^{-n-2s+\delta}$ for any $\delta > 0$. This is stated in the following result.

Proposition 2.4. *Let $s \in (0,1)$, and let L be any operator of the form (1.1)-(1.2). Let $p(t,x)$ be the heat kernel associated to L. Then,*

(a) *For all $\delta > 0$,*

$$\int_{\mathbb{R}^n} \left(1 + |x|^{2s-\delta}\right) p(1,x) dx \leq C. \tag{2.4}$$

(b) *Moreover,*

$$[p(1,x)]_{C^{0,1}(\mathbb{R}^n)} \leq C.$$

The constant C depends only on n, s, δ, and the ellipticity constants (1.2).

Proof. (a) Assume first that $\mu(d\theta) = a(\theta)d\theta$, with $a \in C^\infty(S^{n-1})$.

We claim that the function

$$\varphi(x) = (1 + |x|^2)^{s-\delta}$$

satisfies

$$|L\varphi| \leq C \qquad \text{in all of } \mathbb{R}^n,$$

with C depending only on n, s, and the ellipticity constants λ, Λ in (1.2).

Indeed, observe that for all $\rho \geq 1$, the rescaled function $\varphi_\rho(x) = \rho^{-2s+2\delta}\varphi(\rho x)$ satisfies $\varphi_\rho(x) = (\rho^{-2} + |x|^2)^{s-\delta}$ and $|L\varphi_\rho| \leq C$ in $B_2 \setminus B_1$, with C independent of ρ. Therefore, scaling back we obtain that $|L\varphi| \leq C\rho^{-2\delta}$ in $B_{2\rho} \setminus B_\rho$ for every $\rho \geq 1$. Hence, $L\varphi$ is bounded in all of $\mathbb{R}^n$, as claimed.

Now, we have

$$\int_{\mathbb{R}^n} \varphi(x) p(1, x)\, dx - 1 = \int_{\mathbb{R}^n} \varphi(x) \big(p(1, x) - p(0, x) \big) dx = \int_0^1 dt \int_{\mathbb{R}^n} \varphi(x) p_t(t, x) dx$$

$$= \int_0^1 dt \int_{\mathbb{R}^n} \varphi(x) L p(t, x) dx = \int_0^1 dt \int_{\mathbb{R}^n} L\varphi(x) p(t, x) dx. \tag{2.5}$$

Thus, it follows that

$$\int_{\mathbb{R}^n} \varphi(x) p(1, x)\, dx \leq 1 + \int_0^1 dt \int_{\mathbb{R}^n} |L\varphi(x)| p(t, x) dx \leq C.$$

Notice that in the last integration by parts in (2.5) we used that p and all its derivatives decay (since $a \in C^\infty(S^{n-1})$).

We have proved (2.4) in case $a \in C^\infty(S^{n-1})$, with C depending only on n, s, δ, λ, and Λ. Finally, by an approximation argument the same identity holds for any spectral measure μ, and thus (a) is proved.

(b) Notice that, by (2.2) and by definition of the ellipticity constants (1.2), we clearly have

$$0 < \lambda |\xi|^{2s} \leq A(\xi) \leq \Lambda |\xi|^{2s}.$$

Using this, it follows immediately from the expression (2.1) that the Fourier transform of $p(1, x)$ is rapidly decreasing and, therefore, the result follows. $\qquad \square$

Remark 2.5. In case that L is an operator of the form (1.3) and a belongs to the space $L \log L(S^{n-1})$, Proposition 2.4 (a) is an immediate consequence of the results of Glowacki-Hebisch [16]. Indeed, it was proved in [16] that, under this assumption on a, the heat kernel satisfies $p(1, x) \leq C|x|^{-n-2s} \omega(x/|x|)$ for some function $\omega \in L^1(S^{n-1})$.

2.2. **Proof of Theorem 2.1.** Using Proposition 2.4, we can now give the:

Proof of Theorem 2.1. Given $\rho \geq 1$ let

$$v(x) = \rho^{-\beta} u(\rho x).$$

Then, v clearly satisfies $Lv = 0$ in the whole $\mathbb{R}^n$. Moreover,

$$\|v\|_{L^\infty(B_R)} = \|\rho^{-\beta} u\|_{L^\infty(B_\rho R)} \leq C\rho^{-\beta}(\rho R)^\beta \leq CR^\beta. \tag{2.6}$$

Then, formally we have

$$v - p(1, \cdot) * v = \big[p(t, \cdot) * v \big]_{t=0}^{t=1} = \int_0^1 \partial_t p * v\, dt = \int_0^1 Lp * v\, dt = \int_0^1 p * Lv = 0$$

and thus

$$v \equiv p(1, \cdot) * v. \tag{2.7}$$

This computation is formal, since we did not checked that the integrals defining the convolutions are finite and since Lv is in principle only defined in weak sense (in the sense of distributions).

To prove rigorously (2.7), we have to do the previous computation in the weak formulation, as follows. Let

$$V(x,t) = (p(t, \cdot) * v)(x).$$

Then, using the growth control on v and Proposition 2.4 (a), it follows that V is a weak solution of $V_t = LV$ in $(0, +\infty) \times \mathbb{R}^n$. Thus, for all $\eta \in C_c^\infty((0,1) \times \mathbb{R}^n)$ we have

$$
\begin{aligned}
-\int_0^1 \int_{\mathbb{R}^n} V \eta_t \, dx \, dt &= \int_0^1 \int_{\mathbb{R}^n} V L\eta \, dx \, dt \\
&= \int_0^1 \int_{\mathbb{R}^n} p(t,z) \int_{\mathbb{R}^n} v(x-z) L\eta(x,t) \, dx \, dz \, dt = 0.
\end{aligned}
\tag{2.8}
$$

In the last identity we have used that $\int_{\mathbb{R}^n} v(x-z) L\eta(x,t) dx = 0$ for all x and t, which follows from the fact that v is a weak solution of $Lv = 0$ in the whole $\mathbb{R}^n$.

Let us justify in detail the change in the order of integration in (2.8). First, observe that the growth control of v (2.6) implies that $\int_{\mathbb{R}^n} |v(x-z)| \, |L\eta(x,t)| dx \leq C(1+|z|)^\beta$, with C depending on η and on the constant in the growth control. Therefore,

$$\int_0^1 \int_{\mathbb{R}^n} t^{-\frac{n}{2s}} p(1, zt^{-\frac{1}{2s}}) \int_{\mathbb{R}^n} |v(x-z)| \, |L\eta(x,t)| dx dz dt \leq C \int_{\mathbb{R}^n} p(1,z)(1+|z|)^\beta dz < \infty.$$

Hence, we can use Fubini in (2.8) to change the order of the integrals, as desired. Thus, (2.7) is proved.

Let us now show that

$$[v]_{C^\gamma(B_1)} \leq C \tag{2.9}$$

for some $\gamma > 0$ and C depending only on n, λ, Λ, and β.

Indeed, given $x, x' \in B_1$ with $x \neq x'$, we have

$$|v(x)-v(x')| = |p(1,\cdot) * v(x) - p(1,\cdot) * v(x')| = \left| \int_{\mathbb{R}^n} \big(p(x-y) - p(x'-y)\big) v(y) dy \right|$$

$$\leq \left| \int_{|y|\leq M} \big(p(x-y) - p(x'-y)\big) v(y) dy \right| + 2 \sup_{x \in B_1} \left| \int_{|y|\geq M} p(x-y) v(y) dy \right|.$$

To bound the first term in the right hand side of the inequality, we use Proposition 2.4 (b) and also (2.6) to find

$$\left| \int_{|y|\leq M} \big(p(x-y) - p(x'-y)\big) v(y) dy \right| \leq C M^{n+\beta} |x - x'|.$$

To bound the second term, we use Proposition 2.4 (a), with $\delta > 0$ such that $2\delta = 2s - \beta$. Using also (2.6), we find that

$$\left| \int_{|y|\geq M} p(x-y) v(y) dy \right| \leq \int_{|y|\geq M} p(x-y)(1+|y|)^{2s-\delta} \frac{|v(y)|}{(1+|y|)^{\beta+\delta}} dy \leq C M^{-\delta}.$$

Thus, we have proved

$$|v(x) - v(x')| \leq C M^{n+\beta} |x - x'| + C M^{-\delta}.$$

Since this can be done for any $M > 0$, we may choose

$$M = |x - x'|^{-\gamma/\delta}, \qquad \text{with } 1 - (n + \beta)\gamma/\delta = \gamma.$$

Then, we have

$$|v(x) - v(x')| \leq C|x - x'|^\gamma,$$

and $\gamma > 0$.

This shows (2.9). Equivalently, what we have proved can be written as

$$[u]_{C^\gamma(B_\rho)} \leq C\rho^{\beta-\gamma} \qquad \text{for all } \rho \geq 1.$$

Next we consider the incremental quotient

$$u_h^\gamma = \frac{u(\cdot + h) - u}{|h|^\gamma}$$

which grows (by the last inequality) as $\|u_h^\gamma\|_{L^\infty(B_R)} \leq CR^{\beta-\gamma}$. Then we can repeat the previous argument with v replaced by u_h^γ and β replaced by $\beta - \gamma$ to show that $[u_h^\gamma]_{C^\gamma(B_R)} \leq CR^{\beta-2\gamma}$, and thus

$$[u]_{C^{2\gamma}(B_R)} \leq CR^{\beta-2\gamma}.$$

We used here that the new γ' in the second step will be larger than γ, and thus we may take γ instead of γ'.

Iterating this procedure, after N steps we find

$$[u]_{C^{N \cdot \gamma}(B_R)} \leq CR^{\beta-N \cdot \gamma}.$$

Taking N the least integer such that $\beta - N\gamma < 0$ and sending $R \to +\infty$, we obtain

$$[u]_{C^{N\gamma}(\mathbb{R}^n)} = 0.$$

This implies that u is a polynomial of degree at most $\lfloor \beta \rfloor$. $\qquad \square$

Finally, we give a consequence of Theorem 2.1 that will be also needed in the proof of Theorem 1.1.

Corollary 2.6. *Let $s \in (0, 1)$, $\alpha \in (0, 1)$, and L be any operator of the form (1.1)-(1.2). Let u be any function satisfying, in the weak sense,*

$$L[u(\cdot + h) - u(\cdot)] = 0 \quad \text{in } \mathbb{R}^n, \qquad \text{for all } h \in \mathbb{R}^n.$$

Assume that u satisfies the growth condition

$$[u]_{C^\alpha(B_R)} \leq CR^\beta \qquad \text{for all } R \geq 1,$$

for some $\beta < 2s$.

Then, u is a polynomial of degree at most $\lfloor \beta + \alpha \rfloor$.

Proof. We apply Theorem 2.1 to $v(x) = u(x + h) - u(x)$, and we find that v is a polynomial. Since this can be done for any h, then u is a polynomial. Finally, the growth condition implies that the degree of the polynomial is at most $\lfloor \beta + \alpha \rfloor$. $\quad \square$

3. Interior regularity

The aim of this section is to prove Theorem 1.1. For it, we will use a compactness argument and the Liouville theorems established in the previous section.

We start with the following.

Lemma 3.1. *Let $s \in (0,1)$, and let λ and Λ be fixed positive constants. Let $\{L_k\}_{k \geq 1}$ be any sequence of operators of the form* (1.1) *whose spectral measures satisfy* (1.2).

Then, a subsequence of $\{L_k\}$ converges weakly to an operator L of the form (1.1)- (1.2).

More precisely, if L_k have spectral measures μ_k then, up to a subsequence, the measures μ_k converge to a measure μ satisfying (1.2).

Moreover, assume (u_k) and (f_k) are sequences of functions satisfying in the weak sense

$$L_k u_k = f_k \ \text{in} \ \Omega$$

for a given bounded domain $\Omega \subset \mathbb{R}^n$. Assume also that, for some functions u and f, we have:

(1) *$u_k \to u$ uniformly in compact sets of $\mathbb{R}^n$,*
(2) *$f_k \to f$ uniformly in Ω,*
(3) *$|u_k(x)| \leq C\left(1 + |x|^{2s-\epsilon}\right)$ for some $\epsilon > 0$, and for all $x \in \mathbb{R}^n$.*

Then, u satisfies

$$Lu = f \ \text{in} \ \Omega$$

in the weak sense, where L be the operator associated to μ.

Proof. Let $\{\mu_k\}_{k \geq 1}$ be the spectral measures of the operators L_k. Using the weak compactness of probability measures on the sphere, we find that there is a subsequence μ_{k_m} converging to a measure μ that satisfies (1.2).

Let L be the operator given by (1.1) whose spectral measure is μ.

We have that

$$\int_{\mathbb{R}^n} u_k L_k \eta = \int_{\Omega} f_k \eta, \quad \text{for all } \eta \in C_c^\infty(\Omega).$$

On the other hand, since $|\eta(x+y) + \eta(x-y) - 2\eta(x)| \leq C \min\{1, |y|^2\}$, by the dominated convergence theorem we obtain that $L_k \eta \to L\eta$ uniformly over compact subsets of $\mathbb{R}^n$.

Moreover, η has support in Ω, which yields $|L_k \eta(x)| \leq C(1+|x|^{n+2s})^{-1}$. Combining this with the growth of u_k we get that $|u_k L_k \eta| \leq C(1+|x|^{n+\epsilon})^{-1}$, and therefore, by the dominated convergence theorem

$$\int_{\mathbb{R}^n} u_k L_k \eta \to \int_{\mathbb{R}^n} u L\eta, \quad \text{for all } \eta \in C_c^\infty(\Omega).$$

Since

$$\int_{\Omega} f_k \eta \to \int_{\Omega} f\eta,$$

it then follows that u is a weak solution of $Lu = f$ in Ω, as desired. $\qquad \square$

We next establish the following result, which is the main step towards Theorem 1.1 (b).

Proposition 3.2. *Let $s \in (0, 1)$, and let L be any operator of the form (1.1)-(1.2). Let $\alpha \in (0, 1)$ be such that $\alpha + 2s$ is not an integer. Let $\alpha' \in (0, \alpha)$ be such that $\lfloor \alpha + 2s \rfloor < \alpha' + 2s < \alpha + 2s$ and that $\alpha < \alpha' + 2s$.*

Let w be any $C_c^\infty(\mathbb{R}^n)$ function satisfying $Lw = f$ in B_1, with $f \in C^\alpha(B_1)$. Then, we have the estimate

$$[w]_{C^{\alpha+2s}(B_{1/2})} \leq C\big([f]_{C^\alpha(B_1)} + \|w\|_{C^{\alpha'+2s}(\mathbb{R}^n)}\big). \tag{3.1}$$

The constant C depends only on n, s, α, α', and the ellipticity constants (1.2).

Proof. The proof of (3.1) is by contradiction. If the statement of the proposition is false then, for each integer $k \geq 0$, there exist L_k, w_k, and f_k satisfying:

- $L_k w_k = f_k$ in B_1;
- L_k is of the form (1.1)-(1.2);
- $[f_k]_{C^\alpha(B_1)} + \|w_k\|_{C^{2s+\alpha'}(\mathbb{R}^n)} \leq 1$ (we may always assume this dividing w_k by the previous quantity);
- $\|w_k\|_{C^{2s+\alpha}(B_{1/2})} \geq k$.

In the rest of the proof we denote

$$\nu = \lfloor \alpha + 2s \rfloor.$$

Since $\nu < \alpha' + 2s < \alpha + 2s$ we then have

$$\sup_k \sup_{z \in B_{1/2}} \sup_{r>0} r^{\alpha'-\alpha} [w_k]_{C^{2s+\alpha'}(B_r(z))} = \infty. \tag{3.2}$$

Next, we define

$$\theta(r) := \sup_k \sup_{z \in B_{1/2}} \sup_{r'>r} (r')^{\alpha'-\alpha} [w_k]_{C^{2s+\alpha'}(B_{r'}(z))}.$$

The function θ is monotone nonincreasing, and we have $\theta(r) < \infty$ for $r > 0$ since we are assuming that $\|w_k\|_{C^{2s+\alpha'}(\mathbb{R}^n)} \leq 1$. In addition, by (3.2) we have $\theta(r) \to \infty$ as $r \downarrow 0$.

Now, for every positive integer m, by definition of $\theta(1/m)$ there exist $r'_m \geq 1/m$, k_m, and $z_m \in B_{1/2}$, for which

$$(r'_m)^{\alpha'-\alpha} [w_{k_m}]_{C^{2s+\alpha'}(B_{r'_m}(z_m))} \geq \frac{1}{2}\theta(1/m) \geq \frac{1}{2}\theta(r'_m). \tag{3.3}$$

Here we have used that θ is nonincreasing. Note that we will have $r'_m \downarrow 0$.

Let $p_{k,z,r}(\cdot - z)$ be the polynomial of degree less or equal than ν in the variables $(x - z)$ which best fits u_k in $B_r(z)$ by least squares. That is,

$$p_{k,z,r} := \arg\min_{p \in \mathbb{P}_\nu} \int_{B_r(z)} \big(w_k(x) - p(x - z)\big)^2 \, dx,$$

where $\mathbb{P}_\nu$ denotes the linear space of polynomials of degree at most ν with real coefficients. From now on in this proof we denote

$$p_m = p_{k_m, z_m, r'_m}.$$

We consider the blow up sequence

$$v_m(x) = \frac{w_{k_m}(z_m + r_m x) - p_m(r_m x)}{(r_m)^{2s+\alpha}\theta(r_m)}. \tag{3.4}$$

Note that, for all $m \geq 1$ we have

$$\int_{B_1(0)} v_m(x) q(x)\,dx = 0 \quad \text{for all } q \in \mathbb{P}_\nu. \tag{3.5}$$

This is the optimality condition for least squares. Note also that (3.3) implies the following nondegeneracy condition for all $m \geq 1$:

$$[v_m]_{C^{2s+\alpha'}(B_1)} \geq 1/2. \tag{3.6}$$

Next, we can estimate

$$[v_m]_{C^{2s+\alpha'}(B_R)} = \frac{1}{\theta(r_m)(r_m)^{\alpha-\alpha'}}[w_{k_m}]_{C^{2s+\alpha'}(B_{Rr_m}(z_m))}$$

$$= \frac{R^{\alpha-\alpha'}}{\theta(r_m)(Rr_m)^{\alpha-\alpha'}}[w_{k_m}]_{C^{2s+\alpha'}(B_{Rr_m}(z_m))}.$$

Indeed, the definition of θ and its monotonicity yield the following growth control for the $C^{2s+\alpha'}$ seminorm of v_m

$$[v_m]_{C^{2s+\alpha'}(B_R)} \leq CR^{\alpha-\alpha'} \quad \text{for all } R \geq 1. \tag{3.7}$$

When $R = 1$, (3.7) implies that $\|v_m - q\|_{L^\infty(B_1)} \leq C$, for some $q \in \mathbb{P}_\nu$. Therefore, (3.5) yields

$$\|v_m\|_{L^\infty(B_1)} \leq C. \tag{3.8}$$

Now, we will see that using (3.7)-(3.8) we obtain

$$[v_m]_{C^\gamma(B_R)} \leq CR^{2s+\alpha-\gamma} \quad \text{for all } \gamma \in [0, 2s+\alpha'] \tag{3.9}$$

Indeed, (3.8) implies that for every multiindex l with $|l| \leq \nu$ there is some point $x_* \in B_1$ such that

$$|D^l v_m(x_*)| \leq C, \qquad x_* \in B_1.$$

The existence of such x_* can be shown taking some nonnegative $\eta \in C_c^\infty(B_1)$ with unit mass and observing that

$$\left| \int \eta(x) D^l v_m(x)\,dx \right| \leq C \int |D^l \eta| v_m(x)\,dx \leq C.$$

Hence, using (3.7), for all l with $|l| = \nu$ and $x \in B_R$ we have

$$|D^l v_m(x)| \leq |D^l v_m(x^*)| + CR^{\alpha-\alpha'}|x - x^*|^{2s+\alpha'-\nu} \leq CR^{2s+\alpha-\nu}.$$

Iterating the same argument one can show the corresponding estimate for $|l| = \nu - 1, \nu - 2$, etc. Then, once established (3.9) for all integer $\gamma \in [0, 2s + \alpha']$, the result for all γ follows by interpolation. Thus, (3.9) is proved.

We now claim that, by further rescaling v_m if necessary, we may assume that in addition to (3.6) the following holds

$$\sup_{|l|=\nu} \mathrm{osc}_{B_1} D^l v_m \geq 1/4, \tag{3.10}$$

where l donates a multiindex. Indeed, if (3.6) holds then there are $x_m \in B_1$ and $h_m \in B_{1-|x_m|}$ such that

$$\sup_{|l|=\nu} \frac{\left|D^l v_m(x_m + h_m) - D^l v_m(x_m)\right|}{|h_m|^{2s+\alpha'-\nu}} \geq 1/4$$

and thus we can consider, instead of v_m, the function

$$\tilde{v}_m = \frac{v_m(x_m + |h_m|x) - \tilde{p}_m(x)}{|h_m|^{2s+\alpha'}},$$

where $\tilde{p}_m \in \mathcal{P}_\nu$ is chosen so that $\tilde{v}_m$ satisfies (3.5) (with v_m replaced by $\tilde{v}_m$).

Note that $\tilde{p}_m$ is the polynomial that approximates better (in the L^2 sense) $v_m(x_m + \cdot)$ in $B_{|h_m|}(x_m)$ and since $v_m \in C^{\sigma+\alpha'}$ with the control (3.7) we have

$$\left|v_m(x_m + |h_m|x) - \tilde{p}_m(x)\right| \leq C|h_m|^{2s+\alpha'}|x|^{2s+\alpha'}.$$

Therefore, $\tilde{v}_m$ also satisfies (3.7) and (3.9) (with v_m replaced by $\tilde{v}_m$). Note that $\tilde{v}_m$ would also be of the form (3.4) for new z_m and r'_m defined as $z_m + x_m$ and $|h_m|r'm$, respectively —where we use that $\theta(|h_m|r'_m) \geq \theta(r'_m)$.

In summary, the new sequence $\tilde{v}_m$ satisfies the same properties as v_m and, in addition, (3.10), as desired.

We now prove the following:

Claim. The sequence v_m converges in $C_{\mathrm{loc}}^{(\nu+2s+\alpha')/2}(\mathbb{R}^n)$ to a function $v \in C_{\mathrm{loc}}^{2s+\alpha'}(\mathbb{R}^n)$. This function v satisfies the assumptions of the Liouville-type Corollary 2.6.

The $C^{(\nu+2s+\alpha')/2}$ uniform convergence on compact sets of $\mathbb{R}^n$ of the function v_m to some $v \in C^{2s+\alpha'}(\mathbb{R}^n)$ follows from (3.9) and the Arzelà-Ascoli theorem (and the usual diagonal sequence argument) — the exponent $(\nu+2s+\alpha')/2$ is chosen so that it is less that $2s + \alpha'$ and greater than both ν and $2s$.

Moreover, passing to the limit (3.9) with $\gamma \in (\alpha, 1]$ such that $\gamma \leq \alpha' + 2s$, we find

$$[v]_{C^\gamma(B_R)} \leq CR^\beta \qquad \text{for all } R \geq 1, \tag{3.11}$$

$\beta = 2s + \alpha - \gamma < 2s$. Thus, v satisfies the growth assumption in Corollary 2.6.

On the hand, each w_k satisfies a $L_k w_k = f_k$ in B_1. Thus, recalling that we have $[f_k]_{C^\alpha(B_1)} \leq 1$, we find that

$$\left|L_k w_k(\bar{x} + \bar{h}) - L_k w_k(\bar{x})\right| \leq |\bar{h}|^\alpha \qquad \text{for all } \bar{x} \in B_{1/2}(z) \text{ and } \bar{h} \in B_{1/2}. \tag{3.12}$$

Note now that, since $\nu \leq 2$,

$$\delta^2 p(x + h, y) - \delta^2 p(x, y) = 0 \quad \text{for all } p \in \mathbb{P}_\nu \text{ and for all } x, y, h \text{ in } \mathbb{R}^n. \qquad (3.13)$$

Here, as usual, we have denoted $\delta^2 \varphi(x) = \varphi(x + y) + \varphi(x - y) - 2\varphi(x)$.

Next, taking into account (3.13), we translate (3.12) from w_{k_m} to v_m. Namely, using the definition of v_m in (3.4), and setting $\bar{h} = r_m h$, and $\bar{x} = z_m + r_m x$ in (3.12), we obtain

$$\frac{1}{(r_m)^{2s}} \left| L_{k_m} \left((r_m)^{2s+\alpha} \theta(r_m) \{ v_m(\cdot + h) - v_m \} \right) (x) \right| \leq (r_m)^\alpha |h|^\alpha$$

whenever $|x| \leq \frac{1}{2r_m}$, and thus

$$|L_{k_m} (v_m(\cdot + h) - v_m)(x)| \leq \frac{1}{\theta(r_m)} \quad \text{whenever } |x| \leq \frac{1}{2r_m}. \qquad (3.14)$$

By Lemma 3.1, the operators L_{k_m} converge weakly (up to subsequence) to an operator L. Thus, passing (3.14) to the limit we find that

$$L(v(\cdot + h) - v) = 0 \quad \text{in all of } \mathbb{R}^n.$$

Notice that to be able to pass to the limit $m \to \infty$ on the right hand side of (3.14) we are using that, by (3.9), the functions $v_{k_m}(\cdot + h) - v_{k_m}$ satisfy

$$\| v_{k_m}(\cdot + h) - v_{k_m} \|_{C^{2s+\alpha'}(B_R)} \leq C(R),$$

and also the growth control

$$\| v_{k_m}(\cdot + h) - v_{k_m} \|_{L^\infty(B_R)} \leq C R^{2s-\epsilon} \qquad \text{for all } R \geq 1,$$

for some $\epsilon > 0$ (this follows from (3.9)).

This finishes the proof of Claim.

We have thus proved that the limit function v satisfies the assumptions of Corollary 2.6, and hence we conclude that v is a polynomial of degree ν. On the other hand, passing (3.5) to the limit we obtain that v is orthogonal to every polynomial of degree ν in B_1, and hence it must be $v \equiv 0$. But then passing (3.6) to the limit we obtain that v cannot be constantly zero in B_1; a contradiction. $\qquad \square$

We can now give the:

Proof of Theorem 1.1 (b). Let $\nu = \lfloor \alpha + 2s \rfloor$, and let α' be such that $\nu < \alpha' + 2s$. Such α' exists because $\alpha + 2s$ is not an integer (by assumption). We will deduce the theorem from Proposition 3.2, as follows.

First, it immediately follows from Proposition 3.2 that for any $w \in C_c^\infty(\mathbb{R}^n)$,

$$[w]_{C^{\alpha+2s}(B_{1/2})} \leq C \big([f]_{C^\alpha(B_1)} + [w]_{C^{\alpha'+2s}(B_2)} + \|w\|_{C^\alpha(\mathbb{R}^n)} \big). \qquad (3.15)$$

To prove this, take a cutoff function $\eta \in C_c^\infty(B_2)$ satisfying $\eta \equiv 1$ in $B_{3/2}$, and apply the proposition to the function ηw. One finds

$$[w]_{C^{\alpha+2s}(B_{1/2})} \leq C \big([f]_{C^\alpha(B_1)} + [L(\eta w - w)]_{C^\alpha(B_1)} + \|w\|_{C^{\alpha'+2s}(B_2)} \big).$$

And since the function $\eta w - w$ vanishes in $B_{3/2}$, then we have

$$[L(\eta w - w)]_{C^\alpha(B_1)} \leq C[w]_{C^\alpha(\mathbb{R}^n)}. \tag{3.16}$$

Thus, (3.15) follows.

We recall now the definition of the norms $\|\phi\|_{\gamma;U}^{(\sigma)}$; see Gilbarg-Trudinger [15]. If $\gamma = k + \gamma'$, with k integer and $\gamma' \in (0, 1]$, then

$$[\phi]_{\gamma;U}^{(\sigma)} = \sup_{x,y \in U} \left(d_{x,y}^{\gamma+\sigma} \frac{|D^k\phi(x) - D^k\phi(y)|}{|x - y|^{\gamma'}} \right),$$

and

$$\|\phi\|_{\gamma;U}^{(\sigma)} = \sum_{l=0}^{k} \sup_{x \in U} \left(d_x^{l+\sigma} |D^l\phi(x)| \right) + [\phi]_{\gamma;U}^{(\sigma)}.$$

Here, we denoted

$$d_x = \mathrm{dist}(x, \partial U) \qquad \text{and} \qquad d_{x,y} = \min\{d_x, d_y\}.$$

We will use next these norms. Indeed, we can rescale (3.15) and apply it to any ball B_ρ of radius $\rho > 0$. Then, dividing by ρ^α, and taking the supremum over all the balls B_ρ such that $B_{2\rho} \subset B_1$, we find

$$[w]_{\alpha+2s;B_1}^{(-\alpha)} \leq C \left([f]_{\alpha;B_1}^{(-\alpha+2s)} + \|w\|_{\alpha'+2s;B_1}^{(-\alpha)} + [w]_{C^\alpha(\mathbb{R}^n)} \right).$$

Thus, using that

$$\|w\|_{\gamma+2s;B_1}^{(-\alpha)} \leq \epsilon \|w\|_{\alpha+2s;B_1}^{(-\alpha)} + C(\epsilon)\|w\|_{L^\infty(B_1)} \qquad \text{for } \gamma < \alpha,$$

we deduce

$$\|w\|_{\alpha+2s;B_1}^{(-\alpha)} \leq C \left([f]_{\alpha;B_1}^{(-\alpha+2s)} + \|w\|_{C^\alpha(\mathbb{R}^n)} \right).$$

Moreover, since $[f]_{\alpha;B_1}^{(-\alpha+2s)} \leq \|f\|_{C^\alpha(B_1)}$,

$$\|w\|_{\alpha+2s;B_1}^{(-\alpha)} \leq C \left(\|f\|_{C^\alpha(B_1)} + \|w\|_{C^\alpha(\mathbb{R}^n)} \right).$$

In particular, we have proved that for all $w \in C_c^\infty(\mathbb{R}^n)$, the following inequality holds

$$\|w\|_{C^{\alpha+2s}(B_{1/2})} \leq C \left(\|f\|_{C^\alpha(B_1)} + \|w\|_{C^\alpha(\mathbb{R}^n)} \right).$$

Finally, by using a standard approximation argument (see Remark 2.3), the result follows for any solution $u \in C^\alpha(\mathbb{R}^n)$, and thus we are done. □

We now establish the estimate with a L^∞ right hand side. As before, we prove first a preliminary result.

Proposition 3.3. *Let $s \in (0, 1)$, $s \neq \frac{1}{2}$, and let L be any operator of the form (1.1)-(1.2). Let $\alpha \in (0, 2s)$ be such that $\lfloor 2s \rfloor < \alpha < 2s$.*

Let w be any $C_c^\infty(\mathbb{R}^n)$ function satisfying $Lw = f$ in B_1, with $f \in L^\infty(B_1)$. Then, we have the estimate

$$[w]_{C^{2s}(B_{1/2})} \leq C \left(\|f\|_{L^\infty(B_1)} + \|w\|_{C^\alpha(\mathbb{R}^n)} \right). \tag{3.17}$$

The constant C depends only on n, s, α, and the ellipticity constants (1.2).

Proof. We follow the steps of the proof of Proposition 3.2.

Assume that the statement is false. Then, for each integer $k \geq 0$, there exist L_k, w_k, and f_k satisfying:

- $L_k w_k = f_k$ in B_1;
- L_k is of the form (1.1)-(1.2);
- $\|f_k\|_{L^\infty(B_1)} + \|w_k\|_{C^\alpha(\mathbb{R}^n)} \leq 1$;
- $\|w_k\|_{C^{2s}(B_{1/2})} \geq k$.

In the rest of the proof we denote

$$\nu = \lfloor 2s \rfloor, \qquad \beta = 2s - \alpha.$$

Since $\nu < \alpha < 2s$ we then have

$$\sup_k \sup_{z \in B_{1/2}} \sup_{r>0} r^{-\beta} [w_k]_{C^\alpha(B_r(z))} = \infty. \tag{3.18}$$

Next, we define

$$\theta(r) := \sup_k \sup_{z \in B_{1/2}} \sup_{r'>r} (r')^{-\beta} [w_k]_{C^\alpha(B_{r'}(z))}.$$

The function θ is monotone nonincreasing, and we have $\theta(r) < \infty$ for $r > 0$ since we are assuming that $\|w_k\|_{C^\alpha(\mathbb{R}^n)} \leq 1$. In addition, by (3.18) we have $\theta(r) \to \infty$ as $r \downarrow 0$.

Now, for every positive integer m, by definition of $\theta(1/m)$ there exist $r'_m \geq 1/m$, k_m, and $z_m \in B_{1/2}$, for which

$$(r'_m)^{-\beta} [w_{k_m}]_{C^\alpha(B_{r'_m}(z_m))} \geq \frac{1}{2}\theta(1/m) \geq \frac{1}{2}\theta(r'_m). \tag{3.19}$$

Here we have used that θ is nonincreasing. Note that we will have $r'_m \downarrow 0$.

As in the proof of Proposition 3.2, we define $p_{k,z,r}(\cdot - z)$ as the polynomial of degree less or equal than ν in the variables $(x - z)$ which best fits u_k in $B_r(z)$ by least squares, and we denote $p_m = p_{k_m, z_m, r'_m}$.

We consider the blow up sequence

$$v_m(x) = \frac{w_{k_m}(z_m + r'_m x) - p_m(r'_m x)}{(r'_m)^{\alpha+\beta}\theta(r'_m)}. \tag{3.20}$$

Note that, for all $m \geq 1$ we have

$$\int_{B_1(0)} v_m(x) q(x)\, dx = 0 \quad \text{for all } q \in \mathbb{P}_\nu. \tag{3.21}$$

(Here, as in (3.5), $\mathbb{P}_\nu$ denotes the linear space of polynomials of degree at most ν with real coefficients.) Note also that (3.19) implies the following nondegeneracy condition for all $m \geq 1$:

$$[v_m]_{C^\alpha(B_1)} \geq 1/2. \tag{3.22}$$

Next, as in (3.7), one can show that

$$[v_m]_{C^\alpha(B_R)} \leq CR^\beta \quad \text{for all } R \geq 1. \tag{3.23}$$

When $R = 1$, (3.23) implies that $\|v_m - q\|_{L^\infty(B_1)} \leq C$, for some $q \in \mathbb{P}_\nu$. Therefore, (3.21) yields

$$\|v_m\|_{L^\infty(B_1)} \leq C. \tag{3.24}$$

We now prove the following:

Claim. Given $\epsilon > 0$ small, the sequence v_m converges in $C^{\alpha-\epsilon}_{\text{loc}}(\mathbb{R}^n)$ to a function $v \in C^\alpha_{\text{loc}}(\mathbb{R}^n)$. This function v satisfies the assumptions of the Liouville-type Theorem 2.1.

The $C^{\alpha-\epsilon}$ uniform convergence on compact sets of $\mathbb{R}^n$ of the function v_m to some $v \in C^\alpha(\mathbb{R}^n)$ follows from (3.23) and the Arzelà-Ascoli theorem. Moreover, passing to the limit (3.23), we find that

$$[v]_{C^\alpha(B_R)} \leq CR^\beta \qquad \text{for all } R \geq 1. \tag{3.25}$$

Thus, v satisfies the growth assumption in Theorem 2.1.

On the hand, each w_k satisfies a $L_k w_k = f_k$ in B_1. Thus, recalling that we have $\|f_k\|_{L^\infty(B_1)} \leq 1$, we find that

$$\left| L_k w_k(\bar{x} + \bar{h}) - L_k w_k(\bar{x}) \right| \leq 2 \qquad \text{for all } \bar{x} \in B_{1/2}(z) \text{ and } \bar{h} \in B_{1/2}. \tag{3.26}$$

Next, as is (3.14), one can translate (3.26) from w_{k_m} to v_m. Indeed, setting $\bar{h} = r'_m h$, and $\bar{x} = z_m + r'_m x$ in (3.26), one has

$$\left| L_{k_m} \left(v_m(\,\cdot\, + h) - v_m \right)(x) \right| \leq \frac{2}{\theta(r'_m)} \quad \text{whenever } |x| \leq \frac{1}{2r'_m}. \tag{3.27}$$

By Lemma 3.1, the operators L_{k_m} converge weakly (up to subsequence) to an operator L. Thus, passing (3.27) to the limit we find that

$$L \left(v(\,\cdot\, + h) - v \right) = 0 \quad \text{in all of } \mathbb{R}^n. \tag{3.28}$$

Notice that to be able to pass to the limit $m \to \infty$ on (3.27) we used that, by (3.23), the functions $v_{k_m}(\,\cdot\, + h) - v_{k_m}$ satisfy the growth control

$$\|v_{k_m}(\,\cdot\, + h) - v_{k_m}\|_{L^\infty(B_R)} \leq C|h|^\alpha R^\beta \qquad \text{for all } R \geq 1.$$

This, combined with the locally uniform convergence of $v_{k_m}(\,\cdot\, + h) - v_{k_m}$ and Lemma 3.1, yields (3.28).

This finishes the proof of Claim.

We have thus proved that the limit function v satisfies the assumptions of Theorem 2.1, and hence we conclude that v is a polynomial of degree ν. On the other hand, passing (3.21) to the limit we obtain that v is orthogonal to every polynomial of degree ν in B_1, and hence it must be $v \equiv 0$. But then passing (3.22) to the limit we obtain that v cannot be constantly zero in B_1; a contradiction. □

We also have the following.

Proposition 3.4. *Let $s = \frac{1}{2}$, and let L be any operator of the form (1.1)-(1.2). Let $\alpha \in (0, 2)$ be such that $\lfloor 2s \rfloor < \alpha < 2s$.*

Let w be any $C_c^\infty(\mathbb{R}^n)$ satisfying $Lw = f$ in B_1, with $f \in L^\infty(B_1)$. Then, we have the estimate

$$[w]_{C^{2s-\epsilon}(B_{1/2})} \leq C\big([f]_{L^\infty(B_1)} + \|w\|_{C^\alpha(\mathbb{R}^n)}\big).$$

The constant C depends only on n, s, α, and the ellipticity constants (1.2).

Proof. The proof is minor modification of the one in Proposition 3.3. One only has to take $\beta = 2s - \alpha - \epsilon$ instead of $\beta = 2s - \alpha$, and follow the same steps as in Proposition 3.3. □

Finally, we can give the:

Proof of Theorem 1.1 (a). We prove only the case $s \neq \frac{1}{2}$, the case $s = \frac{1}{2}$ follows with exactly the same argument.

By Proposition 3.3, for all $w \in C_c^\infty(\mathbb{R}^n)$ we have the estimate

$$[w]_{C^{2s}(B_{1/2})} \leq C\big(\|f\|_{L^\infty(B_1)} + \|w\|_{C^\alpha(\mathbb{R}^n)}\big),$$

where α is such that $\lfloor 2s \rfloor < \alpha < 2s$.

Then, multiplying w by a cutoff function, it immediately follows that

$$[w]_{C^{2s}(B_{1/2})} \leq C\big(\|f\|_{L^\infty(B_1)} + \|w\|_{C^\alpha(B_2)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big); \qquad (3.29)$$

see the proof of Theorem 1.1 (b) above.

Now, using the norms $\|\phi\|_{\gamma;U}^{(\sigma)}$ defined before, we can rescale (3.29) and apply it to any ball B_ρ of radius $\rho > 0$. Then, taking the supremum over all the balls B_ρ such that $B_{2\rho} \subset B_1$, we find

$$[w]_{2s;B_1}^{(0)} \leq C\big(\|f\|_{0;B_1}^{(2s)} + \|w\|_{\alpha;B_1}^{(0)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big).$$

Thus, we deduce

$$\|w\|_{2s;B_1}^{(0)} \leq C\big(\|f\|_{0;B_1}^{(2s)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big).$$

In particular, for all $w \in C_c^\infty(\mathbb{R}^n)$, the following inequality holds

$$\|w\|_{C^{2s}(B_{1/2})} \leq C\big(\|f\|_{L^\infty(B_1)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big).$$

Finally, by using an approximation argument (see Remark 2.3), the result follows. □

To end this section, we give an immediate consequence of Theorem 1.1. Notice that here we assume some regularity on the spectral measure a, but the ellipticity constants are the same as before. In particular, we are not assuming positivity of a in all of S^{n-1}.

Corollary 3.5. *Let $s \in (0, 1)$, L be given by (1.3), and assume that*

$$a \in C^\alpha(S^{n-1}).$$

Let u be a solution of (1.7). *Then, if* $f \in C^\alpha(B_1)$ *and* $u \in L^\infty(\mathbb{R}^n)$,

$$\|u\|_{C^{\alpha+2s}(B_{1/2})} \le C \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{C^\alpha(B_1)} \right)$$

whenever $\alpha + 2s$ is not an integer.

   The constant C depends only on n, s, ellipticity constants (1.2), *and* $\|a\|_{C^\alpha(S^{n-1})}$.

Proof. The proof is a minor modification of the proof of Theorem 1.1 (b). Indeed, one only needs to replace the estimate (3.16) therein, by the following one

$$[L(\eta w - w)]_{C^\alpha(B_1)} \le C[w]_{L^\infty(\mathbb{R}^n)},$$

which follows easily using that $a \in C^\alpha(S^{n-1})$ —recall that $\eta \equiv 1$ in B_1 and $\eta \in C_c^\infty(B_2)$. With this modification, the rest of the proof is exactly the same. $\square$

   Finally, we give an immediate consequence of Theorem 1.1 that will be used later.

Corollary 3.6. *Let $s \in (0,1)$, and let L be any operator of the form* (1.1)-(1.2). *Let u be any solution of*

$$Lu = f \quad in \; B_1,$$

with $f \in L^\infty(B_1)$. Then, for any $\epsilon > 0$,

$$\|u\|_{C^{2s}(B_{1/2})} \le C \left(\sup_{R \ge 1} R^{\epsilon-2s} \|u\|_{L^\infty(B_R)} + \|f\|_{L^\infty(B_1)} \right) \quad if \; s \ne \frac{1}{2},$$

and

$$\|u\|_{C^{2s-\epsilon}(B_{1/2})} \le C \left(\sup_{R \ge 1} R^{\epsilon-2s} \|u\|_{L^\infty(B_R)} + \|f\|_{L^\infty(B_1)} \right) \quad if \; s = \frac{1}{2},$$

The constant C depends only on n, s, ϵ, and the ellipticity constants (1.2).

Proof. The proof follows by using that the truncated function $\tilde{u} = u\chi_{B_2}$ satisfies the hypotheses of Theorem 1.1. $\square$

4. A Liouville theorem in the half space

   In this Section we prove the following Liouville-type theorem, which will be needed in the proof of Theorem 1.2.

Theorem 4.1. *Let L be an operator of the form* (1.1)-(1.2). *Let u be any weak solution of*

$$\begin{cases} Lu &= 0 \quad in \; \mathbb{R}^n_+ \\ u &= 0 \quad in \; \mathbb{R}^n_-. \end{cases} \tag{4.1}$$

Assume that, for some $\beta < 2s$, u satisfies the growth control

$$\|u\|_{L^\infty(B_R)} \le CR^\beta \quad for \; all \; R \ge 1.$$

Then,

$$u(x) = K(x_n)^s_+$$

for some constant $K \in \mathbb{R}$.

Notice that Theorem 4.1 is related to Theorem 1.4 in [31]. However, the proofs of the two results are quite different. Indeed, in [31] we first used a method of Caffarelli to obtain a Hölder estimate for u/d^s up to the boundary, and then we iterated this estimate to show the Liouville theorem. Here, instead, we only use estimates for u (and not for u/d^s) to establish Theorem 4.1.

Recall that in the present context we can not use the method of Caffarelli (that we adapted to nonlocal equations in [31, 30]), because the operators (1.1)-(1.2) do not satisfy a Harnack inequality.

4.1. **Barriers.** We next construct supersolutions and subsolutions that are needed in our analysis. We will need them both in the proofs of the Liouville Theorem 4.1 and of Theorem 1.2.

These barriers are essentially the same as the ones constructed in our work [31], however the proofs must be redone so that the ellipticity constants are (1.2).

Before constructing the sub and supersolution, we give two preliminary lemmas. These are the analogues of Lemmas 3.1 and 3.2 in [31].

Lemma 4.2. *Let $s \in (0,1)$, and let L be given by (1.1)-(1.2). Let*

$$\varphi^{(1)}(x) = \big(\operatorname{dist}(x, B_1)\big)^s.$$

Then,

$$0 \le L\varphi^{(1)}(x) \le C\left\{1 + \big|\log(|x| - 1)\big|\right\} \quad in \ B_2 \setminus B_1. \tag{4.2}$$

The constant C depends only on s, n, and the ellipticity constants (1.2).

Proof of Lemma 4.2. We use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. Let us estimate $L\varphi^{(1)}(x_\rho)$ where $x_\rho = (0, 1 + \rho)$ for $\rho \in (0, 1)$. To do it, we subtract the function $\psi(x) = (x_n - 1)_+^s$, which satisfies $L\psi(x_\rho) = 0$. As in [31, Lemma 3.1], we have that

$$0 \le \big(\varphi_1^{(1)} - \psi\big)(x_\rho + y) \le \begin{cases} C\rho^{s-1}|y'|^2 & \text{for } y = (y', y_n) \in B_{\rho/2} \\ C|y'|^{2s} & \text{for } y = (y', y_n) \in B_1 \setminus B_{\rho/2} \\ C|y|^s & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

Therefore,

$$\begin{aligned}
0 \le L\varphi^{(1)}(x_\rho) &= L\big(\varphi^{(1)} - \psi\big)(x_\rho) \\
&= \int_{S^{n-1}} \int_{-\infty}^{\infty} \frac{\big(\varphi_1^{(1)} - \psi\big)(x_\rho + r\theta) + \big(\varphi_1^{(1)} - \psi\big)(x_\rho - r\theta)}{2} \frac{dr}{|r|^{1+2s}} \, d\mu(\theta) \\
&\le C \int_{S^{n-1}} \left(\int_{|r|<\rho/2} \frac{\rho^{s-1}|r|^2 dr}{|r|^{1+2s}} + \int_{\rho/2<|r|<1} \frac{|r|^{2s} dr}{|r|^{1+2s}} + \int_{|r|>1} \frac{|r|^s dr}{|r|^{1+2s}} \right) d\mu \\
&\le C\Lambda\big(1 + |\log \rho|\big).
\end{aligned}$$

Thus, (4.2) follows. $\square$

Lemma 4.3. *Let $s \in (0,1)$, and let L be given by (1.1)-(1.2). Let*

$$\varphi^{(3)}(x) = \big(\text{dist}(x, B_1)\big)^{3s/2}.$$

Then,

$$L\varphi^{(3)}(x) \geq c(|x| - 1)^{-s/2} \quad \text{for all } x \in B_2 \setminus B_1. \tag{4.3}$$

The constants $c > 0$ and C depend only on n, s, and the ellipticity constants (1.2).

Proof. As before, we denote $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. Let us estimate $L\varphi^{(3)}(x_\rho)$, where $x_\rho = (0, 1 + \rho)$ for $\rho \in (0, 1)$. To do it we subtract the function $\psi(x) = (x_n - 1)_+^{3s/2}$. By homogeneity, we have that ψ satisfies $L\psi(x_\rho) = c\rho^{-s/2}$ for some $c > 0$; see [31, Section 2]. We note that

$$\big(\varphi^{(3)} - \psi\big)(x_\rho) = 0$$

and, as in [31, Lemma 3.2],

$$0 \geq \big(\varphi^{(3)} - \psi\big)(x_\rho + y) \geq \begin{cases} -C\rho^{3s/2-1}|y'|^2 & \text{for } y = (y', y_n) \in B_{\rho/2} \\ -C|y'|^{3s} & \text{for } y = (y', y_n) \in B_1 \setminus B_{\rho/2} \\ -C|y|^{3s/2} & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

Then, one finds that

$$L\varphi^{(3)}(x_\rho) - c\rho^{-s/2} \geq -C,$$

which establishes (4.3). □

Using the previous lemma, one can now construct the supersolution that will be used in the next section.

Lemma 4.4 (Supersolution). *Let $s \in (0, 1)$, and let L be given by (1.1)-(1.2). There are positive constants ϵ and C, and a radial, bounded, continuous function φ_1 which is $C^{1,1}$ in $B_{1+\epsilon} \setminus \overline{B_1}$ and satisfies*

$$\begin{cases} L\varphi_1(x) \leq -1 & \text{in } B_{1+\epsilon} \setminus \overline{B_1} \\ \varphi_1(x) = 0 & \text{in } B_1 \\ \varphi_1(x) \leq C\big(|x| - 1\big)^s & \text{in } \mathbb{R}^n \setminus B_1 \\ \varphi_1(x) \geq 1 & \text{in } \mathbb{R}^n \setminus B_{1+\epsilon} \end{cases}$$

The constants ϵ, c and C depend only on n, s, and ellipticity constants.

Proof. See the proof of Lemma 3.3 in [31]. □

4.2. **Hölder regularity up to the boundary for u.** Using the interior estimates and the supersolution constructed above, we find the following.

Proposition 4.5. *Let $s \in (0, 1)$, L be any operator of the form (1.1)-(1.2), and Ω be any bounded Lipschitz domain satisfying the exterior ball condition. Let $f \in L^\infty(\Omega)$, and u be a weak solution of*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.4}$$

Then,

$$\|u\|_{C^s(\overline{\Omega})} \leq C\|f\|_{L^\infty(\Omega)}.$$

The constant C depends only on n, s, Ω, and the ellipticity constants (1.2).

Proof. The proof of this result is quite standard once one has interior estimates (given by Theorem 1.1) and an appropriate barrier (given by Lemma 4.4). For more details, see the proof of Proposition 1.1 in [30], where this was done for the case of the fractional Laplacian. $\square$

We will also need the following version of the estimate.

Proposition 4.6. *Let L be any operator of the form* (1.1)-(1.2). *Let $f \in L^\infty(B_1^+)$, and $u \in L^\infty(\mathbb{R}^n)$ be any bounded solution of*

$$\begin{cases} Lu & = & f & in\ B_1^+ \\ u & = & 0 & in\ B_1^-. \end{cases} \tag{4.5}$$

Then,

$$\|u\|_{C^s(\overline{B_{1/2}})} \leq C\big(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\big).$$

Proof. The proof is the same as the one of Proposition 4.5. $\square$

4.3. **Proof of Theorem 4.1.** Here we prove Theorem 4.1. For it, we will need the following result, established in [31].

Lemma 4.7 ([31]). *Let u satisfy $(-\Delta)^s u = 0$ in $\mathbb{R}_+$ and $u = 0$ in $\mathbb{R}_-$. Assume that, for some $\beta \in (0, 2s)$, u satisfies the growth control $\|u\|_{L^\infty(0,R)} \leq CR^\beta$ for all $R \geq 1$. Then $u(x) = K(x_+)^s$.*

We can now give the:

Proof of Theorem 4.1. Given $\rho \geq 1$, define $v_\rho(x) = \rho^{-\beta} u(\rho x)$. Then, it follows that v_ρ satisfies the same growth condition as u, namely

$$\|v_\rho\|_{L^\infty(B_R)} \leq CR^\beta \quad \text{for all } R \geq 1. \tag{4.6}$$

Indeed, one has

$$\|v_\rho\|_{L^\infty(B_R)} = \rho^{-\beta}\|u\|_{L^\infty(B_{\rho R})} \leq \rho^{-\beta} C(\rho R)^\beta = CR^\beta.$$

Moreover, we know that $Lv_\rho = 0$ in $\mathbb{R}_+^n$ and $v_\rho = 0$ in $\mathbb{R}_-^n$.

In particular, if we consider $\bar{v}_\rho = v_\rho \chi_{B_2}$, then $\bar{v}_\rho \in L^\infty(\mathbb{R}^n)$ satisfies

$$\begin{cases} L\bar{v}_\rho & = & g_\rho & in\ B_1^+ \\ \bar{v}_\rho & = & 0 & in\ B_1^- \end{cases} \tag{4.7}$$

for some $g_\rho \in L^\infty(B_1^+)$. Indeed, thanks to the growth condition (4.6), we have $\|g_\rho\|_{L^\infty(B_1^+)} \leq C_0$ for some C_0 independent of ρ. Then, by Proposition 4.6, it follows that

$$\|v_\rho\|_{C^s(B_{1/2})} = \|\bar{v}_\rho\|_{C^s(B_{1/2})} \leq CC_0.$$

Therefore, we find

$$[u]_{C^s(B_{\rho/2})} = \rho^{-s}[u(\rho x)]_{C^s(B_{1/2})} = \rho^{\beta-s}[v_\rho]_{C^s(B_{1/2})} \le CC_0\rho^{\beta-s}.$$

In other words, we have proved that

$$[u]_{C^s(B_R)} \le CR^{\beta-s} \qquad \text{for all } R \ge 1.$$

Now, given $\tau \in S^{n-1}$ such that $\tau_n = 0$, and given $h > 0$, consider

$$w(x) = \frac{u(x + h\tau) - u(x)}{h^s}.$$

By the previous considerations, we have

$$\|w\|_{L^\infty(B_R)} \le CR^{\beta-s} \qquad \text{for all } R \ge 1.$$

Moreover, we clearly have $Lw = 0$ in $\mathbb{R}^n_+$ and $w = 0$ in $\mathbb{R}^n_-$. Therefore, we can repeat the previous argument (applied to w instead of u), to find that

$$[w]_{C^s(B_R)} \le CR^{\beta-2s} \qquad \text{for all } R \ge 1.$$

Hence, since $\beta < 2s$, letting $R \to \infty$ in the previous inequality we find that

$$w \equiv 0 \qquad \text{in } \mathbb{R}^n.$$

Therefore, $u(x + h\tau) = u(x)$ for all $h > 0$ and for all $\tau \in S^{n-1}$ such that $\tau_n = 0$. Thus, we have that u depends only on the x_n-variable, i.e.,

$$u(x) = \bar{u}(x_n)$$

for some 1D function $\bar{u}$.

But we then have that

$$
\begin{aligned}
Lu(x) &= \int_{S^{n-1}} \int_{-\infty}^{\infty} \big(u(x + \theta r) + u(x - \theta r) - 2u(x)\big) \frac{dr}{|r|^{1+2s}} \, d\mu(\theta) \\
&= \int_{S^{n-1}} \int_{-\infty}^{\infty} \big(\bar{u}(x_n + \theta_n r) + \bar{u}(x_n - \theta_n r) - 2\bar{u}(x_n)\big) \frac{dr}{|r|^{1+2s}} \, d\mu(\theta) \\
&= -c \int_{S^{n-1}} (-\Delta)^s_{\mathbb{R}} \big(\bar{u}(x_n + \theta_n r)\big)\big|_{r=0} \, d\mu(\theta) \\
&= -c \int_{S^{n-1}} |\theta_n|^{2s} (-\Delta)^s_{\mathbb{R}} \bar{u}(x_n) \, d\mu(\theta) \\
&= -c\,(-\Delta)^s \bar{u}(x_n),
\end{aligned}
$$

for some constant $c > 0$. Therefore, $\bar{u}$ solves $(-\Delta)^s\bar{u} = 0$ in $\mathbb{R}_+$, $\bar{u} = 0$ in $\mathbb{R}_-$. Hence, using Lemma 4.7 we finally deduce that $\bar{u}(x_n) = K(x_n)^s_+$, and thus

$$u(x) = K(x_n)^s_+,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5. Boundary regularity

In this section we finally prove Theorem 1.2.

The main ingredient in its proof is the following result. In it, we use the following terminology.

Definition 5.1. We say that Γ is a $C^{1,1}$ surface with radius $\rho_0 > 0$ splitting B_1 into U^+ and U^- if the following happens:

- The two disjoint domains U^+ and U^- partition B_1, i.e., $\overline{B_1} = \overline{U^+} \cup \overline{U^-}$.
- The boundary $\Gamma := \partial U^+ \backslash \partial B_1 = \partial U^- \backslash \partial B_1$ is a $C^{1,1}$ surface with $0 \in \Gamma$.
- All points on $\Gamma \cap \overline{B_{3/4}}$ can be touched by two balls of radii ρ_0, one contained in U^+ and the other contained in U^-.

The result reads as follows.

Proposition 5.2. Let $s \in (0,1)$ and $\beta \in (s, 2s)$ be given constants. Let Γ be a $C^{1,1}$ surface with radius ρ_0 splitting B_1 into U^+ and U^-; see Definition 5.1.

Let $f \in L^\infty(U^+)$, and assume that $u \in L^\infty(\mathbb{R}^n)$ is a solution of

$$\begin{cases} Lu & = & f & in \ U^+ \\ u & = & 0 & in \ U^-, \end{cases}$$

where L is any operator of the form (1.1)-(1.2).

Then, for all $z \in \Gamma \cap \overline{B_{1/2}}$ there is a constant $Q(z)$ with $|Q(z)| \leq CC_0$ for which

$$\left| u(x) - Q(z)\big((x-z) \cdot \nu(z)\big)_+^s \right| \leq CC_0 |x-z|^\beta \qquad for \ all \ x \in B_1,$$

where $\nu(z)$ is the unit normal vector to Γ at z pointing towards U^+ and

$$C_0 = \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(U^+)}.$$

The constant C depends only on n, ρ_0, s, β, and the ellipticity constants (1.2).

In order to show Proposition 5.2, we will need some preliminary lemmas.

First, we will need the following technical result.

Lemma 5.3. Let $\beta > s$ and $\nu \in S^{n-1}$ be some unit vector. Let $u \in C(B_1)$ and define

$$\phi_r(x) := Q_*(r) \, (x \cdot \nu)_+^s, \tag{5.1}$$

where

$$Q_*(r) := \arg\min_{Q \in \mathbb{R}} \int_{B_r} \big(u(x) - Q(x \cdot \nu)_+^s\big)^2 \, dx = \frac{\int_{B_r} u(x) \, (x \cdot \nu)_+^s \, dx}{\int_{B_r} (x \cdot \nu)_+^{2s} \, dx}.$$

Assume that for all $r \in (0,1)$ we have

$$\|u - \phi_r\|_{L^\infty(B_r)} \leq C_0 r^\beta. \tag{5.2}$$

Then, there is $Q \in \mathbb{R}$ satisfying $|Q| \leq C\big(C_0 + \|u\|_{L^\infty(B_1)}\big)$ such that

$$\|u - Q(x \cdot \nu)_+^s\|_{L^\infty(B_r)} \leq CC_0 r^\beta$$

for some constant C depending only on β and s.

Proof. We may assume $\|u\|_{L^\infty(B_1)} = 1$. By (5.2), for all $x' \in B_r$ we have

$$\left|\phi_{2r}(x') - \phi_r(x')\right| \leq \left|u(x') - \phi_{2r}(x')\right| + \left|u(x') - \phi_r(x')\right| \leq CC_0 r^\beta.$$

But this happening for every $x' \in B_r$ yields, recalling (5.1),

$$\left|Q_*(2r) - Q_*(r)\right| \leq CC_0 r^{\beta-s}.$$

In addition, since $\|u\|_{L^\infty(B_1)} = 1$, we clearly have that

$$|Q_*(1)| \leq C. \tag{5.3}$$

Since $\beta > s$, this implies the existence of the limit

$$Q := \lim_{r \searrow 0} Q_*(r).$$

Moreover, using again $\beta - s > 0$,

$$\left|Q - Q_*(r)\right| \leq \sum_{m=0}^\infty \left|Q_*(2^{-m}r) - Q_*(2^{-m-1}r)\right| \leq \sum_{m=0}^\infty CC_0 2^{-m(\beta-s)} r^{\beta-s} \leq CC_0 r^{\beta-s}.$$

In particular, using (5.3) we obtain

$$|Q| \leq C(C_0 + 1). \tag{5.4}$$

We have thus proven that for all $r \in (0,1)$

$$\begin{aligned}
\left\|u - Q(x \cdot \nu)_+^s\right\|_{L^\infty(B_r)} &\leq \|u - Q_*(r)(x \cdot \nu)_+^s\|_{L^\infty(B_r)} + \\
&\quad + \|Q_*(r)(x \cdot \nu)_+^s - Q(x \cdot \nu)_+^s\|_{L^\infty(B_r)} \\
&\leq C_0 r^\beta + |Q_*(r) - Q| r^s \leq C(C_0 + 1) r^\beta.
\end{aligned}$$

$\square$

Second, we will also need the following estimate in order to control the "errors coming from the geometry of the domain".

Lemma 5.4. *Assume that B_1 is divided into two disjoint subdomains Ω_1 and Ω_2 such that $\overline{B_1} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. Assume that $\Gamma := \partial\Omega_1 \setminus \partial B_1 = \partial\Omega_2 \setminus \partial B_1$ is a $C^{0,1}$ surface and that $0 \in \Gamma$. Moreover assume that, for some $\rho_0 > 0$, all the points on $\Gamma \cap \overline{B_{3/4}}$ can be touched by a ball of radius $\rho_0 \in (0, 1/4)$ contained in Ω_2.*

Let $s \in (0,1)$, and let L be any operator of the form (1.1)-(1.2). Let $\alpha \in (0,1)$, $g \in C^\alpha(\overline{\Omega_2})$, $f \in L^\infty(\Omega_1)$, and $u \in C(\overline{B_1})$ satisfying $|u(x)| \leq M(1+|x|)^\beta$ in $\mathbb{R}^n$ for some $\beta < 2s$. Assume that u satisfies in the weak sense

$$Lu = f \ \text{ in } \ \Omega_1, \quad u = g \ \text{ in } \ \Omega_2.$$

Then, there is $\gamma \in (0, \alpha)$ such that $u \in C^\gamma(\overline{B_{1/2}})$ with the estimate

$$\|u\|_{C^\gamma(B_{1/2})} \leq C\big(\|u\|_{L^\infty(B_1)} + \|g\|_{C^\alpha(\Omega_2)} + \|f\|_{L^\infty(\Omega_1)} + M\big).$$

The constants C and γ depend only on n, s, α, ρ_0, β, and ellipticity constants.

Proof. Define $\tilde{u} = u\chi_{B_1}$. Then $\tilde{u}$ satisfies $L\tilde{u} = \tilde{f}$ in $\Omega_1 \cap B_{3/4}$ and $\tilde{u} = g$ in Ω_2, where $\|\tilde{f}\|_{L^\infty(\Omega_1 \cap B_{3/4})} \leq C(\|f\|_{L^\infty(\Omega_1)} + M) := C_0'$. The constant C depends only on n, s, β, and ellipticity constants.

The proof consists of two steps.

First step. We next prove that there are $\delta > 0$ and C such that for all $z \in \Gamma \cap \overline{B_{1/2}}$ it is

$$\|\tilde{u} - g(z)\|_{L^\infty(B_r(z))} \leq Cr^\delta \quad \text{for all } r \in (0, 1), \tag{5.5}$$

where δ and C depend only on n, s_0, C_0', $\|u\|_{L^\infty(B_1)}$, $\|g\|_{C^\alpha(\Omega_2)}$, and ellipticity constants.

Let $z \in \Gamma \cap \overline{B_{1/2}}$. By assumption, for all $R \in (0, \rho_0)$ there $y_R \in \Omega_2$ such that a ball $B_R(y_R) \subset \Omega_2$ touches Γ at z, i.e., $|z - y_R| = R$.

Let φ_1 and $\epsilon > 0$ be the supersolution and the constant in Lemma 4.4. Take

$$\psi(x) = g(y_R) + \|g\|_{C^\alpha(\Omega_2)}\big((1 + \epsilon)R\big)^\alpha + \big(C_0' + \|u\|_{L^\infty(B_1)}\big)\varphi_1\left(\frac{x - y_R}{R}\right).$$

Note that ψ is above $\tilde{u}$ in $\Omega_2 \cap B_{(1+\epsilon)R}$. On the other hand, from the properties of φ_1, it is $M^+\psi \leq -\big(C_0' + \|u\|_{L^\infty(B_1)}\big)R^{-2s} \leq -C_0'$ in the annulus $B_{(1+\epsilon)R}(y_R) \setminus B_R(y_R)$, while $\psi \geq \|u\|_{L^\infty(B_1)} \geq \tilde{u}$ outside $B_{(1+\epsilon)R}(y_R)$. It follows that $\tilde{u} \leq \psi$ and thus we have

$$\tilde{u}(x) - g(z) \leq C\big(R^\alpha + (r/R)^s\big) \quad \text{for all } x \in B_r(z) \quad \text{and for all } r \in (0, \epsilon R) \text{ and } R \in (0, \rho_0).$$

Here, C denotes a constant depending only on n, s_0, C_0', $\|u\|_{L^\infty(B_1)}$, $\|g\|_{C^\alpha(\Omega_2)}$, and ellipticity constants. Taking $R = r^{1/2}$ and repeating the argument up-side down we obtain

$$|\tilde{u}(x) - g(z)| \leq C\big(r^{\alpha/2} + r^{s/2}\big) \leq Cr^\delta \quad \text{for all } x \in B_r(z) \text{ and } r \in (0, \epsilon^{1/2})$$

for $\delta = \frac{1}{2}\min\{\alpha, s_0\}$. Taking a larger constant C, (5.5) follows.

Second step. We now show that (5.5) and the interior estimates in Theorem 1.1 (b) imply $\|u\|_{C^\gamma(B_{1/2})} \leq C$, where C depends only on the same quantities as above.

Indeed, given $x_0 \in \Omega_1 \cap B_{1/2}$, let $z \in \Gamma$ and $r > 0$ be such that

$$d = \text{dist}\,(x_0, \Gamma) = \text{dist}\,(x_0, z).$$

Let us consider

$$v(x) = \tilde{u}\left(x_0 + \frac{d}{2}x\right) - g(z).$$

We clearly have

$$\|v\|_{L^\infty(B_1)} \leq C \quad \text{and} \quad \|v\|_{L^\infty(B_R)} \leq CR^s \quad \text{for } R \geq 1.$$

On the other hand, v satisfies

$$Lv(x) = (d/2)^{2s}L\tilde{u}(x_0 + rx) \quad \text{in } B_1$$

and thus

$$\|Lv\|_{L^\infty(B_1)} \leq C_0' \quad \text{in } B_1.$$

Therefore, Corollary 3.6 yields

$$\|v\|_{C^\alpha(B_{1/2})} \le C$$

or equivalently

$$[u]_{C^\alpha(B_{d/4}(x_0))} \le Cd^{-\alpha}. \tag{5.6}$$

Combining (5.5) and (5.6), using the same argument as in the proof of Proposition 1.1 in [31], we obtain

$$\|u\|_{C^\gamma(\Omega_1 \cap B_{1/2})} \le C,$$

as desired.   □

Using the previous results, and a compactness argument in the spirit of the one in [31], we can give the:

Proof of Proposition 5.2. Assume that there are sequences Γ_k, Ω_k^+, Ω_k^-, f_k, u_k, and L_k that satisfy the assumptions of the proposition, that is,

- Γ_k is a $C^{1,1}$ hyper surface with radius ρ_0 splitting B_1 into Ω_k^+ and Ω_k^-;
- L_k is of the form (1.1) and satisfying (1.2);
- $\|u_k\|_{L^\infty(\mathbb{R}^n)} + \|f_k\|_{L^\infty(\Omega_k^+)} = 1$;
- u_k is a solution of $Lu_k = f_k$ in Ω_k^+ and $u_k = 0$ in Ω_k^-;

but suppose for a contradiction that the conclusion of the proposition does not hold. That is, for all $C > 0$, there are k and $z \in \Gamma_k \cap \overline{B_{1/2}}$ for which no constant $Q \in \mathbb{R}$ satisfies

$$\left| u_k(x) - Q\big((x-z)\cdot\nu_k(z)\big)_+^s \right| \le C|x-z|^\beta \quad \text{for all } x \in B_1. \tag{5.7}$$

Here, $\nu_k(z)$ denotes the unit normal vector to Γ_k at z, pointing towards Ω_k^+.

In particular, using Lemma 5.3,

$$\sup_k \sup_{z \in \Gamma_k \cap B_{1/2}} \sup_{r>0} r^{-\beta} \|u_k - \phi_{k,z,r}\|_{L^\infty(B_r(z))} = \infty, \tag{5.8}$$

where

$$\phi_{k,z,r}(x) = Q_{k,z}(r)\big((x-z)\cdot\nu_k(z)\big)_+^s \tag{5.9}$$

and

$$Q_{k,z}(r) := \arg\min_{Q\in\mathbb{R}} \int_{B_r(z)} \left| u_k(x) - Q\big((x-z)\cdot\nu_k(z)\big)_+^s \right|^2 dx$$

$$= \frac{\int_{B_r(z)} u_k(x)\big((x-z)\cdot\nu_k(z)\big)_+^s dx}{\int_{B_r(z)} \big((x-z)\cdot\nu_k(z)\big)_+^{2s} dx}.$$

Next define the monotone in r quantity

$$\theta(r) := \sup_k \sup_{z \in \Gamma_k \cap B_{1/2}} \sup_{r'>r} (r')^{-\beta} \|u_k - \phi_{k,z,r'}\|_{L^\infty(B_{r'}(x_0))}.$$

We have $\theta(r) < \infty$ for $r > 0$ and $\theta(r) \nearrow \infty$ as $r \searrow 0$. Clearly, there are sequences $r_m \searrow 0$, k_m, and $z_m \to z \in \overline{B}_{1/2}$, for which

$$(r_m)^{-\beta} \|u_{k_m} - \phi_{k_m, z_m, r_m}\|_{L^\infty(B_{r_m}(x_m))} \geq \theta(r_m)/2. \tag{5.10}$$

From now on in this proof we denote $\phi_m = \phi_{k_m, z_m, r_m}$ and $\nu_m = \nu_{k_m}(z_m)$.

In this situation we consider

$$v_m(x) = \frac{u_{k_m}(z_m + r_m x) - \phi_m(z_m + r_m x)}{(r_m)^\beta \theta(r_m)}.$$

Note that, for all $m \geq 1$,

$$\int_{B_1} v_m(x)(x \cdot \nu_m)_+^s \, dx = 0. \tag{5.11}$$

This is the optimality condition for least squares.

Note also that (5.10) is equivalent to

$$\|v_m\|_{L^\infty(B_1)} \geq 1/2, \tag{5.12}$$

which holds for all $m \geq 1$.

In addition, for all k and z we have

$$|Q_{k,z}(2r) - Q_{k,z}(r)| \leq r^{\beta - s} \theta(r).$$

Indeed,

$$\begin{aligned}
|Q_{k,z}(2r) - Q_{k,z}(r)|r^s &= \|\phi_{k,z,2r} - \phi_{k,z,r}\|_{L^\infty(B_r(z))} \\
&\leq \|\phi_{k,z,2r} - u\|_{L^\infty(B_{2r}(z))} + \|u - \phi_{k,z,r}\|_{L^\infty(B_r(z))} \\
&\leq (2r)^\beta \theta(r) + r^\beta \theta(r) = Cr^\beta \theta(r).
\end{aligned}$$

Thus, for $R = 2^N$ we have

$$\begin{aligned}
\frac{r^{s-\beta}|Q_{k,z}(rR) - Q_{k,z}(r)|}{\theta(r)} &\leq \sum_{j=0}^{N-1} 2^{j(\beta - s)} \frac{(2^j r)^{s-\beta}|Q_{k,z}(2^{j+1}r) - Q_{k,z}(2^j r)|}{\theta(r)} \\
&\leq C \sum_{j=0}^{N-1} 2^{j(\beta - s)} \frac{\theta(2^j r))}{\theta(r)} \leq C 2^{N(\beta - s)} = CR^{\beta - s},
\end{aligned}$$

where we have used $\beta > s$.

Moreover, we have

$$\|v_m\|_{L^\infty(B_R)} = \frac{1}{\theta(r_m)(r_m)^\beta}\big\|u_{k_m} - Q_{k_m,z_m}(r_m)\big((x-z_m)\cdot\nu_m\big)_+^{s_m}\big\|_{L^\infty(B_{r_m R})}$$

$$\leq \frac{R^\beta}{\theta(r_m)(r_m R)^\beta}\big\|u_{k_m} - Q_{k_m,z_m}(r_m R)\big((x-z_m)\cdot\nu_m\big)_+^{s_m}\big\|_{L^\infty(B_{r_m R})} +$$

$$+ \frac{1}{\theta(r_m)(r_m)^\beta}\big|Q_{k_m,z_m}(r_m R) - Q_{k_m,z_m}(r_m)\big|(r_m R)^{s_m}$$

$$\leq \frac{R^\beta\theta(r_m R)}{\theta(r_m)} + CR^\beta,$$

and hence v_m satisfy the growth control

$$\|v_m\|_{L^\infty(B_R)} \leq CR^\beta \quad \text{for all } R \geq 1. \tag{5.13}$$

We have used the definition $\theta(r)$ and its monotonicity.

Now, without loss of generality (taking a subsequence), we assume that

$$\nu_m \longrightarrow \nu \in S^{n-1}.$$

Then, the rest of the proof consists mainly in showing the following Claim.

Claim. *A subsequence of v_m converges locally uniformly in $\mathbb{R}^n$ to some function v which satisfies $Lv = 0$ in $\{x \cdot \nu > 0\}$ and $v = 0$ in $\{x \cdot \nu < 0\}$, for some L of the form* (1.1) *satisfying* (1.2).

Once we know this, a contradiction is immediately reached using the Liouville type Theorem 4.1, as seen at the end of the proof.

To prove the Claim, given $R \geq 1$ and m such that $r_m R < 1/2$ define

$$\Omega_{R,m}^+ = \big\{x \in B_R : (z_m + r_m x) \in \Omega_{k_m}^+ \quad \text{and} \quad x \cdot \nu_m(z_m) > 0\big\}.$$

Notice that for all R and k, the origin 0 belongs to the boundary of $\Omega_{R,m}^+$.

We will use that v_m satisfies an elliptic equation in $\Omega_{R,m}^+$. Namely,

$$L_{k_m} v_m(x) = \frac{(r_m)^{2s}}{(r_m)^\beta\theta(r_m)} f_{k_m}(z_m + r_m x) \quad \text{in } \Omega_{R,m}^+. \tag{5.14}$$

This follows from the definition of v_m and the fact that $L_{k_m}\phi_m = 0$ in $\{(x-z)\cdot\nu_m > 0\}$.

Notice that the right hand side of (5.14) converges uniformly to 0 as $r_m \searrow 0$, since $\beta < 2s$ and $\theta(r_m) \uparrow \infty$.

In order to prove the convergence of a subsequence of v_m, we first obtain, for every fixed $R \geq 1$, a uniform in m bound for $\|v_m\|_{C^\delta(B_R)}$, for some small $\delta > 0$. Then the local uniform convergence of a subsequence of v_m follows from the Arzelà-Ascoli theorem.

Let us fix $R \geq 1$ and consider that m is always large enough so that $r_m R < 1/4$.

Let Σ_m^- be the half space which is "tangent" to $\Omega_{k_m}^-$ at z_m, namely,

$$\Sigma_m^- := \big\{(x - z_m) \cdot \nu(z_m) < 0\big\}.$$

The first step is showing that, for all m and for all $r < 1/4$,

$$\big\|u_{k_m} - \phi_m\big\|_{L^\infty\big(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)\big)} \leq Cr^{2s} \leq Cr^{2s} \qquad (5.15)$$

for some constant C depending only on s, ρ_0, ellipticity constants, and dimension.

Indeed, we may rescale and slide the supersolution φ_1 from Lemma 4.4 and use the fact that all points of $\Gamma_{k_m} \cap B_{3/4}$ can be touched by balls of radius ρ_0 contained in $\Omega_{k_m}^-$. We obtain that

$$|u_{k_m}| \leq C\big(\text{dist}\,(x, \Omega_{k_m}^-)\big)^s,$$

with C depending only on n, s, ρ_0, and ellipticity constants. On the other hand, by definition of ϕ_m we have

$$|\phi_m| \leq C\big(\text{dist}\,(x, \Sigma_m^-)\big)^s.$$

But by assumption, points on $\Gamma_k \cap B_{3/4}$ can be also touched by balls of radius ρ_0 from the $\Omega_{k_m}^+$ side, and hence we have a quadratic control (depending only on ρ_0) on on how Γ_{k_m} separates from the hyperplane $\partial\Sigma_m^-$. As a consequence, in $B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)$ we have

$$C\big(\text{dist}\,(x, \Omega_{k_m}^-)\big)^s \leq Cr^{2s} \quad \text{and} \quad C\big(\text{dist}\,(x, \Sigma_m^-)\big)^s \leq Cr^{2s}.$$

Hence, (5.15) holds.

We use now Lemma 5.4 to obtain that, for some small $\gamma \in (0, s)$,

$$\|u_{k_m}\|_{C^\gamma(B_{1/8}(z_m))} \leq C \quad \text{for all } m.$$

On the other hand, clearly

$$\|\phi_m\|_{C^\gamma(B_{1/8}(z_m))} \leq C \quad \text{for all } m.$$

Hence,

$$\big\|u_{k_m} - \phi_m\big\|_{C^\gamma\big(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)\big)} \leq C. \qquad (5.16)$$

Next, interpolating (5.15) and (5.16) we obtain, for some positive $\delta < \gamma$ small enough (depending on γ, s, and β),

$$\big\|u_{k_m} - \phi_m\big\|_{C^\delta\big(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)\big)} \leq Cr^\beta. \qquad (5.17)$$

Therefore, scaling (5.17) we find that

$$\big\|v_m\big\|_{C^\delta\big(B_R \setminus \Omega_{R,m}^+\big)} \leq C \quad \text{for all } m \text{ with } r_m R < 1/4. \qquad (5.18)$$

Next we observe that the boundary points on $\partial\Omega_{R,m}^+ \cap B_{3R/4}$ can be touched by balls of radius $(\rho_0/r_m) \geq \rho_0$ contained in $B_R \setminus \Omega_{R,m}^+$. We then apply Lemma 5.4 (rescaled) to v_m. Indeed, we have that v_m solves (5.14) and satisfies (5.18). Thus, we obtain, for some $\delta' \in (0, \delta)$,

$$\big\|v_m\big\|_{C^{\delta'}(B_{R/2})} \leq C(R), \quad \text{for all } m \text{ with } r_m R < 1/4, \qquad (5.19)$$

where we write $C(R)$ to emphasize the dependence on R of the constant, which also depends on s, ρ_0, ellipticity constants, and dimension, but not on m.

As said above, the Arzelà-Ascoli theorem and the previous uniform (in m) $C^{\delta'}$ estimate (5.19) yield the local uniform convergence in $\mathbb{R}^n$ of a subsequence of v_m to some function v.

In addition, by Lemma 3.1 there is a subsequence of L_{k_m} which converges weakly to some operator L, which is of the form (1.1) and satisfies (1.2). Hence, it follows that $Lv = 0$ in all of $\mathbb{R}^n$, and thus the Claim is proved.

Finally, passing to the limit the growth control (5.13) on v_m we find $\|v\|_{L^\infty(B_R)} \leq R^\beta$ for all $R \geq 1$. Hence, by Theorem 4.1, it must be

$$v(x) = K\big(x \cdot \nu(z)\big)_+^s.$$

Passing (5.11) to the limit, we find

$$\int_{B_1} v(x)\big(x \cdot \nu(z)\big)_+^s \, dx = 0.$$

But passing (5.12) to the limit, we reach the contradiction. Thus, the Proposition is proved. □

Before giving the proof of Theorem 1.2, we prove the following.

Lemma 5.5. *Let Γ be a $C^{1,1}$ surface of radius $\rho_0 > 0$ splitting B_1 into U^+ and U^-; see Definition 5.1. Let $d(x) = \mathrm{dist}\,(x, U^-)$. Let $x_0 \in B_{1/2}$ and $z \in \Gamma$ be such that*

$$\mathrm{dist}\,(x_0, \Gamma) = \mathrm{dist}\,(x_0, z) =: 2r.$$

Then,

$$\left\|\big((x - z) \cdot \nu(z)\big)_+^s - d^s(x)\right\|_{L^\infty(B_r(x_0))} \leq Cr^{2s}, \tag{5.20}$$

$$\left[d^s - \big((x - z) \cdot \nu(z)\big)_+^s\right]_{C^{s-\epsilon}(B_r(x_0))} \leq Cr^s, \tag{5.21}$$

and

$$\left[d^{-s}\right]_{C^{s-\epsilon}(B_r(x_0))} \leq Cr^{-2s+\epsilon}. \tag{5.22}$$

The constant C depends only on ρ_0.

Proof. Let us denote

$$\bar{d}(x) = \big((x - z) \cdot \nu(z)\big)_+.$$

First, since Γ is $C^{1,1}$ with curvature radius bounded below by ρ_0, we have that $|\bar{d} - d| \leq Cr^2$ in $B_r(x_0)$, and thus (5.20) follows.

To prove (5.21) we use on the one hand that

$$\left\|\nabla d - \nabla \bar{d}\right\|_{L^\infty(B_r(x_0))} \leq Cr, \tag{5.23}$$

which also follows from the fact that Γ is $C^{1,1}$. On the other hand, using the inequality $|a^{s-1} - b^{s-1}| \le |a - b| \max\{a^{s-2}, b^{s-2}\}$ for $a, b > 0$, we find

$$\left\| d^{s-1} - \bar{d}^{s-1} \right\|_{L^\infty(B_r(x_0))} \le Cr^2 \max \left\{ \left\| d^{s-2} \right\|_{L^\infty(B_r(x_0))}, \left\| \bar{d}^{s-2} \right\|_{L^\infty(B_r(x_0))} \right\} \le Cr^s. \tag{5.24}$$

Thus, using (5.23) and (5.24), we deduce

$$\left[d^s - \bar{d}^s \right]_{C^{0,1}(B_r(x_0))} = \left\| d^{s-1}\nabla d - \bar{d}^{s-1}\nabla \bar{d} \right\|_{L^\infty(B_r(x_0))} \le Cr^s.$$

Therefore, (5.21) follows.

Finally, interpolating the inequalities

$$\left[d^{-s} \right]_{C^{0,1}(B_r(x_0))} = \left\| d^{-s-1}\nabla d \right\|_{L^\infty(B_r(x_0))} \le Cr^{-s-1} \quad \text{and} \quad \left\| d^{-s} \right\|_{L^\infty(B_r(x_0))} \le Cr^{-s},$$

(5.22) follows. $\square$

We can finally give the

Proof of Theorem 1.2. First, by Proposition 4.4, we have $\|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)}$. We may assume that

$$\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)} \le 1.$$

Let us pick any point on $\partial\Omega$, and let us see that u/d^s is $C^{s-\epsilon}$ around this point. Rescaling and translating Ω if necessary, we may assume that $0 \in \partial\Omega$, and that the sets $U^+ = \Omega \cap B_1$ and $U^- = B_1 \setminus \Omega$ satisfy the conditions in Definition 5.1 (with $\Gamma = B_1 \cap \partial\Omega$).

Then, by Proposition 5.2 we have that, for all $z \in \Gamma \cap \overline{B_{1/2}}$, there is $Q = Q(z)$ such that

$$|Q(z)| \le C \quad \text{and} \quad \|u - Q\left((x - z)\cdot\nu(z)\right)_+^s\|_{L^\infty(B_R(z))} \le CR^{2s-\epsilon} \tag{5.25}$$

for all $R > 0$, where C depends only on n, s, ρ_0, ϵ, and ellipticity constants.

Now, to prove the $C^{s-\epsilon}$ estimate up to the boundary for u/d^s we must combine a C^s interior estimate for u with (5.25).

Let x_0 be a point in $\Omega^+ \cap B_{1/4}$, and let $z \in \Gamma$ be such that

$$2r := \text{dist}\,(x_0, \Gamma) = \text{dist}\,(x_0, z) < \rho_0.$$

Note that $B_r(x_0) \subset B_{2r}(x_0) \subset \Omega^+$ and that $z \in \Gamma \cap B_{1/2}$ (since $0 \in \Gamma$).

We claim now that there is $Q = Q(x_0)$ such that $|Q(x_0)| \le C$,

$$\|u - Qd^s\|_{L^\infty(B_r(x_0))} \le Cr^{2s-\epsilon}, \tag{5.26}$$

and

$$[u - Qd^s]_{C^{s-\epsilon}(B_r(x_0))} \le Cr^s, \tag{5.27}$$

where the constant C depends only on n, s, ϵ, ρ_0, and ellipticity constants.

Indeed, (5.26) follows immediately combining (5.25) and (5.20).

To prove (5.27), let

$$v_r(x) = r^{-s}u(z + rx) - Q\left(x\cdot\nu(z)\right)_+^s.$$

Then, (5.25) implies

$$\|v_r\|_{L^\infty(B_4)} \le Cr^{s-\epsilon}$$

and

$$\|v_r\|_{L^\infty(B_R)} \le Cr^{s-\epsilon}R^s.$$

Moreover, v_r solves the equation

$$Lv_r = r^s f(z + rx) \quad \text{in } B_2(\tilde{x}_0),$$

where $\tilde{x}_0 = (x_0 - z)/r$ satisfies $|\tilde{x}_0 - z| = 2$. Hence, using the interior estimate in Corollary 3.6 we obtain $[v_r]_{C^{s-\epsilon}(B_1(\tilde{x}_0))} \le Cr^{s-\epsilon}$. This yields that

$$r^{s-\epsilon}\left[u - Q\left((x-z)\cdot\nu(z)\right)_+^s\right]_{C^{s-\epsilon}(B_r(x_0))} = r^s[v]_{C^{s-\epsilon}(B_1(\tilde{x}_0))} \le Cr^s r^{s-\epsilon}.$$

Therefore, using (5.21), (5.27) follows.

Let us finally show that (5.26)-(5.27) yield the desired result. Indeed, note that, for all x_1 and x_2 in $B_r(x_0)$,

$$\frac{u}{d^s}(x_1) - \frac{u}{d^s}(x_2) = \frac{(u - Qd^s)(x_1) - (u - Qd^s)(x_2)}{d^s(x_1)} + (u - Qd^s)(x_2)(d^{-s}(x_1) - d^{-s}(x_2)).$$

By (5.27), and using that d is comparable to r in $B_r(x_0)$, we have

$$\frac{\left|(u - Qd^s)(x_1) - (u - Qd^s)(x_2)\right|}{d^s(x_1)} \le C|x_1 - x_2|^{s-\epsilon}.$$

Also, by (5.26) and (5.22),

$$\left|u - Qd^s\right|(x_2)\left|d^{-s}(x_1) - d^{-s}(x_2)\right| \le C|x_1 - x_2|^{s-\epsilon}.$$

Therefore,

$$[u/d^s]_{C^{s-\epsilon}(B_r(x_0))} \le C.$$

From this, we obtain the desired estimate for $\|u/d^s\|_{C^{s-\epsilon}(\Omega^+ \cap B_{1/2})}$ by summing a geometric series, as in the proof of Proposition 1.1 in [30]. $\square$

6. Final comments and remarks

Even for the fractional Laplacian, all the interior regularity results are sharp; see for example Section 7 in [3]. The only difference between Theorem 1.1(b) and the classical interior estimate for the fractional Laplacian is that we need to assume that $u \in C^\alpha(\mathbb{R}^n)$ in order to have a $C^{\alpha+2s}$ estimate in $B_{1/2}$. We show here that this assumption is in fact necessary.

Proposition 6.1. *Let $s \in (0,1)$, and let L be the operator in $\mathbb{R}^2$ given by (1.4). Let $\alpha \in (0, s]$, and $\epsilon > 0$ small.*

Then, there exists a function u satisfying:

(i) *$Lu = 0$ in B_1*
(ii) *$u \in C^{\alpha-\epsilon}(\mathbb{R}^n)$*
(iii) *$u \equiv 0$ in $B_2 \setminus B_1$*

(iv) $u \notin C^{\alpha+2s}(B_{1/2})$

This means that in Theorem $(1.1)(b)$ the $C^\alpha(\mathbb{R}^n)$ norm on the right hand side can not be removed.

Concerning our boundary regularity result, we also expect it to be sharp for general stable operators (1.1)-(1.2). Indeed, while for the fractional Laplacian (and for any operator (1.3) with $a \in C^\infty(S^{n-1})$) one has that $(-\Delta)^s(d^s)$ is $C^\infty(\overline{\Omega})$ whenever Ω is C^∞ (see [18]), in this case we have the following.

Proposition 6.2. *There exists an operator of the form* (1.1)-(1.2) *and a* C^∞ *bounded domain* $\Omega \subset \mathbb{R}^n$ *for which*
$$L(d^s) \notin L^\infty(\Omega),$$
where $d(x)$ *a* C^∞ *function satisfying* $d \equiv 0$ *in* $\mathbb{R}^n \setminus \Omega$, *and that coincides with* $\mathrm{dist}(x, \mathbb{R}^n \setminus \Omega)$ *in a neighborhood of* $\partial\Omega$.

As a consequence of the previous example, we do not expect the estimates in Theorem 1.2 to hold at order s. In other words, we do not expect u/d^s to be $C^s(\overline{\Omega})$.

We next show Propositions 6.1 and 6.2.

Proof of Proposition 6.1. Let
$$u_0(x) = (x_1)_+^{\alpha-\epsilon}\eta(x),$$
where $\eta \in C_c^\infty(B_2(p))$, $p = (0,4)$, and $\eta \equiv 1$ in $B_1(p)$. Let u be the solution to
$$\begin{cases} Lu &= 0 & \text{in } B_1 \\ u &= u_0 & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$
Then, u clearly satisfies (i), (ii), (iii).

Let us show next that $u \notin C^{\alpha+2s}(B_{1/2})$ by contradiction. Assume $u \in C^{\alpha+2s}(B_{1/2})$, and define $u_1 = u\chi_{B_1}$, and $u_2 = u - u_1$. Notice that $u_1 \in C^\alpha(\mathbb{R}^n)$ (by Proposition 4.4, since $Lu_1 = -Lu_2 \in L^\infty(B_1)$ and $\alpha \leq s$) and $u_1 \in C^{\alpha+2s}(B_{1/2})$ (by Theorem 1.1). Thus, we have $Lu_1 \in C^\alpha(B_{1/4})$. Therefore, we also have
$$Lu_2 \in C^\alpha(B_{1/4}).$$
since $Lu_2 = -Lu_1$ in B_1.

Now recall that
$$Lw(a,b) = \int_{-\infty}^{\infty} \frac{w(a,b) - w(a,b+t)}{|t|^{1+2s}}\, dt + \int_{-\infty}^{\infty} \frac{w(a,b) - w(a+t,b)}{|t|^{1+2s}}\, dt.$$
Hence, taking the points $x_1 = (0,0)$ and $x_2 = (\delta, 0)$, with $\delta > 0$ small, we have
$$Lu_2(x_1) - Lu_2(x_2) = \int_{-\infty}^{\infty} \frac{u_2(\delta, t) - u_2(0,t)}{|t|^{1+2s}}\, dt,$$
where we have used that u_2 has support in $B_2(p)$. Also, $u_2(0,t) = 0$ for all t, and hence
$$Lu_2(x_1) - Lu_2(x_2) = \int_{-\infty}^{\infty} \frac{u_2(\delta, t)}{|t|^{1+2s}}\, dt > \int_3^4 \frac{C\delta^{\alpha-\epsilon}}{|t|^{1+2s}}\, dt = c\delta^{\alpha-\epsilon}.$$

Therefore,
$$\frac{Lu_2(x_1) - Lu_2(x_2)}{|x_1 - x_2|^\alpha} > c\delta^{-\epsilon},$$
and hence $Lu_2 \notin C^\alpha(B_{1/4})$, a contradiction. $\square$

We finally give the

Proof of Proposition 6.2. We take Ω to coincide with $\tilde{\Omega} = \{x \in \mathbb{R}^n : |x| > 1\}$ in a neighborhood of $z_0 = (1, 0, ..., 0)$. Then, in a neighborhood of x_0, we have $d^s(x) = (|x| - 1)^s$.

We will show that $L(d^s)$ is not bounded in a neighborhood of z_0. Equivalently, we will show that Lu is not bounded in a neighborhood of z_0, where
$$u(x) = (|x|^2 - 1)^s \eta(x),$$
where η is a smooth function satisfying $\eta \equiv 1$ in $B_\delta(z_0)$ and $\eta \equiv 0$ outside $B_{2\delta}(z_0)$, where $\delta > 0$ is small enough so that $\partial\Omega$ coincides with $\partial\tilde{\Omega}$ in $B_{2\delta}(z_0)$.

We claim that Lu is bounded if and only if $L(d^s)$ is bounded, because the quotient of these two functions is $C^\infty(\overline{\Omega})$. Indeed, let w be any $C^\infty(\mathbb{R}^n)$ extension of $u/d^s|_\Omega$. Then, we have
$$Lu = L(d^s w) = wL(d^s) + d^s Lw - I_L(d^s, w),$$
where I_L is the bilinear form associated to the operator L. Now, w is C^∞ and d^s is C^s, it turns out that Lw and $I_L(d^s, w)$ belong to $L^\infty(\Omega)$. Hence, using that w is bounded by above and below by positive constants, we find that
$$Lu \in L^\infty(\Omega) \iff L(d^s) \in L^\infty(\Omega),$$
as claimed.

Notice now that, since u is bounded at infinity, then to prove the boundedness of $Lu(x)$ it is only important the values of u in a neighborhood of x.

Let $x = (x_1, x')$, with $x' \in \mathbb{R}^{n-1}$. Let us restrict the function d^s to the hyperplane $\{x_1 = 1 + r\}$, with $r > 0$ very small. We find that
$$u(1 + r, x') = \left((1 + r)^2 + |x'|^2 - 1\right)^s \eta(1 + r, x') = (2r + r^2 + |x'|^2)^s \eta(1 + r, x')$$
$$= r^s \left(2 + r + \left|\frac{x'}{\sqrt{r}}\right|^2\right)^s \eta(1 + r, x').$$

Thus, if we choose $L = L_1 + L_2$, with L_1 being the $(n-1)$-dimensional fractional Laplacian in the $(x_2, ..., x_n)$ variables, and L_2 the 1-dimensional fractional Laplacian in the x_1-variable, we find that
$$Lu(1 + r, 0, ..., 0) = L_1 v^{(r)}(0) + L_2 v_2(r),$$
where $v^{(r)}(x') = r^s \left(2 + r + \left|\frac{x'}{\sqrt{r}}\right|^2\right)^s \eta(1 + r, x')$, and $v_2(r) = r_+^s + (-1 - r)_+^s$. Since $L_2[(r_+)^s] = 0$, then $L_2 v_2(r)$ is bounded for $r > 0$. Thus, to prove that Lu is not bounded in Ω it suffices to show that $L_1 v^{(r)}(0) \to \infty$ as $r \downarrow 0$.

But, defining

$$\tilde{v}^{(r)}(y) = \left(2 + r + |y|^2\right)^s \eta(1 + r, ry)$$

we have that

$$L_1 v^{(r)}(0) = L_1 \tilde{v}^{(r)}(0).$$

Finally, as $r \downarrow 0$, we have that $\eta(1 + r, ry)$ converges to the constant function 1 in all of $\mathbb{R}^n$, and hence it is immediate to see that

$$\lim_{r \to 0} L_1 \tilde{v}^{(r)}(0) = \infty,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

References

[1] R. Bass, Z.-H. Chen, *Regularity of Harmonic functions for a class of singular stable-like processes*, Math. Z. 266 (2010), 489-503.

[2] R. Bass, D. Levin, *Harnack inequalities for jump processes*, Potential Anal. 17 (2002), 375–388.

[3] R. Bass, *Regularity results for stable-like operators*, J. Funct. Anal. 257 (2009), 2693–2722.

[4] J. Bertoin, *Lévy Processes*, Cambridge Tracts in Mathematics, 121. Cambridge University Press, Cambridge, 1996.

[5] C. Bjorland, L. Caffarelli, A. Figalli, *Non-Local Gradient Dependent Operators*, Adv. Math. 230 (2012), 1859-1894.

[6] K. Bogdan, P. Sztonyk, *Harnack's inequality for stable Lévy processes*, Potential Anal. 22 (2005), 133-150.

[7] K. Bogdan, P. Sztonyk, *Estimates of the potential kernel and Harnack's inequality for the anisotropic fractional Laplacian*, Studia Math. 181 (2007), 101–123.

[8] T. Byczkowski, J. P. Nolan, B. Rajput, *Approximation of multidimensional stable densities*, J. Multivariate Analysis 46 (1993), 13-31.

[9] L. Caffarelli, L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. 62 (2009), 597-638.

[10] W. Chen, C. Li, L. Zhang, T. Cheng, *A Liouville Theorem for α-harmonic functions in $\mathbb{R}_+^n$*, preprint arXiv (Sep. 2014).

[11] B. N. Cheng, S. T. Rachev, *Multivariate stable future prices*, Math. Finance 5 (1995), 133-153.

[12] J. Dziubanski, *Asymptotic behaviour of densities of stable semigroups of measures*, Probab. Theory Related Fields 87 (1991), 459-467.

[13] M. M. Fall, *Entire s-harmonic functions are affine*, Proc. Amer. Math. Soc., to appear.

[14] M. M. Fall, T. Weth, *Liouville theorems for a general class of nonlocal operators*, preprint arXiv (Apr. 2015).

[15] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations Of Second Order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977.

[16] P. Glowacki, W. Hebisch, *Pointwise estimates for densities of stable semigroups of measures*, Studia Math. 104 (1993), 243-258.

[17] G. Grubb, N. J. Kokholm, *A global calculus of parameter-dependent pseudodifferential boundary problems in L_p Sobolev spaces*, Acta Math. 171 (1993), 165-229.

[18] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators*, Adv. Math. 268 (2015), 478-528.

[19] G. Grubb, *Local and nonlocal boundary conditions for μ-transmission and fractional elliptic pseudodifferential operators*, Anal. PDE 7 (2014), 1649-1682.

[20] A. Janicki, A. Weron, *Simulation And Chaotic Behavior Of α-Stable Stochastic Processes*, Monographs and Textbooks in Pure and Applied Mathematics, 178 (1994), New York.

[21] M. Kassmann, A. Mimica, *Analysis of jump processes with nondegenerate jumping kernels*, Stoch. Process. Appl. 123 (2013), 629-650.

[22] M. Kassmann, M. Rang, R. W. Schwab, *Hölder regularity for integro-differential equations with nonlinear directional dependence*, Indiana Univ. Math. J., to appear.

[23] M. Kassmann, R. W. Schwab, *Regularity results for nonlocal parabolic equations*, Riv. Mat. Univ. Parma 5 (2014), 183-212.

[24] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer, New York, 1972.

[25] P. I. Lizorkin, *Multipliers of Fourier integrals in the space L_p*, Trudy Mat. Inst. Steklov 89 (1967), 247-267; English translation in Proc. Steklov Inst. Math. 89 (1967), 269-290.

[26] R. N. Mantegna, H. E. Stanley, *Scaling behaviour in the dynamics of an economic index*, Nature 376 (1995), 46-49.

[27] J. P. Nolan, *Multivariate Stable Distributions: Approximation, Estimation, Simulation and Identification*, In R. J. Adler, R. E. Feldman, and M. S. Taqqu (Eds.), A Practical Guide to Heavy Tails, 509-526. Boston: Birkhauser (1998).

[28] J. P. Nolan, *Fitting Data and Assessing Goodness-of-fit with stable distributions*, Applications of Heavy Tailed Distributions in Economics, Engineering and Statistics, Washington DC, (1999).

[29] M. Pivato, L. Seco, *Estimating the spectral measure of a multivariate stable distribution via spherical harmonic analysis*, J. Multivar. Anal. 87 (2003), 219-240.

[30] X. Ros-Oton, J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. 101 (2014), 275-302.

[31] X. Ros-Oton, J. Serra, *Boundary regularity for fully nonlinear integro-differential equations*, Duke Math. J., to appear.

[32] G. Samorodnitsky, M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models With Infinite Variance*, Chapman and Hall, New York, 1994.

[33] R. W. Schwab, L. Silvestre, *Regularity for parabolic integro-differential equations with very irregular kernels*, Anal. PDE, to appear.

[34] J. Serra, *Regularity for fully nonlinear nonlocal parabolic equations with rough kernels*, Calc. Var. Partial Differential Equations 54 (2015), 615-629.

[35] J. Serra, *$C^{\sigma+\alpha}$ regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels*, Calc. Var. Partial Differential Equations, to appear.

[36] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. 60 (2007), 67-112.

[37] L. Silvestre, *Hölder estimates for solutions of integro differential equations like the fractional Laplacian*, Indiana Univ. Math. J. 55 (2006), 1155-1174.

[38] E. Stein, *Singular Integrals And Differentiability Properties Of Functions*, Princeton Mathematical Series No. 30, 1970.

[39] P. Sztonyk, *Regularity of harmonic functions for anisotropic fractional Laplacians*, Math. Nachr. 283 (2010), 289-311.

[40] T. Watanabe, *Asymptotic estimates of multi-dimensional stable densities and their applications*, Trans. Amer. Math. Soc. 359 (2007), 2851–2879.

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